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Carroll stories

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Abstract: We study various aspects of the Carroll limit in which the speed of light is sent to zero. A large part of this paper is devoted to the quantization of Carroll field theories. We show that these exhibit infinite degeneracies in the spectrum and may suffer from non-normalizable ground states. As a consequence, partition functions of Carroll systems are ill-defined and do not lead to sensible thermodynamics. These seemingly pathological properties might actually be a virtue in the context of flat space holography.

Better defined is the Carroll *regime*, in which we consider the leading order term in an expansion around vanishing speed of light without taking the strict Carroll limit. Such an expansion may lead to sensible notions of Carroll thermodynamics. An interesting example is a gas of massless particles with an imaginary chemical potential conjugate to the momentum. In the Carroll regime we show that the partition function of such a gas leads to an equation of state with $w = -1$.

As a separate story, we study aspects of Carroll gravity and couplings to Carrollian energy-momentum tensors. We discuss many examples of solutions to Carroll gravity, including wormholes, Maxwell fields, solutions with a cosmological constant, and discuss the

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structure of geodesics in a Carroll geometry. The coupling of matter to Carroll gravity also allows us to derive energy-momentum tensors for hypothetical Carroll fluids from expanding relativistic fluids as well as directly from hydrostatic partition functions.

Keywords: Space-Time Symmetries, Field Theories in Lower Dimensions, Field Theory Hydrodynamics, Black Holes

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1 Introduction

Carroll symmetry arises in the limit of vanishing speed of light, in which the Poincaré algebra is contracted to the Carroll algebra [1–3]. The most notable features of this algebra are that Carroll boosts commute and the Hamiltonian is a central charge. The physics arising in this Carroll limit is full of strange phenomena and mysteries. In contrast to Galilean relativity, Carroll symmetry implies that under boosts, space is absolute and time is relative,

$$t' = t - \vec{b}\cdot\vec{x}\,, \qquad \vec{x}\,' = \vec{x}\,, \tag{1.1}$$

where $\vec{b}$ is the boost parameter. The light cone closes up as $c \to 0$ so particles with timelike worldlines cannot move in the Carroll limit and the theory becomes ultralocal. On the other hand there exist other types of Carroll particles with zero energy but they cannot stand still.[1] They can be understood as the $c \to 0$ limit of relativistic tachyons [8]. As a consequence, the theory seems to have potential problems with causality [1, 8]. Furthermore, as we will see in this paper, Carroll quantum field theories have some pathologies, as they do not seem to have well defined partition functions and there are uncontrolled divergences in perturbation theory that are difficult to regularize. So why is it worth to continue with this research?

Flat space holography. The main motivation for studying systems with Carroll symmetry comes from the expectation that conformal Carroll field theories might be dual to quantum gravity in asymptotically flat spacetime. This expectation is substantiated by the fact that the asymptotic BMS symmetry group [9] of flat space is the conformal extension of the Carroll group living on its null boundary [10–13]. There is by now more evidence that conformal Carrollian field theories play a role in the celestial holography approach to flat space holography [14**?** –17].

If Carrollian field theories are dual to quantum gravity in flat space, their thermal properties should say something about black holes.[2] But it is well known that black holes in flat space do not have well-defined partition functions at non-zero temperature; they are never in thermal equilibrium [23]. Therefore, we expect similar problems with defining partition functions for Carroll field theories, and this is what we will demonstrate in this paper as one of the main results. So the seemingly pathological properties of Carroll quantum field theory may actually be a virtue of being a consequence of flat space holography. It is interesting to contrast this with large black holes in AdS, which can be in thermal equilibrium with the Hawking radiation. Its partition function is well defined, but diverges in the large radius limit in which AdS becomes flat space. The small black hole in AdS is of course unstable, and faces the same problems as in flat space.

[1] In [4, 5] it was demonstrated that non-trivial dynamics for coupled Carroll particles can be realized when introducing interactions. In [6] particles in the extended Carroll group were studied and in [7] fractonic particles.

[2] Carroll symmetries in relation to black holes have also been studied in, e.g., the context of Love numbers [19] and the black hole membrane paradigm [20–22].

Cosmology and dark energy. Carroll symmetry might also be relevant for de Sitter cosmology and inflation [8]. In the Carroll limit where we keep the Hubble constant fixed, the Hubble radius $R = c/H$ goes to zero and outside it recessional velocities are naturally large compared to the speed of light. As we send the speed of light to zero, essentially the entire universe becomes super-Hubble and hence Carrollian. The Hubble radius defines the causal patch of an observer, and as the Hubble radius goes to zero, the theory becomes ultralocal, one of the main characteristics of Carrollian physics. We have already given an example in [8] of a scalar field that in the Carroll limit yields an equation of state corresponding to dark energy, i.e. $w = -1$ and so $\mathcal{E} + P = 0$, leading to a de Sitter universe when coupled to gravity.

In this paper, we look at another example, and start with a Boltzmann gas of relativistic massless particles with a chemical potential (with the dimensions of velocity) conjugate to the momentum. We compute from the partition function the energy and pressure and consider these quantities in the Carroll regime, i.e. to leading order in the small c-expansion. The strict $c = 0$ Carroll limit is not a well-defined statistical system, but any small value of c is. We then show that for imaginary values of the chemical potential, such gasses in the Carroll regime have an equation of state with $w = -1$. This analysis is done in section 5.3.

Carroll gravity. A related aspect of Carrollian physics is Carroll gravity. This is obtained by considering the small speed of light (i.e. ultra-local) limit of General Relativity (GR) which was first considered in [24]. The more general small c expansion [25] can be seen as a perturbative expansion around the (singular) Carroll point, complimentary to the large c expansion that gives rise to Post-Newtonian corrections. Recently [26], this ultra-local expansion of GR was considered using the modern perspective of non-Lorentzian geometry, incorporating that Carroll geometry arises from Lorentzian geometry when taking $c \to 0$. As a result one obtains the electric (time-like) Carroll gravity action at leading order from the Einstein-Hilbert action, while the magnetic (space-like) Carroll gravity action is a truncation of the next-to-leading order term. These theories are considered from a Hamiltonian point of view in [27] (see also [28, 29]). Carroll gravity appears to describe interesting dynamics of limits of important solutions in gravity. For example, it is closely related to the Beliniski-Khalatnikov-Lifshitz limit [30] describing the near-singularity dynamics of general relativity. We also show that wormholes arise as the Carroll limit of black hole solutions in section 4. More generally, various aspects of Carroll gravity and geometry have appeared in a wide variety of recent studies [8, 19, 20, 27, 31–56].

Carroll hydrodynamics. Hydrodynamics is an important framework for computing quantities such as pressure and energy-density for quantum systems. Coupling such a system to curved spacetime allows for effective computation of such quantities, but does not guarantee thermodynamical consistency. In this paper we study two candidates of perfect fluids, timelike and spacelike, that satisfy Carroll symmetries. Using complementary geometric and thermodynamical arguments we highlight that the timelike candidate cannot be a true hydrodynamical fluid.

The study of Carroll symmetry in hydrodynamics has been initiated starting from two distinct notions of Carroll symmetry. The main distinction is that one notion starts from a fully diffeomorphism covariant approach with local tangent space Carroll boosts as hallmark of Carroll symmetry [57], which has a Ward identity that constrains the energy flux to vanish. This notion is adopted in this paper. The other notion of Carroll fluids considers Carroll diffeomorphisms as manifestation of Carroll symmetry, a less restrictive requirement that, e.g., translates to a less constrained energy flux compared to the former approach and was pioneered in [58, 59]. For more details on this comparison we refer to [53].

Outline. In this paper we work out specific aspects of this counterintuitive Carroll setting and report on their non-trivial features, but also possible limitations of our (non-)relativistic intuition or, perhaps, the limitations of Carrollian physics. The examples we study are free quantum models, various solutions to the Carroll version of general relativity, and manifestations of thermodynamics and hydrodynamics. One of the recurring themes we find is that although we can construct models that adhere to the Carroll symmetries, their thermodynamical nature or statistical mechanical realizations seem elusive.

In the context of quantum field theories we work out two models, the so-called electric and magnetic Carroll contractions of a free massive scalar field, and point out features that arise when quantizing such theories. This is the topic of discussion in section 2. In section 3 we review how Carroll geometry and gravity follows from the ultra-local (i.e. small speed of light) expansion of GR and study the coupling to matter in the Carroll limit. We Subsequently report in section 4 on various Carroll solutions that arise for specific models, including a non-trivial solution to Carroll gravity in the presence of matter. Afterwards, energy-momentum tensors are studied via an expansion around zero speed of light and using the hydrostatic partition function, and we furthermore provide an analysis concerning the statistical nature of a Carroll gas. This is done in section 5. Some appendices are added with more material and technical details.

2 Carroll quantum field theory

In this section we explore some properties of Carroll quantum field theories to gain some insight in the structure of their physical Hilbert space, their correlation functions and their thermodynamics.[3] For the sake of the discussion below, we will assume that the fields in a Carroll QFT transform under Carroll coordinate transformations in exactly the same way as they would transform under a more general coordinate transformation. It is conceivable that completely other realizations of the Carroll algebra exist but in those cases the implications would need to be examined on a case-by-case basis, in particular Ward identities would not take the usual form, and it is not clear that these other realizations could, e.g., be applied to a putative theory at future null infinity in flat space holography.

Carroll QFT's can roughly be divided in (i) "electric" theories, (ii) "magnetic" theories and (iii) a combination of these two. This nomenclature has its origin in considering Carrollian limits of the Maxwell equations of motion [31], where in one limit only the electric field survives, and in another only the magnetic field survives. An off-shell formulation was introduced in [8, 27]. Departing from free relativistic theories, heuristically, the electric

[3] Some aspects of Carroll quantum field theories have also been discussed in the recent works [60, 61].

and magnetic limit, respectively, correspond to considering timelike or spacelike excitations before the relativistic causal structure collapses by taking the Carroll limit.

More generally, electric theories are theories which are ultralocal in space and have non-trivial time-dependence, whereas magnetic theories have a very simple time-dependence and non-trivial space-dependence. Theories of type (i) and (ii) can be coupled together to give rise to theories of type (iii).

In what follows, we will consider a simple illustrative example of an electric and magnetic scalar field theory and consider their properties. We will also make some comments on more general electric and magnetic theories, and in particular point out that one can construct electric Carroll theories starting from any quantum mechanical system, and d-dimensional magnetic theories starting from any $(d-1)$-dimensional Euclidean field theory (as was also eluded to in [53]). We postpone a study of theories of type (iii) to future work.

It is interesting whether the electric theory could be describing the "hard" particles and the magnetic theory the "soft" particles as one has in flat space holography. The magnetic theory has arbitrarily soft particles but no normalizable zero energy states as we will see below. It would be interesting to explore the precise connection between the magnetic theory and soft modes in flat space holography in more detail.

In addition to the aforementioned works, conformal Carroll symmetry realizations in field theory were studied in [13, 53, 62–68], Carroll fermions in [69–71], Carroll electrodynamics in [72], Carroll Yang-Mills in [73], SUSY realizations in [74], and fractonic realizations in [75, 76]. For a recent discussion on quantum effects in Carroll field theory see [60].

2.1 Electric scalar theory

A simple example of an electric scalar theory is given by the Lagrangian

$$S = \frac{1}{2}\int \mathrm{d}t\, \mathrm{d}^d\vec{x}(\dot{\phi}^2 - m^2\phi^2)\,. \tag{2.1}$$

This theory is ultralocal, in the sense that there are no spatial derivatives and therefore spatial points appear to be completely independent from each other: the theory consists of an infinite number of harmonic oscillators, one for each value of the coordinate x. In particular, if we were to put the theory on a lattice, there would be no need to add any coupling between different lattice points.

Canonical quantization. The most general solution of the field equation is

$$\phi = e^{imt}\int \mathrm{d}^d\vec{k}\, a_{\vec{k}^\dagger}\, e^{i\vec{k}\cdot\vec{x}} + c.c.\,, \tag{2.2}$$

as was pointed out by [16]. Upon quantization, the canonical commutator

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^{(d)}(\vec{x}-\vec{y})\,, \tag{2.3}$$

implies that

$$[a_{\vec{k}}, a^\dagger_{\vec{l}}] = \frac{1}{2m(2\pi)^d}\delta^{(d)}(\vec{k}-\vec{l})\,, \tag{2.4}$$

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so up to some irrelevant normalization this is indeed just an infinite collection of harmonic oscillators (we assume $m > 0$ for now). The obvious choice for a ground state is to take a state which is the ground state for each harmonic oscillator. To write down Ward identities, we need to make sure that the Carroll symmetry is not spontaneously broken, which follows fairly straightforwardly from the explicit form of the symmetries of the theory which we will now review.

Symmetries. Consider a transformation of the form

$$\delta\phi = \xi^t \partial_t \phi + \xi^i \partial_i \phi + \xi \phi \,. \tag{2.5}$$

If we vary the action, we find the following necessary conditions for this to be a symmetry

$$\begin{aligned} 2\xi + \partial_t \xi^t - \partial_i \xi^i &= 0 \,, \\ \partial_t \xi^i &= 0 \,, \\ m^2 \partial_t \xi^t + m^2 \partial_i \xi^i - \partial_t^2 \xi - 2m^2 \xi &= 0 \,. \end{aligned} \tag{2.6}$$

This has some peculiar m-dependent symmetries which we will not investigate. The symmetries which are m-independent are of the form

$$\delta\phi = a(x^i) \partial_t \phi + b^i(x^i) \partial_i \phi + \frac{1}{2} \partial_i b^i \phi \,, \tag{2.7}$$

which includes Carroll transformations but is in fact a much larger group of transformations, presumably due to the fact that the theory is ultralocal in x and quadratic. Higher order interactions will restrict b to be constant, but the function a remains unconstrained.

In fact, the transformations of the form $\delta\phi = a(x^i)\partial_t \phi$ are reminiscent of supertranslations. There is no analogue of superrotations in this toy model unless we consider the massless “conformal” case. With the mass equal to zero the symmetries become

$$\delta\phi = \Big(a(x^i) + t(\partial_i b^i - 2\xi_0)\Big) \partial_t \phi + b^i(x^i) \partial_i \phi + (\xi_0 + t\xi_1)\phi \,, \tag{2.8}$$

for any ξ_0, ξ_1 depending only on spatial components. This indeed contains terms linear in t in ξ^t, as expected for superrotations. For simplicity, we continue below with the mass turned on.

The conserved currents are found to be

$$\begin{aligned} J^t &= a(\dot\phi^2 - m^2\phi^2) - 2\left(a(x^i)\partial_t\phi + b^i(x^i)\partial_i\phi + \frac{1}{2}\partial_i b^i \phi\right)\dot\phi \,, \\ J^i &= b^i(\dot\phi^2 - m^2\phi^2) \,, \end{aligned} \tag{2.9}$$

which for constant a, b yield the energy-momentum tensor of the system.

The conserved charges read

$$\begin{aligned} Q_a &= \int \mathrm{d}^d x \, a(\dot\phi^2 + m^2\phi^2) \,, \\ Q_b^i &= \int \mathrm{d}^d x \, (2b^i \partial_i \phi \dot\phi + \partial_i b^i \phi \dot\phi) \,. \end{aligned} \tag{2.10}$$

The first conserved charge in terms of modes reads

$$Q_a = 4m^2 \int \mathrm{d}^d x\, \mathrm{d}^d k\, \mathrm{d}^d l\, a_k^\dagger a_l\, e^{i(k-l)\cdot x} a(x)\,, \tag{2.11}$$

so that in particular the Hamiltonian reads

$$H = 2m^2 (2\pi)^d \int \mathrm{d}^d k\, a_k^\dagger a_k\,. \tag{2.12}$$

We get the usual quantization rules with $a^\dagger$ creation and a annihilation operator and we normal order such that the annihilation operator is put to the right.

The Carroll boost charges are (take $a = \epsilon \cdot x$)

$$C_i = -4im^2 (2\pi)^d \int \mathrm{d}^d k\, \mathrm{d}^d l\, a_k^\dagger a_l\, \partial_i \delta(k-l)\,, \tag{2.13}$$

with ∂_i the derivative with respect to k^i, while momenta read

$$P^i = 4m(2\pi)^d \int \mathrm{d}^d k\, k^i a_k^\dagger a_k\,, \tag{2.14}$$

and one can check that these obey the right type of commutator,

$$[P^i, C_j] = 4i\delta^i_j H\,. \tag{2.15}$$

We can then normal order the generators of the Carroll algebra which will preserve their commutation relations and also guarantee that they annihilate the unique ground state.

Spectrum. Therefore for these types of Carroll theories there is a unique normalizable Carroll invariant ground state satisfying $a_k|0\rangle = 0$, and the energies come in multiples of m with infinite degeneracy — all states $a_k^\dagger|0\rangle$ have energy m regardless of the choice of k,

$$H\, a_k^\dagger|0\rangle = m\, a_k^\dagger|0\rangle\,. \tag{2.16}$$

This is just a manifestation of the ultralocality of the theory. States of the form $a_k^\dagger a_l^\dagger|0\rangle$ are all degenerate with energy eigenvalue $2m$, etc.

2.1.1 Correlation functions

We can compute the commutator

$$[\phi(t,\vec{x}), \phi(t',\vec{x}\,')] = -\frac{i}{m}\sin[m(t-t')]\delta^d(\vec{x}-\vec{x}\,')\,, \tag{2.17}$$

which agrees with the $c \to 0$ limit of the commutator of a relativistic massive scalar field. The main difference is the appearance of the delta function in space, so the commutator vanishes for any two separated points in space.

We can also compute correlators in this theory. The time-ordered correlator is computed to be

$$\begin{aligned}\langle 0|\mathrm{T}(\phi(t,\vec{x})\phi(t',\vec{x}\,'))|0\rangle &= \frac{1}{2m}\Big(e^{-im(t-t')}\theta(t-t') + e^{im(t-t')}\theta(t'-t)\Big)\delta^{(d)}(\vec{x}-\vec{x}\,')\,,\\ &= \frac{1}{(2\pi i)^{d+1}}\int \mathrm{d}q\, \mathrm{d}^d\vec{p}\, \frac{e^{-iq(t-t')+i\vec{p}\cdot(\vec{x}-\vec{x}')}}{-q^2+m^2-i\varepsilon}\,. \end{aligned} \tag{2.18}$$

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This has a form which agrees with the Carroll Ward identity which allows solutions of the form $f(t-t')\delta^{(d)}(\vec{x}-\vec{x}^{\,\prime})$ [8], which is explicitly Carroll boost invariant.

Notice that the second line is the (electric) Carroll limit of the correlator of relativistic scalar field, in which the spatial momenta are suppressed in the $c \to 0$ limit. The integral over spatial momenta then trivially gives rise to the ultralocality in space, in the form of the delta function. Furthermore, notice that from the first line, it is easy to see that the two-point correlation function is again Carroll boost invariant. This may seem somewhat surprising, since it would mean that propagation is causal in any boosted frame. Causality is however not guaranteed in Carroll systems, as was already noticed in the original paper [1]. This is because in general, time-ordering is not a Carroll-boost invariant notion. The reason is that under Carroll boosts, we have the transformation laws

$$\Delta t' = \Delta t - \vec{b} \cdot \Delta \vec{x}\,. \tag{2.19}$$

For large enough boost parameter $|\vec{b}|$, time ordering between any two events can change sign between boosted Carroll observers, at least if $\Delta x \neq 0$. Fortunately, in the above correlator, when $\Delta x = x' - x$ is non-zero, the entire correlator vanishes and so time ordering is boost invariant and causality at the level of the propagator is satisfied because the theory is ultralocal.

