A scattering theory of ultrarelativistic solitons

Amin, M.A.; Lim, E.A.; Yang, I.S.

Published in:
Physical Review D. Particles, Fields, Gravitation, and Cosmology

DOI:
10.1103/PhysRevD.88.105024

Citation for published version (APA):
A scattering theory of ultrarelativistic solitons

Mustafa A. Amin, Eugene A. Lim, and I-Sheng Yang

Kavli Institute for Cosmology and Institute of Astronomy, Madingley Road, Cambridge CB3 0HA, United Kingdom
Theoretical Particle Physics and Cosmology Group, Physics Department, King’s College London, Strand, London WC2R 2LS, United Kingdom
IOP and GRAPPA, Universiteit van Amsterdam, Science Park 904, 1090 GL Amsterdam, Netherlands

I. INTRODUCTION

In 1834, Russell discovered solitary waves [1]: spatially localized configurations of fields that propagate without any distortion. More than 150 years after their discovery, solitary waves are still actively studied for their often counterintuitive, yet elegant properties [2–4]. They have found applications in disparate fields such as atomic physics [5], superconductivity [6], field theory [7], biology [8], condensed matter physics [9,10], nonlinear optics [11], quantum chromodynamics [12], cosmology [13–15] and neuroscience [16]. Our paper is an attempt to further the study of interactions of these fascinating objects.

Solitary waves have a particular subset called solitons [2] which possess a rather intriguing property. Solitons are not only stable on their own, collisions between solitons leave them completely unchanged apart from a phase shift. A tremendous amount of work exists on understanding solitons and their interactions [17–23]. However, examples of systems with true solitons are limited (for example, see [2]). Such systems are often integrable and for such integrable systems powerful techniques exist to write down their multisoliton solutions analytically [17,18]. However, it is difficult to know a priori whether the solitary waves in a theory are also solitons and whether such systems admit analytic multisoliton solutions.

We would like to predict the results of general solitary wave collisions. Such collisions include, but are not limited to, the elastic case, characteristic of true solitons in integrable systems. We focus on solitary waves in relativistic scalar field theories with canonical kinetic terms and potentials with effectively single minima (including periodic potentials with multiple minima). Such theories and their solitary wave solutions appear naturally in cosmology and high energy physics [24,25]. We are also motivated by the fact that in certain relativistic scalar field theories, the solitary waves effectively pass through each other when colliding at ultrarelativistic velocities [29–34].

In this paper, we provide a general perturbative framework to study ultrarelativistic solitary wave collisions. Assuming that the colliding solitary waves effectively pass through each other at zeroth order, we provide a perturbative framework to calculate the corrections to this “free passage” behavior. We then show that the corrections are indeed small and present an order by order prescription to calculate the full result.

The essential ingredient of our perturbative framework is as follows. The colliding solitary waves interact significantly only when they overlap. In the rest frame of one solitary wave where the other approaches with velocity \( v \rightarrow 1 \), the space-time area of such overlap \( A_{\text{int}} \approx (\gamma v)^{-1} \), where \( \gamma = (1 - v^2)^{-1/2} \approx 1 \). We use this property to provide a perturbative framework to calculate the effects of collisions, with \((\gamma v)^{-1}\) as the small expansion parameter.

In Sec. II we discuss the conditions that the solitary waves must satisfy for our framework to be applicable and state some simplifying assumptions. Despite these simplifications, our framework should be applicable to

---

1Some condensed matter systems near critical points also admit a similar structure [26]. One simply replaces the speed of light by the Fermi velocity [27,28].

