Kissing the cheeks of Huygens

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Abstract
How well do the so-called cheeks of Huygens enforce a constant time of oscillation on a pendulum, independent of the amplitude of the pendulum swing? We use a video measurement tool to test it with a homemade experimental setup that can be fruitfully used in classroom and physics laboratory settings. Data fits reveal how well the setup confirms the usefulness of the cheeks of Huygens. Comparisons of computer model results with recorded data have been made for the same purpose. The equation of motion of a pendulum with a string of arbitrary length that winds along two surfaces of cycloidal cheeks is derived, too. Numerical solutions of this equation are used for the error analysis of measurements with the homemade setup.

Keywords: pendulum motion, cheeks of Huygens, free pendulum formulas, modelling, experimentation, error analysis

Supplementary material for this article is available online

1. Introduction
In school physics and mathematics, the pendulum swing is a popular object of study. Already in lower secondary education, pupils carry out experiments with a small diameter bob suspended on a low mass string and they discover how the time of a full swing, in physics commonly referred to as the period of the pendulum, depends on the length of the string, the mass of the bob, and the size of the initial amplitude. They also discover that a change in the mass of the bob does not affect the period, provided that the length of the pendulum is not thereby affected. Ignoring friction, and not taking into account the mass of the string and the radius of the bob, the relationship between the period \(T\) and the length \(l\) of the pendulum is given by

\[
T = 2\pi \sqrt{\frac{l}{g}}
\]  

(1)

where \(g\) is the acceleration of gravity. There is always a warning in instructional materials that this formula, discovered by Christiaan Huygens [1, 2], only works well for small amplitudes. For amplitudes larger than an angle of \(4^\circ\) it already...
notably underestimates the period of the pendulum. Yet in laboratory work, the above relationship is often used by students to estimate the acceleration of gravity through experiments in which they determine periods of pendulums of various lengths, plot $T^2$ against $l$, and determine the slope of the linear regression line.

In the seventeenth century, scientific discovery as well as nautical discovery and overseas trade meant an increasing need for accuracy in determining longitude at sea. The difference between local time and time at the home port, recorded by an accurate clock on board of the ship, would make it possible to determine the longitude of the current position. In his attempts to construct a pendulum clock that accurately measures time, Huygens solved in 1659 the problem of making the period of a pendulum truly constant. Using classical geometry methods he proved that the bob must swing along the arc of an inverted cycloid instead of along a circular arc. To make this possible, a pivot is needed to shorten the length of the pendulum for larger amplitudes. Huygens realised this by placing two curved metal plates of cycloidal shape, along which the string of the pendulum winds. These curved plates are nowadays known as the cheeks of Huygens. You can see them in figure 1 at the top part of the rear side of one of the first pendulum clocks built according to the Huygens’ design. In 1673 he published this design with all mathematical underpinnings in his book *Horologium Oscillatorium* [3].

In the Dutch nationwide physics exam at higher general vocational education (havo) level in 2013 [4], students were asked to explain how the cheeks compensate the effect of a larger initial amplitude on the period of the pendulum. The following answer of the exam question was provided for marking purposes: ‘When the pendulum makes contact with the cheeks the (free part of the) pendulum gets shorter. Then the period gets smaller (because the period increases for larger initial amplitudes, this increase is compensated).’

A calculus-based derivation of the isochronous property of cycloidal pendulum of Huygens requires too advanced mathematical concepts and techniques for secondary school students, but not for undergraduate physics students [5]. We discuss this in the next section.

Figure 1. Rear side of a pendulum clock built by Salomon Coster in 1659 (Collection Rijksmuseum Boerhaave, Leiden). Reproduced with permission from National Museum Boerhaave, Leiden.

On the principle that ‘to measure is to know’ and inspired by earlier work in teacher education [6], we have built a setup with which the compensating effect of the cheeks of Huygens on the period of a pendulum can be tested. We explain the construction of the setup and describe it in more detail in a supplement named *Construction manual* to inspire physics teachers interested in building such a setup for demonstration purpose in a classroom setting or for use in a student laboratory session. We describe some experiments that can be performed with the setup. We present results that we obtained and compare them with theoretical and computer modelling results.

