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# An integral representation for the product of parabolic cylinder functions

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## ABSTRACT

This paper uses the convolution theorem of the Laplace transform to derive an inverse Laplace transform for the product of two parabolic cylinder functions in which the orders as well as the arguments differ. This result subsequently is used to obtain an integral representation for the product of two parabolic cylinder functions  $D_\nu(x)D_\mu(y)$ . The integrand in the latter representation contains the Gaussian hypergeometric function or alternatively can be expressed in terms of the associated Legendre function of the first kind.

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## 1. Introduction

The study of Nicholson–type integrals for the product of two parabolic cylinder functions has recently gained renewed interest. In 2003, Malyshev [1] derived an integral representation for the product of two parabolic cylinder functions with the same order and identical or opposite arguments,  $D_\nu(x)D_\nu(\pm x)$ . More recently, Glasser [2] obtained a representation for the product of two parabolic cylinder functions with identical orders but unrelated arguments,  $D_\nu(x)D_\nu(y)$ . Integral representations, on the other hand, for the case of unrelated orders but identical or opposite arguments,  $D_\nu(\pm x)D_{\nu+\mu-1}(x)$ , were derived in [3]. The approach in [4] offered a first attempt at allowing simultaneously for differing arguments as well as orders. However, this approach still required the orders in each of the separate integral representations to be linearly related. More in particular, Veestraeten [4] used the results of [5] to first obtain integral representations for  $D_\nu(x)D_\nu(y)$  and  $D_\nu(x)D_{\nu-1}(y)$  that in a next step, via the recurrence relation for the parabolic cylinder function, yielded separate representations for other, linearly related orders such as  $D_\nu(x)D_{\nu+1}(y)$  and  $D_\nu(x)D_{\nu-2}(y)$ . This paper generalizes existing results by deriving an integral representation in which both the arguments as well as the orders are (linearly) unrelated, that is, for  $D_\nu(x)D_\mu(y)$ .

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The convolution theorem of the Laplace transform is used to obtain an inverse Laplace transform for the product of two parabolic cylinder functions with different arguments and orders. This derivation starts from two inverse Laplace transforms for single parabolic cylinder functions that are documented in [6]. The latter expressions are selected in view of their ability, via the integral representation of the Appell function  $F_1(a, b_1, b_2; c; x, y)$ , to ultimately yield a Gaussian hypergeometric function within the integrand of the inverse Laplace transform. The resulting transform is also specialized into the inverse Laplace transform for the product of two complementary error functions that was documented in [7]. In a final step, the inverse Laplace transform is used to obtain an integral representation for  $D_\nu(x)D_\mu(y)$  in which the integrand contains a Gaussian hypergeometric function or alternatively can be transformed into an associated Legendre function of the first kind.

## 2. An inverse Laplace transform for the product of two parabolic cylinder functions with different arguments and orders

The parabolic cylinder function in Whittaker's notation, see [8], is denoted by  $D_\nu(z)$  where  $\nu$  and  $z$  represent the order and argument, respectively, with  $\nu$  and  $z$  being complex numbers. This function is related to Weber's parabolic cylinder function  $U(\nu, z)$  via the identity  $U(\nu, z) = D_{-\nu-1/2}(z)$ , see [9].

The central inverse Laplace transform is presented in Theorem 2.1. In its derivation, the convolution theorem is applied to two inverse Laplace transforms for single parabolic cylinder functions that are specified in [6].

**Theorem 2.1:** *Let  $\nu$  and  $\mu$  be two complex numbers such that  $\operatorname{Re}(\nu + \mu) < 1$ . Then, for all complex numbers  $x$  and  $y$  with  $|\arg x| < \pi$ ,  $|\arg y| < \pi$  and  $|\arg x + \arg y| < \pi$ , we have*

$$\begin{aligned} & p^{-1/2} \exp\left(\frac{1}{2}p(x+y)\right) D_\nu(2^{1/2}x^{1/2}p^{1/2})D_\mu(2^{1/2}y^{1/2}p^{1/2}) \\ &= \frac{2^{(\nu+\mu)/2}}{\Gamma((1-\nu-\mu)/2)} \int_0^\infty \exp(-pt)t^{-(1+\nu+\mu)/2}(y+t)^{\mu/2}(x+t)^{\nu/2} \\ & \times F\left(-\frac{\nu}{2}, -\frac{\mu}{2}; \frac{1}{2}(1-\nu-\mu); \frac{t(x+y+t)}{(x+t)(y+t)}\right) dt \\ & [\operatorname{Re} p > 0]. \end{aligned} \tag{2.1}$$

