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## On the effective action of chiral $W_3$ gravity \*

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We perform a perturbative analysis of the effective action, as defined in the paper, of quantum chiral  $W_3$  gravity. The results are given as a  $1/c$  expansion, where  $c$  is the central charge of the matter system that drives the  $W_3$  gravity dynamics. Certain nonlocalities in the Ward identities for the induced action, which we found previously, are cancelled by 1-loop contributions from propagating spin-2 and spin-3 gauge fields. On the basis of these results, we propose an exact all-order result for the effective action.

Theories of (quantum)  $D = 2$  W-gravity are believed to be integrable in the same sense in which  $D = 2$  Wess–Zumino–Witten (WZW) models and  $D = 2$  gravity can be viewed as integrable theories [1]. One would therefore expect that it should be possible to obtain exact, all-order results for some of the quantities of interest in these quantum field theories. One such quantity is the effective action, which is related to the generating functional of connected Green functions of the quantum gauge fields by a Legendre transformation. Indeed, both for WZW models and pure gravity the effective action is explicitly known [1].

The defining current algebra of a theory of W-gravity with a finite number of higher spins present is non-linear. Due to this, the issue of integrability becomes much more subtle. This became apparent from the results obtained in ref. [2] for the *induced* action of  $W_3$  gravity. As defined below, the induced action is an intermediate result of the fully quantized theory. It takes full account of the quantum matter fields, but it treats the  $W_3$  gauge fields as classical external sources. This induced action is controlled by a Ward Identity (WI), which takes the form of an anomalous conservation law for the quantum effective currents  $T_{\text{eff}}$  and  $W_{\text{eff}}$ . In WZW models and in pure gravity, the corresponding WIs are local linear differential equations which can be solved explicitly. In the  $c \rightarrow \pm\infty$  limit of  $W_3$  gravity the WIs are local, albeit non-linear differential equations, which can again

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be solved explicitly by constraining an  $Sl(3, \mathbb{R})$  gauge theory, see ref. [3]. However, as was shown in ref. [2], the locality of the  $W_3$  WIs for the induced action is spoiled by certain order- $1/c$  corrections to the  $c \rightarrow \pm\infty$  results. (The simplest non-localities arise at the level of four external spin-3 fields, and correspond to 3-loop Feynman diagrams if one represents the  $W_3$  currents  $T$  and  $W$  in terms of scalar matter fields.) Thus it seems that in  $W_3$  gravity, contrary to WZW models and  $D = 2$  pure gravity, the induced action cannot be determined explicitly. The WI for the induced action of  $w_\infty$  gravity [4], which is related to the quantum algebra  $W_\infty$ , is local and linear and does not show the complications one finds for  $W_N$  with finite  $N$ .

In this paper we will argue that the obstruction for finding an exact result for the *induced* action for  $W_3$  gravity is automatically resolved if one considers the full *effective* action of the theory! In particular, we will show that the non-localities found in the original WIs for the induced action do not carry over to the WIs for the effective action since they are precisely cancelled by  $1/c$  corrections coming from the quantization of the  $W_3$  gauge fields. Moreover, we find evidence that these local WIs for the effective action at finite  $c$  are related to the local WIs for the induced action at  $c \rightarrow \pm\infty$  by a finite renormalization of some constants, which are the level  $k = c/24 + \dots$  and  $Z$ -factors for the  $W_3$  source fields. As was mentioned above and explicitly worked out in [3], the latter WIs can be solved explicitly.

We now turn to a derivation of these results. Our starting point are (abstract) conserved currents  $T$  and  $W$ , satisfying  $\bar{\partial}T = \bar{\partial}W = 0$ , which obey the Operator Product Expansions (OPEs) of the quantum  $W_3$  algebra [5]

$$\begin{aligned}
 T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots, \\
 T(z)W(w) &= \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{(z-w)} + \dots, \\
 W(z)W(w) &= \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
 &\quad + \frac{1}{(z-w)^2} \left[ 2\beta\Lambda(w) + \frac{3}{10}\partial^2 T(w) \right] \\
 &\quad + \frac{1}{(z-w)} \left[ \beta\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w) \right] + \dots, \tag{1}
 \end{aligned}$$

where  $\beta = 16/(22 + 5c)$  and  $A(w)$  is given by

$$A(w) = (TT)(w) - \frac{3}{10}\partial^2 T(w). \tag{2}$$

The central charge  $c$  is left as a free parameter.

Concrete realizations of such currents are provided by conformal matter systems  $\phi$ , with action functional  $S[\phi]$ , which possess exact  $W_3$  symmetry at the quantum level. A simple example is a theory of two real scalar fields [6], with central charge  $c = 2$ . By including a background charge in this theory, one can obtain general values of  $c$ .