2. Huygens ‘revisited’

Let us first describe Huygens’ construction of a cycloidal pendulum, presented by him through geometrical methods [3], in modern calculus- and physics-based language. The description consists of three parts: (i) proof that an inverted cycloid is
an isochronous curve; (ii) proof that the bob of a pendulum can be forced to wind along an inverted cycloid by letting the string of the pendulum move along a shifted version of the inverted cycloid; and (iii) derivation of the equation of motion for the angle of the free part of the cycloidal pendulum with the vertical at the tangential point at the inverted cycloid steering the motion. We will show how this equation of motion differs from the one for a pendulum with a fixed suspension point and that it cannot be derived by simply using the equation of a swinging pendulum with varying length.

2.1. Isochronous motion along an inverted cycloid

Instead of considering a bob hanging on a string and swinging in a fixed plane of motion about its point of suspension, we make a Gestalt switch and imagine a mass in the form of a bead gliding under gravity frictionless along a wire profile. Here, we consider a wire profile in the form of an inverted cycloid. We show that this is an isochronous curve, that is, wherever on this curve a mass starts to move, it will always take the same time for the mass to return to its initial position. We follow the approach of Broer [7].

A cycloid is the trajectory drawn by a fixed point on a wheel when the latter rolls on the ground without slipping. It becomes an inverted cycloid when the wheel rolls along a ceiling as shown in Figure 2.

Figure 2. Inverted cycloid, parameterised by a rolling angle $\varphi$.

Let $r$ be the radius of the wheel and $\varphi$ be the rolling angle (in radians and measured with respect to the downward vertical spoke position). Choose the coordinate system with the horizontal axis parallel to the ceiling and the position of the fixed point on the wheel for $\varphi = 0$ as the origin. Then the inverted cycloid $I$ is parameterised by

$$x_I(\varphi) = r(\varphi + \sin \varphi) \quad \text{and} \quad y_I(\varphi) = r(1 - \cos \varphi)$$

where $\varphi$ is in the interval $[-\pi, \pi]$. Next, we consider the distance $s_I(\varphi)$ travelled by the fixed point on the wheel as a function of $\varphi$.

For the infinitesimal change $d\varphi$ of the rolling angle, Pythagoras’ theorem and the double angle cosine formula give the corresponding infinitesimal change $ds_I$ as

$$ds_I = \sqrt{dx_I^2 + dy_I^2} = \sqrt{\left(\frac{dx_I}{d\varphi}\right)^2 + \left(\frac{dy_I}{d\varphi}\right)^2} d\varphi = r\sqrt{2(1 + \cos \varphi)} d\varphi = 2r \cos \left(\frac{1}{2} \varphi\right) d\varphi.$$

Integration leads to

$$s_I(\varphi) = 4r \sin \left(\frac{1}{2} \varphi\right) + c.$$  

The vertical height of the fixed point can be expressed as

$$y_I(\varphi) = r(1 - \cos \varphi) = 2r \sin^2 \left(\frac{1}{2} \varphi\right) = \frac{1}{8r} s_I(\varphi)^2.$$  

The above formulas allow us to determine the equation of motion for a mass $m$ moving along the inverted cycloid under gravity. The gravitational potential $U$, with zero value set for $y = 0$, is traditionally given by $U = mgy$. Written as a function of distance $s_I$ it becomes

$$U(s) = \frac{mg}{8r} s_I^2.$$  

Because

$$F = -\frac{dU}{ds_I}$$  

is valid for the force $F$ acting on the mass, we get the following equation of motion:

$$m \frac{d^2 s_I}{dt^2} = -\frac{mg}{4r} s_I.$$  

Written in the form

$$\frac{d^2 s_I}{dt^2} + \omega^2 s_I = 0$$  

where $\omega^2 = \frac{mg}{4r}$.
with angular frequency
\[ \omega = \sqrt{\frac{g}{4r}} \]
we see that this is the differential equation of harmonic oscillation and we can conclude that the motion along the cycloid is isochronous. No matter where the bob is released, its distance travelled along the inverted cycloid is always represented by a sinusoid with the same period. The period of the cycloidal pendulum of Huygens is given by
\[ T = 2\pi \sqrt{\frac{4r}{g}}. \quad (5) \]
Note that the above formula for the period corresponds with the classical formula \((1)\) for a pendulum of length \(l = 4r\) swinging over a small angle.