**Proof:** We start from the two inverse Laplace transforms for single parabolic cylinder functions that are specified in (5) and (6) on p. 290 in [6]

$$\Gamma(\nu) \exp\left(\frac{1}{2}ap\right) D_{-2\nu}(2^{1/2}a^{1/2}p^{1/2}) = \int_0^\infty \exp(-pt)2^{-\nu}a^{1/2}t^{\nu-1}(t+a)^{-\nu-1/2} dt$$

$[\operatorname{Re} p > 0, \operatorname{Re} \nu > 0, |\arg a| < \pi]$

and

$$\Gamma(\nu)p^{-1/2} \exp\left(\frac{1}{2}ap\right) D_{1-2\nu}(2^{1/2}a^{1/2}p^{1/2}) = \int_0^\infty \exp(-pt)2^{1/2-\nu}t^{\nu-1}(t+a)^{1/2-\nu} dt$$

$[\operatorname{Re} p > 0, \operatorname{Re} \nu > 0, |\arg a| < \pi].$

These two inverse Laplace transforms, in the notation of Theorem 2.1, are rewritten as

$$\Gamma(-\nu/2) \exp\left(\frac{1}{2}xp\right) D_\nu(2^{1/2}x^{1/2}p^{1/2}) = \int_0^\infty \exp(-pt) 2^{\nu/2} x^{1/2} t^{-\nu/2-1} (t+x)^{(\nu-1)/2} dt$$

$$[\operatorname{Re} p > 0, \operatorname{Re} \nu < 0, |\arg x| < \pi] \quad (2.2)$$

and

$$\Gamma((1-\mu)/2) p^{-1/2} \exp\left(\frac{1}{2}yp\right) D_\mu(2^{1/2}y^{1/2}p^{1/2})$$

$$= \int_0^\infty \exp(-pt) 2^{\mu/2} t^{-(\mu+1)/2} (t+y)^{\mu/2} dt$$

$$[\operatorname{Re} p > 0, \operatorname{Re} \mu < 1, |\arg y| < \pi]. \quad (2.3)$$

Define  $\bar{f}_1(p)$  and  $\bar{f}_2(p)$  as the Laplace transforms of the original functions  $f_1(t)$  and  $f_2(t)$ , respectively

$$\bar{f}_1(p) = \int_0^\infty \exp(-pt) f_1(t) dt,$$

$$\bar{f}_2(p) = \int_0^\infty \exp(-pt) f_2(t) dt,$$

for which  $\operatorname{Re} p > 0$ . The convolution theorem of the Laplace transform, see [10,11], then gives

$$\bar{f}_1(p)\bar{f}_2(p) = \int_0^\infty \exp(-pt) f_1(t) * f_2(t) dt, \quad (2.4)$$

where  $f_1(t) * f_2(t)$  is the convolution of the original functions  $f_1(t)$  and  $f_2(t)$ . This convolution is defined as

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau.$$

The functions  $f_1(t)$  and  $f_2(t)$  in the convolution theorem (2.4) are taken from the inverse transforms (2.2) and (2.3), respectively

$$f_1(t) = 2^{\nu/2} x^{1/2} t^{-\nu/2-1} (t+x)^{(\nu-1)/2} \quad \text{and} \quad f_2(t) = 2^{\mu/2} t^{-(\mu+1)/2} (t+y)^{\mu/2}.$$

The convolution integral then is given by

$$f_1(t) * f_2(t) = \int_0^t 2^{\nu/2} x^{1/2} \tau^{-\nu/2-1} (\tau+x)^{(\nu-1)/2} 2^{\mu/2} (t-\tau)^{-(\mu+1)/2} ((t-\tau)+y)^{\mu/2} d\tau.$$

The substitution  $\tau = tu$  allows to rewrite this integral as

$$f_1(t) * f_2(t) = 2^{(\nu+\mu)/2} x^{\nu/2} t^{-(1+\nu+\mu)/2} (y+t)^{\mu/2}$$

$$\times \int_0^1 u^{-\nu/2-1} (1-u)^{-(1+\mu)/2} \left(1 + \frac{t}{x}u\right)^{(\nu-1)/2} \left(1 - \frac{t}{t+y}u\right)^{\mu/2} du.$$

