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A remark on the irreducible characters and fake degrees of finite real reflection groups

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Summary. Beynon and Lusztig have shown that the fake degrees of almost all irreducible characters of finite real reflection groups are palindromes, and that the exceptions to this rule correspond to the non rational characters of the generic ring A defined over $R = \mathbb{C}[q]$. Their proof consists of a case-by-case check. In this note we give an explanation for this phenomenon and some related facts about fake degrees. Moreover, in the situation where we allow for distinct parameters q_x in the definition of A , we shall give a simple uniform proof of the fact that all the central idempotents of $A^{\tilde{K}}$ are elements of $A^{\tilde{K}}$, where $\tilde{K} = \mathbb{C}[\sqrt{q_x}]$.

Let G be a finite real reflection group acting on a Euclidean vector space V of dimension n . The G -action on the complexification $V_{\mathbb{C}}$ of V has a natural extension to the symmetric algebra $\mathcal{S}$ of $V_{\mathbb{C}}$. Let $\mathcal{I}$ denote the ideal of $\mathcal{S}$ that is generated by the subalgebra $\mathcal{S}_+^G$ that consists of linear combinations of G -invariant elements of positive homogeneous degree. It is well known that the coinvariant algebra $\mathcal{S}/\mathcal{I}$ affords the regular representation when considered as a representation space for G . The G -action on $\mathcal{S}/\mathcal{I}$ is compatible with the natural grading of $\mathcal{S}/\mathcal{I}$ by homogeneous degree. For each irreducible representation $\tau \in \hat{G}$ of degree $d_{\tau} = \chi_{\tau}(1)$ we let $p_{\tau}^1, \dots, p_{\tau}^{d_{\tau}}$ denote the embedding degrees of τ in $\mathcal{S}/\mathcal{I}$, written in nondecreasing order. The positive integers p_{τ}^i are usually called the τ -exponents.

Definition 1. Let $\tau \in \hat{G}$. The fake degree $f_{\tau} \in \mathbb{C}[T]$ of τ is defined by

$$f_{\tau}(T) = \sum_{i=1}^{d_{\tau}} T^{p_{\tau}^i}.$$

The fake degrees are relevant for the study of representations of finite groups of Lie type. They have been determined explicitly for all irreducible

finite real reflection groups due to work of Steinberg [20] (the case of the symmetric groups S_n), Lusztig [17] (the other classical groups), Macdonald (unpublished) (F_4), Beynon and Lusztig [3] (the other exceptional Weyl groups), and Alvis and Lusztig [1] (H_4) (the remaining cases being trivial). The fundamental formula that makes it possible to find these fake degrees explicitly in the exceptional cases is the following:

$$f_\tau(T) = |G|^{-1} \prod_{i=1}^n (1 - T^{d_i}) \sum_{g \in G} \frac{\chi_\tau(g)}{\det_V(1 - Tg)} \quad (1)$$

where $d_1, \dots, d_n$ are the fundamental degrees of G . Obviously, in order to make (1) effective, explicit character tables (cf. [10, 11]) and computer aid for the expansion of (1) have to be used.

We shall now describe some striking facts that emerge from the explicit lists of the fake degrees. These observations are all due to Beynon and Lusztig [3] in the Weyl group cases. From [1] it is clear that their observations also apply to the remaining cases. First of all, the f_τ are palindromic in an overwhelming majority of cases. In those cases one has the following formula for the center of symmetry of f_τ in terms of the values of the character χ_τ on the reflections of G . For any $\tau \in \widehat{G}$ we define

$$N_\tau = d_\tau^{-1} \sum \chi_\tau(r)$$

where the sum runs over the set of all reflections in G . If N denotes the number of reflections in G and $\tau \in \widehat{G}$ is such that f_τ is palindromic, then

$$f_\tau(T^{-1}) = T^{N-N_\tau} f_\tau(T).$$

Moreover, if f_τ is not palindromic then there exists a unique $j(\tau) \in \widehat{G}$ such that

$$f_\tau(T^{-1}) = T^{N-N_\tau} f_{j(\tau)}(T). \quad (2)$$

It turns out that $N_\tau = N_{j(\tau)}$ or, equivalently, $j^2(\tau) = \tau$. If one defines $j(\tau) = \tau$ for all τ such that f_τ is a palindrome then $j: \widehat{G} \rightarrow \widehat{G}$ is an involution such that (2) holds for all $\tau \in \widehat{G}$.