2.2. The cycloidal form of the cheeks of Huygens
Huygens got the bright idea to let the string wind from its point of suspension along the surfaces of curved metal plates shaped such that the bob is forced to follow the path of an inverted cycloid. He called the curve that prescribes the shape of the metal plates the evolute of the inverted cycloid, which he called on its turn the involute. Huygens developed his theory of evolute and involute, and he proved in a rigorous geometrical way that in this particular case the evolute is a translated version of the inverted cycloid \([3]\). We verify this result for what is called the cycloidal pendulum of Huygens.

Figure 3 shows the light blue evolute at the top and the black involute at the bottom. The string of the pendulum is decomposed as the curve from \(S\) to \(P_E\) on the evolute and the straight line from point \(P_E\) to \(P_I\).

The evolute can be parameterised as a transformed version of the involute: the point \(P_E = (x_E(\varphi), y_E(\varphi))\) can be defined as
\[ x_E(\varphi) = x_I(\varphi + \pi) - \pi r \quad \text{and} \quad y_E(\varphi) = y_I(\varphi + \pi) + 2r, \]
that is,
\[ x_E(\varphi) = r(\varphi - \sin \varphi) \quad \text{and} \quad y_E(\varphi) = r(3 + \cos \varphi) \]
with \(|\varphi| \leq \pi\). A rather technical, but straightforward calculation shows that the line segment \(P_E P_I\) is tangent to the evolute \(E\) and perpendicular to the involute \(I\) at any angle \(\varphi\). What remains to be shown is that the length of the curve from \(S\) to \(P_I\) is constant. The length of the curved part from \(S\) to \(P_E\) can be computed in the same way as we did for the distance \(s_I(\varphi)\) (see the supplement Equation of motion). In this case we have the infinitesimal change
\[ ds_E = 2r \sin \left( \frac{1}{2} \varphi \right) d\varphi \]
and integration leads to
\[ s_E(\varphi) = 4r - 4r\cos\left(\frac{1}{2}\varphi\right). \] (6)

This formula gives the distance along the evolute from \( S \) to \( P_E \).

For symmetry reasons, the length of the line segment \( P_EP_I \) is twice the length of the segment \( PP_I \). Note that we have an isosceles triangle \( PP_IM_I \) and this explains why the angle at \( P \) and at \( P_I \) equal \( \frac{1}{2}\varphi \). The cosine formula in trigonometry and the double angle cosine rule lead to the length of \( PP_I \), namely \( 2r\cos\left(\frac{1}{2}\varphi\right) \). Thus, the length of the segment \( P_EP_I \) is given by
\[ |P_EP_I| = 4r\cos\left(\frac{1}{2}\varphi\right). \] (7)

This is the length of the free part of the string of the pendulum. Adding formulas (6) and (7), we conclude that the length of the dark blue curve, consisting of a curved and straight part, is constant and equal to \( 4r \), and that the terminal point of this curve lies on an inverted cycloid. The above derivation also makes clear that cycloidal cheeks of Huygens only work well for a pendulum of specific length: the length of the pendulum must be equal to four times the radius of the wheel that generates the cycloidal shape of the cheeks.