The integral in the latter expression will be specified in terms of the Gaussian hypergeometric function  $F(a, b; c; z)$ . Hereto, two steps need to be applied. First, the integral

is expressed in terms of the Appell function  $F_1(a, b_1, b_2; c; x, y)$  by using its integral representation

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F_1(a, b_1, b_2; c; x, y) = \int_0^1 u^{a-1} (1-u)^{c-a-1} (1-xu)^{-b_1} (1-yu)^{-b_2} du,$$

for  $\operatorname{Re} c > \operatorname{Re} a > 0$ , see (8.2.5) in [12] and (42) on p. 450 in [13]. Second, the Appell function  $F_1(a, b_1, b_2; c; x, y)$  is reduced into the Gaussian hypergeometric function for  $b_1 + b_2 = c$  via

$$F_1(a, b_1, b_2; b_1 + b_2; x, y) = (1-y)^{-a} F\left(a, b_1; b_1 + b_2; \frac{x-y}{1-y}\right),$$

see (8.3.1.2) in [12] and (63) on p. 452 in [13].

Applying these two steps allows to express the convolution integral as

$$f_1(t) * f_2(t) = 2^{(v+\mu)/2} (xy)^{v/2} t^{-(1+v+\mu)/2} (y+t)^{(\mu-v)/2} \frac{\Gamma(-v/2)\Gamma((1-\mu)/2)}{\Gamma((1-v-\mu)/2)} \\ \times F\left(-\frac{v}{2}, \frac{1-v}{2}; \frac{1}{2}(1-v-\mu); -\frac{t(x+y+t)}{xy}\right).$$

In order to obtain an inverse Laplace transform that also applies to  $x=0$  and  $y=0$ , the above expression will be rewritten via the following linear transformation formula for the Gaussian hypergeometric function

$$F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right),$$

see (15.3.4) in [14]. Using the latter formula then gives

$$f_1(t) * f_2(t) = 2^{(v+\mu)/2} t^{-(1+v+\mu)/2} (y+t)^{\mu/2} (x+t)^{v/2} \frac{\Gamma(-v/2)\Gamma((1-\mu)/2)}{\Gamma((1-v-\mu)/2)} \\ \times F\left(-\frac{v}{2}, -\frac{\mu}{2}; \frac{1}{2}(1-v-\mu); \frac{t(x+y+t)}{(x+t)(y+t)}\right),$$

and the inverse Laplace transform (2.1). ■

**Remark 1:** The above inverse Laplace transform for the product of two parabolic cylinder functions can be specialized for the product of two complementary error functions,  $\operatorname{erfc}$ , via the identity  $D_{-1}(z) = \exp(z^2/4)\sqrt{\pi/2} \operatorname{erfc}(z/\sqrt{2})$ , see (9.254.1) in [15]. Setting  $v = \mu = -1$  in (2.1) and using the identity  $F(1/2, 1/2; 3/2; z) = z^{-1/2} \arcsin(z^{1/2})$ , see (76) on p. 473 in [13], gives

$$p^{-1/2} \exp(p(x+y)) \operatorname{erfc}(x^{1/2} p^{1/2}) \operatorname{erfc}(y^{1/2} p^{1/2}) \\ = \frac{2}{\pi^{3/2}} \int_0^\infty \exp(-pt) (x+y+t)^{-1/2} \arcsin\left(\sqrt{\frac{t(x+y+t)}{(x+t)(y+t)}}\right) dt$$

$$[|\arg x| < \pi, |\arg y| < \pi, |\arg x + \arg y| < \pi],$$

which corresponds with the inverse Laplace transform (3.7.8.1) in [7].

### 3. A Nicholson-type integral representation for the product of two parabolic cylinder functions with different arguments and orders

Setting  $p=1$  in the inverse Laplace transform (2.1) and rescaling  $2^{1/2}x^{1/2}$  and  $2^{1/2}y^{1/2}$  to  $x$  and  $y$ , respectively, gives the following integral representation for the product of two parabolic cylinder functions in which both the arguments and the orders are allowed to differ

$$\begin{aligned}
 D_\nu(x)D_\mu(y) &= \frac{\exp\left(-\frac{1}{4}(x^2 + y^2)\right)}{\Gamma((1 - \nu - \mu)/2)} \int_0^\infty \exp(-t)t^{-(1+\nu+\mu)/2}(x^2 + 2t)^{\nu/2}(y^2 + 2t)^{\mu/2} \\
 &\quad \times F\left(-\frac{\nu}{2}, -\frac{\mu}{2}; \frac{1}{2}(1 - \nu - \mu); \frac{2t(x^2 + y^2 + 2t)}{(x^2 + 2t)(y^2 + 2t)}\right) dt \\
 &\quad \left[ \operatorname{Re}(\nu + \mu) < 1, |\arg x| < \frac{\pi}{2}, |\arg y| < \frac{\pi}{2}, |\arg x + \arg y| < \frac{\pi}{2} \right]. \quad (3.1)
 \end{aligned}$$