We now proceed and use these  $W_3$  conformal field theories to define induced  $W_3$  gravity. We will work in the chiral gauge, where the only  $W_3$  gauge fields are  $h = h_{++}$  and  $b = b_{+++}$ , which can be viewed as two lightcone components of covariant spin-2 and spin-3 vielbein fields  $e_\mu^\pm$  and  $b_\mu^{\pm\pm}$  [7]. We work in euclidean space with coordinates  $z, \bar{z}$ . The induced action  $\Gamma_{\text{ind}}[h, b]$  is defined by the following path-integral

$$\begin{aligned} \exp(\Gamma_{\text{ind}}[h, b]) &= \int \mathcal{D}\phi \exp\left(S[\phi] - \frac{1}{\pi} \int (hT + bW)\right) \\ &= \left\langle \exp\left(-\frac{1}{\pi} \int (hT + bW)\right) \right\rangle, \end{aligned} \tag{3}$$

where the brackets denote the expectation value with respect to a vacuum in which  $\langle T \rangle = \langle W \rangle = 0$ . The exponent in the first line of (3) contains the classical action with minimal coupling to  $W_3$  gravity through the currents  $T$  and  $W$ . (In the two-scalar realization without background charge, the currents  $T$  and  $W$  are actually the Noether currents for the  $\epsilon$  and  $\lambda$   $W_3$  gauge symmetries; in that case the exponent possesses exact  $W_3$  gauge invariance [8]). It is important to realize that for evaluating the induced action, all the information needed is in the OPEs (1); further microscopic details of the matter system are not relevant. We can therefore abstract from the specific matter system  $\phi$  and consider the dynamics that are generated by the induced action (3) with parameter  $c$ .

An important feature of the induced action for  $W_3$  gravity is that it actually describes *propagating*  $h$  and  $b$  fields. This can be inferred from the following two leading terms in the expression for the induced action, which were already given by Matsuo in [9],

$$\Gamma_{\text{ind}}[h, b] = -\frac{c}{24\pi} \int h \frac{\partial^3}{\partial} h - \frac{c}{720\pi} \int b \frac{\partial^5}{\partial} b + \dots \tag{4}$$

At the classical level the fields  $h$  and  $b$  entered the action as Lagrange multipliers. The fact that they become propagating at the quantum level is a manifestation of

an anomaly in the  $W_3$  gauge symmetries. If  $c = 100$ , this anomaly can consistently be cancelled by the inclusion of ghosts; in that case one ends up with a *critical*  $W_3$  gravity theory with no propagating gauge degrees of freedom. In all other cases, the anomaly remains and we are dealing with a *non-critical* theory.

Let us now briefly summarize the results for the induced action that were obtained in ref. [2]. In the scalar field realization for  $c = 2$ , the terms in the perturbative expansion of the induced action correspond to Feynman diagrams whose internal lines represent the propagating scalar fields. In the general case, the various terms are directly derived from the fundamental OPEs (1).

The induced action of  $W_3$  gravity can be expanded in even powers of the field  $b$ . The full result for the  $b$ -independent terms in  $\Gamma_{\text{ind}}[h, b]$  is given by

$$\Gamma_{\text{ind}}^{(0)}[h, b] = \frac{-c}{24\pi} \int \partial^2 h \frac{1}{\bar{\partial}} \frac{1}{1 - h(\partial/\bar{\partial})} \partial h. \quad (5)$$

The exact result for the terms quadratic in  $b$  reads

$$\Gamma_{\text{ind}}^{(2)}[h, b] = -\frac{c}{720\pi} \int \partial Y \bar{\nabla} Y, \quad (6)$$

where  $\bar{\nabla} = \bar{\partial} - h\partial$  and the scalar field  $Y$  is given by

$$Y = \frac{1}{\bar{\nabla}} \left[ \partial + \left( \frac{\partial}{\bar{\partial}} \frac{1}{1 - h(\partial/\bar{\partial})} \partial h \right) \right] \left[ \partial b + 2b \left( \frac{\partial}{\bar{\partial}} \frac{1}{1 - h(\partial/\bar{\partial})} \partial h \right) \right]. \quad (7)$$

(We derived this result to all orders in  $h$  by means of a tensor calculus based on the  $\epsilon$  gauge symmetry [2], which is reminiscent of the familiar tensor calculus used in general relativity.) We finally present the leading terms quartic in  $b$ , which will play an important role in our discussion below. In the scalar field realization, they correspond to certain 3-loop diagrams with four external  $b$ -lines, which were explicitly computed in ref. [2]. For general  $c$ , these terms directly follow from the expression for the 4-point function  $\langle WWWW \rangle$ , which can be derived from the fundamental OPEs (1). One finds

$$\Gamma_{\text{ind}}^{(4)}[h = 0, b] = -\frac{c}{60 \cdot 6!} \frac{1}{\pi} [\text{I}] - \frac{2\beta c}{5 \cdot 7!} \frac{1}{\pi} [\text{II}], \quad (8)$$