2.3. The equation of motion of a cycloidal pendulum of Huygens

We denote the angle between the free part \( P_EP_I \) of the string of the cycloidal pendulum and the vertical at the point \( P_E \) (see figure 3) by \( \theta \). Thus: \( \theta = \frac{1}{4}\varphi \). In section 2.1 we have derived the following differential equation (formula (4)):
\[ \frac{d^2s_I}{dt^2} + \frac{g}{4r}s_I = 0 \]
where \( s_I = 4r\sin\theta \) (formula (3)). The chain rule of differentiation gives
\[ \frac{ds_I}{dt} = \frac{ds_I}{d\theta} \frac{d\theta}{dt} \]
that is,
\[ \frac{ds_I}{dt} = 4r\dot{\theta}\cos\theta \]
where we use Newton’s dot notation. Then, the second derivative is given by the rules of differentiation as
\[ \frac{d^2s_I}{dt^2} = 4r\ddot{\theta}\cos\theta - 4r\dot{\theta}^2\sin\theta. \]

Substitution of this derivative and \( s_I = 4r\sin(\theta) \) (formula (3)) in the differential equation of \( s_I \), lead after some rewriting to
\[ \ddot{\theta} + \frac{g}{l(\theta)}\sin\theta = \dot{\theta}^2\tan\theta \] (8)
where \( l(\theta) = 4rcos\theta \) is the length of the free part \( P_EP_I \) of the string of the cycloidal pendulum (formula (7)). The nonzero term on the right-hand side shows that the equation of motion is not the same as the one obtained by applying the equation of motion of a free pendulum with a fixed suspension point and a given length to a cycloidal pendulum with a moving suspension point \( P_E \) and a varying length \( l(\theta) = |P_EP_I| \). It is a misconception that one could derive in this way the equation of motion for a pendulum with a nonuniformly moving suspension point.

When the equation of motion is written as
\[ \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta + \frac{g}{4r}\sin\theta = 0 \]
it can be simplified by calculation rules of differentiation as
\[ \frac{d^2}{dt^2}\sin\theta + \frac{g}{4r}\sin\theta = 0. \]

In other words, \( \sin(\theta) \) satisfies a differential equation of harmonic oscillation. This equation of motion of the cycloidal pendulum of Huygens is commonly derived by a Lagrangian approach in advanced classical mechanics courses at under-graduate level (e.g. [8]).

Lagrangian mechanics is certainly most convenient when one wishes to derive the equation of motion for a pendulum with a string of arbitrary length \( L \) that winds along an inverted cycloid. Then the differential equation becomes (see the supplement Equation of motion):
\[ \ddot{\theta} + \frac{g}{l(\theta)}\sin\theta = \dot{\theta}^2 - \frac{4r\sin\theta}{l(\theta)} \] (9)
where \( l(\theta) = L - 4r + 4r \cos \theta \) is the length of the free part of the string of the pendulum. In section 5.4 we will use this equation to study the pendulum motion for a length of the string that does not correspond with the unique length leading to an isochronous motion. Through simulations we will study the effect of a small deviation from the perfect length of a cycloidal pendulum of Huygens on the period of oscillatory motion. In other words, we will use a computer model to carry out an error analysis of the experimental results with respect to length measurements in the setup.

3. The experimental setup

Prior to the construction of the setup we made a wish list. The size of the setup must be suitable for use in a school laboratory space and in the classroom for a demonstration experiment. It must be possible to collect data through video measurements and through experiments with sensors, such as an ultrasonic distance sensor or a photogate. To enlarge the variety of experiments, at least two different sets of cheeks of Huygens must be mounted. The setup must be able to withstand repeated experiments. Furthermore, the total costs must remain within a school budget for physics apparatus.