2In the case of single minimum or periodic potentials, effectively passing through each other is captured by a linear superposition of the two solitary wave solutions. In multiple minima potentials which are not periodic, this is not true and further modification of the solitary wave profiles has to be taken into account. As a result, we restrict ourselves to single minimum or periodic potentials in this paper.
many well-known objects like oscillons [35], Q-balls [36], and domain walls. We then present the general framework of our perturbation theory to calculate the effects of the collision, paying particular attention to how the \((\gamma \nu)^{-1}\) expansion emerges. Since we are interested in the effect of the collisions, not necessarily the subsequent evolution of the perturbations, we always evaluate the perturbations soon after the collision. The long time behavior of the perturbations is discussed in Appendix A. For simplicity, we only consider linearized perturbations in the main body of the text. Nonlinear effects are discussed in Appendix B.

After this general discussion, in Secs. III and IV we focus on a simple example in \((1+1)\) dimensions—kinks in a single scalar field theory with periodic potentials: \(V(\phi) = V(\phi + \Delta \phi)\) for all \(\phi\). We explicitly calculate the effects of the kink-kink collision at leading order in our expansion parameter \((\gamma \nu)^{-1}\). These leading order results already reveal a few important and surprising facts.

In the rest frame of the stationary kink which collided with an incoming kink, we find the following:

(i) The velocity change of the stationary kink is zero (therefore its velocity remains zero).

\[ \Delta \nu = 0 + O((\gamma \nu)^{-2}). \] (1.1)

(ii) The stationary kink acquires a phase shift (spatial translation)\(^4\) given by

\[
\Delta x = \frac{1}{2(\gamma \nu) M} \int_0^{\Delta \phi} \int_0^{\Delta \phi} d\phi_1 d\phi_2 \left[ \frac{V(\phi_1) + V(\phi_2) - V(\phi_1 + \phi_2)}{\sqrt{V(\phi_1)V(\phi_2)}} \right] + O((\gamma \nu)^{-2}). \] (1.2)

where \(M\) is the energy of the stationary kink.

We claim that the above expressions provide the closeform answer for the phase shift and velocity change [to leading order in \(1/(\gamma \nu)\)] for collisions between any pair of periodic kinks in \((1 + 1)\) dimensions. Importantly, the phase shift can be calculated simply by an integral over the potential. We do not need a solution for the field profile or the detailed dynamics of the interaction during the collision. We note that in a companion paper [37] we derive the above results for the position and velocity shift using a slightly different approach.

In Sec. V we show that these results are consistent with the exact results for the integrable Sine-Gordon model. We also show that they agree extremely well with our detailed numerical simulations in models that are far from being integrable. We discuss our results and possible future directions in Sec. VI. Finally, in Appendix C we describe the possible usefulness of an optical theorem for our framework. We hope that this work will serve as a stepping stone to a more general scattering theory of solitary waves.

II. GENERAL PERTURBATIVE EXPANSION

In this section we describe our general framework for understanding collisions of solitary waves. We focus on solitary waves in scalar field theories; however, the general idea is more widely applicable. We will discuss simplifying assumptions regarding the scalar field potential and the solitary waves under consideration. A large class of theories and their corresponding solitary wave solutions such as Q-balls [36,38], oscillons [35,39–42], kinks and domain walls are consistent with these assumptions. We will then show that the effect of collisions between ultrarelativistic solitary waves can be understood and calculated in a controlled fashion.

We keep the discussion quite general in this section. The reader craving more concreteness can refer to Secs. III and IV where we apply our framework to \((1+1)\) dimensional kinks. Some of the arguments below will be repeated there.

A. Scalar field potential

Consider a canonical scalar field with a Lagrangian

\[ L = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi). \] (2.1)

The equation of motion is\(^5\)

\[ \Box \phi = V'(\phi). \] (2.2)

We assume that the potential has a minimum (vacuum) at \(\phi = 0\) with \(V(0) = 0\). The potential can have multiple vacua as long as it is periodic: \(V(\phi) = V(\phi + \Delta \phi)\). Therefore, every vacuum is effectively the same. An example of such a potential is shown in Fig. 1. Our assumption regarding the potential is driven by practical considerations. We need a calculable background solution for perturbation theory to be easily applicable. For the potentials above, a linear superposition of two solitary waves provides a good background solution during collisions.\(^6\) Thus if individual solitary wave solutions are available (analytically or numerically), a good background solution can be easily constructed.