It is interesting to consider the massless limit, which corresponds to a free conformal Carroll scalar. After subtracting a diverging constant, we find for the $m \to 0$ limit of (2.18)

$$\langle 0|\mathrm{T}(\phi(t,\vec{x})\phi(t',\vec{x}^{\,\prime}))|0\rangle = -\frac{i}{2}|t-t'|\,\delta^{(d)}(\vec{x}-\vec{x}^{\,\prime})\,. \tag{2.20}$$

The massless case is most interesting in the context of flat space holography. The Feynman propagator is then

$$G_F(x,x') = -i\langle 0|\mathrm{T}(\phi(t,\vec{x})\phi(t',\vec{x}^{\,\prime}))|0\rangle = -\frac{1}{2}|t-t'|\,\delta^{(d)}(\vec{x}-\vec{x}^{\,\prime})\,, \tag{2.21}$$

whereas the retarded propagator is

$$G_{\mathrm{ret}}(x,x') = i\langle 0|[\phi(t,\vec{x}),\phi(t',\vec{x}^{\,\prime})]|0\rangle\theta(t-t') = (t-t')\theta(t-t')\delta^{d}(\vec{x}-\vec{x}^{\,\prime})\,. \tag{2.22}$$

2.1.2 Scale invariance

We can replace the mass term in (2.1) by a general potential $V(\phi)$. This will generically lead to an equation for symmetries which looks like

$$(\partial_t\xi^t + \partial_i\xi^i)V(\phi) - \phi V'(\phi)\xi = 0\,, \tag{2.23}$$

which forces $\xi = 0$ unless the potential is proportional to a single power of ϕ. So for generic potentials we still have all the symmetries as above, but with $\partial_i b^i = 0$ (so these do not fix b to be constant in higher dimension), and if V is a pure power we get more symmetry, in particular a dilation symmetry. For example, for $\dot{\phi}^2 - \phi^p$ there is a symmetry under which $t \to \lambda^z t$, $x \to \lambda x$ and $\phi \to \lambda^\xi \phi$ with

$$z = \left(1 - \frac{p}{2}\right)\xi, \qquad d = -\left(1 + \frac{p}{2}\right)\xi\,, \tag{2.24}$$

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and the corresponding generator will extend the Carroll algebra as

$$[D, H] = -zH, \qquad [D, P_i] = -P_i, \qquad [D, C_i] = (1 - z)C_i. \tag{2.25}$$

Conformal Carroll symmetry has $z = 1$, obtained from a limit of relativistic scale invariance where space and time scale the same way. The boosts then commute with dilations, and we furthermore have

$$p = 2\,\frac{d+1}{d-1}\,. \tag{2.26}$$

This constrains the values of p for a given dimension, for instance we have $p = 4$ for three spatial dimensions and $p = 6$ for $d = 2$. The case $d = 1$, so two spacetime dimensions, is special as there is no solution for p when $z = 1$; only the free scalar with only a kinetic term is scale invariant.

2.1.3 General electric theories

A much larger class of electric theories is obtained as follows. We take a d-dimensional spatial manifold M and some quantum mechanical system with Hilbert space $\mathcal{H}$ and Hamiltonian H, and associate one copy of the quantum mechanical system to each point in M. If the quantum mechanical system has a Lagrangian description, then we can simply let all the fields depend on the spatial coordinates x and write $\mathcal{L} = \int_M \mathrm{d}x L_{QM}[\phi(t,x)]$ for the full Lagrangian of the system. Here $\mathrm{d}x$ represents the measure on M, in coordinates it would read $\mathrm{d}^d x\sqrt{g}$ with g the metric on M.

These theories have conserved charges similar to those in (2.10) where

$$Q_a = \int_M \mathrm{d}x\, a(x) H_x\,, \tag{2.27}$$

with H_x the Hamiltonian of the quantum mechanical system at the point x. Since $[H_x, H_y] = 0$ these charges all commute with each other for all space-dependent functions $a(x)$. There are also charges Q_b associated to the isometries of M that are a bit less straightforward to write down, but they simply translate the system along M. In particular, when exponentiated these will act on Q_a as

$$Q_a \to Q_{a'} \equiv \int_M \mathrm{d}x\, a(x-b) H_x\,. \tag{2.28}$$

If M is simply $\mathbb{R}^d$ then these theories are in particular invariant under the standard Carroll algebra.

The finite energy eigenstates of the system consist of the somewhat singular states which are a tensor product of the ground state for almost all x times a finite number of finite energy states at a finite number of points on x. The energy eigenvalue is the finite sum of the individual non-zero energies and excited states are infinitely degenerate. The ground state is the tensor product of the ground states for all x. It is clearly annihilated by both Q_a and Q_b. The first excited state is the tensor product of ground states times the first excited state of the quantum mechanical system at one point $x \in M$. These states are degenerate and there is an $L^2(M)$ worth of such states. We saw a momentum space version of these states in our scalar example earlier.

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Correlation functions of operators with vanishing one-point functions in the ground state $\langle 0|O_{x_1}\dots O_{x_n}|0\rangle$ with the operators O_{x_i} part of the quantum mechanical theory associated to the point x_i will vanish unless for each x_i there is at least another x_j with $x_i = x_j$. In other words, these correlators will contains products of delta-functions in the spatial coordinates. This is therefore a general feature of these electric theories. One can also get derivatives of delta-functions if one considers explicit realizations in terms of fields and if one constructs operators involving spatial derivatives but the theory remains ultra-local. Since these theories are by assumption built from well-defined quantum mechanical building blocks, they require no further regularization or renormalization.

2.2 Magnetic scalar theory

We now consider the magnetic theory

$$S = \int \mathrm{d}t\, \mathrm{d}^d x \,\left(\chi\dot{\phi} + \mathcal{L}(\phi)\right)\,, \tag{2.29}$$

where $\mathcal{L}$ can be any Lagrangian depending on ϕ. One could consider different versions of this theory, e.g. one where the leading term is $\dot{\phi}_1\dot{\phi}_0$ (as one obtains in Taylor series expansions around $c = 0$ [8]), but this theory has a phase space which is twice as large as the original theory and the theory with a Lagrange multiplier does not have this feature.

Canonical quantization. The field equations are $\dot{\phi} = 0$ and $\dot{\chi} = \mathcal{L}'(\phi)$ where $\int \delta\mathcal{L} \equiv \int \delta\phi\mathcal{L}'(\phi)$ so $\mathcal{L}'$ is just shorthand for the equations of motion obtained from $\mathcal{L}$. The general solution of the field equations is

$$\begin{aligned}\phi &= \phi(x)\,,\\ \chi &= \chi(x) + t\mathcal{L}'(\phi(x))\,.\end{aligned} \tag{2.30}$$

The canonical momenta are

$$\begin{aligned}\pi_\chi &= 0\,,\\ \pi_\phi &= \chi + \pi_{\mathcal{L}}(\phi)\,,\end{aligned} \tag{2.31}$$

which show that this is a system with constraints. As the solutions of the field equation show, the complete phase space is spanned by $\phi(x)$ and $\chi(x)$ so it is sufficient to find their Poisson bracket and commutator. To make life a little bit simpler, we will assume that with a suitable shift of χ we can always absorb all terms with time derivatives in $\mathcal{L}$, so that $\mathcal{L}$ no longer contains time derivatives and $\pi_{\mathcal{L}} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0$. Then the definition of the canonical momenta take the form of two second-class constraints and using a standard Dirac bracket for systems with constraints we obtain

$$[\chi(x), \phi(y)] = -i\delta^{(d)}(x-y)\,. \tag{2.32}$$

If we write

$$\phi = \int \mathrm{d}^d k\, e^{ikx}\, a_k\,, \qquad \chi = \int \mathrm{d}^d k\, e^{ikx}\, b_k\,. \tag{2.33}$$

Then

$$[a_k, b_l] = \frac{i}{(2\pi)^d}\delta^{(d)}(k+l) \tag{2.34}$$

with $a_k^\dagger = a_{-k}$ and similar for b_k.

Symmetries. We next examine the global symmetries of the magnetic scalar theory using an ansatz similar to the electric case

$$\delta\phi = \xi^t \partial_t \phi + \xi^i \partial_i \phi + \xi\phi\,. \tag{2.35}$$

The variation of the action becomes

$$\delta S = \int \mathrm{d}t\mathrm{d}^d x \left(\delta\chi\dot{\phi} - \dot{\chi}(\xi^t\partial_t\phi + \xi^i\partial_i\phi + \xi\phi) + \xi^t\partial_t\phi\mathcal{L}'(\phi) + \xi^i\partial_i\phi\mathcal{L}'(\phi) + \xi\phi\mathcal{L}'(\phi)\right). \tag{2.36}$$

For generic $\mathcal{L}$ the last term cannot be canceled against anything and we must choose $\xi = 0$. For special $\mathcal{L}$ we may be able to choose a ξ so that the theory has an additional scale symmetry but we will ignore that possibility for now. The term $\xi^i\partial_i\phi\mathcal{L}'$ can also not be canceled by anything, and to cancel this term ξ^i must be a symmetry of $\mathcal{L}$. The term $\xi^t\partial_t\phi\mathcal{L}'(\phi)$ can be rewritten as[4]

$$\xi^t\partial_t\phi\mathcal{L}'(\phi) = -\mathcal{L}\partial_t\xi^t + \dot{\phi}\partial_i\xi^t\frac{\partial\mathcal{L}}{\partial\partial_i\phi}\,, \tag{2.37}$$

up to total derivative terms. The term $\mathcal{L}\partial_t\xi^t$ cannot be canceled and so we must set $\partial_t\xi^t = 0$. The second term in (2.37) can be cancelled by assigning an appropriate transformation law to the χ field. Finally, the term $\dot{\chi}\xi^i\partial_i\phi$ can also not be canceled unless $\partial_t\xi^i = 0$. There are no further restrictions so we get the following symmetries (with ξ^i a symmetry of $\mathcal{L}$)

$$\delta\phi = \xi^t\partial_t\phi + \xi^i\partial_i\phi\,, \quad \delta\chi = \xi^t\partial_t\chi + \partial_i(\chi\xi^i) - \partial_i\xi^t\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\,, \quad \partial_t\xi^i = \partial_t\xi^t = 0\,. \tag{2.38}$$

Interestingly, we once more get many more symmetries than just Carroll, we seem to get supertranslations, just like what we got in the electric case.

The conserved currents are found to be (assuming again $\pi_{\mathcal{L}} = 0$)

$$\begin{aligned} J^t &= \xi^t\mathcal{L} - \chi\xi^i\partial_i\phi\,, \\ J^k &= \chi\xi^k\dot{\phi} - \xi^t\dot{\phi}\frac{\partial\mathcal{L}}{\partial(\partial_k\phi)} + \Lambda^k[\xi^i] - \xi^i\partial_i\phi\frac{\partial\mathcal{L}}{\partial(\partial_k\phi)}\,, \end{aligned} \tag{2.39}$$

where $\Lambda^k[\xi^i]$ is the total derivative obtained by varying $\mathcal{L}$ with respect to ξ^i. Recall that we assumed that ξ^i is a symmetry, so $\delta_{\xi^i}\mathcal{L} = \partial_k\Lambda^k[\xi^i]$ by assumption. One can explicitly check, using the field equations, that this current is conserved.

The conserved charges are therefore given by

$$Q[\xi] = \int \mathrm{d}^d x \left(\xi^t\mathcal{L} - \chi\xi^i\partial_i\phi\right). \tag{2.40}$$

We could go ahead and express the charges in terms of the modes (2.33) of ϕ and χ but this is not particularly instructive. It is more insightful to write the quantum charges in a Schrödinger representation as

$$Q[\xi] = \int \mathrm{d}^d x \left(\xi^t\mathcal{L} + i\xi^i\partial_i\phi\frac{\delta}{\delta\phi}\right) \tag{2.41}$$

[4]The term $\xi^t\partial_t\phi\mathcal{L}'(\phi)$ could have been canceled by a suitable transformation of χ. The reason we have to rewrite this term is because otherwise ξ^t corresponds to a trivial gauge transformation parameter due to an equation of motion symmetry of the action whereby χ transforms into the equation of motion of ϕ, and ϕ into the equation of motion of χ.

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which indeed have the same commutation relations as the vector fields $\xi^\mu \partial_\mu$. We could also have put the functional derivative to the left of $\partial_i \phi$ with a similar result.

In particular, the Hamiltonian of the system is

$$H = -\int \mathrm{d}^d x \, \mathcal{L} \,, \tag{2.42}$$

where we introduced the correct sign to be in agreement with convention chosen in the starting Lagrangian (2.29). It is easy to see with the bracket given above that this indeed generates time translations of the solutions of the field equations.

Spectrum. To analyze the spectrum of the theory we will first consider a special case where the volume is finite (so momenta are discrete) and where $\mathcal{L}$ is quadratic and contains a term proportional to ϕ^2. Then in the zero mode sector we have a structure of the form $[a_0, b_0] = i$ (we dropped an irrelevant normalization factor) and $H \sim a_0^2$. We see that a_0 and b_0 are like coordinates and momenta. If we take b_0 to be like momentum and a_0 like position we can consider position eigenstates $|a_0\rangle$ which are delta-function normalizable. In this notation, the energy eigenstates for $E \geq 0$ are $|\sqrt{E}\rangle$ and $|-\sqrt{E}\rangle$. We therefore get a continuous spectrum with delta-function normalizable eigenstates. In particlar, there is no normalizable ground state.[5]

For non-zero modes we pick some momentum k and consider the modes with momentum $\pm k$. The Hamiltonian will be proportional to $a_k a_{-k}$. There might be a k-dependent prefactor but will ignore that prefactor as it is just the overall normalization. So the structure that we have is (dropping irrelevant factors)

$$[b, a^\dagger] = [b^\dagger, a] = -i, \qquad H = a a^\dagger. \tag{2.43}$$

We might be able to directly study this in coordinate space but it is instructive to do it in terms of harmonic oscillators as well. We can redefine

$$b = -\frac{i}{2}c + \frac{i}{2}d^\dagger, \quad b^\dagger = \frac{i}{2}c^\dagger - \frac{i}{2}d, \quad a = c + d^\dagger, \quad a^\dagger = c^\dagger + d, \tag{2.44}$$

which results in standard harmonic oscillators $[c, c^\dagger] = [d, d^\dagger] = 1$ and a Hamiltonian $H = (c + d^\dagger)(c^\dagger + d)$. It is an amusing exercise to find the spectrum of this system whose details we defer to appendix A. The result of this computation is that the spectrum is infinitely degenerate and continuous and does not possess a normalizable ground state.

A more general analysis could proceed as follows using the Schrödinger representation of states as wave functionals $\Psi[\phi(x)]$. Consider the classical solution space to the equation $-\int \mathrm{d}^d x \mathcal{L}(\phi) = E$. Any Schrödinger wave functional on the space of $\phi(x)$ with support on this subspace will be an energy eigenstate. Since the support is on a submanifold of function space (because the Hamiltonian does not depend on the momenta) we expect these states to be at best delta-function normalizable (certainly with respect to other energy eigenstates). In particular, there is no normalizable ground state (unless we have the trivial case $\mathcal{L} = 0$).

[5]Note that in the relativistic case, there is an extra term in the action proportional to $c^2\chi^2$, which makes the ground state normalizable.

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If the equation $\mathcal{L}(\phi) = 0$ has $\phi = 0$ as a solution, then because $\mathcal{L} = 0$ and $\partial_i \phi = 0$ the wave functional with delta-functional support at $\phi = 0$ appears to be invariant under the Carroll symmetries, but this is a purely formal statement due to the non-normalizability of this ground state.

2.2.1 Correlation functions

Consider the simplest case of a magnetic scalar theory, given by

$$S = \int \mathrm{d}t\, \mathrm{d}^d x\, \chi \dot{\phi} \,. \tag{2.45}$$

Under Carroll boosts, the two scalars transform as

$$\delta\chi = \vec{b} \cdot \vec{x}\, \dot{\chi} \,, \qquad \delta\phi = \vec{b} \cdot \vec{x}\, \dot{\phi} \,, \tag{2.46}$$

and the Lagrangian transforms into a total time derivative.

The Green's function is now

$$G_{\chi\phi}(t, \vec{x}\,; t', \vec{x}^{\,\prime}) = \frac{i}{2} \mathrm{sgn}(t - t')\, \delta^d(\vec{x} - \vec{x}^{\,\prime}) \,, \tag{2.47}$$

and satisfies $\partial_t G(t, t') = i\delta(t - t')\delta^d(\vec{x} - \vec{x}^{\,\prime})$. The result is basically the Fourier transform of $1/\omega$, and the pole at $\omega = 0$ produces the discontinuity at $t = t'$ appearing in the sign function. This pole, which appears generically in magnetic field theories, could be interpreted as the dual of a bulk soft mode propagating to the Carroll boundary.

One can also consider more general Lagrangians for the magnetic theory, e.g. by adding spatial derivatives to $\mathcal{L}$, but we will not discuss these theories further here. We will make some more comments about the structure of the correlation functions when we discuss general magnetic theories below. The form of the correlation functions for the free electric and magnetic scalar theories have also been discussed in [68].

2.2.2 Scale symmetry

Just as in the electric case, we can consider theories with additional scale symmetries. Assume for example that under $x \to \lambda x$ and $\phi \to \lambda^{\xi}\phi$ the scaling of $\mathcal{L}$ reads $\mathcal{L} \to \lambda^{\alpha}\mathcal{L}$, then the action is scale invariant under the additional assignment $t \to \lambda^{-\alpha-d} t$ and $\chi \to \lambda^{-\xi-d}\chi$. A simple theory with $\mathcal{L} \sim (\partial_i\phi)^2$ would have a scale invariance with $\alpha = 2\xi - 2$. It is therefore easy to construct scale invariant magnetic Carroll theories starting from a suitable scale covariant $\mathcal{L}$.

2.2.3 General magnetic theories

A general construction of magnetic theories starts from any d-dimensional Euclidean field theory $-\mathcal{L}(\phi_a)$ which depends on fields $\phi_a(x)$. We can then write down the following magnetic theory

$$S = \int \mathrm{d}t \mathrm{d}^d x\, (\chi^a \dot{\phi}_a + \mathcal{L}(\phi_a)) \,. \tag{2.48}$$

If $\mathcal{L}$ has a more complicated field content we similarly need to double the field content and add first order couplings which force the fields in $\mathcal{L}$ to be time-independent, and which force the additional fields to be linear in time on-shell, and such that the new fields serve as canonical momenta for the fields in $\mathcal{L}$. For example, if $-\mathcal{L}$ is Euclidean Maxwell theory, the theory will take the form [8]

$$S = \int \mathrm{d}t\mathrm{d}^d x \left(\chi_i E_i - \frac{1}{4}F_{ij}^2\right), \quad E_i = \partial_i A_t - \partial_t A_i\,, \quad F_{ij} = \partial_i A_j - \partial_j A_i\,, \tag{2.49}$$

where A is a 1-form. In the remainder we will restrict attention to a Euclidean field theory with a single scalar field for simplicity. We already discussed various aspects of such theories above, including the symmetries and the spectrum of the theory. They are most easily understood in a Schrödinger wave functional formalism, where states are wave functionals $\Psi[\phi(x)]$ with inner product given by the path integral

$$\langle\Psi|\Psi\rangle = \int \mathcal{D}\phi\, |\Psi[\phi(x)]|^2\,. \tag{2.50}$$

Since the Hamiltonian is given by $H = -\int \mathrm{d}^d x \mathcal{L}(\phi(x))$, the equation for energy eigenstates reads

$$-\int \mathrm{d}^d x \mathcal{L}(\phi(x))\Psi[\phi(x)] = E\Psi[\phi(x)]\,, \tag{2.51}$$

which means that the wave function must have support on the space of solutions of the equation $-\int \mathrm{d}^d x \mathcal{L}(\phi(x)) = E$ only. This shows that the spectrum will be generically continuous and infinitely degenerate.

Correlation functions of $\chi \equiv -i\frac{\delta}{\delta\phi} + t\mathcal{L}'(\phi)$ (see (2.30)) and ϕ take the form

$$\langle\Psi|F(\chi,\phi)|\Psi\rangle \to \int \mathcal{D}\phi\, \Psi^*[\phi(x)] F\left(-i\frac{\delta}{\delta\phi} + t\mathcal{L}'(\phi), \phi\right)\Psi[\phi(x)]\,. \tag{2.52}$$

From this we observe a few general properties.

- Correlation functions involving only ϕ are time-independent in any state and therefore invariant under time-translations and Carroll boosts.
- Correlation functions involving a finite number of χ-fields will be polynomial in t, up to possible theta-functions associated to a choice of time-ordering.
- Correlation functions of ϕ's in states ψ_E which are invariant under translations and rotations will obey all Carroll Ward identities even if ψ_E is not annihilated by all Carroll generators.
- For wave-functionals of the form $\psi_E = N\exp(-S_E[\phi]/2)$ with some auxiliary Euclidean "action" S_E and normalization factor N, correlation functions will be given by correlation functions in an auxiliary Euclidean QFT with action S_E.