We decided to construct a wooden housing with two cheeks made for a pendulum with a period of 1 s and two cheeks made for a pendulum with a period of 1.5 s; see figure 4. The corresponding lengths of the pendulums (theoretically equal to \( 4r \)) are 24.85 cm and 55.91 cm, respectively. We used the formula of the inverted cycloid (formula (2)) to create templates printable on paper. We refer to the two sets as the small and large cheeks of Huygens. In addition, the housing has a third point of suspension for a freely moving pendulum. For aesthetic reasons, but also for the option to compare the orbit of the bob of the pendulum with the arc of the cheeks, two full arcs of the small cheeks have been mounted. This determines the width of the housing: 80 cm. A height of 70 cm leaves space for measurement equipment beneath a pendulum of 56 cm used for the large cheeks and having a swing time of 1.5 s. Adjustable legs allow levelling of the housing, which is crucial for the isochronous motion of the cycloidal pendulum. The string of the pendulum is chosen as flexible, thin, and of low mass as possible; a nylon string with a diameter of 0.3 mm is suitable. To guarantee a precise winding of the string along the cheeks, the cheeks are mounted one diameter of the string apart from each other. In order to give the string as little space for moving back and forth at the springing points of the cheeks, we used a hypodermic needle for guiding the string through the housing. This also ensures the endurance of the passthrough and the onset of the cheeks with repetition of experiments. Within the set of bobs available to us, we experimentally picked the one that worked best: a bob weighing 14 gram and having a diameter of 15 mm. In practice, it is easier to release this bob without early misalignment in the pendulum motion (i.e. an early change of the plane of motion) than a larger and heavier bob. With a bob of low weight it is also easier to fix the desired string length (e.g. with a clothes-peg), as shown in figure 4. A sliding calliper is used to release the bob at the correct distance from the back wall of the housing so that the pendulum swings nicely along the surfaces of the cheeks. For a full description of our setup we refer to the supplement Construction manual.

4. Experiments

We used a high speed camera to record the motion of a pendulum with 200 frames per second. This implies a time resolution of 0.005 s. To minimise image distortion, the camera was positioned as far
Figure 5. Screen shot of a COACH video measurement and data analysis.

away from the setup as possible and perpendicular to the back wall of the housing. We used the tracking option of the video analysis tool of the COACH environment [9, 10] to measure the position of the bob during its motion in a coordinate system with the point of suspension as origin. For length calibration of the coordinate system we used a line segment of known length in the plane of motion: (i) the total width of the pair of cheeks for the pendulum swing along the small cheeks; (ii) the distance between the two large cheeks (dark red in figure 5, and behind the dark blue small cheeks) at their lowest points for a measurement of pendulum swing along the large cheeks; and (iii) the width of the housing for the free pendulum.

A good reason for choosing video analysis of high speed recorded video clips of pendulum motions is that one gets in this way more than enough data to apply Fourier analysis for a quick determination of the period. In addition, COACH offers a state-of-the-art sinusoidal regression method [10] so that one can visually compare the regression curve built up from one or more sinusoids with the data plot. In most of our experiments, we collected data for the first ten periods of the pendulum motion and subsequently made a sinusoidal fit of the recorded horizontal position of the bob. From the regression formula we computed the period of the pendulum swing. This works well in cases where the distance travelled by the bob along the inverted cycloid and its horizontal position, computed by formulas (3) and (2), respectively, are almost the same. The reason is that the distance travelled by the bob can certainly be represented by a sinusoid as it satisfies the differential equation of harmonic motion (formula (4)). This assumption is fulfilled in most of our experiments: for example, the relative difference between the horizontal position of the bob and its distance travelled is less than 3% for angles $\theta$ less than 25°. For larger angles we verified that the horizontal position of the bob, which can be approximated as a period function by a finite Fourier series, is in fact well represented by a single sinusoid and that adding more components to the Fourier series would not lead to a different value of the period.

Figure 5 is a screen shot of a COACH activity in which the pendulum motion along the small cheeks with a matching string length (i.e. with
an expected period of 1 s) has been recorded for several minutes with 40 data points per second. The upper left window shows the setup and an experimenter releasing the bob of the pendulum. In the lower left window, the measured horizontal position of the bob of the pendulum is shown for the first ten periods. The sinusoidal fit of these data is shown in the lower right window of the screen shot and the angular frequency of 6.2482 s$^{-1}$ corresponds to a period of 1.0056 s. So the deviation from the theoretical period is within the time resolution. And not just that: the red graph in the upper right window, which is the extension of the sinusoidal fit for the first 10 s to other time intervals, shows that this period is still valid after three minutes: only the amplitude is wrong because damping of the pendular motion is ignored. But it gets even better: in COACH we fitted the data recorded in the first 30 s to a damped sinusoid and the turquoise graph in the upper right window shows that this fit still perfectly describes the trajectory of the bob after three minutes. It has the same frequency as the sinusoidal fit.