In regard to formula (2) it is interesting to note that we also have the following formula, due to a recent observation of De Jeu [15]:

$$2 \sum_{i=1}^{d_\tau} p_\tau^i = d_\tau(N - N_\tau).$$

This follows from (1) by differentiation at $T = 1$. It explains why the involution j of $\widehat{G}$ has the following property:

$$\sum_{i=1}^{d_\tau} p_\tau^i = \sum_{i=1}^{d_\tau} p_{j(\tau)}^i. \quad (3)$$

From [1] and [3] it is known precisely for which τ the fake degree is not palindromic. These exceptional characters are: the two characters of E_7 of degree 512, the four characters of E_8 of degree 4096, the two characters of H_3 of degree 4 and finally the four characters of H_4 of degree 16 that are denoted by χ_{18} , χ_{19} , χ_{20} , χ_{21} in the list of Grove [12]. As was observed in [3], these characters also distinguish themselves from the other characters by means of another property they have in common: these characters do not correspond to a rational character of the generic algebra (also see [4], section 11.3, and [18]).

The method that we shall employ to study these matters is based on an explicit realization of representations of the generic ring via certain systems of differential equations that are defined on $G \backslash V^{\text{reg}}$, and which were studied extensively in [8], [16], [19]. Let us recall the main results for our purposes in this paper. Let $\mathcal{R} \subset V^*$ be the normalized root system for G . We decompose $\mathcal{R}$ into orbits $\mathcal{R}_1, \dots, \mathcal{R}_m$ for the action of G . Let $k_1, \dots, k_m$ be complex indeterminates. Moreover, we shall use the notation $q_i = -\exp(-2\pi\sqrt{-1}k_i)$ throughout this paper. If $\alpha \in \mathcal{R}_i$ then we define $k_\alpha = k_i$ and $q_\alpha = q_i$. The reflection in G corresponding to α will be denoted by r_α . The foremost important operator in this theory is the well known Dunkl operator $T_\xi(k)$ [7]. It acts on functions that are defined on a G -invariant subset of V , and is defined by the following formula ($\xi \in V$, $\underline{k} = (k_1, \dots, k_m)$):

$$T_\xi(\underline{k})(f) = \partial_\xi f + \sum_{\alpha \in \mathcal{R}_+} k_\alpha \alpha(\xi) \alpha^{-1}(f - f^{r_\alpha}).$$

The T_ξ commute [7], hence we can define an operator T_s for each element $s \in \mathcal{S}$. The T_ξ are equivariant for the G -action. Therefore, if $s \in \mathcal{S}^G$ then T_s maps G -invariant functions to G -invariant functions. Moreover, the restriction of T_s to G -invariant functions on V^{reg} is in fact a linear partial differential operator D_s with polynomial coefficients on $G \backslash V$ [13]. This gives rise to the following system of differential equations for elements f of $\mathcal{O}_p$, the space of holomorphic germs at a point p of the regular orbit space $G \backslash V^{\text{reg}}$ (see [19]):

$$\mathcal{L}_p(\underline{k}) = \{f \in \mathcal{O}_p \mid D_s(\underline{k})(f) = 0 \forall s \in \mathcal{S}_+^G\}.$$

This system of equations reduces to the system of equations defining the harmonic polynomials on V when we substitute $\underline{k} = 0$. In this special case we see that $\mathcal{L}(0)$ is a local system on $G \backslash V^{\text{reg}}$ of rank $|G|$. Its monodromy can easily be described as follows. The fundamental group of $G \backslash V^{\text{reg}}$ with respect to some base point p is isomorphic to the braid group B of G . The monodromy action $\mu(0)$ on $\mathcal{L}_p(0)$ factors through the natural map $B \rightarrow G$ and this results in a representation $\nu(0)$ of G , which is equivalent to the regular representation (because it is just the G -action on harmonic polynomials).

It was shown in [19] that this situation for $k = 0$ is typical of the situation for generic $\underline{k}$. To fix notations, let R denote the ring of complex polynomials in the q_i , and let S be the ring of entire functions of the indeterminates k_i . Let their fields of fractions be denoted by K and L respectively. The generic algebra A of the Coxeter system (G, Σ) (where Σ denotes a set of simple

reflections) is by definition the free R -algebra with generators a_s , ($s \in \Sigma$) that satisfy the braid relations as usual and the (slightly unconventional) quadratic relations

$$\forall s \in \mathcal{S} : (a_s - 1)(a_s - q_s) = 0.$$

We shall use the notations $A(\underline{q})$ for the specialization of A at $\underline{q}$ and A^S for the extension of scalars by S . From [19] we have:

Theorem 2. *The solution space $\mathcal{L}_p$ is a free S module of rank $|G|$ ($\forall p \in G \setminus V^{\text{reg}}$) and its monodromy μ factors through the natural map $SB \rightarrow A^S$. This results in a representation v of A^S with the property that $v^{\bar{L}}$ is equivalent to the regular representation of $A^{\bar{L}}$.*

Actually, for all $\tau \in \widehat{G}$ one can define a locally free sheaf of S modules over $G \setminus V^{\text{reg}}$ for which the monodromy affords a representation of A^S that specializes to τ at $\underline{k} = 0$. This sheaf is the sheaf of flat sections of the Kniznik–Zamolodchikov–Dunkl connection with values in V_τ , the representation space of τ (cf. [8]). An important consequence of this construction is the fact that A^L is a split semisimple algebra. If $\tau \in \widehat{G}$ we shall use the notation e_τ for the central primitive idempotent of A^L that corresponds to τ when we specialize at $\underline{k} = 0$. The corresponding irreducible character of A^L is denoted by χ_τ . Clearly, $\chi_\tau(a_g) \in S \forall g \in G$, a fact that we shall use later on.

Note that if $\underline{l} \in \mathbf{Z}^m$ then

$$e_\tau(\underline{k} + \underline{l}) = e_{\tau'}(\underline{k})$$

for some $\tau' \in \widehat{G}$, because the structural constants of A are periodic functions with period lattice $\mathbf{Z}^m$ (being elements of K). This leads us to the following:

Definition 3. *Let $\text{Per}(\widehat{G})$ denote the group of permutations of $\widehat{G}$. We denote by*

$$\rho : \mathbf{Z}^m \rightarrow \text{Per}(\widehat{G})$$

the homomorphism defined by ($\tau \in \widehat{G}$ and $\underline{l} \in \mathbf{Z}^m$) :

$$e_\tau(\underline{k} + \underline{l}) = e_{\rho_l(\tau)}(\underline{k}).$$

The defining differential equations for $\mathcal{L}$ possess certain symmetry properties with respect to the group of linear characters of G , and these will be of crucial importance to us. Let us denote the group of linear characters of G by G^* . Given $\varepsilon \in G^*$ we define $\underline{l}(\varepsilon) \in \{0, 1\}^m$ by the rule $\varepsilon(r_\alpha) = (-1)^{l_i(\varepsilon)} \forall \alpha \in \mathcal{R}_i$. This defines an isomorphism from G^* onto $(\mathbf{Z}/2\mathbf{Z})^m$. Next we define an action of G^* on the indeterminates k_i by means of the formula

$$i_\varepsilon(k_i) = l_i(\varepsilon) + (-1)^{l_i(\varepsilon)} k_i.$$

Observe that $i_\varepsilon(k_i) = k_i + l_i(\varepsilon) \bmod(2\mathbf{Z})$ if $k_i \in \mathbf{Z}$. Now we can formulate the symmetries of the differential equations. These results are very elementary and can be found in [19], Proposition 3.9.

Proposition 4. *Let $\pi(\underline{k}) = \prod_{\alpha \in \mathfrak{H}_+} \alpha^{k_\alpha}$. This is a multivalued function on $G \setminus V^{\text{reg}}$ of determination order one. Conjugation of a differential operator by such a function is a well defined operation, and we have ($s \in S^G$, $\varepsilon \in G^*$):*

$$D_s(i_\varepsilon(\underline{k})) = \pi(\underline{k} - i_\varepsilon(\underline{k})) \circ D_s(\underline{k}) \circ \pi(i_\varepsilon(\underline{k}) - \underline{k}).$$

The next Theorem is well known if we specialize to the situation $q_i = q \forall i$ by the results of [1, 2, 5, 18]. However, since our proof is surprisingly simple and covers the two parameter case as well we decided to include this here.

Theorem 5. *The irreducible characters χ_τ , $\tau \in \widehat{G}$ of the split semisimple algebra $A^{\widetilde{K}}$ have the following property:*

$$\forall g \in G : \chi_\tau(a_g) \in \widetilde{R}$$

where $\widetilde{R} = \mathbf{C}[\sqrt{q_i}]$. Consequently, the central idempotents e_τ of $A^{\widetilde{K}}$ belong to $A^{\widetilde{K}}$ where $\widetilde{K}$ is the quotient field of $\widetilde{R}$.