Finally we remark that, complimentary to the general construction of magnetic theories described above, there is another method. This is based on a map that uses as input a (magnetic) Galilean and an electric Carroll action and generates a corresponding magnetic Carroll action. This is described in appendix B (see in particular equation (B.11)). According to this method the action (2.29) with $\mathcal{L}$ depending on ϕ and its spatial derivatives $\partial_i\phi$ is a magnetic Galilean theory to which we add a suitable constraint and appendix B explains why the resulting theory is Carroll invariant.

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2.3 Thermodynamics

In section 5 we will discuss energy-momentum tensors for Carrollian fluids. Such energy-momentum tensors have made frequent appearance in the literature, but in order for these energy-momentum tensors to be the actual energy-momentum tensor of a well-defined microscopic quantum system, we should find Carrollian quantum systems with well-defined thermodynamics and a well-defined equation of state. As we will see, it is very problematic to find such systems in the strict $c \to 0$ limit. These problems also manifest themselves in our discussion in section 5.3 when we consider the $c \to 0$ limit of partition functions of ideal gases.

2.3.1 Representation theory and partition function

The first place where we see a potential problem in defining thermodynamics for Carroll systems is in the representation theory which we discussed in [8]. The Carroll algebra contains commutators of the type $[C, P] = H$ with C the Carroll boost, P a momentum generator, and H the Hamiltonian. The representations of the subalgebra spanned by C, P, H are very simple. H is a central element so we can take it to be a given number. For $H \neq 0$ this commutator is like the commutator for a single coordinate and momentum in quantum mechanics, and the relevant Hilbert space is therefore $L^2(\mathbb{R})$. For $H = 0$ we simply fix C and P to a particular value since now all three generators commute.

We therefore see that energy-eigenstates with $E \neq 0$ are necessarily infinitely degenerate. Theories with only $E = 0$ states have a partition function which is temperature-independent and equal to the dimension of the Hilbert space, which will be infinite in a local QFT, and we will not consider this pathological case in what follows. To have an infinitely degenerate spectrum is not necessarily problematic if we are in flat space, because thermodynamics is only supposed to be finite in finite volume and to become extensive in the large volume limit where everything becomes proportional to volume. One can therefore ask whether the infinite degenerate finite energy eigenstates can be resolved with the help of an IR regulator. Here, there are two possibilities. If the IR regulator does preserve the Carroll algebra, it will typically make the spectrum of P discrete rather than continuous. However, since the energy does not depend on P at all, energy levels remain infinitely degenerate, leading to a divergent partition function even in finite volume. Notice that for standard quantum systems the energy will typically always depend non-trivially on P and this pathology does therefore not arise.

It is also possible that the IR regulator breaks the Carroll algebra. If we denote the IR regulator by some length scale L, energies can depend non-trivially on L as $E = E_0 + L^{-\alpha} f(P) + \ldots$ with $\alpha > 0$ so that we recover the infinitely degenerate spectrum in the limit $L \to \infty$ where we remove the IR regulator. One also expects that the IR regulator makes the momenta discrete in units of $1/L$ so we will write $P = n/L$. If there are d momenta this yields a contribution to the partition function which heuristically looks like

$$Z \ni L^d \sum_{n \in \mathbb{Z}^d} e^{-\beta(E_0 + L^{-\alpha} f(n/L) + \ldots)} . \tag{2.53}$$

The prefactor is what in the usual case gives rise to the extensive behavior of the free energy In order for the sum over n to be regulated by $f(n/L)$ we would need to introduce a new temperature $\beta^* = L^{-\alpha}\beta$ and keep this fixed as $L \to \infty$. This would however make the partition function vanish due to the factor $e^{-\beta E_0}$. Even if we would ignore this fact, the resulting theory would no longer be extensive due to the extra L scaling in the temperature. This extra L scaling would effectively drive the theory to infinite temperature in the $L \to 0$ limit.

Let us exemplify this starting with the free massive relativistic particle with Hamiltonian $H = \sqrt{c^2\vec{p}^2 + m^2c^4}$. In the Carroll limit $c \to 0$ with $E_0 = mc^2$ fixed — one can call this the electric limit since the one particle spectrum has non-zero energy — we can make the expansion

$$H = E_0 + \frac{1}{2}\frac{c^2\vec{p}^2}{E_0} + \cdots . \tag{2.54}$$

The second term in the Hamiltonian breaks Carroll symmetry, but it vanishes in the $c \to 0$ limit. It can be used as a regulator and we can now compute the partition function quite easily. There is no real need to discretize the momenta and we find

$$Z = \frac{V}{h^d} e^{-\beta E_0} \left(\frac{2\pi E_0}{\beta c^2}\right)^{d/2} . \tag{2.55}$$

The result is diverging as $1/c^d$ in the Carroll limit, as expected.[6] In terms of the dimensionless quantities

$$x \equiv \frac{\beta hc}{R} , \qquad y \equiv \beta E_0 , \tag{2.56}$$

with length scale $R^d \equiv V$, we can write the partition function as

$$Z = x^{-d} e^{-y} (2\pi y)^{d/2} . \tag{2.57}$$

The Carroll limit can now be taken on the dimensionless quantity $x \to 0$, and so it has a pole of order d. More details on the Carroll limit of relativistic particles are given in 5.3.

While the above argument is admittedly rather sketchy, the general structure of the electric and magnetic theories that we described above implies that in both cases energy levels remain infinitely degenerate even in finite volume leading to pathological thermodynamics. As far as a possible relation to flat space holography goes, however, this may be a feature rather than a bug because there is no finite temperature of flat space either.

One can also ask whether there are other ways to regulate the infinities in the finite temperature partition functions. For the general electric theories it is not clear how to do that, but for a general magnetic theory one can do this as follows. Since the Hamiltonian is $H = -\int \mathrm{d}^d x\, \mathcal{L}(\phi)$, the canonical partition function can be expressed as (notice that in our conventions $\mathcal{L}$ is negative definite)

$$Z = \mathrm{Tr}(e^{-\beta H}) = \int D\Phi(x) \langle \phi(x) | e^{\beta \int \mathrm{d}^d x\, \mathcal{L}(\phi)} | \phi(x) \rangle = \int D\Phi(x) e^{\beta \int \mathrm{d}^d x\, \mathcal{L}(\phi)} . \tag{2.58}$$

[6]If one would rescale the temperature as $\beta^* = \beta c^2$, then $e^{-\beta E_0} = e^{-\beta^* E_0/c^2} \to 0$, which would make the partition function vanish for any finite β^*, and so no good thermodynamics.

In other words, the partition function of the magnetic theory is equal to the Euclidean partition function of theory $-\mathcal{L}$ seen as a Euclidean theory, seen as a theory in its own right, with a prefactor β. It is interesting that the temperature shows up in the prefactor and not as the periodicity of an imaginary time direction. In fact, this description is somewhat reminiscent of stochastic quantization, where Euclidean theories are viewed as finite temperature statistical systems in one-dimension higher, and with Carrollian theories being a concrete realization of the higher-dimensional theory. While the regularized Euclidean partition function can potentially be computed, it is not clear whether this regularization spoils the microscopic thermodynamics interpretation, nor is there any a priori reason why the thermodynamics obtained from the regularized partition function should be compatible with Carroll symmetry. We will explore this issue in a simple scalar example in some more detail below.

We have certainly not exhausted all possibilities in the above. For theories with scale symmetry (and more generally for Carroll theories which have more symmetries beyond the Carroll algebra) one could consider partition functions that are not based on the Hamiltonian but on the generator of scale symmetry. This is precisely what we do when we consider standard CFT's on the plane, where the dilatation generator maps to the time translation on the cylinder under a conformal transformation. For Carroll theories there are several issues with this perspective: it is not clear the generator of scale symmetry has a discrete spectrum, it is not clear whether we can map the plane to the cylinder in such theories, and the Hamiltonian of Carrollian theories on the cylinder is still infinitely degenerate. An object that might have a better chance of being well-defined is to write a function which counts the number of independent local operators with a given scaling dimension, $Z \sim \sum_{\mathcal{O}} N_{\Delta_{\mathcal{O}}} e^{-\beta\Delta_{\mathcal{O}}}$. Without a suitable operator-state correspondence, where scaling dimensions are somehow related to time translations in a possibly different geometry, it is not clear whether this function has a thermodynamics interpretation, but it appears to be well-defined, and it would be interesting to study it further.

2.3.2 Scalar example

To illustrate some of the issues in finding microscopic Carroll thermodynamics, we will consider the explicit example of a free scalar field in two dimensions with an action of the form

$$S = \int \mathrm{d}^2x(a^2\dot{\phi}^2 - b^2(\partial_x\phi)^2 - m^2\phi^2)\,, \tag{2.59}$$

where we take the periodicity of x to be $2\pi R$. We have introduced two parameters a and b. For the relativistic scalar, we have $a = 1/c$ and $b = 1$. The electric theory can be obtained in the limit $b \to 0$, the magnetic theory is obtained in the limit $a \to \infty$ which can be seen easily from rewriting the Lagrangian with an auxiliary field χ with $L_\chi = \chi\dot{\phi} - \frac{1}{4a^2}\chi^2$. The magnetic limit is a bit subtle as we will see below.

Quantization of the theory is straightforward. The partition function of the theory is

$$Z = e^{-\beta E_C} \prod_{k\in\mathbb{Z}} \frac{1}{1 - e^{-\beta E_k}}\,, \tag{2.60}$$

where the energy of each harmonic oscillator is

$$E_k = \sqrt{\frac{b^2k^2/R^2 + m^2}{a^2}}\,, \tag{2.61}$$

and E_C is the vacuum Casimir energy of the theory on the cylinder, which is equal to a suitably regulated sum of zero-point energies, $E_C = \frac{1}{2}\sum_k E_k$. It is not clear how important this zero point energy is, but interestingly it is UV divergent for nonzero masses. One way to see this is to expand E_k in powers of m^2, so we can write

$$E_C = -\frac{b}{12Ra} + \frac{m}{2a} + \frac{Rm^2}{2ab}\sum_{k>0}\frac{1}{k} - \frac{R^3m^4}{8ab^3}\zeta(3) + \dots \tag{2.62}$$

where we used $\sum_{k>0} k = -1/12$. The sum $\sum_{k>0}\frac{1}{k}$ is UV divergent and can for example also not be zeta function regularized. So

$$E_C = -\frac{b}{12Ra} + \frac{m}{2a} + \frac{Rm^2}{2ab}\log(R\Lambda/b) - \frac{R^3m^4}{8ab^3}\zeta(3) + \dots \tag{2.63}$$

where Λ is some energy UV cutoff. We will ignore the Casimir energy for the time being.

Before taking any limit, we notice that the partition function only depends on the dimensionless quantities $x = b\beta/Ra$ and $y = \beta m/a$.

- Electric case: In the electric case with $b \to 0$, the partition function must end up being a function of y alone. It can therefore not depend on R and one can therefore not obtain extensive thermodynamics.

- Magnetic case: In the magnetic case with $a \to \infty$, the ration x/y remains finite so the partition function can only be a function of this ratio. But this ratio does not depend on temperature, and we would therefore end up with a temperature-independent partition function. We will confirm that this indeed is what happens if we define the partition function through the Euclidean path integral (2.58).

- Conformal case: As an aside, in the conformal limit with $m \to 0$ the partition function can only be a function of x. In order for it to be extensive, $\log Z$ must be linear in R and therefore proportional to $1/x$. This is indeed the correct answer for a 2d CFT, where for large R and/or high temperature we get the Cardy answer $\log Z \sim 1/x$.

One can try to define modified electric and magnetic limits in which one does not only send $a \to \infty$ or $b \to 0$ but at the same time scales some other parameters in the theory as well. If both x and y remain finite in this limit one is not really taking a limit but merely redefining the units of the theory, so that case is not very interesting. If we also demand that the theory is extensive (so $\log Z$ is linear in R) the logarithm of the partition function must be proportional to $1/x$.

To explore the constraint of extensivity, we first consider the standard $R \to \infty$ limit of the theory in which we write

$$\log Z = \sum_{k,n>0} \frac{1}{n} e^{-n\beta E_k}\,, \tag{2.64}$$

and approximate the sum by an integral to extract the term linear in R. This yields

$$\log Z \simeq \sum_{n>0} \frac{Rm}{nb} f\left(\frac{\beta nm}{a}\right) = \sum_{n>0} \frac{y}{nx} f(ny) \tag{2.65}$$

where

$$f(\xi) = \int dz e^{-\xi\sqrt{z^2+1}} = \frac{2}{\xi} + \dots \tag{2.66}$$

and where we also included the leading term in the small ξ expansion. Therefore, the partition function is approximately equal to

$$\log Z \simeq \frac{Ra\pi^2}{3b\beta} = \frac{\pi^2}{3x} \tag{2.67}$$

which is valid in the limit where $ny = \beta nm/a$ becomes small and just the standard Cardy answer for a $c = 1$ theory. We clearly see that the partition function in the CFT regime diverges in the magnetic limit $a \to \infty$ and also in the electric limit $b \to 0$.

Turning back to (2.65), one can verify that $\sum_{n>0} \frac{y}{n} f(ny)$ is constant for $y \to 0$ and decays exponentially for large y. There is no other scaling regime where its behavior is different and well-behaved which is what would be required for non-trivial extensive Carrollian thermodynamics.

We therefore find no evidence for the existence of any limit, even a rather contrived one, of the partition function which yields extensive and Carroll invariant thermodynamics, and both the standard electric and magnetic limits seem to give rise to somewhat pathological answers.

We conclude this section by comparing the magnetic limit of the partition function to the Euclidean path integral representation of the magnetic partition function in (2.58). When $a \to \infty$ the expression in (2.60) becomes

$$Z \sim \prod_k \frac{1}{\beta E_k} = \frac{1}{y} \left(\prod_{k>0} \frac{1}{x^2k^2}\right) \left(\prod_{k>0} \frac{1}{1+\frac{y^2}{k^2x^2}}\right) = \frac{2\pi}{x} \left(\prod_{k>0} \frac{1}{x^2k^2}\right) \frac{1}{2\sinh \pi y/x} \tag{2.68}$$

where in the last step we use the infinite product representation of the sinh function. In the $a \to 0$ limit $x \to 0$ so the x-dependent formal prefactor in the partition function is badly divergent but it does not depend on the variable y/x which we keep fixed and we could decide to remove this prefactor. This would leave us with a finite partition function

$$Z \sim \frac{1}{2\sinh \pi y/x} \tag{2.69}$$

which also happens to be the partition function of an ordinary harmonic oscillator. This harmonic oscillator result is precisely what one would obtain from the expression (2.58), as we recognize that in the magnetic case this 1d Euclidean theory is simply the Euclidean theory of a quantum mechanical harmonic oscillator. If we translate variables more precisely we get $b^2\beta = m_{ho}$, $2\pi R = \beta_{ho}$ and $\beta m^2 = m_{ho}\omega_{ho}^2$, with β_{ho}, m_{ho} and ω_{ho} standard harmonic oscillator variables. The partition function is then equal that of an ordinary harmonic oscillator so that

$$Z = \frac{1}{2\sinh \beta_{ho}\omega_{ho}/2} = \frac{1}{2\sinh \pi Rm/b} = \frac{1}{2\sinh \pi y/x} \tag{2.70}$$

which agrees with (2.68) and indeed does not depend on the temperature β as this just appears as a prefactor in the Euclidean action in agreement with our general scaling symmetry analysis. While one could argue that this is the "correct" partition function for the magnetic theory, it does not give rise to Carrollian thermodynamics, which may be due to the implicit regularization which has been employed and which apparently breaks the Carroll symmetry of the problem.

2.3.3 Stress tensor of the scalar example

For completeness, we point out a few other peculiar features of the scalar example. As we pointed out in [57] the stress-tensor of a general non-boost invariant fluid (we assumed the existence of a consistent thermodynamic description to obtain this form) reads

$$T^t{}_t = -\mathcal{E}, \quad T^i{}_t = -(\mathcal{E}+P)v^i, \quad T^t{}_j = \mathcal{P}_j, \quad T^i{}_j = P\delta^i_j + v^i\mathcal{P}_j\,. \tag{2.71}$$

For $v^i \neq 0$ Carroll symmetry implies $\mathcal{E}+P=0$ as an additional constraint. If we also assume that the momentum density $\mathcal{P}_j$ is proportional to the velocity then the statement $\mathcal{E}+P=0$ also holds when $v^i=0$ [8]. If we do not make that assumption and put $v^i=0$ the requirement $\mathcal{E}+P=0$ no longer applies.

The stress tensor of the electric theory was given in (4.19) in [8] and that of the magnetic theory in (4.29) in that same paper. One sees that the electric theory is an example of a stress tensor of the form (2.71) with $v^i=0$ but with $\mathcal{E}+P\neq 0$. Carroll symmetry therefore does not impose any additional constraints on the partition function in this case. The stress tensor of the magnetic theory is also of the form (2.71) but with non-zero velocity proportional to the gradient of the scalar field and indeed $\mathcal{E}+P=0$.

Some of these observations may sound contradictory, however since neither the electric nor the magnetic theory has a well-defined thermodynamic description, the assumptions that were used to derive (2.71) do not apply anyway.

2.3.4 Two theories with BMS$_3$ symmetry

We briefly consider the conformal electric and magnetic theories in two dimensions. The symmetries for the massless electric theory were given in (2.8). The subset of transformation of the form

$$\delta\phi = a(x)\partial_t\phi + tb'(x)\partial_t\phi + b\partial_x\phi\,, \tag{2.72}$$

form a BMS$_3$ algebra. The modes of a are usually denoted by M_m and those of b by L_m. They form the algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{m+n} + c_L(n^3-n)\delta_{m+n,0}\\
[L_n, M_m] &= (n-m)M_{m+n} + c_M(n^3-n)\delta_{m+n,0}\\
[M_n, M_m] &= 0\,,
\end{aligned} \tag{2.73}$$

with $c_M=0$ and $c_L=2$ [77].

The $2d$ massless magnetic theory with

$$S = \int \mathrm{d}t\mathrm{d}x\,\left(\chi\partial_t\phi - \frac{1}{2}(\partial_x\phi)^2\right)\,, \tag{2.74}$$

also has a BMS$_3$ symmetry given by the same transformation (2.72) for ϕ together with

$$\delta\chi = (a(x) + tb'(x))\partial_t\chi + \partial_x(b(x)\chi) + (a'(x) + tb''(x))\partial_x\phi. \tag{2.75}$$

The relevant conserved charges are given in (2.40). In particular, the charge at $t = 0$ for the b-transformations is $Q \sim \int \mathrm{d}x\, b\chi\partial_x\phi$ so that the stress tensor which generates the Virasoro transformations is $T \sim \chi\partial_x\phi$. We can think of χ and ϕ as a bosonic beta-gamma system where χ has dimension one and ϕ has dimension zero. The central charge of this beta-gamma system is $c = 2$. The generators M are the modes of the spin two operator $\partial_x\phi\partial_x\phi$, and interpreting this also in terms of a beta-gamma system we see that there is no central term between the stress-tensor and this spin-two current. We conclude that $c_L = 2$ and $c_M = 0$ just like in the electric case.

It is interesting to consider the decomposition of the Hilbert spaces of the electric and magnetic theory in terms of representations of the BMS$_3$ algebra. Our theories are unitary, and since the BMS$_3$ does not admit unitary highest-weight representations, the electric and magnetic theory will not contain such representations. For completeness, we briefly review the argument why BMS$_3$ does not have unitary highest weight representations.

Consider a highest weight state with $L_0 = \Delta$ and $M_0 = \xi$. At the first excited level there are two states obtained by acting with L_{-1} and M_{-1}. The matrix of inner products is (see (2.14) in [78]) is

$$\begin{pmatrix} 2\Delta & 2\xi \\ 2\xi & 0 \end{pmatrix}.$$

This matrix has one positive and one negative eigenvalue for $\xi \neq 0$ because the determinant is $-4\xi^2 < 0$. Therefore the inner product is not positive definite unless $\xi = 0$. If $\xi = 0$, there is a null vector obtained by acting with M_{-1}. Similarly, at higher levels, all states which involve at least one M raising operator are null states when $\xi = 0$. So we either have negative norm states, or we have a unitary highest weight representation of Virasoro where all M_n map all states to zero. The latter case is a bit pathological but it would be an example where all states have zero energy.