While this restriction on the potential might appear severe, note that the potential only has to be periodic for the field range explored by the solitary wave solutions. For example, individual kinks probe adjacent minima whereas two kinks interacting during a collision can probe multiple nearby minima. In contrast, small amplitude oscillons only

\(^3\)A stationary kink is a static solution that interpolates between adjacent minima of the potential. In higher dimensions they would be domain walls.

\(^4\)Note that the spatial translation is only meaningful in the absence of the velocity change.

\(^5\)We use the “mostly plus” convention for the Minkowski metric.

\(^6\)We verify this by calculating the correction to this superposition. The fact that superposition is a good solution before the collision comes from our assumption of localization discussed in the next subsection.

105024-2
probe a single minimum of the potential. Examples of field profiles of these individual solitary waves are shown in Fig. 2.

For some nonperiodic potentials (given the conditions specified in [32]), a modification on top of the superposition is required to obtain a good background solution. If a good background solution is available, our perturbative framework may be generalized to include these cases.

**B. Localized solutions**

We require that the solitary waves under consideration be spatially localized along the direction of collision. Along the orthogonal direction we require the objects to be either localized or possess a symmetry that renders the infinite directions redundant. For example, domain walls often possess such a symmetry.

A solitary wave solution \( \phi_\ell(x, t) \) is localized if for every \( \epsilon \), there is a center \( x_0 \) and a size \( L \) with \( |\phi_\ell(x, t) - \phi_\ell(x)| < \epsilon \) for all \( |x - x_0| > L/2 \).\(^7\) Here \( \phi_\ell = N\Delta\phi \) with \( N = 0, \pm 1, \ldots \) is the vacuum value of the field. One can picture a localized solution as having a small tail that rapidly approaches a minimum of \( V(\phi) \) beyond some length scale \( \sim L \) away from its center.

In this paper we assume exponentially suppressed tails. This has the advantage that along with the field profiles, other quantities derived from them (for example spatial integrals or derivatives of the profile) also have exponentially suppressed tails. In many cases, power law tails can be sufficient as well, but a more detailed statement has to be made about when they can be ignored. We leave that for future work.

\(^7\)Recall that it is fine to have spatial directions along which this criterion is not satisfied as long as there exists a symmetry that renders those directions redundant. The symmetry can be an approximate one. For example, as long as the radius of curvature of a domain wall is much larger than its thickness, it effectively has a planar symmetry.
Before the two solitary waves overlap, their linear superposition \((\phi_A + \phi_B)\) satisfies Eq. (2.2) if we ignore the exponentially suppressed tails (see lower plot in Fig. 4). Our claim is that \((\phi_A + \phi_B)\) continues to provide a good approximation even after the collision. We justify our claim by calculating the corrections to it and showing that they are indeed small. We calculate the corrections using a perturbative framework that uses \(1/(\gamma v)\) as the small parameter.

First, we write the general solution as
\[
\phi(x, t) = \phi_A(x, t) + \phi_B(x, t) + h(x, t). \tag{2.4}
\]
The linearized equation of motion for \(h\) is \(^8\)
\[
\Box - W_0(x, t)h = S(x, t) + \Delta W(x, t)h, \tag{2.5}
\]
where for future convenience we have defined
\[
S(x, t) \equiv V'(\phi_A + \phi_B) - V'(\phi_A) - V'(\phi_B), \tag{2.6}
\]
\[
W_0(x, t) \equiv V''(\phi_A), \tag{2.7}
\]
\[
\Delta W(x, t) \equiv V''(\phi_A + \phi_B) - V''(\phi_A). \tag{2.8}
\]
We refer to \(S\) as the source, \(W_0\) as the mass term (related to the stationary solitary wave) and \(\Delta W\) as the change in mass due to the fast moving solitary wave. Note that \(W_0\) is nonzero in the space-time area occupied by \(\phi_A\) [the entire green strip in Fig. 3(b)] and \(\Delta W\) is nonzero in the space-time area occupied by \(\phi_B\) [see orange strip in Fig. 3(b)].