Note that the integrand in the integral representation (3.1) can alternatively also be expressed via the associated Legendre function of the first kind  $P_b^a(x)$  by using the identity

$$F(a, b; a + b + 1/2; z) = 2^{a+b-1/2} \Gamma(a + b + 1/2) z^{(1-2a-2b)/4} P_{a-b-1/2}^{1/2-a-b}(\sqrt{1-z}),$$

see [16]. This gives

$$\begin{aligned}
 D_\nu(x)D_\mu(y) &= \exp\left(-\frac{1}{4}(x^2 + y^2)\right) \int_0^\infty \exp(-t)(2t)^{-(1+\nu+\mu)/4}(x^2 + 2t)^{(\nu-\mu-1)/4} \\
 &\quad \times (y^2 + 2t)^{(\mu-\nu-1)/4}(x^2 + y^2 + 2t)^{(1+\nu+\mu)/4} \\
 &\quad \times P_{(\mu-\nu-1)/2}^{(1+\nu+\mu)/2}\left(\frac{xy}{\sqrt{(x^2 + 2t)(y^2 + 2t)}}\right) dt \\
 &\quad \left[ \operatorname{Re}(\nu + \mu) < 1, |\arg x| < \frac{\pi}{2}, |\arg y| < \frac{\pi}{2}, |\arg x + \arg y| < \frac{\pi}{2} \right].
 \end{aligned}$$

Specializing the integral representation (3.1) for identical orders and/or arguments gives alternative specifications for the results derived in [1–4]. Setting  $y=x$  in the representation (3.1) yields an expression for  $D_\nu(x)D_\mu(x)$  that differs from the integral in (2.36) in [3] on account of the different conditions on the orders, namely  $\operatorname{Re}(\nu + \mu) < 1$  versus  $\operatorname{Re} \nu < 0$  in [3]. Also, the specializations toward the expressions for  $D_\nu(x)D_\nu(y)$  in (2.10) in [2] and (2.2) in [4] and  $D_\nu(x)D_\nu(x)$  in (23) in [1] hold under different conditions for  $\nu$ , namely for  $\operatorname{Re} \nu < 1/2$  versus  $\operatorname{Re} \nu < 0$  in [1,2,4]. Using  $\mu = -\nu - 1$  and  $y=x$  in the representation (3.1) gives

$$\begin{aligned}
 D_\nu(x)D_{-\nu-1}(x) &= \exp\left(-\frac{1}{2}x^2\right) \int_0^\infty \exp(-t)(x^2 + 2t)^{-1/2} P_{-\nu-1}\left(\frac{x^2}{x^2 + 2t}\right) dt \\
 &\quad \left[ |\arg x| < \frac{\pi}{4} \right].
 \end{aligned}$$

The integrand in the latter result contains the Legendre function  $P_a(x)$ . Moreover, this representation imposes no restriction on  $\nu$ , whereas the expression for  $D_\nu(x)D_{-\nu-1}(x)$  in

(7) on p. 120 in [17] requires  $\operatorname{Re} \nu > -1$ . Finally, using  $\mu = \nu - 1$  and  $y = x$  and applying the identity

$$F(a, a + 1/2; 2a + 1; z) = \left( \frac{2}{1 + \sqrt{1 - z}} \right)^{2a},$$

see (105) on p. 461 in [13], returns a representation of which the integrand no longer comprises special functions

$$D_\nu(x)D_{\nu-1}(x) = \frac{\exp\left(-\frac{1}{2}x^2\right)}{\Gamma(1-\nu)} \int_0^\infty \exp(-t)t^{-\nu}(x^2 + 2t)^{-1/2}(x^2 + t)^\nu dt$$

$$\left[ \operatorname{Re} \nu < 1, |\arg x| < \frac{\pi}{4} \right].$$

The latter representation is also an alternative to the expressions (2.3) and (2.4) in [4] that both require  $\operatorname{Re} \nu < 0$ .

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## Disclosure statement

No potential conflict of interest was reported by the author.

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