As we mentioned in the wish list at the beginning of section 3 we wanted a versatile setup for student experiments. The results of the video analysis shown in figure 5 were certainly not the first ones obtained. When everything of the setup had been sawn, planed, screwed, put together, and levelled with care, we naturally want to know how well it works. First we measured the swing time with increasing initial displacements of a pendulum with a length of 25 cm that winds along the small cheeks. This length is close to the theoretical length of 24.85 cm leading to a period of 1 s. As we will see in the section 5.1, the period of the pendulum remained very convincingly constant! We also checked the isochronous property of the setup for the large cheeks with a pendulum of length 56 cm (close to the theoretical length of 55.91 cm that leads to a period of 1.5 s) and the result was equally convincing. In order to compare these results with a free pendulum we collected data for this motion for increasing initial angles. All measurement series were repeated twice from scratch to get an impression of the measurement error. The data collected for the freely swinging pendulum were also used to see whether they could be described well by a Taylor series approximation of the exact formula for the period of a free pendulum, or by other mathematical expressions mentioned in the literature for a large-angle pendulum period [11–13].

Finally, students may not be aware that a set of cheeks of Huygens only works for a pendulum of which the string has a particular length, or they may not have spent a lot of thought on the effect of a ‘wrong’ length of the pendulum. In a Predict-Observe-Explain instructional approach one could ask students first to brainstorm on what happens when one uses a too small or too long pendulum given some set of cheeks of Huygens, come up with a prediction and justify this. Hereafter they could do a set of experiments to find out what happens in reality. Finally they try to reconcile any conflict between their prediction and measurement. This approach is expected to enhance the students’ understanding of pendulum motion and how cheeks of Huygens influence the period of the oscillation.

5. Experimental results

We present and discuss our experimental results regarding cycloidal and free pendulums.

5.1. Isochronous pendulum motion through a set of cheeks of Huygens

In the first series of experiments we determined the period of a pendulum of length close to 25 cm that winds along the small cheeks for increasingly large initial amplitudes. The results are shown in figure 6, together with a horizontal regression curve. The results of a series of experiments for a freely moving pendulum of the same length are added to the diagram for comparison. The conclusion is evident: for a freely moving pendulum, the period increases with increasing amplitude, and for a pendulum moving along the small cheeks of Huygens, the period does not depend on the initial amplitude. The average period of 1.005 ± 0.002 s is in very good agreement with the period of 1.003 s for a small-angle pendulum (formula (1)) with a length as measured by the video tool.

Comparable experiments for a pendulum moving along the large cheeks led to the average period of 1.502 ± 0.001 s, which matched the expected period again.
Kissing the cheeks of Huygens

Figure 6. Data plot with measured period vs initial amplitude for a pendulum with and without the use of the small cheeks of Huygens in our experimental setup.

Figure 7. The orbit of the bob of the pendulum and the cheeks of Huygens have the same cycloidal shape.

Figure 8. Data plot with measured period vs the square \( \theta^2 \) of the measured maximum angle \( \theta \) and a linear regression curve.

5.2. Large-angle pendulum formulas put to the test

The period of oscillation of a pendulum varies with the amplitude. This can be observed in rather simple experiments, but as Holmes and Wieman [14] report, only in so-called structured quantitative inquiry labs that offer students opportunities to make many experimental decisions, such as how many repeated trials to take and how many times to let the pendulum swing for each measurement. These authors state that, when the students measure the period across several amplitudes with sufficient precision, they discover the second-order quadratic behaviour that deviates from the small-angle approximation, but only if they properly understand and trust the quality of their data.