There is a different way to see that there cannot be unitary representations with $\xi \neq 0$. If we look at the action of M_0 on a basis of states of a given level we find a triangular matrix with ξ on the diagonal and only non-trivial upper triangular matrix elements. See (3.11) and below in [78]. Such a matrix has only one proper eigenvector (similar to matrices in Jordan normal form), but if we had a positive definite inner product with respect to which M_0 would be hermitian, we should be able to find a complete basis of eigenvectors. Therefore, there cannot exist a positive definite inner product.[7]

In the electric theory, the conserved currents whose modes correspond to L_m and M_m are $M(x) \sim \dot{\phi}^2$ and $L(x) \sim \dot{\phi}\partial_x(1 - t\partial_t)\phi$ which are both time-independent on-shell. On-shell we can write

$$\phi = \gamma(x) + t\beta(x), \quad M(x) \sim \beta(x)\beta(x), \quad L(x) \sim \beta\partial_x\gamma\,. \tag{2.76}$$

[7]Though maybe not obvious, triangular matrices can be self-adjoint with respect to mixed signature inner products.

Just like in the magnetic case, the structure is reminiscent of a beta-gamma system with a field of dimension one and a field of dimension zero. The difference with the magnetic case is that here M is expressed in terms of the weight-one degree of freedom, whereas in the magnetic case it was expressed in terms of the weight-zero degree of freedom.

To write an explicit basis, we are going to decompose the fields in Fourier modes along the spatial S^1. The non-zero modes come in pairs with opposite momenta along the circle, and we will decompose each pair in a radial and an angular variable. Moreover, we will Fourier transform wavefunctionals of the modes with respect to the angular variables of the non-zero modes in order to isolate the eigenvalues under rotations which corresponds to the L_0 generator. For each pair of non-zero spatial Fourier modes, this will yield sets of states of the form $|p,m\rangle$ which represents the wave function $\delta(|z|-p)e^{im\phi}$ on the complex plane, with inner product $\langle p,m|p',m'\rangle = 2\pi p\delta(p-p')\delta_{m,m'}$. The reason why we use these somewhat peculiar basis states is that the electric and magnetic theory resemble a free particle rather than a harmonic oscillator. The most natural representation of the Hilbert space therefore uses states with continuous coordinate or momentum labels rather than raising and creation operators. We now summarize the result that one obtains for the spectrum of L_0 and M_0. The Hilbert space will be given by states of the form

$$\mathcal{H} = |p_0\rangle \otimes \otimes_{k>0} |p_k, m_k\rangle \tag{2.77}$$

using basis states as described above, with $p_0 \in \mathbb{R}$, $p_k \geq 0$, and $m_k \in \mathbb{Z}$. For the magnetic theory, the numbers will refer to the modes of the field itself, and for the electric theory the modes will refer to momentum conjugate to the scalar, but we will keep the same notation for either case.

We then find that

$$L_0^{\text{electric}} = L_0^{\text{magnetic}} = \sum_{k>0} k m_k \,, \tag{2.78}$$

and

$$\begin{aligned} M_0^{\text{electric}} &= p_0^2 + \sum_{k>0} p_k^2 \,, \\ M_0^{\text{magnetic}} &= \sum_{k>0} k^2 p_k^2 \,. \end{aligned} \tag{2.79}$$

One could consider the contribution of the momentum sector to a partition function $\mathrm{tr}(e^{-\beta M_0 + i\theta L_0})$ which keeps track of both quantum numbers. This yields (up to some numerical factors)

$$\begin{aligned} Z_{\text{electric}} &= \frac{1}{\sqrt{\beta}} \prod_{k>0} \frac{\delta(k\theta)}{\beta} \,, \\ Z_{\text{magnetic}} &= \int dp_0 \prod_{k>0} \frac{\delta(k\theta)}{k^2\beta} \,. \end{aligned} \tag{2.80}$$

Both partition functions are rather pathological in agreement with our earlier observations that Carroll partition functions tend to not be well-behaved.

If we focus only on the eigenvalues of L_0 we can try to formally separate $m_k > 0$ and $m_k < 0$ by introducing $q = e^{i\theta}$ and $\tilde{q} = e^{-i\theta}$ and by writing

$$\mathrm{Tr}(e^{i\theta L_0}) = \prod_{k>0} \frac{1 - q^k \tilde{q}^k}{(1 - q^k)(1 - \tilde{q}^k)} \,. \tag{2.81}$$

We could subsequently consider regulating this expression by taking $|q| < 1$ and $|\tilde{q}| < 1$. It is however unclear from the present perspective what the physical meaning of this procedure is. With L_0 generating a compact U(1), a generalized character would naturally be a distribution on the group (as our delta-functions above) and the introduction of q and $\tilde{q}$ seems somewhat arbitrary and not in line with generalized group characters.

Expressions similar to (2.81) which are reminiscent of standard free boson/Virasoro characters appear when computing characters for the BMS algebra in non-unitary highest-weight representations [77–79] and also when considering characters for induced representations [80] and in the partition function for thermal flat space [81]. In the latter two cases the characters are formally infinite and require a regularization similar to the one above. The lack of suitable Carroll thermodynamics in our examples suggests that any regularization which makes the partition function well-behaved will also automatically break the Carroll symmetry. It therefore remains unclear what the precise physical meaning of these regulated characters and corresponding partition functions is.

Finally, we notice that the electric and magnetic theory do not appear to enjoy a form of modular invariance due to the asymmetric treatment of space and time. If anything, the electric and magnetic theory could be related to each other under a modular transformation, but such a relation is not manifest in our ill-defined product formulas for the partition function.

3 Carroll geometry and energy momentum tensor from small c expansion

In this section we first review how Carroll geometry arises from expanding Lorentzian geometry around $c = 0$. For a primer on non-Lorentzian gravity theories we refer to [82]. We then discuss the dynamics of Carroll gravity and its coupling to generic Carrollian field theories, including the notion of a Carrollian energy momentum tensor. For the gravitational part we will use the results of [26] for the action and equations of motion of electric/magnetic Carroll gravity obtained from the ultra-local expansion of General Relativity. See also [27] for work on this in the Hamiltonian formalism.

3.1 Ultra-local expansion of General Relativity

Following [26], we start by briefly reviewing the geometry and dynamics obtained from expanding Lorentzian geometry around $c = 0$, yielding Carroll geometry to leading order. This parallels the non-relativistic expansion around $c = \infty$ [83–85].

Consider the expansion of a Lorentzian metric $g_{\mu\nu}$ around $c = 0$, which is called the Carrollian or ultra-local expansion. The starting point is to split time and space in a covariant way by writing

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}\,, \tag{3.1}$$

where T_μ is the time-like vielbein and $\Pi_{\mu\nu} = \delta_{ab} E^a_\mu E^b_\nu$ the spatial part of the metric expressed in terms of the spatial vielbeins E^a_μ, with Latin indices running over the tangent space spatial directions only. Local (boost) Lorentz transformations correspond in this form to

$$\delta T_\mu = c^{-2}\Lambda_a E^a_\mu\,, \qquad \delta\Pi_{\mu\nu} = \Lambda_a T_\mu E^a_\nu + \Lambda_a T_\nu E^a_\mu\,. \tag{3.2}$$

For use below we note that $\sqrt{-g} = cE$ with $E = \det(T_\mu, E^a_\mu)$, which is invariant under local Lorentz boosts. The inverse metric can be written as

$$g^{\mu\nu} = -c^{-2} T^\mu T^\nu + \Pi^{\mu\nu}\,, \tag{3.3}$$

where

$$T^\mu T_\mu = -1\,, \qquad T^\mu \Pi_{\mu\nu} = 0\,, \qquad \Pi^{\mu\nu} T_\nu = 0\,, \qquad \Pi^{\mu\rho}\Pi_{\rho\nu} = \delta^\mu_\nu + T^\mu T_\nu\,. \tag{3.4}$$

By assumption the fields introduced above start at order c^0 in a Taylor expansion around $c = 0$, so that the vielbeins and their inverses may be expanded as[8,9]

$$T_\mu = \tau_\mu + \mathcal{O}(c^2)\,, \qquad E^a_\mu = e^a_\mu + c^2 \pi^a_\mu + \mathcal{O}(c^4)\,, \tag{3.5}$$
$$T^\mu = v^\mu + c^2 M^\mu + \mathcal{O}(c^4)\,, \quad E^\mu_a = e^\mu_a + \mathcal{O}(c^2)\,. \tag{3.6}$$

Expanding the local Lorentz transformations (3.2) leads to the local Carroll boost transformations (arising from $\Lambda_a = O(c^2)$ which follows from the requirement that the form of the $c = 0$ expansion is preserved by the transformation (3.2))

$$\delta\tau_\mu = \lambda^a e^a_\mu\,, \qquad \delta e^a_\mu = 0\,, \qquad \delta\pi^a_\mu = \lambda^a \tau_\mu\,, \tag{3.7}$$
$$\delta v^\mu = 0\,, \qquad \delta M^\mu = \lambda^a e^\mu_a\,, \qquad \delta e^\mu_a = \lambda_a v^\mu\,. \tag{3.8}$$

The corresponding expansion for the metric and its inverse is then

$$g_{\mu\nu} = h_{\mu\nu} + c^2(\Phi_{\mu\nu} - \tau_\mu\tau_\nu) + \mathcal{O}(c^4)\,, \tag{3.9}$$

$$g^{\mu\nu} = -\frac{1}{c^2} v^\mu v^\nu + h^{\mu\nu} - 2v^{(\mu}M^{\nu)} + \mathcal{O}(c^2)\,, \tag{3.10}$$

where we have defined

$$h_{\mu\nu} := \delta_{ab} e^a_\mu e^b_\nu\,, \qquad h^{\mu\nu} := \delta^{ab} e^\mu_a e^\nu_b\,, \qquad \Phi_{\mu\nu} := 2e^a_{(\mu}\pi^a_{\nu)}\,. \tag{3.11}$$

Note that $\Phi_{\mu\nu}$ obeys the property that $v^\mu v^\nu \Phi_{\mu\nu} = 0$.

As a direct consequence of local Lorentz invariance, one may check that the terms appearing at a given order in c^2 in both the metric and inverse metric given above are

[8] As in [84, 85] we make the self-consistent choice of only even powers in c.

[9] We note that the field M^μ was already introduced in ref. [33] using the relation between Carrollian geometry and null hypersurfaces (in order to construct appropriate Carroll boost invariants). Here we see that it also arises naturally from the small c expansion of a Lorentzian metric [26].

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invariant under Carroll boosts. For use below we record here that while $h_{\mu\nu}$ is Carroll invariant we have that

$$\delta h^{\mu\nu} = 2v^{(\mu}e_a^{\nu)}\lambda^a\,, \tag{3.12}$$

which is consistent with the completeness relation $-v^\mu\tau_\nu + h^{\mu\rho}h_{\rho\nu} = \delta^\mu_\nu$. We also introduce the following Carroll boost invariant combinations

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - M^\mu v^\nu - M^\nu v^\mu\,, \tag{3.13}$$

$$\hat{\tau}_\mu = \tau_\mu - h_{\mu\nu}M^\nu\,, \tag{3.14}$$

$$\bar{\Phi}_{\mu\nu} = \Phi_{\mu\nu} - \tau_\mu\tau_\nu\,. \tag{3.15}$$

Next we introduce a Carroll metric-compatible 'connection' $\tilde{\nabla}_\mu$ satisfying

$$\tilde{\nabla}_\mu v^\nu = 0, \qquad \tilde{\nabla}_\rho h_{\mu\nu} = 0. \tag{3.16}$$

We will assume that the torsion is purely intrinsic, i.e. expressed in terms of $K_{\mu\nu}$ which is defined as

$$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_v h_{\mu\nu}\,. \tag{3.17}$$

This tensor is also purely spatial, since it satisfies $v^\mu K_{\mu\nu} = 0$. We put the word connection in quotation marks because the ones we work with are not Carroll boost invariant. A convenient choice is [26, 33, 34][10]

$$\begin{aligned}\tilde{\Gamma}^\rho_{\mu\nu} &= -v^\rho\partial_{(\mu}\tau_{\nu)} - v^\rho\tau_{(\mu}\mathcal{L}_v\tau_{\nu)} \\ &\quad + \frac{1}{2}h^{\rho\lambda}\left[\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\right] - h^{\rho\lambda}\tau_\nu K_{\mu\lambda}.\end{aligned} \tag{3.18}$$

Note that this connection is constructed to have only intrinsic torsion

$$2\tilde{\Gamma}^\rho_{[\mu\nu]} = 2h^{\rho\lambda}\tau_{[\mu}K_{\nu]\lambda}. \tag{3.19}$$

This reflects the result that the intrinsic torsion of a Carroll metric-compatible connection is determined by the extrinsic curvature $K_{\mu\nu}$ [86].

Using the expansion of the metric, one can obtain the expansion of the Riemann curvature tensor in GR and subsequently expand the Einstein-Hilbert action (see [26] for details).

From the expansion of the Levi-Civita connection we derive the leading order behavior of the Ricci tensor $R_{\mu\rho} = R_{\mu\sigma\rho}{}^\sigma$ where the Riemann tensor is

$$R_{\mu\nu\rho}{}^\sigma = -\partial_\mu\Gamma^\sigma_{\nu\rho} + \partial_\nu\Gamma^\sigma_{\mu\rho} - \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\rho} + \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\rho}\,. \tag{3.20}$$

A straightforward calculation then gives

$$R_{\mu\rho} = O(c^{-2})\,, \qquad v^\mu R_{\mu\rho} = O(1)\,. \tag{3.21}$$

[10] This connection can be obtained [26] from the small speed of light expansion of the Levi-Civita connection. It also appeared in [34] and is a special case of the general class of Carroll connections satisfying the compatibility requirements (3.16), which was determined in [33, 34].

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This implies that the Ricci scalar is at most of order c^{-2} so that for the Einstein tensor we obtain the same behaviour as for the Ricci tensor, namely

$$G_{\mu\rho} = O(c^{-2})\,, \qquad v^\mu G_{\mu\rho} = O(1)\,. \tag{3.22}$$

This means that the leading order behavior of $G_{\mu\nu}$ is a c^{-2} term that is orthogonal to v^μ, i.e. that is purely spatial. In the next subsection we will use these results together with the Einstein equations to infer what the behaviour of the energy-momentum tensor should be as $c \to 0$.

Using the expansions reviewed in this section, one can compute the corresponding small c expansion of the Einstein-Hilbert action. This was done in [26] and in section 4.1 we will collect the results that are needed for the purposes of this paper.

3.2 Expanding the energy-momentum tensor around $c = 0$

Now that we have decomposed the metric tensor around $c = 0$ in section 3.1, we are able to consider the expansion of a general relativistic energy-momentum tensor around $c = 0$ to obtain the leading order non-trivial Carroll energy-momentum tensor. This will be relevant in section 4 when we consider some examples of solutions of Carrollian gravity coupled to matter. Moreover, it will be the starting point in section 5 when we consider Carrollian perfect fluid stress tensors as obtained from the small c expansion of relativistic perfect fluids.

Consider the Lagrangian

$$\mathcal{L} = \frac{c^3}{16\pi G_N}\sqrt{-g}R + \mathcal{L}_{\text{mat}}\,. \tag{3.23}$$

Note that we use c^3 (as opposed to c^4) because we have a put a factor of c^2 into $g_{\mu\nu}$ which amounts to rescaling $\sqrt{-g}$ by a factor of c. The Einstein equation then gives $G_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}$ where

$$\delta\mathcal{L}_{\text{mat}} = \frac{1}{2}c^{-1}\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}\,. \tag{3.24}$$

In the previous section we concluded that the Ricci scalar is $O(c^{-2})$. This means that the Einstein-Hilbert Lagrangian is $O(c^2/G_N)$. We allow ourselves the possibility that Newton's constant can scale with powers of c.

Consider the Einstein equation with the indices up, i.e. $G^{\mu\nu} = \frac{8\pi G_N}{c^4}T^{\mu\nu}$. We can use (3.22) to write down the general structure of $G^{\mu\nu}$ to leading order in c for an arbitrary geometry. By using the expansion of the inverse metric (3.3) and (3.22) we learn that

$$G^{\mu\nu} = c^{-4}v^\mu v^\nu G_{(-4)} + c^{-2}G^{\mu\nu}_{(-2)} + c^{-2}v^\mu v^\nu G_{(-2)} + O(1)\,, \tag{3.25}$$

where $G^{\mu\nu}_{(-2)}$ has no $v^\mu v^\nu$ component as we explicitly split that part off and called it $G_{(-2)}v^\mu v^\nu$. The reason for this is that $G_{(-2)}v^\mu v^\nu$ is a subleading correction whereas $G^{\mu\nu}_{(-2)}$ is a leading order term. Since we will only care about leading order terms we will ignore the $G_{(-2)}v^\mu v^\nu$ part. Using the Einstein equations we can now infer what the expansion of $T^{\mu\nu}$ should be around $c = 0$ for an arbitrary geometry. We thus conclude that we must have

$$T^{\mu\nu} = c^{-N}\left(-\mathcal{T}v^\mu v^\nu + c^2\hat{\mathcal{T}}^{\mu\nu} + O(c^4)\right)\,, \tag{3.26}$$

where $\hat{\mathcal{T}}^{\mu\nu}$ is defined up to the addition of a term proportional to $v^\mu v^\mu$ and where furthermore

$$G_N c^{-N} \qquad \text{is independent of } c \text{ for some integer } N. \tag{3.27}$$

We can interpret the objects $\mathcal{T}$ and $\hat{\mathcal{T}}^{\mu\nu}$ appearing in the expansion of the energy-momentum tensor as arising from variation with respect to the Carroll objects $\hat{\tau}_\mu$ and $h_{\mu\nu}$. To see this we substitute (3.26) into the right hand side of (3.24) and expand $\delta g_{\mu\nu}$ as

$$\delta g_{\mu\nu} = \delta h_{\mu\nu} + c^2 \delta\bar{\Phi}_{\mu\nu} + O(c^4)\,. \tag{3.28}$$

This leads to

$$\begin{aligned} \delta\mathcal{L}_{\text{mat}} &= e c^{2-N}\left(-\mathcal{T} v^\mu \delta\hat{\tau}_\mu + \frac{1}{2}\hat{\mathcal{T}}^{\mu\nu}\delta h_{\mu\nu} + O(c^2)\right) \\ &= e c^{2-N}\left(-\mathcal{T} v^\mu \delta\tau_\mu + \frac{1}{2}\left(\hat{\mathcal{T}}^{\mu\nu} + \mathcal{T}\left[v^\mu M^\nu + v^\nu M^\mu\right]\right)\delta h_{\mu\nu} + O(c^2)\right)\,, \end{aligned} \tag{3.29}$$

where $-e^2 = \det\left(-\tau_\mu\tau_\nu + h_{\mu\nu}\right)$. In deriving the above we used that $v^\mu\bar{\Phi}_{\mu\nu} = \hat{\tau}_\nu$ so that $\frac{1}{2}v^\mu v^\nu \delta\bar{\Phi}_{\mu\nu} = v^\mu\delta\hat{\tau}_\mu$.

Note that $v^\mu v^\nu \delta h_{\mu\nu} = 0$, which follows because $v^\mu h_{\mu\nu} = 0$, and hence the response to varying $h_{\mu\nu}$ gives a symmetric $(0,2)$ tensor that is defined up to a part proportional to $v^\mu v^\nu$, which is in agreement with the $c = 0$ expansion of the Einstein tensor results above.