However, \(S\) is nonzero only in the overlap region \(A_{\text{int}}\). Moreover, the maximum and minimum values of \(S\), \(W_0\) and \(\Delta W\) do not depend on \(v\), a direct consequence of dealing with a Lorentz invariant scalar field theory. Our definition of \(W_0\) in terms of \(\phi_A\) and the way we have arranged Eq. (2.5) reflects our choice to concentrate on the effects of the collision on \(\phi_A\). \(^9\) We concentrate on evaluating \(h(x, t)\) in the region \(|x| \leq A/2\) and \(t \sim t_{\text{int}}\), where \(t_{\text{int}} = v^{-1}(A + \gamma^{-1}B)\) marks the end of the collision [upper dashed line in Fig. 3(b)].

Let us first see why \(h(x, t)\) is expected to be small in ultrarelativistic collisions. Before the collision begins, i.e. for \(t < -t_{\text{int}}\), \(h(x, t) = 0\) is a consistent solution of Eq. (2.5). Then the solution for \(t \sim t_{\text{int}}\) is
\[
h(x, t) \sim \int dt' dx' G(x, t; x', t') S(x', t') \sim \frac{1}{(\gamma v)}, \tag{2.9}
\]
where \(G\) is a Green’s function satisfying
\[
\Box - W_0(x, t) G(x, t; x', t') = \delta(x - x', t - t'). \tag{2.10}
\]
To understand the above expression for \(h\) and the scaling with \((\gamma v)\) recall that \(S\) is nonzero only when the solitary waves overlap, i.e. in the interaction area \(A_{\text{int}} \propto 1/(\gamma v)\), and that the magnitude of \(S\) does not depend on \(\gamma v\) in this interaction area. These two properties of \(S\) yield the desired scaling of \(h \sim 1/(\gamma v)\). Note that in the above discussion we have ignored the \(\Delta Wh\) term because this term becomes

\(^{9}\)This focus on the stationary solitary wave is partially driven by the fact that in the special case where the solitary waves are static in their rest frame, their perturbations can be expanded in terms of a separable eigenmode basis.

---

\(^{8}\)The nonlinear effects from higher order terms in \(h\) are discussed in Appendix B.

---

FIG. 3 (color online). (a) The figure shows a collision between a stationary solitary wave \(\phi_A\) and a fast moving one \(\phi_B\) as a function of space and time (not an actual simulation). Notice the Lorentz contraction in the spatial width of the fast moving soliton. Also note the small perturbations generated by the collision. (b) A simplified but useful view of the collision, highlighting the space-time area occupied by the solitary waves. The strips show the space-time area where the field deviates significantly from its vacuum value(s). The vertical (green) strip represents the space-time area occupied by the stationary solitary wave, whereas the thin diagonal strip (orange) represents the area occupied by a fast moving one. The effective overlap area, or the area of interaction \(A_{\text{int}}\), is shown as a black parallelogram in the middle. The time \(t = t_{\text{int}}\) marks the end of the collision. When the incoming solitary wave is ultrarelativistic, the interaction area is suppressed by \(1/(\gamma v)\).
From the above expressions, it is easy to see that at 