where the two basic structures [I] and [II] are given by

$$\begin{aligned} [\text{I}] &= \int \left( 2b \frac{\partial^3}{\bar{\partial}} b - 3\partial b \frac{\partial^2}{\bar{\partial}} b + 3\partial^2 b \frac{\partial}{\bar{\partial}} b - 2\partial^3 b \frac{1}{\bar{\partial}} b \right) \frac{1}{\bar{\partial}} \left( 2b \frac{\partial^6}{\bar{\partial}} b + 3\partial b \frac{\partial^5}{\bar{\partial}} b \right), \\ [\text{II}] &= \int \left( b \frac{\partial}{\bar{\partial}} b - \partial b \frac{1}{\bar{\partial}} b \right) \frac{1}{\bar{\partial}} \left( b \frac{\partial^8}{\bar{\partial}} b + 6\partial b \frac{\partial^7}{\bar{\partial}} b + 14\partial^2 b \frac{\partial^6}{\bar{\partial}} b + 14\partial^3 b \frac{\partial^5}{\bar{\partial}} b \right). \end{aligned} \quad (9)$$

Let us now consider the Ward Identities satisfied by the induced action. As in the case of pure gravity, one derives these WIs from the OPEs satisfied by the basic currents, which we gave in (1). However, in this case special care is needed to deal with the non-linearity of the  $WW$  OPE. The final result is given in terms of variables  $u$  and  $v$  defined by

$$u = -\frac{12\pi}{c} \frac{\delta}{\delta h} \Gamma_{\text{ind}}[h, b], \quad v = -\frac{360\pi}{c} \frac{\delta}{\delta b} \Gamma_{\text{ind}}[h, b].$$

We find

$$\begin{aligned} \bar{\partial}u &= D_1 h + \frac{1}{30} [3v\partial + 2(\partial v)] b, \\ \bar{\partial}v &= [3v\partial + (\partial v)] h + D_2 b \\ &\quad + \frac{4\beta}{35} \left( 2\partial b \frac{1}{\bar{\partial}} Q_{bb}^8 + b \frac{\partial}{\bar{\partial}} Q_{bb}^8 \right) + O(b^3 h^{\geq 1}, b^5, \dots). \end{aligned} \tag{10}$$

In here,  $D_1$  and  $D_2$  are the 3rd and 5th order Gelfand–Dickey operators given by

$$\begin{aligned} D_1 &= \partial^3 + 2u\partial + u', \\ D_2 &= \partial^5 + 10u\partial^3 + 15u'\partial^2 + 9u''\partial + 2u''' + 16u^2\partial + 16uu', \end{aligned} \tag{11}$$

where the primes denote  $\partial$ . The expression  $Q_{XY}^8$  is given by

$$Q_{XY}^8 = X \frac{\partial^8}{\bar{\partial}} Y + 6\partial X \frac{\partial^7}{\bar{\partial}} Y + 14\partial^2 X \frac{\partial^6}{\bar{\partial}} Y + 14\partial^3 X \frac{\partial^5}{\bar{\partial}} Y. \tag{12}$$

The reason why the structure [I] does not appear explicitly in the WI is that the second factor in [I], including the  $1/\bar{\partial}$ , is proportional to (the  $b^2$  terms in)  $u$ , so that the corresponding contribution to  $\bar{\partial}v$  appears as the local terms proportional to  $u$  in  $D_2 b$ . The contribution to  $\bar{\partial}v$  from the structure [II] cannot be expressed as a local expression in  $u$  and  $v$  and leads to the explicit  $Q_{bb}^8$  terms in (10).

Before we proceed to the effective action, we will first rephrase the above findings for the induced action. For this purpose we define a reference functional  $\Gamma_L[h, b]$  by the property that

$$u = -\pi \frac{\delta}{\delta h} \Gamma_L, \quad v = -30\pi \frac{\delta}{\delta b} \Gamma_L \tag{13}$$

satisfy the equations

$$\begin{aligned} \bar{\partial}u &= D_1 h + \frac{1}{30} [3v\partial + 2(\partial v)] b, \\ \bar{\partial}v &= [3v\partial + (\partial v)] h + D_2 b. \end{aligned} \tag{14}$$

A related functional  $W_L[u, v]$  is obtained from  $\Gamma_L[h, b]$  by a Legendre transformation

$$W_L[u, v] = \Gamma_L[h(u, v), b(u, v)] + \frac{1}{\pi} \int (hu + \frac{1}{30}bv), \quad (15)$$

where  $h(u, v)$  and  $b(u, v)$ , which we denote by  $h_L(u, v)$  and  $b_L(u, v)$  for later reference, are determined through the relations in (13) and we have that

$$h_L(u, v) = \pi \frac{\delta}{\delta u} W_L, \quad b_L(u, v) = 30\pi \frac{\delta}{\delta v} W_L. \quad (16)$$

In ref. [3], we showed how to obtain  $\Gamma_L$  and  $W_L$  by reducing a  $Sl(3, \mathbb{R})$  WZW action (see also refs. [10–14]) and we obtained an explicit and local expression for  $\Gamma_L$  in terms of variables  $r$  and  $s$ , which generalize Polyakov's variable  $f$  [15].