The Carroll energy-momentum tensor is then

$$(T_{\text{Car}})^\mu{}_\nu = -\mathcal{T} v^\mu \hat{\tau}_\nu + \hat{\mathcal{T}}^{\mu\rho} h_{\rho\nu} = -\mathcal{T} v^\mu \tau_\nu + \mathcal{T}^{\mu\rho} h_{\rho\nu}\,, \tag{3.30}$$

where

$$\mathcal{T}^{\mu\rho} = \hat{\mathcal{T}}^{\mu\rho} + \mathcal{T}\left[v^\mu M^\rho + v^\rho M^\mu\right]\,. \tag{3.31}$$

This does not depend on the undetermined $v^\mu v^\nu$ term in the response to varying $h_{\mu\nu}$. The response to varying τ_μ is the energy current which in the case of a Carrollian field theory is of the form $-\mathcal{T} v^\mu$ where $\mathcal{T}$ is the energy density. We see that this current has no components in the spatial vielbein directions. This is a consequence of local Carroll boost invariance. If we would write $-\mathcal{T}^\mu$ for the variation with respect to τ_μ (keeping $h_{\mu\nu}$ fixed) then demanding invariance under Carroll boosts $\delta\tau_\mu = \lambda^a e^a_\mu$ forces the spatial projections of $\mathcal{T}^\mu$ to vanish. Hence the most general energy current $\mathcal{T}^\mu$ is of the form $-\mathcal{T} v^\mu$. The Carroll boost Ward identity is

$$v^\nu h_{\mu\rho}\,(T_{\text{Car}})^\mu{}_\nu = 0\,, \tag{3.32}$$

i.e. the condition that the energy flux is zero, [8, 57]. From (3.26), by lowering one index with $g_{\mu\nu}$ and expanding, we can conclude that

$$T^\mu{}_\nu = c^{2-N}\left((T_{\text{Car}})^\mu{}_\nu + O(c^2)\right)\,. \tag{3.33}$$

Hence, the Carroll energy-momentum tensor is simply the leading order term of $T^\mu{}_\nu$.

4 Carrollian gravity, solutions and geodesics

In this section we consider Carrollian gravity and its coupling to matter. We focus mostly on electric (timelike) gravity theory but also comment on the magnetic (spacelike) case. As a specific example we consider the coupling to electric Carrollian electrodynamics, and in particular describe the resulting equations of motion. We subsequently discuss various solutions: vacuum, non-zero cosmological constant and novel solutions arising from the coupling to Carrollian electrodynamics. Finally, we discuss the properties of geodesics in a Carrollian spacetime.

4.1 Electric (timelike) Carroll gravity coupled to Carrollian matter

We follow here [26]. The electric Carroll gravity (ECG) action is

$$S_{\text{ECG}} = \frac{c^2}{16\pi G_N}\int_M d^{d+1}xe\left[K^{\mu\nu}K_{\mu\nu} - K^2\right] , \tag{4.1}$$

where $K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_v h_{\mu\nu}$ is the extrinsic curvature, which is spatial, since it satisfies $v^\mu K_{\mu\nu} = 0$ and where $K^{\mu\nu} = h^{\mu\rho}h^{\nu\sigma}K_{\rho\sigma}$. Varying the action with respect to v^μ and $h^{\mu\nu}$ we have

$$\delta S_{\text{ECG}} = \frac{c^2}{8\pi G_N}\int_M d^{d+1}xe\left[\mathcal{G}^v_\mu \delta v^\mu + \frac{1}{2}\mathcal{G}^h_{\mu\nu}\delta h^{\mu\nu}\right] . \tag{4.2}$$

This leads to the equations of motion $\mathcal{G}^v_\mu = 0$ and $\mathcal{G}^h_{\mu\nu} = 0$, where

$$\mathcal{G}^v_\mu = -\frac{1}{2}\tau_\mu\left(K^{\rho\sigma}K_{\rho\sigma} - K^2\right) + \left(\tilde{\nabla}_\rho - \mathcal{L}_v\tau_\rho\right)h^{\rho\nu}\left(K_{\mu\nu} - Kh_{\mu\nu}\right), \tag{4.3a}$$

$$\mathcal{G}^h_{\mu\nu} = -\frac{1}{2}h_{\mu\nu}\left(K^{\rho\sigma}K_{\rho\sigma} - K^2\right) + K\left(K_{\mu\nu} - Kh_{\mu\nu}\right) - v^\rho\tilde{\nabla}_\rho\left(K_{\mu\nu} - Kh_{\mu\nu}\right) + A\tau_\mu\tau_\nu , \tag{4.3b}$$

where A is an undetermined scalar. This is because $\tau_\mu\tau_\nu\delta h^{\mu\nu} = 0$. Note that here the covariant derivative is taken with respect to the Carroll metric-compatible connection (3.18).

Projecting out the time and space components of each equation using v^μ and $h^{\mu\nu}$, we see that the time-space component of $\mathcal{G}^h_{\mu\nu}$ vanishes ($v^\mu h^{\nu\rho}\mathcal{G}^h_{\mu\nu} = 0$ is a consequence of local Carroll boost invariance). Ignoring $v^\mu v^\nu \mathcal{G}^h_{\mu\nu}$ (which is not an equation of motion and thus plays no role whatsoever), the equations of motion can be written as

$$K^{\mu\nu}K_{\mu\nu} - K^2 = 0 , \tag{4.4a}$$

$$\left(\tilde{\nabla}_\rho - \mathcal{L}_v\tau_\rho\right)h^{\rho\sigma}\left(K_{\sigma\mu} - Kh_{\sigma\mu}\right) = 0 , \tag{4.4b}$$

$$-\mathcal{L}_v K_{\mu\nu} - 2K_\mu{}^\rho K_{\rho\nu} + KK_{\mu\nu} = 0 . \tag{4.4c}$$

These have the form of constraint and evolution equations. The derivation of (4.4c) will be detailed further below.

Adding Carroll matter. The leading order Carroll gravity action is order c^2/G_N where we allow the possibility that G_N scales with c in a nontrivial way when expanding around $c = 0$. In order to couple this to Carroll matter we need to make sure that the Carroll matter Lagrangian is of the same order in c as the LO gravity action. Suppose that the Carroll matter Lagrangian is order c^M for some $M \in \mathbb{Z}$ then the two theories couple provided that $G_N \sim c^{2-M}$.

In general, a Carroll invariant matter action is a functional of matter fields ϕ with Carroll metric sources, i.e. $S_{\rm M}[\phi; v^\mu, h^{\mu\nu}]$. Varying the action gives rise to two currents

$$\delta S_{\rm M} = c^M \int_M d^{d+1}x e \left[-T^v_\mu \delta v^\mu - \frac{1}{2} T^h_{\mu\nu} \delta h^{\mu\nu} \right] , \tag{4.5}$$

where we are agnostic about M and where the currents T^v_μ and $T^h_{\mu\nu}$ are c independent.

These currents can be combined into the Carroll energy-momentum tensor

$$T^\mu{}_\nu = v^\mu T^v_\nu + h^{\mu\rho} T^h_{\rho\nu} . \tag{4.6}$$

We can alternatively define the responses to varying τ_μ and $h_{\mu\nu}$. If we define

$$\delta S_{\rm M} = c^M \int_M d^{d+1}x e \left[-\mathcal{T}^\mu \delta \tau_\mu + \frac{1}{2} \mathcal{T}^{\mu\nu} \delta h_{\mu\nu} \right] , \tag{4.7}$$

then the Carroll energy-momentum tensor is given by

$$T^\mu{}_\nu = -\mathcal{T}^\mu \tau_\nu + \mathcal{T}^{\mu\rho} h_{\rho\nu} , \tag{4.8}$$

where we have the relation between the currents

$$T^v_\rho = \tau_\rho \tau_\mu \mathcal{T}^\mu - h_{\rho\nu} \tau_\mu \mathcal{T}^{\mu\nu} , \tag{4.9}$$

$$T^h_{\mu\nu} = h_{\mu\rho} h_{\nu\sigma} \mathcal{T}^{\rho\sigma} - \tau_\rho h_{\sigma\mu} \mathcal{T}^\mu - \tau_\sigma h_{\rho\mu} \mathcal{T}^\mu . \tag{4.10}$$

The sourced Carroll gravity equations of motion are thus

$$\mathcal{G}^v_\mu = 8\pi G_N c^{M-2} T^v_\mu , \quad \mathcal{G}^h_{\mu\nu} = 8\pi G_N c^{M-2} T^h_{\mu\nu} . \tag{4.11}$$

Contracting the first of these two sourced equations with v^μ and $h^{\mu\nu}$ we obtain

$$K^{\mu\nu} K_{\mu\nu} - K^2 = 16\pi G_N c^{M-2} v^\mu T^v_\mu , \tag{4.12a}$$

$$h^{\mu\nu} \left(\tilde{\nabla}_\rho - \mathcal{L}_v \tau_\rho \right) h^{\rho\sigma} (K_{\sigma\mu} - K h_{\sigma\mu}) = 8\pi G_N c^{M-2} h^{\mu\nu} T^v_\mu . \tag{4.12b}$$

The second equation in (4.11) can be written as

$$\begin{aligned} & h_{\mu\nu} \left(\mathcal{L}_v K - K^2 \right) - \frac{1}{2} h_{\mu\nu} \left(K_{\rho\sigma} K^{\rho\sigma} - K^2 \right) - \mathcal{L}_v K_{\mu\nu} + K K_{\mu\nu} - 2 h^{\rho\sigma} K_{\mu\rho} K_{\nu\sigma} + A \tau_\mu \tau_\nu \\ & \quad = 8\pi G_N c^{M-2} T^h_{\mu\nu} , \end{aligned} \tag{4.13}$$

where we used the identity

$$v^\rho \tilde{\nabla}_\rho K_{\mu\nu} = \mathcal{L}_v K_{\mu\nu} + 2 h^{\rho\sigma} K_{\mu\rho} K_{\nu\sigma} . \tag{4.14}$$

Taking the trace of (4.13) with respect to $h^{\mu\nu}$ we find

$$\mathcal{L}_v K - K^2 = 8\pi G_N c^{2-M} \left(\frac{d}{d-1} v^\mu T^v_\mu + \frac{1}{d-1} h^{\mu\nu} T^h_{\mu\nu} \right) , \tag{4.15}$$

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where d is the number of spatial dimensions and where we used the identity

$$K_{\mu\nu}\mathcal{L}_v h^{\mu\nu} = 2K^{\mu\nu}K_{\mu\nu}\,. \tag{4.16}$$

We note that $T^h_{\mu\nu}$ is only determined up to a term proportional to $\tau_\mu\tau_\nu$. Furthermore, Carroll boost symmetry tells us that the $v^\mu h^{\nu\lambda}$ projection of (4.13) vanishes. Without loss of generality we can contract (4.13) with $h^{\mu\sigma}$. Using (4.12a) and (4.15) we thus obtain

$$h^{\mu\sigma}\left[-\mathcal{L}_v K_{\sigma\nu} + KK_{\sigma\nu} - 2K_\sigma{}^\rho K_{\nu\rho}\right] = 8\pi G_N c^{M-2} h^{\mu\sigma}\left[T^h_{\sigma\nu} - \frac{1}{d-1}h_{\sigma\nu}\left(v^\rho T^v_\rho + h^{\kappa\lambda}T^h_{\kappa\lambda}\right)\right]. \tag{4.17}$$

If we take the trace of this equation we recover (4.15). In the absence of sources we recover (4.4c).

4.2 Coupling to electric Carrollian electrodynamics

We start by obtaining the action of the electric Carroll Maxwell action coupled to curved (Carrrollian) spacetime (see [8, 31] for Carrollian electrodynamics on flat space). One starts with the Maxwell action coupled to GR

$$S_{\text{Maxwell}} = -\frac{1}{4c\mu_0}\int d^{d+1}x\sqrt{-g}\,g^{\mu\rho}g^{\nu\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\,, \tag{4.18}$$

and expands $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + \mathcal{O}(c^2)$. The constant μ_0 is the vacuum magnetic permeability. The leading order action is the electric Carroll Maxwell theory (ECM)

$$S_{\text{ECM}} = \frac{1}{2c^2\mu_0}\int d^{d+1}x\,e v^\mu v^\rho h^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \tag{4.19}$$

where we used equations (3.9) and (3.10). As expected this is essentially the electric field squared. We can compute the relevant components of the energy momentum tensor. Using (4.5) we obtain $M = -2$ and

$$T^v_\mu = -\frac{1}{2\mu_0}\left(2h^{\nu\sigma}v^\rho F_{\mu\nu}F_{\rho\sigma} + \tau_\mu[v^\lambda v^\rho h^{\nu\sigma}F_{\lambda\nu}F_{\rho\sigma}]\right)\,, \tag{4.20}$$

$$T^h_{\mu\nu} = -\frac{1}{\mu_0}\left(v^\rho v^\sigma F_{\rho\mu}F_{\sigma\nu} - \frac{1}{2}h_{\mu\nu}[v^\lambda v^\rho h^{\alpha\sigma}F_{\lambda\alpha}F_{\rho\sigma}]\right) + A\tau_\mu\tau_\nu\,, \tag{4.21}$$

where A is undetermined. Note that the Ward identity $v^\mu h^{\nu\rho}T^h_{\mu\nu} = 0$ is correctly satisfied as well as the fact that the energy-momentum tensor (3.30) is traceless for $d = 3$. The equations of motion (4.12a), (4.12b) and (4.17) become in this case

$$K^{\mu\nu}K_{\mu\nu} - K^2 = -\frac{8\pi G_N}{c^4\mu_0}E^2\,, \tag{4.22}$$

$$h^{\mu\nu}\left(\tilde{\nabla}_\rho - \mathcal{L}_v\tau_\rho\right)h^{\rho\sigma}\left(K_{\sigma\mu} - Kh_{\sigma\mu}\right) = \frac{8\pi G_N}{c^4\mu_0}h^{\mu\nu}h^{\lambda\sigma}F_{\mu\lambda}E_\sigma\,, \tag{4.23}$$

$$h^{\mu\sigma}\left[-\mathcal{L}_v K_{\sigma\nu} + KK_{\sigma\nu} - 2K_\sigma{}^\rho K_{\nu\rho}\right] = \frac{8\pi G_N}{c^4\mu_0}h^{\mu\sigma}\left[-E_\sigma E_\nu + \frac{1}{d-1}h_{\sigma\nu}E^2\right]\,, \tag{4.24}$$

where we defined the electric field $E_\mu = -v^\rho F_{\rho\mu}$ and $E^2 = h^{\mu\nu}E_\mu E_\nu$.

4.3 Magnetic Carroll gravity and magnetic Carroll Maxwell

Both for the gravity side as well as the Maxwell side one can also consider the magnetic limit. Together with the electric theories, this then gives rise to four different sets of sourced equations of motions as one can couple electric/magnetic Carroll gravity to electric/magnetic Carroll Maxwell. We will not spell out in detail the equations of motion, but give below the corresponding actions.

Magnetic Carroll gravity. The LO action in the $c = 0$ expansion of GR is at order c^2 and defines the electric theory. The magnetic theory is obtained by going to the NLO action at order c^4 and adding a Lagrange multiplier that kills the LO action. The details can be found in [26]. The resulting theory is the magnetic Carroll gravity theory (MCG):

$$S_{\mathrm{MCG}} = \frac{c^4}{16\pi G_N}\int \mathrm{d}^{d+1}x\, e\left(\phi^{\mu\nu}K_{\mu\nu} + h^{\mu\nu}\tilde{R}_{\mu\nu}\right) , \tag{4.25}$$

where the Lagrange multiplier is $\phi^{\mu\nu}$. Here $\tilde{R}_{\mu\nu}$ denotes the Ricci tensor associated with the connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$.

Magnetic Carroll Maxwell. The action for magnetic Carroll Maxwell (MCM) coupled to Carroll gravity is given by

$$S_{\mathrm{MCM}} = \frac{1}{2\mu_0}\int \mathrm{d}^{d+1}x e\left[\chi_\mu h^{\mu\nu}v^\sigma F_{\nu\sigma} - \frac{1}{4}h^{\mu\rho}h^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\right] , \tag{4.26}$$

where χ_μ is a Lagrange multiplier setting the electric field to zero. In this case we have $M = 0$ (cf. equation (4.5)). This theory can also be coupled to the electric Carroll gravity theory, but we refrain from working out the details.

4.4 Carroll spacetimes: examples

In this subection, we study some examples[11] and consider Carrollian limits of Schwarzschild, Reissner-Nordström, de Sitter, and anti de Sitter metrics. We also discuss the generic structure of the geodesic equation in Carrollian geometries.

4.4.1 Schwarzschild black holes

Here we consider an example of a Carroll spacetime whose connection has torsion. It arises in the Carroll limit of the Schwarzschild metric

$$g = -c^2\left(1 - \frac{R}{r}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{R}{r}}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 , \tag{4.27}$$

where the Schwarzschild radius in terms of the mass is given by

$$R = \frac{2MG_N}{c^2} , \tag{4.28}$$

and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$.

[11] Some of these examples were discussed earlier in [8, 26, 54] and are included here with further details.

Electric limit and the Kasner spacetime. In this limit one keeps fixed the black hole energy, or more precisely the combination MG_N. This is similar to [8] if we write $MG_N = EG_C^{(el)}$, with

$$E = Mc^2 \,, \qquad G_C^{(el)} \equiv \frac{G_N}{c^2} \,, \tag{4.29}$$

which can both be kept fixed in the Carroll limit. Effectively, this limit describes the region inside the black hole where gravity is strong. Perhaps the most proper way of doing this, is to use Kruskal coordinates, but it is instructive to proceed with Schwarzschild coordinates in the region $r < R$. The first step in the Carroll limit then amounts to taking $R/r \gg 1$ and one finds the metric

$$\mathrm{d}s^2 = \frac{2MG_N}{r}\mathrm{d}t^2 - \frac{r}{2MG_N}c^2\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,. \tag{4.30}$$

Notice that effectively, this limit is also produced by the expansion around the singularity at $r = 0$, where the coordinates r and t reverse their role of space and time coordinates. It is known that the region close to the singularity is described by a Kasner-like metric [87]. Indeed, if we redefine

$$\tau = \frac{1}{\sqrt{2MG_N}} r^{3/2} \,, \tag{4.31}$$

we find

$$\mathrm{d}s^2 = -\frac{4}{9}c^2\mathrm{d}\tau^2 + \frac{1}{(H\tau)^{2/3}}\mathrm{d}t^2 + \frac{\tau^{4/3}}{H^{2/3}}\mathrm{d}\Omega^2 \,, \tag{4.32}$$

where $H \equiv \frac{1}{2MG_N}$, and the dimensions of the quantities are $[\tau] = [t] = s$ and $[H] = s^2 m^{-3}$. This is a Kasner-like metric with Kasner exponents

$$p_1 = -\frac{1}{3} \,, \qquad p_2 = p_3 = \frac{2}{3} \,. \tag{4.33}$$

When taking the Carroll limit, we keep H fixed and obtain the following quantities:

$$v^\tau = -\frac{3}{2} \,, \qquad h = \frac{1}{(H\tau)^{2/3}}\mathrm{d}t^2 + \frac{\tau^{4/3}}{H^{2/3}}\mathrm{d}\Omega^2 \,, \qquad K = -\frac{1}{2H^{2/3}\tau^{5/3}}\mathrm{d}t^2 + \frac{\tau^{1/3}}{H^{2/3}}\mathrm{d}\Omega^2 \,. \tag{4.34}$$

Notice that, in contrast to the magnetic limit, the extrinsic curvature $K = K_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ is nonzero. We checked that this solution solves equations (4.4). More generally, this solution falls into the class of general vacuum solutions to the electric theory given in [26].[12]

Magnetic limit and Carroll wormholes. There is another limit we can take, by not keeping MG_N fixed in the Carroll limit, but instead the Schwarzschild radius R. In terms of the mass and Newton's constant, such a limit can be taken by keeping the quantities

$$E = Mc^2 \,, \qquad G_C^{(m)} \equiv \frac{G_N}{c^4} \,, \tag{4.35}$$

fixed. We call this the magnetic limit and we read off the following quantities

$$v = v^t\partial_t \,, \qquad v^t = -\sqrt{\frac{r}{r-R}} \,, \qquad K_{\mu\nu} = 0 \,. \tag{4.36}$$

[12] We also note that this case of Kasner spacetime solutions was independently observed in [88] and we thank Marc Henneaux for a discussion on the appearance of Kasner geometry in the electric Carroll limit.

The extrinsic curvature vanishes as a consequence of the fact that the Carroll metric

$$h = \frac{1}{1 - \frac{R}{r}} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \,, \tag{4.37}$$

is static, so the Lie-derivative along v is zero. This metric is that of a constant time-slice of the Schwarzschild black hole.