Proof. Since the central idempotents e_τ are defined over L , we can decompose $\mathcal{L}$ into isotypical components:

$$\mathcal{L}(\underline{k}) = \bigoplus_{\tau \in \widehat{G}} \mathcal{L}_\tau(\underline{k}).$$

By Proposition 4 we have:

$$\mathcal{L}(i_\varepsilon(\underline{k})) = \pi(\underline{k} - i_\varepsilon(\underline{k})) \cdot \mathcal{L}(\underline{k}).$$

Comparing the isotypical decompositions on both sides of this equation, we obtain:

$$\mathcal{L}_\tau(i_\varepsilon(\underline{k})) = \pi(\underline{k} - i_\varepsilon(\underline{k})) \cdot \mathcal{L}_{\tau'}(\underline{k}) \tag{4}$$

for some $\tau' \in \widehat{G}$. We calculate τ' in two different ways: first we substitute $\underline{k} = 0$ and use Definition 3. We obtain:

$$\rho_{\underline{l}(\varepsilon)}(\tau) = \varepsilon \otimes \tau'. \tag{5}$$

On the other hand, substitution of $\underline{k} = \underline{l}(\varepsilon)$ yields

$$\tau = \varepsilon \otimes \rho_{\underline{l}(\varepsilon)}(\tau').$$

Comparison of the two expressions gives $\rho_{\underline{l}(\varepsilon)} = \rho_{-\underline{l}(\varepsilon)}$. This means that all idempotents e_τ are periodic with a period lattice that contains $(2\mathbf{Z})^m$. Since we are working in the symmetric split semisimple algebra A^L we may apply Fossum's formula relating central primitive idempotents and irreducible characters [9]. Hence $\chi_\tau(a_g)$ is also periodic with a period lattice that contains $(2\mathbf{Z})^m$. Consequently, $\chi_\tau(a_g) \in S$ is an entire function of $\sqrt{q_1}, \dots, \sqrt{q_m}$. Moreover, it is well known that $\chi_\tau(a_g) \in S$ is integral over R , hence certainly integral over $\widetilde{R}$. It follows that $\chi_\tau(a_g) \in \widetilde{R}$, by an elementary argument from the theory of entire functions. $\square$

By means of the map $\varepsilon \rightarrow \underline{l}(\varepsilon)$ we defined an isomorphism $G^* \simeq (\mathbf{Z}/2\mathbf{Z})^m$. Because of the previous theorem we now know that $(2\mathbf{Z})^m \subset \text{Ker}(\rho)$. So we can now compose $\varepsilon \rightarrow \underline{l}(\varepsilon)$ with ρ to obtain the following definition:

Definition 6. Let $j : G^* \rightarrow \text{Per}(\widehat{G})$ be the homomorphism such that $\forall \varepsilon \in G^*$ and $\forall \tau \in \widehat{G}$:

$$j_\varepsilon(\tau) = \rho_{\underline{l}(\varepsilon)}(\tau). \quad (6)$$

Notice that $G^* \simeq (\mathbf{Z}/2\mathbf{Z})^m \simeq \text{Gal}(\widetilde{K}/K)$ and that in this sense j is the natural Galois action. Observe that $j_\varepsilon = \rho(i_\varepsilon(\underline{k}) - \underline{k})$ if $\underline{k} \in \mathbf{Z}^m$.

We return to the study of $\mathcal{L}$ in order to say more about the fake degrees. As was mentioned before, the specialization $\mathcal{L}(0)$ is in fact just the space of harmonic polynomials and $v(0)$ is the natural G -action on these polynomials. The information about the τ -exponents is given by the eigenspace decomposition of $\mathcal{L}(0)$ with respect to the Euler vector field $E = \sum x_i \partial_{x_i}$ on V . This can be generalized to $\mathcal{L}$. Since the equations defining $\mathcal{L}$ are homogeneous it is clear that E acts on $\mathcal{L}$. Moreover, it is clear that this action commutes with the monodromy action on $\mathcal{L}$. Hence it makes sense to extend the notion of τ -exponent by means of the eigenvalues of E on $\mathcal{L}_\tau = e_\tau \cdot \mathcal{L}$. These eigenvalues were studied in [19], Theorem 7.10. The results are as follows:

Theorem 7. Let $\mathcal{K}$ be the largest open set of values of $\underline{k}$ such that $A^S(\underline{k})$ is semisimple, and all central idempotents of $A^{\overline{K}}$ are regular at $\underline{k}$ (this is the complement of a closed analytic subset of codimension one in $\mathbf{C}^m$). If $\underline{k} \in \mathcal{K}$ then $\mathcal{L}_\tau(\underline{k})$ is well defined. Moreover, E acts semisimply on $\mathcal{L}_\tau(\underline{k})$, and its eigenvalues are:

$$p_\tau^i(\underline{k}) = p_\tau^i - \sum_{\alpha \in \mathcal{R}_+} k_\alpha \left(1 - \frac{\chi_\tau(r_\alpha)}{d_\tau} \right) \quad (7)$$

(where r_α is the reflection in G that corresponds to α).