The above results for the induced action can now be summarized by the following formula

$$\Gamma_{\text{ind}}[h, b] = \frac{1}{12}c\Gamma_L[h, b] - \frac{32}{25 \cdot 7!} \frac{1}{\pi} [\text{II}] + \mathcal{O}(b^4 h^{\geq 1}, b^6, \dots) + \mathcal{O}\left(\frac{1}{c}\right), \quad (17)$$

where the consecutive terms are of order  $c$ , 1 and  $1/c$ , respectively. Note that the difference between the induced action and the reference functional  $\Gamma_L$  is of relative order  $1/c$  in a large- $c$  expansion.

We now perform the second step in our quantization procedure, which is to quantize the fields  $h$  and  $b$ . We define the generating functional  $W[t, w]$  of connected Green functions by

$$\exp(W[t, w]) = \int \mathcal{D}h \mathcal{D}b \exp\left(\Gamma_{\text{ind}}[h, b] + \frac{1}{\pi} \int (ht + bw)\right). \quad (18)$$

The functional  $W[t, w]$  is related to the effective action  $\Gamma_{\text{eff}}$  by a Legendre transformation. The latter corresponds to all one-particle irreducible graphs with propagating  $h$  and  $b$  fields and  $h$  and  $b$  fields on the external lines. In terms of diagrams, the complete perturbative evaluation of  $W[t, w]$  involves two independent loop-expansions, one with matter loops due to the path integral over  $\phi$  and the second with gauge field loops due to the integral over  $h$  and  $b$ . The net result can be analyzed as follows in terms of a  $1/c$  expansion for large  $c$  (which is the weak coupling regime).

The path integral (18) can be approximated by the saddle-point contribution. This leads to the leading term in  $W[t, w]$ , which is simply the Legendre transform of the induced action. This tree result should then be corrected by further terms coming from diagrams with  $h$  and  $b$  loops. We now observe that, as is obvious

from (4), the kinetic terms for  $h$  and  $b$  in the induced action are proportional to  $c$ , such that  $1/c$  plays the role of Planck's constant in the path integral (18), while the interaction terms in the induced action are of order  $c$  or sub-leading with extra powers of  $1/c$  (see (5), (6), (8) and (17)). From this it follows that the loop-corrections to the saddle-point result are suppressed by a strictly positive power of  $1/c$  as compared to the leading terms in the saddle-point result.

The  $1/c$  expansion of the induced action  $\Gamma_{\text{ind}}[h, b]$  in (17) leads to a  $1/c$  expansion of the saddle-point contribution to  $W[t, w]$  when the latter is viewed as a function of  $t/c$  and  $w/c$ , i.e.

$$\begin{aligned}
 W[t, w] = & \frac{6c}{\pi} \int \left(\frac{t}{c}\right) \frac{\bar{\partial}}{\partial^3} \left(\frac{t}{c}\right) + \frac{180c}{\pi} \int \left(\frac{w}{c}\right) \frac{\bar{\partial}}{\partial^5} \left(\frac{w}{c}\right) + \dots \\
 & - \frac{32}{25 \cdot 7!} \frac{1}{\pi} [\text{II}] \left[ b \rightarrow \frac{\bar{\partial}}{\partial^5} \left(\frac{360}{c} w\right) \right] + \dots
 \end{aligned} \tag{19}$$

Through the orders  $c$  and  $c^0$ , this saddle-point contribution can be written as

$$\Gamma_{\text{ind}}[h, b] + \frac{1}{\pi} \int (ht + bw), \tag{20}$$

with  $h(t, w) = h_{\text{L}}((12/c)t, (360/c)w)$  and  $b(t, w) = b_{\text{L}}((12/c)t, (360/c)w)$  as in (16). Using the relation (15), one finds that the leading term  $\frac{1}{12}c\Gamma_{\text{L}}[h, b]$  in  $\Gamma_{\text{ind}}[h, b]$  leads to the term  $(c/12)W_{\text{L}}[(12/c)t, (360/c)w]$  in  $W[t, w]$ .