We can now transform metric to isotropic coordinates, defined by

$$\rho \equiv \frac{1 + \sqrt{1 - \frac{R}{r}}}{1 - \sqrt{1 - \frac{R}{r}}} \,, \qquad r = \frac{R}{4} \left(\sqrt{\rho} + \frac{1}{\sqrt{\rho}} \right)^2 \,. \tag{4.38}$$

It was shown in [26] that this is indeed a solution of the magnetic Carroll gravity theory. In the patch on the outside of the black hole horizon, we have $\rho \in \ (1, \infty)$, but we extend it to $\rho \in \ (0, \infty)$. This defines an extension of the Carrollian Schwarzschild geometry. Then the resulting Carroll metric becomes conformally flat

$$h = \left(\frac{(\rho + 1)^2 R}{4\rho^2} \right)^2 \left(\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega^2 \right) \,. \tag{4.39}$$

The vector field v in these coordinates becomes

$$v = -\frac{\rho + 1}{\rho - 1} \partial_t \,. \tag{4.40}$$

Notice that h is invariant under $\rho \mapsto 1/\rho$ and that v changes sign under this map. This is the two-fold $\mathbb{Z}_2$-symmetry that gives us the familiar diagram of the Einstein-Rosen bridge [89]. Inspired by this, we refer to (4.39) as the Carroll wormhole.

It is easy to recover the original black hole entropy from the Carroll wormhole. One looks at the point where the neck of the wormhole is smallest, which is at the $\mathbb{Z}_2$ fixed point $\rho = 1$, corresponding to the original location of the horizon, $r = R$. The area of the sphere at $\rho = 1$ then gives the entropy of the original black hole via $S = \frac{A}{4}$, as $h(\rho = 1) = R^2 \mathrm{d}\Omega^2$.[13]

4.4.2 Solutions with cosmological constant

de Sitter. The de Sitter spacetime is one that has multiple widely used coordinate systems. We will consider comoving and static coordinates. The reason that we consider both coordinate systems is because the comoving coordinates make use of the Hubble constant H, while the static coordinates include the Hubble radius

$$R_H = \frac{c}{H} \,. \tag{4.41}$$

Similar to the Schwarzschild black hole, we can take two limits, depending on whether we take H or R_H fixed in the Carroll limit.

[13] The entropy and temperature of the Schwarzschild black hole in units of magnetic Carroll gravity, i.e. in terms of R and $G_C^{(m)}$, are given by $S = \frac{k_B}{c\hbar} \frac{\pi R^2}{G_C^{(m)}}$ and $T = \frac{c\hbar}{k_B} \frac{1}{4\pi R}$, where k_B is Boltzmann's constant and $\hbar$ is Planck's constant. We refer to [90] for more details about the thermodynamics and definition of Carrroll black hole geometries such as the one described here.

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The coordinate transformation between these two coordinate systems depends on c, which becomes singular in the Carroll limit. Therefore, this results in two inequivalent (by diffeomorphisms and local Carroll boosts) Carroll limits, dubbed the electric and magnetic limits.

Electric limit — Inflationary/comoving coordinates. The de Sitter metric in comoving coordinates is

$$g = -c^2 \mathrm{d}t^2 + \mathrm{e}^{2Ht}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) . \tag{4.42}$$

In the Carroll limit we keep H fixed and obtain the following quantities:

$$v^t = -1 , \qquad h = \mathrm{e}^{2Ht}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) , \qquad K_{ij} = H\mathrm{e}^{2Ht}\delta_{ij} . \tag{4.43}$$

Note that the extrinsic curvature is non-zero and $i = 1, 2, 3$ runs over spatial indices only. The Carroll metric is again conformally flat. If we add a cosmological constant to the LO action (4.1) then the above solution solves the corresponding equations of motion.

The electric limit of de Sitter was used in [8]. In this limit, the cosmological constant was kept fixed, as well as the rescaled Newton constant $G_C^{(el)} = G_N/c^2$, similar as in the electric limit of the Schwarzschild black hole. The precise relation between the relevant quantities is

$$H^2 = \frac{8\pi G_C^{(el)}}{3}\Lambda , \tag{4.44}$$

and Λ has dimensions of energy. Differently from the black hole, is that the Schwarzschild radius goes to infinity, whereas the de Sitter radius $R_H = c/H$ goes to zero.

Electric and magnetic limit — Static coordinates. To get the de Sitter metric in static coordinates, we start with the metric in comoving coordinates. The first step is to convert to spherical coordinates on the spatial part of the metric, and substitute $H = \frac{c}{R_H}$:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{e}^{2ct/R_H}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right) . \tag{4.45}$$

Now we perform the coordinate transformation:

$$\rho := r\mathrm{e}^{ct/R_H}, \qquad \tau := t - \frac{R_H}{2c}\log\left(-1 + \frac{r^2}{R_H^2}\mathrm{e}^{2ct/R_H}\right) , \tag{4.46}$$

to get the de Sitter metric in static coordinates:

$$\mathrm{d}s^2 = -\left(1 - \frac{\rho^2}{R_H^2}\right)c^2\mathrm{d}\tau^2 + \frac{\mathrm{d}\rho^2}{1 - \frac{\rho^2}{R_H^2}} + \rho^2\mathrm{d}\Omega^2 . \tag{4.47}$$

We note that these coordinates are only valid for $0 < \rho < R_H$. Note also that the expressions of the static coordinates in terms of the comoving coordinates depend on c, and are not well defined in the $c \to 0$ limit. For the metric (4.47), we have the following quantities:

$$\tau_\tau = \sqrt{1 - \frac{\rho^2}{R_H^2}} , \qquad v^\tau = -\sqrt{\frac{R_H^2}{R_H^2 - \rho^2}} , \qquad K_{\mu\nu} = 0 . \tag{4.48}$$

Note that now the extrinsic curvature of de Sitter spacetime is equal to 0. This is caused by the fact that we have taken a different Carroll limit, namely one in which R_H instead of H is kept fixed. In this limit, the Carroll metric is

$$h = \frac{\mathrm{d}\rho^2}{1 - \frac{\rho^2}{R_H^2}} + \rho^2 \mathrm{d}\Omega^2 \,. \tag{4.49}$$

The relation between the Hubble radius and the cosmological constant is

$$R_H^2 = \frac{3}{8\pi G_C^{(m)} \Lambda} \,, \qquad G_C^{(m)} = \frac{G_N}{c^4} \,. \tag{4.50}$$

Notice that, because now we kept $G_C^{(m)}$ fixed, we can also keep the positive energy density Λ fixed in the Carroll limit.

One can also take the electric limit of the static dS patch and this gives yet another space (not diffeomorphic to the electric limit of the FLRW form of the dS metric). This would give $R_H = cH^{-1}$ and

$$v = -H\rho\partial_\rho \,, \qquad h = H^2 \mathrm{d}\tau^2 + \rho^2 \mathrm{d}\Omega^2 \,. \tag{4.51}$$

If we define $R = \tau$ and $T = H^{-1} \log \rho$, then we have

$$v = -\partial_T \,, \qquad h = e^{2HT} \left(H^2 \mathrm{d}R^2 + \mathrm{d}\Omega^2 \right) \,. \tag{4.52}$$

This Carroll geometry is not diffeomorphic to the electric limit of the dS metric in comoving coordinates. This is a consequence of the fact that the coordinate transformation (4.46) is not analytic in c.

Anti-de Sitter. Let us consider the case of anti-de Sitter (AdS) spacetimes. Global coordinates for AdS can be obtained by considering (4.47) and taking Λ negative such that $0 < \rho < \infty$ and

$$\mathrm{d}s^2 = -\left(1 + \frac{\rho^2}{R_{\mathrm{AdS}}^2}\right) c^2 \mathrm{d}\tau^2 + \frac{\mathrm{d}\rho^2}{1 + \frac{\rho^2}{R_{\mathrm{AdS}}^2}} + \rho^2 \mathrm{d}\Omega^2 \,, \tag{4.53}$$

where now

$$R_{\mathrm{AdS}}^2 = -\frac{3c^4}{8\pi G_N \Lambda} \,, \tag{4.54}$$

and where Λ is the negative energy density of the space, which we will treat as independent of c.

In the **electric limit** $G_C^{(el)} = G_N/c^2$ is kept fixed as $c \to 0$, which results in $\tilde{H} := c/R_{\mathrm{AdS}}$ being constant. As a result we find at leading order

$$v = -\tilde{H}\rho\partial_\rho \,, \quad h = -\tilde{H}^2 \rho^2 \mathrm{d}\tau^2 + \rho^2 \mathrm{d}\Omega^2 \,, \tag{4.55}$$

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which is the Poincaré patch with spherical slicing. Note however that h now has signature $(0,-1,1,1)$ (as opposed to $(0,1,1,1)$). These spacetimes are also known as pseudo-Carrollian.[14]

For the **magnetic limit** we keep $G_C^{(m)}=\frac{G_N}{c^4}$ fixed as $c\to 0$, which means that $R_{\rm AdS}$ will remain constant. In this limit we find

$$v=-\left(1+\frac{\rho^2}{R_{\rm AdS}^2}\right)^{-1/2}\partial_\tau\,,\quad h=\frac{\mathrm{d}\rho^2}{1+\frac{\rho^2}{R_{\rm AdS}^2}}+\rho^2\mathrm{d}\Omega^2\,. \tag{4.56}$$

4.4.3 Reissner-Nordström

Considering the Carroll limit in the context of the Schwarzschild metric enables one to study a resulting metric outside the Schwarzschild horizon (magnetic limit) and a metric inside the Schwarzschild horizon (electric limit). Here we point out that for a Reissner-Nordström black hole, which classically has an inner and an outer horizon, the electric limit applies to the geometry between the inner and the outer horizon. This last metric yields a charged deformation of the Kasner metric.

The Reissner-Nordström metric is given by

$$\mathrm{d}s^2=-\left(1-\frac{R_S}{r}+\frac{R_{Q,P}^2}{r^2}\right)c^2\mathrm{d}t^2+\left(1-\frac{R_S}{r}+\frac{R_{Q,P}^2}{r^2}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2\,, \tag{4.57}$$

where

$$R_S=\frac{2G_NM}{c^2}\,,\quad R_{Q,P}^2=\frac{1}{4\pi}\left(\frac{Q^2}{\epsilon_0}+P^2\mu_0\right)\frac{G_N}{c^4}\,. \tag{4.58}$$

In here ϵ_0 and μ_0 are the electric and magnetic constants and Q and P the electric and magnetic charges respectively, and the units for Q is Coulomb (C), and for P it is $C\,m\,s^{-1}$. The geometry is supported by a gauge field that is given by

$$A=-\frac{1}{4\pi\epsilon_0}\frac{Q}{r}\mathrm{d}t-\frac{\mu_0}{4\pi}P\cos\theta\mathrm{d}\phi\,. \tag{4.59}$$

The speed of light is given by $c=1/\sqrt{\epsilon_0\mu_0}$ and so the Carroll limit $c\to 0$ can be reached by taking either $\epsilon_0\to\infty$ or $\mu_0\to\infty$, keeping the magnetic or electric constant fixed respectively, and in such a way that the quantization condition for the charges is preserved. These two limits are called magnetic and electric Carroll limits, which we discuss now.

In section 4.2 we looked at electric and magnetic limits of Maxwell. We expanded the gauge field such that it is $O(1)$ plus corrections. The LO action only sees the $O(1)$ part of the electric part of the gauge field, whereas the NLO action sees the $O(1)$ part of the magnetic part of the gauge field, essentially the spatial part of A. The coupling constant of the LO electric theory is $c^2\mu_0=\epsilon_0^{-1}$ and the coupling constant of the magnetic theory is μ_0.

[14] In this case the $c=0$ expansion of the Lorentzian metric reads

$$g_{\mu\nu}=h_{\mu\nu}+c^2\left(\tau_\mu\tau_\nu-\Phi_{\mu\nu}\right)+\mathcal{O}(c^4)\,,$$

which should be contrasted with (3.9). The plus sign in front of $\tau_\mu\tau_\nu$ makes it that now $h_{\mu\nu}$ has signature $(0,-1,1,1)$. We still have $v^\mu h_{\mu\nu}=0$.

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Comparing this with the results of the current section we see that the gauge field in (4.59) needs to be order $O(1)$. This can be achieved either by keeping $\epsilon_0 c^2$ fixed which implies that $\epsilon_0 \to \infty$ and μ_0 is fixed. This is the magnetic limit. Or, alternatively, we keep ϵ_0 fixed and take $P = c^2 \tilde{P}$, so that $c \to 0$ implies $\mu_0 \to \infty$. This is the electric limit.

Electric limit. In this case we take Q^2/ϵ_0 to be constant when taking the Carroll limit and consider $\mu_0 \to \infty$. This seems to only make sense when the magnetic charge is zero, so henceforth we set $P = 0$. As before, when taking the electric limit, we introduce $E = Mc^2$ and $G_C^{(el)} = G_N/c^2$ which we keep fixed. Then we find

$$v = -\left[\frac{2G_C E}{r}\left(1 - \frac{Q^2}{8\pi\epsilon_0 E}\frac{1}{r}\right)\right]^{1/2}\frac{\partial}{\partial r}\,, \tag{4.60}$$

$$h = \frac{2EG_C}{r}\left(1 - \frac{Q^2}{8\pi\epsilon_0 E}\frac{1}{r}\right)\mathrm{d}t^2 + r^2\mathrm{d}\Omega^2\,. \tag{4.61}$$

The Carroll data v and h are not defined at $r = b := Q^2/(8\pi\epsilon_0 E)$ and we need to restrict $r \in (b, \infty)$ in order that h is positive semi-definite. This limit describes the region between the inner and outer horizon of the RN black hole with the outer horizon sent to infinity. If we define $a = 2EG_C$ then the inner and outer horizons of the RN metric are at $r_\pm$ given by

$$r_\pm = \frac{a}{2c^2}\left(1 \pm \sqrt{1 - 4c^2 b/a}\right)\,. \tag{4.62}$$

Expanding this around $c = 0$ we see that indeed r_+ goes to infinity while r_- goes to b.

Let us perform the following coordinate transformation, $(t, r) \mapsto (\rho, T)$, defined by

$$\rho = t\,, \qquad \frac{\partial r}{\partial T} = a^{1/2}\frac{\sqrt{r-b}}{r}\,, \tag{4.63}$$

then we find for v and h,

$$v = -\frac{\partial}{\partial T}\,, \tag{4.64}$$

$$h = \frac{a}{r}\left(1 - \frac{b}{r}\right)\mathrm{d}\rho^2 + r^2\mathrm{d}\Omega^2\,, \tag{4.65}$$

where $r = r(T)$. The (electric) Carroll gauge field becomes

$$A = -\frac{1}{4\pi\epsilon_0}\frac{Q}{r(T)}d\rho\,. \tag{4.66}$$

This geometry is a kind of "charge deformation" of the Kasner geometry. It has a non-trivial electric field since the vector potential has a radial component which depends on time. It can be checked that equations (4.66) and (4.66) satisfy equations (4.22)–(4.24).

Magnetic limit. In the magnetic limit we keep both R_S fixed and send $\epsilon_0 \to \infty$ keeping μ_0 and the charges Q and P fixed. Keeping R_S fixed can be achieved by keeping $E = Mc^2$ and $G_C^{(m)} = G_N/c^4$ fixed as before. In this limit, we get

$$R^2_{Q,P} \to R^2_P = \frac{\mu_0}{4\pi}P^2\,G_C^{(m)}\,. \tag{4.67}$$

Notice that the electric charge has dropped out and only the magnetic charge survives, as expected from taking a magnetic limit. The Carroll metric obtained in this limit then is

$$v = -\left(1 - \frac{R_S}{r} + \frac{R_P^2}{r^2}\right)^{-1/2} \frac{\partial}{\partial t}\,, \tag{4.68}$$

$$h = \left(1 - \frac{R_S}{r} + \frac{R_P^2}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\,, \tag{4.69}$$

so we find a case similar to the Schwarzschild wormhole. Assuming that $1 - \frac{R_S}{r} + \frac{R_P^2}{r^2}$ has two distinct real roots (which will be the case provided $R_S^2 > 4R_P^2$), we see that h is positive semi-definite and v is real for $r > r_+$ and $r < r_-$, where r_+ is the outer and r_- the inner horizon.

The extrinsic curvature vanishes again. The geometry is then supported by a magnetic field only, given by $B = \frac{\mu_0 P}{4\pi} \frac{1}{r^2} dr$, and survives in the Carroll limit where μ_0 and P are fixed.

We define a new radial coordinate ρ such that

$$\frac{\mathrm{d}\rho}{\rho} = \frac{\mathrm{d}r}{\sqrt{(r - r_-)(r - r_+)}}\,. \tag{4.70}$$

This allows us to write

$$h = \frac{r^2(\rho)}{\rho^2} \left(\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega^2\right)\,, \tag{4.71}$$

where r is now a function of ρ. This function can be found by integrating (4.70) and inverting which leads to

$$r = \frac{1}{2}\left(r_+(x+1) - r_-(x-1)\right)\,, \tag{4.72}$$

$$x = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\,. \tag{4.73}$$

In terms of ρ we can write v as

$$v = -\frac{2\left(r_+(x+1) - r_-(x-1)\right)}{\left(r_+ - r_-\right)\left(\rho - \frac{1}{\rho}\right)} \partial_t\,. \tag{4.74}$$

The region outside the outer horizon corresponds to $r > r_+$. This is equivalent to demanding $x > 1$. If we take $\rho > 1$ then this captures the region $r > r_+$. Since x is symmetric under $\rho \leftrightarrow 1/\rho$. We see that we can extend the ρ coordinate to also cover $0 < \rho < 1$. This gives us again the wormhole geometry. We see that h is invariant under $\rho \leftrightarrow 1/\rho$ whereas v changes sign.

We finally note that the magnetically charged wormhole solution described above is expected to be a solution of MCG coupled to MCM. It would be an interesting exercise to explicitly check this using the actions (4.25) and (4.26).

4.5 Geodesics

To discuss the properties of the geodesics, we start with the action of a massive relativistic particle coupled to a pseudo-Riemannian metric

$$S = \mp|m|c \int \sqrt{\mp g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}\, \mathrm{d}\sigma = \int L\, \mathrm{d}\sigma\,, \tag{4.75}$$

where σ is a worldline parameter and there is worldline diffeomorphism invariance of the action. The upper signs will be chosen for timelike geodesics, and the lower signs for spacelike geodesics. The momenta $p_\mu \equiv \partial L/\partial \dot{x}^\mu$ (the dot denotes the derivative with respect to σ) satisfy

$$p^\mu p_\mu \pm |m|^2 c^2 = 0\,. \tag{4.76}$$

One can write this in the more familiar way, $p^\mu p_\mu + m^2c^2 = 0$, if for spacelike geodesics we let the particle be tachyonic with $m^2 = -|m|^2 < 0$.

The equations of motion are

$$g_{\alpha\nu} \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} + \frac{1}{2}\left[\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right] \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0\,, \tag{4.77}$$

where, up to rescaling and translation, τ is the proper time for a timelike geodesic or the proper length s for a spacelike geodesic ($\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2$). For the proper time/length, one has

$$\varepsilon \equiv -g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = c^2, 0, -1\,, \tag{4.78}$$

for timelike, null, or spacelike geodesics, respectively. Notice the factor of c^2 for timelike geodesics which will vanish in the Carroll limit. This implies that timelike geodesics become null in the Carroll limit, and since $c \to 0$, light cones close up and a timelike Carroll particle in this class can no longer move.

We can therefore concentrate on the spacelike geodesics, for which one should read (4.78) with τ replaced by s. The units therefore are different and ε is dimensionless and equal to unity. From here on, we therefore use s whenever we talk about spacelike geodesics.

We could contract the equations of motion (4.77) with the inverse metric to get the more familiar geodesic equation

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}s^2} + \Gamma^\rho_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0\,, \tag{4.79}$$

but we don't do so below in order to make the c-expansion easier. This way, we do not need to refer to the connection and any of its properties, but instead expand the metric again as in (3.9),

$$g_{\mu\nu} = h_{\mu\nu} + c^2 \overline{\Phi}_{\mu\nu} + O(c^4)\,. \tag{4.80}$$

Furthermore, we assume that, given a solution $x^\rho(s)$, that $\frac{\mathrm{d}x^\rho}{\mathrm{d}s}$ has a Taylor expansion around $c = 0$,

$$\frac{\mathrm{d}x^\rho}{\mathrm{d}s} = \sum_{i=0}^{\infty} c^i \left.\frac{\mathrm{d}x^\rho}{\mathrm{d}s}\right|_{(i)}\,, \tag{4.81}$$

where (i) indicates the expansion order. For spacelike geodesics an expression like the one above should always be possible whenever a Carroll limit exists. Now we can derive new equations order by order in powers of c. In the Carroll limit, we are interested in the leading term. To lowest order, we find

$$h_{\alpha\nu} \left.\frac{\mathrm{d}^2 x^\nu}{\mathrm{d}s^2}\right|_{(0)} + \frac{1}{2}\left(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}\right) \left.\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right|_{(0)} \left.\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\right|_{(0)} = 0\,. \tag{4.82}$$

This equation follows from an action given by (from now on we will drop the subindex (0))

$$S_{\text{Carroll}} = p_0 \int \sqrt{h_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}\,\mathrm{d}\sigma\,, \tag{4.83}$$

where p_0 is a constant with the dimensions of a momentum, independent of c, and which we take to be positive.