$G$

using the Green’s function

$A$

As stated earlier, $S$ is only active in the interaction area $A_{\text{int}}$. $\Delta W h$ will be active in a larger (presumably infinite) area, but only contributes to the integral below $t$ because of the Green’s function. For $t \sim t_{\text{int}}$, this contribution comes from the dashed orange parallelogram along with the solid black one in Fig. 3(b). This active area is similarly suppressed: $A_{\text{int}}^{\prime} \sim A_{\text{int}} \approx 1/(\gamma \nu)$. Thus, if we only care about the value of $h$ up to time $t \sim t_{\text{int}}$, our recursive expansion $h^{(n)}$ has the following convenient scaling:

\begin{equation}
\begin{split}
    h^{(1)}(x, t) &= \int_{A_{\text{int}}} dt' dx' G(x, t; x', t') S(x', t') \sim \frac{1}{(\gamma \nu)}, \\
    h^{(n)}(x, t) &= \int_{A_{\text{int}}} dt' dx' G(x, t; x', t') \Delta W(x', t') h^{(n-1)}(x, t) \\
    &\sim \frac{1}{(\gamma \nu)^n}, \quad n \geq 2.
\end{split}
\end{equation}

From the above expressions, it is easy to see that at $t \sim t_{\text{int}}$ and $(\gamma \nu) \gg 1$: $h^{(n)} \approx A_{\text{int}} h^{(n-1)} \approx 1/(\gamma \nu)^n$. The long term behavior of the perturbations, $h(t \gg t_{\text{int}})$, is a bit more subtle and is discussed in Appendix A.

In summary, from the point of view of perturbations on top of the solitary wave $\phi_A$, the “collision” is nothing but an “overlap” of the background. The overlap results in some temporary nontrivial dynamics. When $(\gamma \nu) \gg 1$, the nontrivial dynamics lasts a short time and happens in a small spatial region. Thus, it leaves a small effect when the incoming solitary wave leaves.\textsuperscript{10} We draw that pictorially in Fig. 4. In the upcoming sections we will calculate $\delta h$ for a collision between a stationary and incoming kink. Although not evident from the notation, we will always assume that we are evaluating the perturbation in the vicinity of the stationary solitary wave and in the time interval $t \sim t_{\text{int}}$.

\textbf{III. (1 + 1) DIMENSIONAL KINKS}

In this section we discuss an example of solitary waves, namely isolated kinks in (1 + 1) dimensional scalar field theories. We will study their collisions in Sec. IV. First, we review general properties of individual kinks. We then show why they can be considered localized in the sense discussed in Sec. II, which in turn allows us to apply our general framework to study their collision. We chose (1 + 1) dimensional kinks because their properties are well understood. Moreover, unlike intrinsically time-dependent solitary waves like oscillons, stationary kinks are time independent. This allows us to decompose linear perturbations around the kinks into a convenient eigenmode basis, which we use in calculating and expressing the physical meaning of $h^{(i)}$.

To avoid possible confusion in the upcoming sections, we pause briefly to clarify our notation. Primes $'$ on a function with a single argument denote partial derivative with respect to that argument, so $f'(u) = \partial_u f$, where $u = \phi, x, t$ or $\gamma(x + vt)$ (for example). We will always denote the argument when using the prime notation. When there is an ambiguity, we will restore the partial derivatives.

\textbf{A. Background}

Consider a 1 + 1 dimensional, canonical scalar field theory with a periodic potential $V(\phi) = V(\phi + \Delta \phi)$ (see Fig. 1). The equation of motion for the scalar field is given by

\begin{equation}
\partial_t^2 \phi(x, t) - \partial_x^2 \phi(x, t) + V'[\phi(x, t)] = 0. \quad (3.1)
\end{equation}

A classic (though somewhat special) example of such a scalar field theory is the Sine-Gordon theory with $V(\phi) = m^2(1 - \cos \phi)$ and $\Delta \phi = 2\pi$. Note that in (1 + 1) dimensions, $\phi$ is dimensionless, and $V(\phi)$ has

\textsuperscript{10}This is reminiscent of the sudden collision approximation in heavy ion collisions. We thank David Seery and Cliff Burgess for pointing this out to us. A large $\gamma$ limit has also been used in gravitational self-force calculations [43].
the dimensions of (energy)$^2$. We assume our potential has
degenerate minima at $\phi = N\Delta \phi$ for integer $N$ with $V(N\Delta \phi) = 0$. We also assume that the minima are
quadratic and convex, that is, $V''(N\Delta \phi) = m^2 > 0$.