We shall now concentrate on the leading  $c^0$  terms in the full expression for  $W[t, w]$ . These come from two sources: (i) the second term in the induced action (17), evaluated at the saddle point, and (ii) the 1-loop corrections in the second path-integral (18). The latter can be computed by standard determinant techniques, as was demonstrated by Polyakov (unpublished, see ref. [16]) for the case of pure gravity. The relevant determinant is the determinant of the matrix of second derivatives of the induced action  $\Gamma_{\text{ind}}[h, b]$  with respect to  $h$  and  $b$ , evaluated at the saddle-point. In here we may again replace  $h$  and  $b$  by their leading parts  $h_{\text{L}}((12/c)t, (360/c)w)$  and  $b_{\text{L}}((12/c)t, (360/c)w)$  (we will assume that this has been done in what follows). One finds the following contribution to  $W[t, w]$  (this is up to a  $c$ -dependent additive constant)

$$W[t, w; \text{1-loop}] = -\frac{1}{2} \log \det \begin{pmatrix} \frac{\delta u}{\delta h} & \frac{\delta v}{\delta h} \\ \frac{\delta u}{\delta b} & \frac{\delta v}{\delta b} \end{pmatrix}. \tag{21}$$

Since we are only interested in the terms independent of  $c$ , we can in this 1-loop calculation replace the full induced action by the leading term  $c/12 \Gamma_{\text{L}}[h, b]$ . This



means that we can use the WIs (14) to work out the determinant. The important observation is that, with (14), the determinant factorizes into two factors, which are both given in terms of *local* differential operators. We have

$$W[t, w; 1\text{-loop}] = \frac{1}{2} \log \det \begin{pmatrix} \bar{V}_2 & -\frac{1}{10}(\partial b) - \frac{1}{15}b\partial \\ L & \bar{V}_3 \end{pmatrix} - \frac{1}{2} \log \det \begin{pmatrix} D_1 & \frac{1}{10}v\partial + \frac{1}{15}(\partial v) \\ 3v\partial + (\partial v) & D_2 \end{pmatrix}, \quad (22)$$

where  $\bar{V}_j = \bar{\delta} - h\partial - j(\partial h)$  and the operator  $L$  is given by

$$L = -(10\partial^3 b + 15\partial^2 b\partial + 9\partial b\partial^2 + 2b\partial^3 + 32u\partial b + 16b\partial u + 16bu\partial). \quad (23)$$

Our task is now to evaluate these two fundamental determinants. It is known in the literature [16,17] that the basic covariant differential operators  $\bar{V}_j$  and  $D_j$ , which can be defined for all integers  $j$ , can be expressed in terms of the reference functionals  $\Gamma_L^{\text{grav}}[h]$  and  $W_L^{\text{grav}}[u]$ , defined by the relations

$$u = -\pi \frac{\delta}{\delta h} \Gamma_L^{\text{grav}}, \quad h = \pi \frac{\delta}{\delta u} W_L^{\text{grav}}, \quad \bar{\delta}u = D_1 h, \quad (24)$$

according to

$$\log \det \bar{V}_j = -\frac{(6j^2 - 6j + 1)}{6} \Gamma_L^{\text{grav}}[h], \quad (25)$$

$$\log \det D_j = \frac{j(2j + 1)(2j + 2)}{6} W_L^{\text{grav}}[u].$$

One would therefore expect that, at least to leading order, similar formulas can be established for our determinants of  $2 \times 2$  operators, this time in terms of the  $W_3$  reference functionals  $\Gamma_L[h, b]$  and  $W_L[u, v]$ . In order to actually compute the determinants, we shall use a representation in terms of auxiliary  $b$ - $c$  systems and make use, once more, of current algebra.

If we define two  $b$ - $c$  systems by the following actions,

$$S[b_{1,2}, c_{1,2}; h, b] = \frac{1}{\pi} \int (b_1 b_2) \begin{pmatrix} \bar{V}_2 & -\frac{1}{10}(\partial b) - \frac{1}{15}b\partial \\ L & \bar{V}_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

$$S[B_{1,2}, C_{1,2}; u, v] = \frac{1}{\pi} \int (B_1 B_2) \begin{pmatrix} D_1 & \frac{1}{10}v\partial + \frac{1}{15}(\partial v) \\ 3v\partial + (\partial v) & D_2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad (26)$$

then the determinants in eq. (22) are just the partition functions of these auxiliary field theories. If we split the above actions into kinetic and interaction parts, we arrive at the following representation of the determinants

$$\det \begin{pmatrix} \bar{V}_2 & -\frac{1}{10}(\partial b) - \frac{1}{15}b\partial \\ L & \bar{V}_3 \end{pmatrix} = \langle \exp -\frac{1}{\pi} \int (hT_h + bT_b + buT_{bu} + b'uT_{b'u}) \rangle_{bc}, \tag{27}$$

$$\det \begin{pmatrix} D_1 & \frac{1}{10}v\partial + \frac{1}{15}(\partial v) \\ 3v\partial + (\partial v) & D_2 \end{pmatrix} = \langle \exp \frac{1}{\pi} \int (uH_u + vH_v + u^2H_{uu}) \rangle_{BC}. \tag{28}$$