In contrast to the Lorentzian case, the action is positive definite, since the metric $h_{\mu\nu}$ is of rank $D-1$ and positive definite along the spatial directions. The minima of the action then correspond to the case where the action vanishes. Those are the particles at rest. In appropriate coordinates $x^\mu = \{t, x^i\}$, the particles at rest satisfy

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}s} = 0\,, \tag{4.84}$$

and $\mathrm{d}t/\mathrm{d}s$ is undetermined because h is of rank $D-1$. These Carroll geodesics correspond to the limit of timelike geodesics.

The other set of solutions is less trivial, and corresponds to non-trivial solutions of (4.82). By contracting this equation with v^α, one easily finds other identities, such as

$$K_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0\,. \tag{4.85}$$

In terms of the Carroll connection (3.18), it is an easy exercise to rewrite (4.82) as follows,

$$h_{\sigma\rho}\left[\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}s^2} + \tilde{\Gamma}^\rho_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\right] = -K_{\sigma\mu}\tau_\nu\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\,, \tag{4.86}$$

where we made use of (4.85).

Notice that for the magnetic Carroll limit, $K_{\mu\nu} = 0$ by definition and the geodesic equation in (4.86) takes a more familiar form.

Notice furthermore that $t(s)$ is again undetermined when $K_{\mu\nu} = 0$, as a consequence of h having only spatial components. In fact, for this magnetic case, t does not appear in the action.

The momenta for the action (4.83) satisfy

$$p_\mu h^{\mu\nu} p_\nu = p_0^2\,, \tag{4.87}$$

as one can easily check. In the adapted coordinates $x^\mu = \{t, x^i\}$, it implies $\vec{p}^{\,2} = p_i h^{ij} p_j = p_0^2$ together with the constraint $E = 0$ as the Hamiltonian vanishes on-shell. These correspond precisely to the type of particles found in [8], obtained from the Carroll limit of relativistic tachyonic particles, but now generalized to arbitrary Carroll geometries.

Examples of geodesics were worked out in [91]. It includes particles traveling through the Carroll wormhole.

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5 The energy-momentum tensor of a putative Carroll fluid

Despite the fact that, as far as we know, there currently does not exist a bona fide microscopic quantum system with well-defined Carrollian thermodynamcs, the notion of Carrollian fluids and energy-momentum tensors that allegedly describe such fluids frequently appears in the literature.

To facilitate comparison with the literature we will consider some aspects of energy-momentum tensors for would-be Carrollian fluids. Despite their purely hypothetical nature, we will in this section (and only in this section) use the term "Carrollian fluid" to refer to this possibly empty set of quantum systems. Energy-momentum tensors that take a perfect Carrollian fluid[15] form can also appear as expectation values of the energy-momentum tensor in particular states in well-defined quantum theories and/or curved backgrounds. These energy-momentum tensors are perfectly fine but one should not interpret quantities such as energy density and pressure as actual thermodynamical quantities.

In this section we construct two distinct types of Carroll perfect fluid energy-momentum tensors using two different methods that give coinciding results: an expansion around $c = 0$ starting from the relativistic fluid energy-momentum tensor and by employing the hydrostatic partition function.

5.1 Carroll expansion of perfect fluid energy-momentum tensors

To illustrate these statements let us take a look at the $c = 0$ expansion of the energy-momentum tensor of a relativistic perfect fluid on an arbitrary curved background. Consider the energy-momentum tensor of relativistic perfect fluid

$$T^{\mu}{}_{\nu} = \mp \frac{\tilde{\mathcal{E}} + P}{c^2} U^{\mu} U_{\nu} + P \delta^{\mu}_{\nu} \,, \tag{5.1}$$

where $U^2 = \pm c^2$ and $\tilde{\mathcal{E}}$ and P are the energy and pressure associated to the fluid, respectively. The upper (lower) sign corresponds to a fluid with spacelike (timelike) velocities. We parameterize the velocity U^{μ} in the following explicit manner

$$U^{\mu} = \frac{u^{\mu}}{\sqrt{\mp\left(1 - \frac{u^2}{c^2}\right)}} \,, \tag{5.2}$$

where $u^2 = \Pi_{\mu\nu} u^{\mu} u^{\nu}$, $T_{\mu} u^{\mu} = 1$ and $u^{\mu} \Pi_{\mu\nu}$ is the 'three'-velocity.[16]

We can consider two distinct expansions for the relativistic fluid vector U^{μ}, depending on whether we use the upper or lower sign. For the upper sign (spacelike case) we expand

$$U^{\mu} = c \frac{u^{\mu}}{u} + O(c^3) \,, \tag{5.3}$$

[15] Transport was studied in for example [92–94].

[16] As a consequence of the $U^2 = \pm c^2$ normalization choice, U is real for either sign. Alternatively one can always choose the normalization $U^2 = -c^2$, with the consequence that U^{μ} is imaginary for the spacelike choice.

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where $u = \sqrt{u^2}$. For the lower sign (timelike case) we need to take into account that $u < c$, even when we take $c \to 0$. As such we assume the leading order expansion around zero of $u = \sqrt{h_{\mu\nu} u^\mu u^\nu}$ to be of order c^2. We find

$$U^\mu = -v^\mu + c^2 U^\mu_{(2)} + O(c^4)\,, \tag{5.4}$$

where $U^\mu_{(2)}$ is some subleading term and $v^\mu \tau_\mu = -1$ and $v^\mu h_{\mu\nu} = 0$.

In both cases (both signs) we take $\tilde{\mathcal{E}}$ and P to be of order c^{2-N} and we will denote the leading order terms in the expansion of the energy density and the pressure by the same symbols. Here N is defined in (3.26). We then obtain for the upper sign the following Carroll energy-momentum tensor

$$\text{spacelike}: \qquad (T_{\text{Car}})^\mu{}_\nu = -\frac{\tilde{\mathcal{E}} + P}{u^2} u^\mu h_{\nu\rho} u^\rho + P \delta^\mu_\nu\,. \tag{5.5}$$

This agrees with a result found in [57]. For the lower sign we obtain

$$\text{timelike}: \qquad (T_{\text{Car}})^\mu{}_\nu = (\tilde{\mathcal{E}} + P) v^\mu \left(\hat{\tau}_\nu - h_{\nu\rho} U^\rho_{(2)}\right) + P \delta^\mu_\nu\,. \tag{5.6}$$

The combination $\hat{\tau}_\nu - h_{\nu\rho} U^\rho_{(2)}$ is the leading order term in the expansion of U_ν for which we have

$$U_\nu = c^2 U_{(2)\nu} + O(c^4)\,, \tag{5.7}$$

with $U_{(2)\nu} = -\left(\hat{\tau}_\nu - h_{\nu\rho} U^\rho_{(2)}\right)$.

5.2 Hydrostatic partition function

The hydrostatic partition function [95, 96] is a thermal partition function evaluated on a weakly curved and stationary background geometry. Due to the stationarity of the background, one can construct the hydrostatic partition function explicitly by relating thermodynamical quantities to the background geometry and a corresponding Killing vector. This method has been applied to non-Lorentzian setups in, e.g., [97, 98].

It is instructive to derive the two distinct Carroll invariant perfect fluid energy-momentum tensors given in (5.5) and (5.6), respectively, directly using the hydrostatic partition function in a Carroll geometry and taking an expansion around $c \to 0$ of the relativistic hydrostatic partition function. Let us first review the relativistic setup.

Relativistic hydrostatic partition function. Let β^μ be a Killing vector of the relativistic geometry

$$\mathcal{L}_\beta g_{\mu\nu} = 0\,. \tag{5.8}$$

The choice of β^μ enables one to make a choice of local frame invariant temperature $\tilde{T}$, where the hydrostatic partition function takes the form $\mathcal{L} = eP(\tilde{T})$ where $-e^2 = \det g_{\mu\nu}$ and

$$g_{\mu\nu}\beta^\mu\beta^\nu = \pm\frac{c^2}{\tilde{T}^2}\,, \tag{5.9}$$

where we choose $\beta^\mu = U^\mu/\tilde{T}$ to be spacelike or timelike oriented corresponding to the four-velocity normalization $U^2 = \pm c^2$. To arrive at the energy-momentum tensor corresponding to the hydrostatic setup, we vary with respect to the metric keeping β^μ fixed. On top of that we impose the thermodynamic relations $\frac{\partial P}{\partial \tilde{T}} = \tilde{s}$ and $\tilde{s}\tilde{T} = \tilde{\mathcal{E}} + P$. Defining the energy-momentum tensor through $\delta\mathcal{L} = \frac{1}{2} e T^{\mu\nu}\delta g_{\mu\nu}$ we find the spacelike (upper sign) or timelike (lower sign) relativistic perfect fluid energy-momentum tensor:

$$T^{\mu\nu} = \mp \frac{\tilde{\mathcal{E}} + P}{c^2} U^\mu U^\nu + P g^{\mu\nu} \,, \tag{5.10}$$

which was presented in (5.1).

Spacelike Carroll case. To consider Carroll perfect fluids we require β^μ to be a Killing vector of the Carroll geometry, i.e.

$$\mathcal{L}_\beta h_{\mu\nu} = 0 \,, \qquad \mathcal{L}_\beta v^\mu = 0 \,. \tag{5.11}$$

The hydrostatic partition function has a Lagrangian of the form $\mathcal{L} = eP(\tilde{T})$ where $-e^2 = \det(-\tau_\mu\tau_\nu + h_{\mu\nu})$ and where $\tilde{T}$ is related to β^μ. There are two options, either β^μ spacelike oriented

$$h_{\mu\nu}\beta^\mu\beta^\nu = \frac{c^2}{\tilde{T}^2} \,, \tag{5.12}$$

or we say that β^μ is proportional to v^μ, i.e. timelike oriented, and

$$\tau_\mu \beta^\mu = \frac{1}{\tilde{T}} \,. \tag{5.13}$$

This condition is Carroll boost invariant because $h_{\mu\nu}\beta^\nu = 0$.

For the spacelike case we take $\beta^\mu = c\beta^\mu_{(1)} + O(c^3)$ where $\beta^\mu_{(1)} = u^\mu/(\sqrt{u^2}\tilde{T})$ and subsequently drop the (1) subscript. The condition $T_\mu u^\mu = 1$ implies $\tau_\mu u^\mu = 1$. This leads to the Carrollian hydrostatic partition function

$$\mathcal{L} = eP(\tilde{T}) \,, \tag{5.14}$$

where $\tilde{T}$ is defined as in (5.12).

The variation of the hydrostatic Lagrangian with respect to the Carroll geometry (keeping β^μ fixed), as derived in (3.29), can be written as

$$\delta\mathcal{L} = e\left(-\mathcal{T}^\mu\delta\tau_\mu + \frac{1}{2}\mathcal{T}^{\mu\nu}\delta h_{\mu\nu}\right) \,. \tag{5.15}$$

For (5.12) and (5.14) we find

$$\mathcal{T}^\mu = P v^\mu \,, \qquad \mathcal{T}^{\mu\nu} = P h^{\mu\nu} - \frac{\tilde{\mathcal{E}} + P}{u^2} u^\mu u^\nu \,, \tag{5.16}$$

where we used that $\frac{\partial P}{\partial \tilde{T}} = \tilde{s}$ and that $\tilde{s}\tilde{T} = \tilde{\mathcal{E}} + P$. The Carroll energy-momentum tensor thus is

$$(T_{\text{Car}})^\mu{}_\nu = -\mathcal{T}^\mu\tau_\nu + \mathcal{T}^{\mu\rho}h_{\rho\nu} = P\delta^\mu{}_\nu - \frac{\tilde{\mathcal{E}} + P}{u^2} u^\mu u^\rho h_{\rho\nu} \,. \tag{5.17}$$

This reproduces the general form of the energy-momentum tensor obtained in (5.5) from the $c = 0$ expansion of the spacelike fluid.

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Timelike Carroll case. If we view the timelike case as an expansion from the relativistic case, we take $\beta^\mu = \beta^\mu_{(0)} + c^2\beta^\mu_{(2)} + O(c^4)$. In this case we have

$$g_{\mu\nu}\beta^\mu\beta^\nu = h_{\mu\nu}\beta^\mu_{(0)}\beta^\nu_{(0)} + c^2\bar{\Phi}_{\mu\nu}\beta^\mu_{(0)}\beta^\nu_{(0)} + 2c^2h_{\mu\nu}\beta^\mu_{(0)}\beta^\nu_{(2)} + O(c^4) = -c^2\tilde{T}^{-2}\,. \tag{5.18}$$

This requires the constraint $h_{\mu\nu}\beta^\mu_{(0)}\beta^\nu_{(0)} = 0$ so that $\beta^\mu_{(0)}$ is proportional to v^μ. Using $\bar{\Phi}_{\mu\nu} = \Phi_{\mu\nu} - \tau_\mu\tau_\nu$ with $v^\mu v^\nu\Phi_{\mu\nu} = 0$ we then find $\tau_\mu\beta^\mu_{(0)} = \tilde{T}^{-1}$ where we have taken the positive root. This leads to (5.13) after dropping the subscript (0). Hence, in order to do the timelike expansion we need to supplement the hydrostatic partition function with a Lagrange multiplier term that enforces $h_{\mu\nu}\beta^\nu = 0$. In the timelike case we thus end up with the Carrollian 'hydrostatic' partition function

$$\mathcal{L} = eP(\tilde{T}) + e\chi^\mu h_{\mu\nu}u^\nu\,, \tag{5.19}$$

where χ^μ is a Lagrange multiplier field and where $\tilde{T}$ is defined as in (5.13) and $\beta^\mu\tilde{T} = u^\mu$. In the previous sentence, we wrote hydrostatic partition function in quotation marks since as we will see shortly it does not actually define a perfect fluid energy-momentum tensor.

The term involving the Lagrange multiplier field can also be interpreted as required in the context of the relativistic expansion. Namely, if we use (5.18) for the temperature $\tilde{T}$, there is the risk that the leading order term becomes positive. The introduced constraint makes sure that this is circumvented.

Using a similar approach for (5.13) and (5.19) we find, supplied with $h_{\mu\nu}\beta^\nu = 0$ coming from varying χ, that

$$\mathcal{T}^\mu = -\tilde{\mathcal{E}}v^\mu\,, \qquad \mathcal{T}^{\mu\nu} = Ph^{\mu\nu} + 2\chi^{(\mu}v^{\nu)}, \tag{5.20}$$

where we used that $\frac{\partial P}{\partial\tilde{T}} = \tilde{s}$, $\tilde{s}\tilde{T} = \tilde{\mathcal{E}} + P$ and $\beta^\mu\tilde{T} = u^\mu = -v^\mu$. Combining these currents, the Carroll energy-momentum tensor thus is

$$(T_{\text{Car}})^\mu{}_\nu = -(\tilde{\mathcal{E}} + P)v^\mu U_{(2)\nu} + P\delta^\mu{}_\nu\,, \tag{5.21}$$

where we introduced $U_{(2)\nu} = -(\tau_\nu + \frac{\chi^\rho h_{\rho\nu}}{\tilde{\mathcal{E}}+P})$. This reproduces the general form of the energy-momentum tensor obtained in (5.6) from the $c = 0$ expansion of the timelike fluid. From the fact that $U_{(2)\nu}$ is in fact a Lagrange multiplier, an additional hydrodynamic quantity that is not reflected in the thermodynamics, it is clear that this energy-momentum tensor falls outside the scope of conventional fluids.

5.3 On microscopic Carroll gasses

Before we consider Carroll gasses in this subsection, let us briefly review the description of a classical Boltzmann gas of free relativistic particles. We express the relativistic velocity U^μ and relativistic momentum P_μ to spatial momentum p_i and energy E via

$$U^\mu = \gamma(1, v^i)\,, \quad P_\mu = (-E, p_i)\,, \quad P^2 = -m^2c^2\,, \quad \Rightarrow \quad E^2 = c^2\vec{p}^2 + m^2c^4\,, \tag{5.22}$$

where γ is the Lorentz factor and m the mass of the particles. The single-particle partition function[17] is defined as

$$Z_1(T,V,v^i) = \frac{1}{h^d}\int \mathrm{d}^d x \int \mathrm{d}^d p\, e^{\tilde{\beta} U^\mu P_\mu} = \frac{V}{h^d}\int \mathrm{d}^d p\, e^{-\beta H_1(p) + \beta v^i p_i}\,, \tag{5.23}$$

where $H_1(p)$ is the single-particle Hamiltonian, h represents Planck's constant and V the volume of space. The chemical potential v^i conjugated to the momentum p_i can be understood as the average total velocity, see e.g. [57]. The introduced inverse temperature

$$\beta = \frac{1}{k_B T} = \gamma\tilde{\beta} = \frac{\gamma}{k_B \tilde{T}}\,, \tag{5.24}$$

is constructed in such a way that $\tilde{\beta}$ is the boost invariant rest-frame inverse temperature (see e.g. [57]). We remind the reader that Z_1 is not required to be boost invariant, but can be related to the boost invariant Lorentz scalar pressure P via the grand canonical potential Ω (see appendix C for more details)

$$\Omega = -\frac{Z_1}{\beta} e^{\beta\mu}\,, \quad P = \frac{Z_1}{V\beta} e^{\beta\mu}\,, \tag{5.25}$$

where $\mu = \tilde{\mu}/\gamma$ is the chemical potential and $\tilde{\mu}$ is the rest-frame chemical potential and the grand potential is related to the pressure P via $\Omega = -PV$.

For simplicity, we focus on a gas of massless particles, with hamiltonian

$$H_1(p) = (\vec{p}^{\,2} c^2)^{1/2} = |\vec{p}|c\,, \tag{5.26}$$

and specify to the case of three spatial dimensions ($d = 3$). The single particle partition function can be worked out to be

$$Z_1 = \frac{V}{h^3}\frac{4\pi}{\beta v}\int_0^\infty \mathrm{d}p\, p \sinh(\beta v p)\, e^{-\beta p c} = \frac{8\pi V \gamma^4}{h^3 c^3 \beta^3}\,, \tag{5.27}$$

which is the partition function of massless relativistic particles with Boltzmann statistics. This result is obvious when $v < c$, and in a relativistic theory we would only consider this case. But, in anticipation of the Carroll limit, we will also consider other cases. Notice first that, for $v > c$ with real velocity v, the integral diverges and is not well defined. There is a way to define it if we consider β and v both to lie in the complex plane. In that case we find that for the integral to converge we require $\mathrm{Re}(\beta v) < \mathrm{Re}(\beta c)$ which leads to

$$\mathrm{Re}\,\beta\ \mathrm{Re}\,(v - c) - \mathrm{Im}\,\beta\ \mathrm{Im}\,v < 0\,. \tag{5.28}$$

When this condition is satisfied, the resulting partition function is still given by (5.27) with complexified parameters for β and v. This expression for the partition function is in fact also well-defined for $v > c$ with real v and β, but then the inequality (5.28) is not satisfied and therefore this case cannot be interpreted as coming from a partition function.

There are many solutions in the complex plane satisfying (5.28), all related by analytic continuation. One particular solution that leads to a real partition function is obtained by

[17] This agrees with the ideal gas model used in [57] in which $z = 1$ and $\lambda = c$.

taking β real and v purely imaginary with $c \neq 0$. This can be realized by taking the chemical potentials v^i purely imaginary and hence the partition function is a Fourier transform. A purely imaginary v can no longer be interpreted as the average total velocity, but there is still a quantity with the dimension of a velocity, and we can therefore consider a Carroll regime where this velocity is much larger than the speed of light. This is one of the cases that we will work out below (spacelike case).