Students can verify how well the approximation formula of Daniel Bernoulli, which he determined in 1749, works for relating the period of the pendulum with the theoretical small-angle period \( T_0 \) and the maximum pendulum angle \( \theta \):  

\[
T = T_0 \left( 1 + \frac{1}{16} \theta^2 \right).
\]

We call this the classical Bernoulli formula. Students can plot the measured period \( T \) against a new quantity \( 1 + \frac{1}{16} \theta^2 \) and fit the data points to a straight line through the origin to find the estimate of \( T_0 \). We have done this for collected data of a pendulum of length 56 cm, having a theoretical value of \( T_0 = 1.501 \) s: we found \( T_0 = 1.5053 \) and we concluded from a visual inspection that the fit describes the data well.

In addition, we plotted the measured period \( T \) against the square \( \theta^2 \) of the measured maximum angle \( \theta \) and did a linear fit to find the estimates of \( T_0 \) and the coefficient in front of \( \theta^2 \). The result is shown in figure 8 and the formula found

\[
T = 1.5058 \left( 1 + \frac{1}{16.65} \theta^2 \right)
\]

is remarkably close to Bernoulli’s formula.

Bernoulli also found the approximation formula

\[
T = T_0 \left( 1 + \frac{1}{4} \sin^2 \left( \frac{\theta}{2} \right) \right).
\]
Data plot with measured period vs the measured maximum angle $\theta$ and the following two regression curves: $T = 1.5055 (1 + \frac{1}{4} \theta^2)$ and $T = 1.5055 (1 + \frac{1}{4} \sin^2 (\frac{\theta}{2}))$.

We call this the sine Bernoulli formula. We plotted the measured period $T$ against $\sin^2 (\theta/2)$ of the measured maximum angle $\theta$ and did a linear fit to find the estimates of $T_0$ and the coefficient in front of $\sin^2 (\theta/2)$. We found for our data set the values 1.5057 and 1/4.05, respectively. Figure 9 illustrates how well the classical Bernoulli approximation and the sine Bernoulli approximation describe our data set.

Data could be fitted with other formulas mentioned in the literature [12–14], but this does not lead to much improvement. For example, a data fit with the formula of Carvalhaes and Suppes [15]

$$T = \frac{2T_0}{1 + \cos (\frac{\theta}{2})}$$

hardly differs from the fit with the classical Bernoulli formula. Actually, from instructional point of view, it is an advantage that so many formulas lead to good results because this gives students food for thought: all formulas work well, but why do scientists conclude that some of these formulas have better properties? What criteria do they use? Does physics play a role herein?

### 5.3. Swinging against cycloidal cheeks of Huygens for a pendulum with a nonmatching string length

The best way to explore the effect of a ‘wrong’ length of a pendulum string winding along cycloidal cheeks of Huygens is to let the pendulum start with the largest possible initial displacement. Table 1 shows large-displacement data for three measurements of a pendulum swing against the set of small cheeks: one with a matching pendulum length of 25 cm, a second one with a too short length of 12.5 cm and third one with a too long length of 56 cm. The data illustrate that the cheeks undercompensate in case of a too long pendulum and make the period larger than the theoretical small-angle period computed with formula (5). In case of a too short pendulum, the cheeks overcompensate and the period becomes less than the theoretical small-angle period. Actually, measurements show that the period gets smaller when the initial displacement becomes larger. The reasoning behind this is not considered to be beyond the reach of secondary school students as the earlier discussed example of a question in a Dutch nationwide physics exam in 2013 [4] illustrates (see section 1).

### 5.4. Comparing the experimental results with modelling results for the motion of a cycloidal pendulum

So far, our experimental results concerned the period of a free and a cycloidal pendulum. But with the equation of motion for the angle of a cycloidal pendulum (formula (8)) we can compare the measured orbit of the bob with the computer modelling result. We used the system dynamics based graphical modelling tool of Coach [9, 10].

In a graphical model, variables, parameters, and relationships between them are represented by means of a system of icons in a diagrammatic picture. The system of differential equations is in this case

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = \alpha \end{cases}$$

where the angular acceleration $\alpha$ is given by

$$\alpha = \omega^2 \tan \theta - \frac{k}{4r} \tan \theta.$$
Table 1. Comparison of the measured large-displacement period with the theoretical small-angle period for pendulums of various lengths swinging against the small cheeks of Huygens.