The basic OPEs for the  $b$ - $c$  systems are

$$\begin{aligned} b_1(z)c_1(w) &= \frac{-1}{(z-w)}, & b_2(z)c_2(w) &= \frac{-1}{(z-w)}, \\ B_1(z)C_1(w) &= -\frac{1}{2} \frac{(z-w)^2}{(\bar{z}-\bar{w})}, & B_2(z)C_2(w) &= -\frac{1}{24} \frac{(z-w)^4}{(\bar{z}-\bar{w})} \end{aligned} \tag{29}$$

and the currents in eqs. (27) and (28) are given by

$$\begin{aligned} T_h &= -b_1\partial c_1 - 2\partial b_1c_1 - 2b_2\partial c_2 - 3\partial b_2c_2, \\ T_b &= -\frac{1}{30}(b_1\partial c_2 + 3\partial b_1c_2) + (10\partial^3 b_2c_1 + 15\partial^2 b_2\partial c_1 + 9\partial b_2\partial^2 c_1 + 2b_2\partial^3 c_1), \\ T_{bu} &= -16(\partial b_2)c_1, \\ T_{b'u} &= 16b_2c_1, \end{aligned} \tag{30}$$

and

$$\begin{aligned} H_u &= (B_1\partial C_1 - \partial B_1C_1) + (2B_2\partial^3 C_2 - 3\partial B_2\partial^2 C_2 + 3\partial^2 B_2\partial C_2 - 2\partial^3 B_2C_2), \\ H_v &= \frac{1}{30}(B_1\partial C_2 - 2\partial B_1C_2) + (2B_2\partial C_1 - \partial B_2C_1), \\ H_{uu} &= 8(B_2\partial C_2 - \partial B_2C_2). \end{aligned} \tag{31}$$

Let us first focus on the determinant (27). Up to a change in notation the currents  $T_h$  and  $T_b$  are very similar to the the currents  $T^{\text{gh}}$  and  $W^{\text{gh}}$  that one finds in a BRST treatment of critical ( $c = 100$ )  $W_3$  gravity [8,18,19]. However, in this case the current  $T_b$  is *not* a primary current with respect to  $T_h$ , and the  $T_b T_b$  OPE is modified accordingly. One finds

$$\begin{aligned}
 T_h(z)T_h(w) &= \frac{-50}{(z-w)^4} + \frac{2T_h(w)}{(z-w)^2} + \frac{\partial T_h(w)}{(z-w)} + \dots, \\
 T_h(z)T_b(w) &= \frac{388b_2c_1(w)}{(z-w)^5} + \frac{96(b_2\partial c_1 + 2\partial b_2c_1)(w)}{(z-w)^4} + \frac{3T_b(w)}{(z-w)^2} \\
 &\quad + \frac{\partial T_b(w)}{(z-w)} + \dots, \\
 T_b(z)T_b(w) &= -\frac{348}{5} \frac{1}{(z-w)^6} + \frac{2T_h(w)}{(z-w)^4} + \frac{\partial T_h(w)}{(z-w)^3} \\
 &\quad + \frac{1}{(z-w)^2} \left[ \frac{3}{10}\partial^2 T_h(w) - \frac{16}{15}\partial^3 b_1c_1(w) \right] \\
 &\quad + \frac{1}{(z-w)} \left[ \frac{1}{15}\partial^3 T_h(w) - \frac{8}{15}\partial(\partial^3 b_1c_1)(w) \right] + \dots \tag{32}
 \end{aligned}$$

The  $T_h T_h$  OPE shows that the functional  $-\frac{100}{12}\Gamma_L[h, b]$  correctly reproduces the  $b$  independent terms in the logarithm of the determinant (see [17]). However, since the current algebra (32) is significantly different from the exact  $W_3$  algebra at  $c = -100$ , we can not expect that  $-\frac{100}{12}\Gamma_L[h, b]$  is the exact result to all orders in  $b$ . For example, in the exact  $c = -100$   $W_3$  algebra, the coefficient of the pole  $(z-w)^{-6}$  in the OPE  $T_b T_b$  is  $-100/3$ , rather than the value  $-348/5$  that we find here. This implies that already at the level of the terms quadratic in  $b$ , the logarithm of (27) deviates from  $-\frac{100}{12}\Gamma_L[h, b]$ . The extra terms quadratic in  $b$  can be written in the following suggestive way

$$\frac{1}{240} \left( \frac{348}{5} - \frac{100}{3} \right) \frac{1}{\pi} \int b \frac{\partial^5}{\partial} b = \frac{272}{5\pi c} \int w(h, b) b + \dots, \tag{33}$$

since we have that at the saddle-point  $w(h, b) = (c/360) (\partial^5/\bar{\partial})b + \dots$ . We will later see that also certain higher-orders in  $b$  can be reproduced by the right-hand side of eq. (33).