Timelike Carroll case. We consider the option

$$v = 0\,, \qquad \gamma \to 1\,, \tag{5.29}$$

as $c \to 0$. Plugging (5.27) into (5.25) we then find

$$P = \frac{8\pi}{h^3 c^3 \beta^4} e^{\beta\mu}\,. \tag{5.30}$$

Notice that this result diverges in the strict Carroll limit as $P \sim c^{-3}$. It turns out that the leading order pressure will not produce consistent thermodynamical relations with the energy-momentum tensor. Let us show this by redoing the limit and setting v identically equal to zero, so we can take $v^i = c^2 v^i_{(2)} + O(c^4)$. Let us consider this case for a Boltzmann gas of free massless relativistic particles. The pressure is (5.25) combined with (5.27). This is a function of T, μ, v^i. The first law states that

$$\mathrm{d}P = s\mathrm{d}T + n\mathrm{d}\mu + \mathcal{P}_i \mathrm{d}v^i = \frac{\tilde{\mathcal{E}} + P}{T}\mathrm{d}T + nT\mathrm{d}\frac{\mu}{T} + \mathcal{P}_i \mathrm{d}v^i\,, \tag{5.31}$$

where we used $\tilde{\mathcal{E}} + P = sT + n\mu$ with $\tilde{\mathcal{E}}$ the internal energy.[18] The momentum density is computed to be

$$\mathcal{P}_i = \frac{4}{c^2}\gamma^2 P v^i\,. \tag{5.32}$$

The LAB frame energy-momentum tensor is[19]

$$T^0{}_0 = -\mathcal{E}\,, \qquad T^i{}_0 = -(\mathcal{E} + P)\,v^i\,, \qquad T^0{}_j = \mathcal{P}_j\,, \qquad T^i{}_j = P\delta^i_j + \mathcal{P}_j v^i\,, \tag{5.33}$$

where $\mathcal{E} = \tilde{\mathcal{E}} + \mathcal{P}_i v^i = 3P + \frac{4}{c^2}\gamma^2 P v^2$. This is the equation of state of a scale invariant system, thus $\tilde{\mathcal{E}} = 3P$ which obeys the ideal gas law $P = k_B n T$ (see appendix C for more details). If we expand this around $c = 0$ by setting $v^i = c^2 v^i_{(2)} + O(c^4)$ and taking T and μ to be $O(1)$ with the leading order terms again denoted by T and μ we find that the pressure becomes the one given in (5.30), while the energy-momentum tensor becomes

$$T^0{}_0 = -\tilde{\mathcal{E}} = -3P\,, \qquad T^i{}_0 = 0\,, \qquad T^0{}_j = 4Pv^j_{(2)}\,, \qquad T^i{}_j = P\delta^i_j\,. \tag{5.34}$$

Note that the pressure (5.30) does not depend on $v^i_{(2)}$. This energy-momentum tensor is not thermodynamic since it cannot be obtained from a first law applied to P in (5.30).

[18]The LAB frame energy density $\mathcal{E}$ depends on the extensive conserved quantities $s, n, \mathcal{P}_i$. The internal energy, or rest-frame energy, is the energy density $\tilde{\mathcal{E}}$ which depends on s, n and v^i. For a boost invariant system the v^i dependence can be absorbed in s and n, so that the internal energy only depends on the rest-frame entropy and particle number densities. The internal and LAB frame energies are thus Legendre transforms of each other where $\mathcal{P}_i$-dependence is traded for v^i-dependence and vice versa, i.e. we have $\mathcal{E} = \tilde{\mathcal{E}} + \mathcal{P}_i v^i$.

[19]There is also a U(1) current $J^0 = n$ and $J^i = nv^i$ but this will play no role in our discussion.

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Spacelike Carroll case. In this case we take the chemical potential v to be purely imaginary. In the small $|c/v|$ limit, i.e. the Carroll regime, we have that $\gamma \to |c/v|$ and we find for the pressure

$$P = \frac{8\pi c}{h^3\beta^4 v^4} e^{\beta\mu} . \tag{5.35}$$

This result can only be obtained for $v \neq 0$. The Carroll regime now yields an energy-momentum tensor of the form of a spacelike Carroll fluid as presented in (5.5) with thermodynamical relations that are consistent with the leading order pressure in (5.35). This fluid explicitly satisfies also $\mathcal{E} = -P$, as can be shown explicitly by using (5.27) to compute

$$-\mathcal{E} = \frac{N}{V}\partial_\beta \log Z_1 - \frac{N}{\beta V} v^i \partial_{v^i} \log Z_1 = \frac{N}{\beta V} = P . \tag{5.36}$$

For more details we refer the reader to appendix C.

It would be interesting to understand better the physical interpretation of having imaginary chemical potentials v^i. If this can be justified physically, our construction of the partition function gives a microscopic description of a system with an equation of state with $w = -1$.

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A Quantum mechanical toy model

In this appendix we consider a quantum mechanical toy model that is representative for the type of Hamiltonian one obtains in case of the magnetic Carroll scalar field theory.

Consider two commuting harmonic oscillators X and Y, i.e. $[X, X^\dagger] = [Y, Y^\dagger] = 1$ and the operator

$$H = (X + Y^\dagger)(X^\dagger + Y) . \tag{A.1}$$

What are the eigenstates of this operator? Notice that $\langle\psi|H|\psi\rangle \geq 0$ so eigenvalues of H will be non-negative. Notice that $H = e^{-X^\dagger Y^\dagger} XY e^{X^\dagger Y^\dagger}$ so we might as well ask about the eigenstates of the operator XY (though one might need to worry about normalizability).

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Consider the states

$$|\psi\rangle_{E,k} = \sum_{l=0}^{\infty} E^l \frac{(X^\dagger)^{k+l}(Y^\dagger)^l}{(k+l)!l!}|0\rangle \tag{A.2}$$

and similariy

$$|\chi\rangle_{E,k} = \sum_{l=0}^{\infty} E^l \frac{(X^\dagger)^{l}(Y^\dagger)^{k+l}}{l!(k+l)!}|0\rangle\,, \tag{A.3}$$

where $|0\rangle$ is the groundstate of XY'. These series also appear in modified Bessel functions. Because

$$XY(X^\dagger)^k(Y^\dagger)^l|0\rangle = kl(X^\dagger)^{k-1}(Y^\dagger)^{l-1}|0\rangle\,, \tag{A.4}$$

we immediately see that both states are eigenstates of XY with eigenvalue E. For $k=0$ the two states agree. For $E=0$ only the first term survives leading to states of the form $(X^\dagger)^k|0\rangle$ and $(Y^\dagger)^k|0\rangle$. We conclude that the eigenstates of H with eigenvalue E are of the form $e^{-X^\dagger Y^\dagger}|\psi\rangle_{E,k}$ and $e^{-X^\dagger Y^\dagger}|\chi\rangle_{E,k}$.

We can try to compute the overlap of these states, so let us consider

$$Z = {}_{E_1,k}\langle\psi|e^{-XY}e^{-X^\dagger Y^\dagger}|\psi\rangle_{E_2,k} \tag{A.5}$$

Expanding the exponentials leads to

$$Z = \sum_{l,l',p,q} \langle 0|(-1)^{p+q}E_1^l E_2^{l'} \frac{X^{k+l+p}Y^{l+p}(X^\dagger)^{k+l'+q}(Y^\dagger)^{l'+q}}{(k+l)!l!p!(k+l')!l'!q!}|0\rangle \tag{A.6}$$

which evaluates to $(l+p=l'+q=m)$

$$\sum_{l,l',m} \frac{(-1)^{l+l'}E_1^l E_2^{l'}(k+m)!m!}{(k+l)!l!(m-l)!(k+l')!l'!(m-l')} \tag{A.7}$$

The sum over m always diverges. We can already see this when we compute the norm of $e^{-X^\dagger Y^\dagger}|0\rangle$ which is infinite. So at best these energy eigenstates are delta-function normalizable.

We can also do the sums over l and l' first which leads to

$$\sum_m \frac{(k+m)!}{k!^2 m!}\,{}_1F_1(-m,k+1,E_1)\,{}_1F_1(-m,k+1,E_2)\,, \tag{A.8}$$

or equivalently in terms of Laguerre polynomials

$$\sum_m \frac{m!}{(m+k)!}L_m^{(k)}(E_1)L_m^{(k)}(E_2)\,. \tag{A.9}$$

Happily, Laguerre polynomials form a set of orthogonal polynomials on $[0,\infty)$ with measure $x^k e^{-x}$, so that

$$\int_0^\infty \mathrm{d}E\, E^k e^{-E} L_m^{(k)}(E) L_n^{(k)}(E) = \frac{(n+k)!}{n!}\delta_{n,m}\,, \tag{A.10}$$

which implies in particular that

$$E_1^k e^{-E_1}\sum_m \frac{m!}{(m+k)!}L_m^{(k)}(E_1)L_m^{(k)}(E_2) = \delta(E_1-E_2)\,, \tag{A.11}$$

so that the states are indeed delta-function normalizable.

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To summarize: the spectrum of the theory consists of states labeled by $E \geq 0$ and $k \in \mathbb{Z}$ with

$$|E,k\rangle = E^{|k|/2} e^{-E/2} e^{-X^\dagger Y^\dagger} \sum_{l=0}^{\infty} E^l \frac{(X^\dagger)^{k+l}(Y^\dagger)^l}{(k+l)!l!}|0\rangle\,, \tag{A.12}$$

for $k \geq 0$ and a similar expression with X and Y interchanged for $k < 0$. The inner product of these states is

$$\langle E_1, k_1|E_2, k_2\rangle = \delta(E_1 - E_2)\delta_{k_1,k_2}\,. \tag{A.13}$$

As a check, we compute the overlap of the ground state with energy eigenstates giving

$$\langle 0|E,k\rangle = e^{-E/2}\delta_{k,0}\,. \tag{A.14}$$

Therefore, $|0\rangle = \int \mathrm{d}E e^{-E/2}|E,0\rangle$ and we easily check that the norm of this state is $\int_0^\infty e^{-E}\mathrm{d}E = 1$. The states with negative energy do not appear (and are presumably not even delta-function normalizable) given the general argument above.

B From Galilean to Carrollian theories and back

In this appendix we present a Lagrangian method which produces a magnetic Carroll theory from a given Galilean theory (for a closely related construction of non-Lorentzian models from a seed Lagrangian see [99]). For simplicity, we give the argument here for a scalar field but we expect the method to generalize to other fields.

Consider a Lagrangian $\mathcal{L}$ for a real scalar field ϕ that is Lorentz invariant and where ϕ is a Lorentz scalar, i.e.

$$\mathcal{L} = \mathcal{L}(\phi, \dot{\phi}, \partial_i\phi)\,, \tag{B.1}$$

where $\mathcal{L}$ infinitesimally transforms as

$$\delta_L \mathcal{L} = \xi_L^\mu \partial_\mu \mathcal{L}\,, \tag{B.2}$$

with

$$\xi_L^\mu \partial_\mu = b^i x^i \partial_t + t b^i \partial_i\,. \tag{B.3}$$

We have set $c = 1$. The Lagrangian is a Lorentz scalar and $\delta_L\phi = \xi_L^\mu \partial_\mu \phi$. We assume that $\mathcal{L}$ can be written as the sum of a Lagrangian that is Galilean invariant and one that is Carroll invariant, i.e. we assume that

$$\mathcal{L} = \mathcal{L}_C(\dot{\phi}, \phi) + \mathcal{L}_G(\partial_i\phi, \phi)\,. \tag{B.4}$$

In this split $\mathcal{L}_C(\dot{\phi}, \phi)$ is an electric Carroll theory and $\mathcal{L}_G(\partial_i\phi, \phi)$ a magnetic Galilean theory. This assumption does not apply to higher-derivative theories but maybe the argument can be generalised to those cases. We can write infinitesimally any Lorentz transformation (B.3) as the sum of a Galilean and a Carroll transformation,

$$\xi_L^\mu \partial_\mu = b^i x^i \partial_t + t b^i \partial_i = \xi_C^\mu \partial_\mu + \xi_G^\mu \partial_\mu\,. \tag{B.5}$$

Write (B.2) as

$$(\delta_C + \delta_G)(\mathcal{L}_C + \mathcal{L}_G) = (\xi_C^\mu + \xi_G^\mu)\,\partial_\mu (\mathcal{L}_C + \mathcal{L}_G)\,. \tag{B.6}$$

The fact that $\mathcal{L}_C$ and $\mathcal{L}_G$ are Carroll and Galilean scalars means that they obey the property that

$$\delta_C \mathcal{L}_C = \xi_C^\mu \partial_\mu \mathcal{L}_C \,, \tag{B.7}$$

where we transform ϕ as a Carroll scalar, $\delta_C\phi = \xi_C^\mu\partial_\mu\phi$, and a similar statement applies to the Galilean Lagrangian. After some algebra we then find

$$\delta_G \mathcal{L}_C + \delta_C \mathcal{L}_G = \xi_G^\mu \partial_\mu \mathcal{L}_C + \xi_C^\mu \partial_\mu \mathcal{L}_G \,. \tag{B.8}$$

Isolating $\delta_C\mathcal{L}_G$ and using that $\mathcal{L}_G = \mathcal{L}_G(\phi, \partial_i\phi)$ and that $\mathcal{L}_C = \mathcal{L}_C(\phi, \dot\phi)$ and computing the variations and derivatives using the chain rule, we obtain

$$\delta_C \mathcal{L}_G = -\frac{\partial \mathcal{L}_C}{\partial \dot\phi} b^i \partial_i \phi + \xi_C^\mu \partial_\mu \mathcal{L}_G \,. \tag{B.9}$$

It can be shown that $\frac{\partial \mathcal{L}_C}{\partial \dot\phi}$ is a Carroll scalar,

$$\delta_C \left(\frac{\partial \mathcal{L}_C}{\partial \dot\phi}\right) = \frac{\partial^2 \mathcal{L}_C}{\partial\phi\partial\dot\phi}\delta_C\phi + \frac{\partial^2 \mathcal{L}_C}{\partial\dot\phi\partial\dot\phi}\delta_C\dot\phi = \xi_C^\mu\partial_\mu \left(\frac{\partial \mathcal{L}_C}{\partial \dot\phi}\right) \,. \tag{B.10}$$

Equation (B.9) is the main observation from which the rest follows. One can construct a new Carroll theory by starting with $\mathcal{L}_G$ and adding to it a Lagrange multiplier term proportional to $\frac{\partial \mathcal{L}_C}{\partial \dot\phi}$. In other words, define

$$\tilde{\mathcal{L}}_C = \mathcal{L}_G + \chi \frac{\partial \mathcal{L}_C}{\partial \dot\phi} \,. \tag{B.11}$$

This new Lagrangian will be a Carroll scalar if χ transform as

$$\delta\chi = \xi_C^\mu \partial_\mu \chi + b^i \partial_i \phi \,. \tag{B.12}$$

This works because $\frac{\partial \mathcal{L}_C}{\partial \dot\phi}$ is a Carroll scalar.

In particular, applying this to the case for which one chooses the Carroll action to be the one of the electric theory and the standard Galilean action

$$\mathcal{L}_C = \frac{1}{2}\dot\phi^2 - V(\phi) \,, \qquad \mathcal{L}_G = -\frac{1}{2}\partial_i\phi\partial_i\phi \,, \tag{B.13}$$

it follows that (B.11) generates the magnetic Carroll theory.

This idea also works in the other direction, so that it is possible to create a new Galilean theory from a Carroll theory by writing

$$\tilde{\mathcal{L}}_G = \mathcal{L}_C + \chi^i \frac{\partial \mathcal{L}_G}{\partial\partial_i \phi} \,. \tag{B.14}$$

We expect the procedure above to be easily applicable to Maxwell actions. Furthermore, in the same spirit probably GR can also be viewed as an appropriate sum of a Carrollian and Galilean gravity theory. However, since in this case there are no global symmetries we expect the details to be somewhat different. It would be interesting to examine this further as well as what happens with higher-derivative theories such as Born-Infeld.

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C Details on gasses

In this appendix we review massless and massive relativistic Boltzmann gasses and verify that their resulting energy momentum tensors reproduce the expected perfect fluid energy-momentum tensors. The single particle partition function is given by (5.23):

$$Z_1(T, V, v^i) = \frac{V}{h^d} \int \mathrm{d}^d p \, e^{-\beta H_1(p) + \beta v^i p_i} \,. \tag{C.1}$$

The canonical N particle partition function for a Blotzmann gas of N free particles is given by $Z = (Z_1^N)/N!$ and the grand canonical partition function $\mathcal{Z}$ can be written as

$$\log \mathcal{Z} = e^{\beta\mu} Z_1 \,, \tag{C.2}$$

where μ is the chemical potential. This allows us to write the pressure and grand potential Ω as[20]

$$P = -\frac{\Omega}{V} = \frac{1}{V\beta} \log \mathcal{Z} = \frac{1}{V\beta} e^{\beta\mu} Z_1 \,. \tag{C.3}$$

The particle number is given by

$$N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{V,T,v^i} = e^{\beta\mu} Z_1 \,, \tag{C.4}$$

which when inserting the pressure as given in (C.3) we readily recognize the ideal gas law:

$$PV = \frac{N}{\beta} \tag{C.5}$$

The energy density of the system follows from

$$\tilde{\mathcal{E}} = -\frac{\partial}{\partial \beta} \log Z = N\langle H_1 \rangle_1 - N v^i \langle p_i \rangle_1 \,, \tag{C.6}$$

where the brackets indicate with subscript 1 indicate average with respect to the single particle partition function and p_i is the momentum that appears in the integral. Furthermore we'll denote $-T^0{}_0 = \mathcal{E} = N\langle H_1 \rangle_1$ such that $\tilde{\mathcal{E}} + P_i v^i = \mathcal{E}$, where $T^0{}_j = P_j = N\langle p_j \rangle_1$ is generalized momentum. We can furthermore compute

$$\langle p_i \rangle_1 = \frac{1}{\beta} \frac{\partial}{\partial v^i} \log\left(Z_1\right) \,. \tag{C.7}$$

Let us now compute the spatial stress tensor $T^i{}_j = N\langle p_i \frac{\partial H_1}{\partial p_j} \rangle_1$ and energy flux $T^i{}_0 = N\langle H_1 \frac{\partial H_1}{\partial p_j} \rangle_1$. We observe

$$T^i{}_j = -\frac{N}{\beta} \langle p_i \frac{\partial}{\partial p_j} \rangle_1 + N\langle p_j \rangle v^i = PV\delta^i{}_j + P_j v^i \,, \tag{C.8}$$

$$T^i{}_0 = -\frac{N}{\beta} \langle H_1 \frac{\partial}{\partial p_i} \rangle_1 + N\langle H_1 \rangle_1 v_i = PV \langle \frac{\partial H_1}{\partial p_i} \rangle_1 + \mathcal{E} v_i \,, \tag{C.9}$$

[20] Using $\mathrm{d}\Omega = -S\mathrm{d}T - P\mathrm{d}V - P_i \mathrm{d}v^i - N\mathrm{d}\mu$.

where $\langle \frac{\partial H_1}{\partial p_i} \rangle_1 = -\frac{1}{\beta} \langle \frac{\partial}{\partial p_i} \rangle_1 v^i$, where the first term vanishes as it is a total derivative. We now collect all the entries of the energy momentum tensor and divide by volume V:

$$T^0{}_0 = -\mathcal{E}\,, \tag{C.10}$$

$$T^i{}_0 = (P + \mathcal{E}) v_i\,, \tag{C.11}$$

$$T^0{}_j = \mathcal{P}_j\,, \tag{C.12}$$

$$T^i{}_j = P \delta^i{}_j + \mathcal{P}_j v^i\,. \tag{C.13}$$

Taking the trace we find

$$T^0{}_0 + T^i{}_i = dP - \tilde{\mathcal{E}}\,. \tag{C.14}$$

As an example, consider a d-dimensional massless relativistic gas, which has the following single particle partition function:

$$Z_1 = 2^d V \frac{\pi^{\frac{d-1}{2}}}{h^d} \frac{\gamma^{d+1}}{(c\beta)^d} \Gamma \left[\frac{d+1}{2} \right]. \tag{C.15}$$

Using (C.6) we find

$$\tilde{\mathcal{E}} = -N \frac{\partial}{\partial \beta} \log Z_1 = \frac{d}{\beta} N = dPV\,, \quad \Leftrightarrow \quad \tilde{\mathcal{E}} - dP = 0\,, \tag{C.16}$$

where in the last equality we used the idea gas law (C.5). This moreover implies that the energy momentum tensor is traceless.

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