<table>
<thead>
<tr>
<th>Pendulum length (m)</th>
<th>Theoretical small-angle period (s)</th>
<th>Maximum displacement tried out (m)</th>
<th>Measured period (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.125</td>
<td>0.709</td>
<td>0.065</td>
<td>0.698</td>
</tr>
<tr>
<td>0.250</td>
<td>1.003</td>
<td>0.174</td>
<td>1.006</td>
</tr>
<tr>
<td>0.560</td>
<td>1.501</td>
<td>0.290</td>
<td>1.519</td>
</tr>
</tbody>
</table>

Figure 10. Screen shot of a Coach modelling activity in which the horizontal position of the bob of a cycloidal pendulum with a period of 1 s is computed by the graphical model in the upper window and plotted as the orange curve in the diagram in the lower window. This graph matches the dark blue background graph of measured data very well.

For a pendulum with a string of arbitrary length \( L \) winding along cycloid cheeks one only needs to add to the above graphical model a parameter and to replace the angular acceleration \( \alpha \) by the following formula:

\[
\alpha = \omega^2 \cdot \frac{4r \sin \theta}{l(\theta)} - \frac{g}{l(\theta)} \sin \theta,
\]

where

\[
l(\theta) = L - 4r + 4r \cos \theta.
\]

This follows from the equation of motion given by formula (9) (see also, the supplement Equation of motion). We used this computer model to simulate various situations and to compare simulation results with experimental results. For example, simulating the third situation in the third row of table 1, that is, simulating a pendulum of length \( L = 56 \text{ cm} \) swinging against the small cheeks of Huygens specified by \( r = 6.25 \text{ cm} \) led
to a computed period of 1.518 s which is in perfect agreement with the measured period of 1.519 s listed in table 1.

One can also use simulations to explore the effect of measurement errors on the period of the pendulum. For example, for a pendulum swinging against the small cheeks of Huygens with a theoretical period of 1 s, a measurement error of 1 mm in the length of the pendulum leads to an error of 0.002 s in the period of the pendulum.

6. Conclusions and discussion
The small amplitude pendulum has been extensively used in school or undergraduate laboratory work as a classic example of simple harmonic motion with a period $T = 2\pi \sqrt{\frac{l}{g}}$, where $g$ is the acceleration of gravity and $l$ is the length of the pendulum. Many students do experiments with pendulums of varying length to determine the local acceleration of gravity. History shows a keen interest in extending the theory to large-angle pendulum motion, on the one hand, and in designing devices that would make the period independent of the amplitude, on the other hand. We started this experimental study with the question how well the cheeks of Huygens are able to enforce a pendulum motion with a period independent of the initial amplitude, and whether this could be tested in a school setting with a homemade setup. The presented work shows that this leads to astonishing good results, provided that the setup is constructed with great care. Student can achieve such marvellous results as well, provided that they work with great care and apply mathematical competencies to reach a good analysis of their experimental data.

Looking back at Huygens’ work on the isochronous pendulum motion, we believe that it can be effectively used to let students experience the strong connection between physics and mathematics. History can help design a learning trajectory for studying pendulum motion. The mathematics presented in section 2 of this paper is certainly not beyond undergraduate physics level.

Due to the time-consuming nature of the laboratory work, students are not required to evaluate uncertainties and errors. But when several student groups do the same experiments, they can jointly study this aspect of data analysis. Anyhow, students must be able to justify good agreement of experimental results with theoretical results obtained by a simple or more advanced theoretical model. We have done this successfully in our comparison of experimental large-angle pendulum data with formulas for the period of a pendulum of such type. We believe that secondary school students and certainly undergraduate students can do the same and enhance in this way their competencies in mathematics and physics.

Data availability statement
All data that support the findings of this study are included within the article (and any supplementary files).

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