If we now look at the terms of order  $b^2h$ , we find contributions from (i) the 3-point function  $\langle T_b T_b T_h \rangle$  and (ii) the 2-point functions  $\langle T_b T_{b'u} \rangle$  and  $\langle T_b T_{bu} \rangle$ , where in the latter two cases we use that at the saddle-point  $u(h, b) = (\partial^3/\partial)h + \dots$ . It so turns out that the terms (ii) precisely cancel the terms in (i) that come from the anomalous part of the OPE  $T_h T_b$ . The remaining terms are precisely described by  $-\frac{100}{12}\Gamma_L[h, b]$  plus the  $b^2h$  terms in the correction term (33).

In a similar way one can compute, order by order, higher terms in the logarithm of the determinant (27). As a last example, which will be crucial for us later on, we discuss the terms of order  $b^4$ . These arise from (i) the 4-point function  $\langle T_b T_b T_b T_b \rangle$  and (ii) the 2-point functions  $\langle T_b T_{b'u} \rangle$  and  $\langle T_b T_{bu} \rangle$ , where in the latter we now use the part of the saddle-point value of  $u(h, v)$  that is quadratic in  $b$  [2]

$$u(h, b) = \dots + \frac{1}{30} \frac{1}{\partial} \left( 3\partial b \frac{\partial^5}{\partial} b + 2b \frac{\partial^6}{\partial} b \right) + \dots \tag{34}$$

After a straightforward though rather lengthy computation one finds that all the  $b^4$  terms can be collected in two terms proportional to the structures [I] and [II] given in (9)! One finds

$$\frac{397}{75 \cdot 6!} \frac{1}{\pi} [\text{I}] + \frac{64}{25 \cdot 7!} \frac{1}{\pi} [\text{II}]. \tag{35}$$

The terms proportional to [I] are precisely reproduced by the two terms in the logarithm of the determinant (27) which we identified above, leaving us with the terms proportional to structure [II].

In summary, we find the following result, which is exact through the orders  $h^n$ ,  $b^2$ ,  $b^2h$  and  $b^4$

$$\begin{aligned} \log \det \begin{pmatrix} \bar{V}_2 & -\frac{1}{10}(\partial b) - \frac{1}{15}b\partial \\ L & \bar{V}_3 \end{pmatrix} \\ = -\frac{100}{12}\Gamma_L[h, b] + \frac{272}{5\pi c} \int w(h, b)b + \frac{64}{25 \cdot 7!} \frac{1}{\pi} [\text{II}] + \dots, \end{aligned} \tag{36}$$

where the dots will not affect the abovementioned orders in  $h, b$ .

For the computation of the other determinant, we consider the current algebra of the currents  $H_u$  and  $H_v$ . We find the following OPEs

$$\begin{aligned} H_u(z)H_u(w) = -6 \frac{(z-w)^2}{(\bar{z}-\bar{w})^2} - \frac{(z-w)}{(\bar{z}-\bar{w})} H_u(w) - \frac{1}{2} \frac{(z-w)^2}{(\bar{z}-\bar{w})} \partial H_u(w) \\ - 2\delta(z-w)H_{uu}(w) + \dots, \end{aligned}$$

$$\begin{aligned}
H_u(z)H_v(w) &= -2\frac{(z-w)}{(\bar{z}-\bar{w})}H_v(w) - \frac{1}{2}\frac{(z-w)^2}{(\bar{z}-\bar{w})}\partial H_v(w) + \dots, \\
H_v(z)H_v(w) &= \frac{-1}{60}\frac{(z-w)^4}{(\bar{z}-\bar{w})^2} - \frac{1}{180}\frac{(z-w)^3}{(\bar{z}-\bar{w})}H_u(w) \\
&\quad - \frac{1}{360}\frac{(z-w)^4}{(\bar{z}-\bar{w})}\partial H_u(w) + \dots
\end{aligned} \tag{37}$$

Note that the singularities in these OPEs are not just functions of the form  $(z-w)^n/(\bar{z}-\bar{w})^m$  (as is the case for pure gravity), but also take the form of bare delta-functions. These lead to terms in the determinant containing a factor  $u^2$ , which combine with similar terms coming from the coupling  $u^2 H_{uu}$  in (28). One already encounters this complication when evaluating the determinant of the operator  $D_2$  defined in (11), which is a part of our more complicated operator. It was suggested in ref. [16] that in the computation of  $\log \det D_2$  the different  $u^2$  terms precisely cancel, so that the final result is simply proportional to  $W_L^{\text{grav}}[u]$  as in (25). We checked this claim for the contributions of the form  $\int u^2(\bar{\partial}/\partial^5)u$  and  $\int u^2(\bar{\partial}/\partial^7)u^2$  and found this to be correct. In our case we do not expect a complete cancellation of the  $u^2$  terms, since the WIs (14) that determine the form of the reference functional  $W_L[u, v]$  explicitly contain  $u^2$  terms.