In scalar field theories with periodic potentials, there
exist static, minimum-energy field configurations called
kinks: $\phi_K(x)$. In such configurations, the field values
interpolate between neighboring minima of the potential.
Taking the field values at the two minima to be 0 and $\Delta \phi$, such a kink satisfies the relations

$$\phi_K''(x) = V'(\phi_K(x)).$$  \hfill (3.2)

$$\lim_{x \to -\infty} \phi_K(x) = 0,$$  \hfill (3.3)

$$\lim_{x \to +\infty} \phi_K(x) = \Delta \phi.$$  \hfill (3.4)

In general, the field profile changes significantly only in
an interval of length $\sim L$ around the center of the kink (see
Fig. 5). Thereafter, the field decays exponentially because
of the mass term $V''(N\Delta \phi) = m^2 > 0$:

$$\phi_K(x) \mod \Delta \phi \sim e^{-m|x|} \leq e^{-mL/2} \quad \text{for} \quad |x| > L/2. \quad (3.5)$$

Such field configurations, where the field deviates
from its vacuum values in a finite region $L$ and thereafter
decays exponentially, are precisely the kind of objects
where our general framework of Sec. II is applicable. As
discussed in that section, after picking an $L$ we will ignore
the tails beyond it. Explicitly, we make the following
pragmatic assumption which significantly simplifies our
analysis:

$$\phi_K(x) \mod \Delta \phi \sim e^{-m|x|} \leq e^{-mL/2} \quad \text{for} \quad |x| > L/2. \quad (3.5)$$

As we are primarily interested in investigating the effects
of kink-kink and kink-antikink collisions where short-
range interactions dominate, the long-range interaction
arising from exponentially suppressed tails can be safely
ignored. For the interested reader, the effects arising from
the exponentially suppressed tails have been studied in the
literature, notably in [44].

The total energy [sometimes colloquially called mass—
not to be mistaken with the mass $V''(\phi)$ term of a potential]
of a stationary kink is

$$M = \int_{-\infty}^{\infty} dx \left( \frac{1}{2} \phi_K''(x)^2 + V[\phi_K(x)] \right) = \int_0^{\Delta \phi} d\phi \sqrt{2V(\phi)}.$$  \hfill (3.7)

We have used $\phi_K'(x) = \sqrt{2V[\phi_K(x)]}$, which can be
obtained by integrating the equation of motion $\phi_K''(x) = V'[\phi_K(x)]$ by parts. Ignoring the exponentially damped tails,
$M = \int_{-L/2}^{L/2} dx \left( \frac{1}{2} \phi_K''(x)^2 + V[\phi_K(x)] \right)$.

Now, let us consider a kink moving at a constant speed $v$
from the positive to negative direction. Since our theory is
Lorentz invariant, such a moving kink is simply given by
boosting the stationary kink profile:

$$\phi(x, t) = \phi_K[\gamma(x + vt)],$$  \hfill (3.8)

where $\gamma = (1 - v^2)^{-1/2}$ is the Lorentz boost factor. The total energy of this moving kink is

$$\int_{-\infty}^{\infty} dx \left( \frac{1}{2} \left( \partial_\tau \phi_K[\gamma(x + vt)] \right)^2 + \frac{1}{2} \left( \partial_\tau \phi_K[\gamma(x + vt)] \right)^2 \right)$$

$$+ V(\phi_K[\gamma(x + vt)]) = \gamma M.$$  \hfill (3.9)

Hence its energy behaves like the energy of a point
particle.