We are now ready to present the main result of this paper:

Theorem 8. Let $\varepsilon \in G^*$ and $\tau \in \widehat{G}$. Put

$$N_{\tau, \varepsilon} = d_\tau^{-1} \sum_{\alpha \in \mathcal{R}_+} l_\alpha(\varepsilon) \chi_\tau(r_\alpha).$$

Then:

$$p_\tau^i = p_{\varepsilon \otimes j_\tau(\tau)}^i - N_{\tau, \varepsilon}$$

or, in other words,

$$f_\tau(T) = T^{-N_{\tau, \varepsilon}} f_{\varepsilon \otimes j_\tau(\tau)}(T).$$

Proof. By (4), (5) and (6) the eigenvalues of E must be the same on $\mathcal{L}_\tau$ and $\pi(\underline{k} - i_\varepsilon(\underline{k})) \cdot \mathcal{L}_{\varepsilon \otimes j_\tau(\tau)}$. Now use the derivation property of E and (7) and equate. $\square$

When we take $\varepsilon = \delta$, the linear character defined by $\delta(g) = \det(g)$, we arrive at the aforementioned observations by Lusztig and Beynon [3]:

Corollary 9. The fake degrees satisfy:

$$f_\tau(T) = T^{N - N_{\tau, \delta}} f_{j_\delta(\tau)}(T^{-1}).$$

Proof. From Theorem 8 we have:

$$f_{\tau}(T) = T^{-N_{\tau,\delta}} f_{\delta \otimes j_{\delta}(\tau)}(T)$$

where N is the number of reflections in G . Now use the well known and elementary fact

$$f_{\delta \otimes \tau}(T) = T^N f_{\tau}(T^{-1})$$

to finish the argument. $\square$

Let us describe this action of G^* on $\widehat{G}$ for each of the individual irreducible groups, based on results of Alvis and Lusztig [1], Benson and Curtis [2], Curtis, Iwahori and Kilmoyer [6], and Lusztig [18]. The action is trivial for A_n , D_n , E_6 , and for the dihedral groups $I_2(2n)$ with n odd. In the cases E_7 , E_8 the action of j_{δ} ($\delta(g) = \det(g)$) coincides with the involution i as was described by Beynon and Lusztig in [3]. If G is of type H_3 or H_4 we can determine the action just using the explicit list of fake degrees in combination with (3) and Corollary 9 (the list of fake degrees of H_4 can be found in [1]; we thank G.J. Heckman for providing us with such a list for H_3). The results are as follows. In the case H_3 , j_{δ} interchanges the two characters of degree 4 (so on these characters j_{δ} is the same as multiplication by δ) and is trivial on the other characters. If G is H_4 then $j_{\delta}(\chi_{18}) = \chi_{20}$ and $j_{\delta}(\chi_{19}) = \chi_{21}$ (whereas $\delta \otimes \chi_{18} = \chi_{19}$ and $\delta \otimes \chi_{20} = \chi_{21}$), and is trivial on all other characters of H_4 . In the cases B_n , F_4 the action is trivial. This follows from the results of the sections 6 and 10 of [2], where it is shown that for these groups each irreducible representation η is determined by the multiplicities $(\eta, 1_{W_J}^W)$ if W_J runs through the set of all parabolic subgroups. (The B_n case also follows from the thesis of Hoefsmit [14], where the irreducible characters of the Hecke algebra are constructed explicitly.) In the remaining even dihedral cases $I_2(2n)$, $n \in 2\mathbb{Z}$, $n \geq 6$, j_{ε} is trivial on the linear characters and has the effect of tensoring by ε on the characters of degree two ($\forall \varepsilon \in G^*$).

It is a natural and intriguing question how much of the above survives when G is a complex reflection group. We have been able to define the Dunkl operators in this situation. Consequently we can define a local system with the right properties and we can calculate the eigenvalues of the Euler vector field acting on this local system. Unfortunately we do not know at present how to prove the necessary symmetry properties for the defining differential equations.

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