As an aside, we remark that the OPEs (37), *without* the delta-function singularities, can be rephrased as  $\text{Sl}(3, \mathbb{R})$  current algebra at level  $k=24$  as follows. By using  $\partial^3 H_u = 0$  and  $\partial^5 H_v = 0$ , one can extract  $8 = 3 + 5$  chiral modes  $j^1(\bar{z}), \dots, j^8(\bar{z})$  from these fields. One then shows that the OPEs of these modes precisely agree with the  $\text{Sl}(3, \mathbb{R})$  current algebra. Clearly, this linear “hidden”  $\text{Sl}(3, \mathbb{R})$  has no direct connection with the WIs (14), which are explicitly non-linear.

We used the OPEs (37) to explicitly compute the leading terms in the logarithm of the determinant (28), which are all consistent, up to a factor of 12, with the WIs (14) of the reference functional  $W_L[u, v]$ , so that

$$\log \det \begin{pmatrix} D_1 & \frac{1}{10}v\partial + \frac{1}{15}(\partial v) \\ 3v\partial + (\partial v) & D_2 \end{pmatrix} = 12W_L[u, v] + \dots \tag{38}$$

Using the saddle-point expressions for  $u(h, b)$  and  $v(h, b)$  we can express the result in terms of  $h$  and  $b$ ; we checked that (38) is exact through the orders  $h^n, b^2, b^2h$  and  $b^4$ .

We remark that in simpler determinant computations, for example for the operators  $\bar{V}_j(h)$ , the OPE formulation can be used to derive a Ward Identity for the determinant, which can then be solved to all orders in  $h$ . In our computations the same is possible in principle, but it turns out that, just as in the identities (10)

for the induced action, explicit non-local terms turn up in these equations. Apart from this, there are the complications with the local quadratic terms in the second determinant. We therefore did not pursue this approach, but instead computed some explicit contributions to sufficiently large order to see if, after combining the induced action and the determinant corrections, simplifications occur.

We are now ready to combine the results (17), (36) and (38) into an expression for the effective action, which is exact through the orders  $h^n$ ,  $b^2$ ,  $b^2h$  and  $b^4$  in the leading  $1/c$  correction to the saddle-point result. To our great satisfaction, we find that the explicit non-local structure [II] precisely cancels between the induced action and the determinant corrections. The remaining terms are

$$\begin{aligned}
 W[t, w] = & \frac{c}{12} W_L \left[ \frac{12}{c} t, \frac{360}{c} w \right] - 6W_L \left[ \frac{12}{c} t, \frac{360}{c} w \right] - \frac{50}{12} \Gamma_L[h, b] \\
 & + \frac{1}{\pi c} \frac{272}{10} \int w b + \dots
 \end{aligned} \tag{39}$$

Once more using the saddle-point equations, we can rewrite this as

$$W[t, w] = \frac{c}{12} \left( 1 - \frac{122}{c} + \dots \right) W_L \left[ \frac{12}{c} \left( 1 + \frac{50}{c} + \dots \right) t, \frac{360}{c} \left( 1 + \frac{386}{5c} + \dots \right) w \right]. \tag{40}$$

We thus find that the computed result for  $W[t, w]$  can be summarized by the simple formula (40). Remembering our original goal to recognize an integrable structure in this theory of chiral  $W_3$  gravity, we now propose that the exact, all-order result for the functional  $W[t, w]$  can be gotten by simply completing the  $1/c$  expansions indicated by the dots in (40). This leads to the formula

$$W[t, w] = 2k W_L[Z^{(t)}t, Z^{(w)}w], \tag{41}$$

where  $k$ ,  $Z^{(t)}$  and  $Z^{(w)}$  are functions of  $c$  that allow the  $1/c$  expansions

$$\begin{aligned}
 k &= \frac{c}{24} \left( 1 - \frac{122}{c} + \dots \right), \\
 Z^{(t)} &= \frac{12}{c} \left( 1 + \frac{50}{c} + \dots \right), \\
 Z^{(w)} &= \frac{360}{c} \left( 1 + \frac{386}{5c} + \dots \right).
 \end{aligned} \tag{42}$$

Obviously, the way we arrived at this proposal is rather cumbersome and one

would expect that more streamlined derivations should be possible. However, we are convinced that the non-trivial cancellations that occurred in our computations are a true sign of the integrability of this quantum field theory.

We remark that the result for  $k$  is consistent (in the limit  $c \rightarrow -\infty$ ) with the formula

$$-48(k+3) = 50 - c + \sqrt{(c-2)(c-98)}, \quad (43)$$

which is the *conjectured* outcome of a KPZ type analysis of constraints in a more covariant formulation of  $W_3$  gravity [9,12].

For pure gravity a formula similar to (41) was proposed in ref. [16] on the basis of the results in ref. [20].

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