B. Perturbations around an isolated kink

Before moving on to collisions, we discuss linearized
perturbations around isolated kinks and decompose them
into a convenient eigenmode basis. As noted before, this
eigenmode basis will simplify calculations and allow
for a useful physical interpretation of the perturbations.
Consider a small perturbation on top of a single static kink:

$$\phi(x, t) = \phi_K(x) + h(x, t).$$  \hfill (3.10)

It obeys the equation of motion (at linear order in $h$

$$\partial_\tau^2 h - \partial_\tau^2 h + W_0(x) h = 0,$$  \hfill (3.11)

where

$$W_0(x) \equiv V''[\phi_K(x)].$$  \hfill (3.12)

The general solution can be decomposed into discrete
(localized) $f_i(x)$ and continuous (free) $f_w(x)$ eigenmodes

FIG. 5 (color online). The orange curve is the field profile
of the kink for the potential given in Fig. 1. We can treat it as an
object of finite size $L$, since away from this central region the
field profile exponentially approaches the vacuum value, 0 or
$\Delta \phi$. The green curve is the normalized spatial derivative of the
profile.
with eigenvalue $E$. The normalized eigenmode that solves Eq. (3.14)

$$[-\partial_t^2 + W_0(x)]f_a(x) = E_a f_a(x)$$  \hspace{1cm} (3.14)

and form a complete orthonormal basis labeled by “$a$” in Eq. (3.13):

$$\int_{-\infty}^{\infty} f_i(x)f_j(x)dx = \delta_{ij}, \hspace{1cm} (3.15)$$

$$\int_{-\infty}^{\infty} f_{w_1}(x)f_{w_2}(x)dx = \delta(w_1 - w_2), \hspace{1cm} (3.16)$$

$$\int_{-\infty}^{\infty} f_0(x)f_{w_0}(x)dx = 0. \hspace{1cm} (3.17)$$

Plugging Eq. (3.13) into Eq. (3.11) and using the orthonormality of $\{f_a(x)\}$, we have an equation of motion for $g_a$:

$$g''_a(t) + E_ag_a(t) = 0. \hspace{1cm} (3.18)$$

Since $W_0(x)$ is time independent, these eigenmodes represent “stationary” excitations on top of a single kink akin to the stationary states of a time-independent Schrödinger equation. Hence it is convenient to express any small perturbation as a sum of eigenmodes on top of the kink.

The eigenmode with the smallest eigenvalue is the zero mode with eigenvalue $E_0 = 0$. As we will show below, the $E_0 = 0$ mode is associated with the boost and translational symmetry of the original equation of motion—the action of the zero mode is to shift the phase and the velocity of the kink. The normalized eigenmode that solves Eq. (3.14) with $E_0 = 0$ is given by

$$f_0(x) = \frac{1}{\sqrt{M}} \phi_K(x), \hspace{1cm} (3.19)$$

where $M$ is the mass of the kink. This can be checked immediately using Eqs. (3.2) and (3.7). The time dependence of the zero mode can be obtained from Eq. (3.18) as

$$g_0(t) = A_0 + B_0t. \hspace{1cm} (3.20)$$

To understand the physical meaning of $A_0$ and $B_0$, consider a small, spatial translation $(\Delta x)$ of the stationary kink profile:

$$\phi_K[x - (\Delta x)] = \phi_K(x) - \phi'_K(x)(\Delta x) + O(\Delta x)^2 + \cdots = \phi_K(x) - \sqrt{M}f_0(x)(\Delta x). \hspace{1cm} (3.21)$$

Similarly, a small velocity perturbation $(\Delta v)$ of the stationary kink profile yields [with $\gamma = (1 - \Delta v^2)^{-1/2} = 1 + (\Delta v^2)/2 + \cdots$]

$$\phi_K[\gamma(x - (\Delta v)t)] = \phi_K(x) - \phi'_K(x)(\Delta v)t + O(\Delta v)^2 + \cdots = \phi_K(x) - \sqrt{M}f_0(x)(\Delta v)t. \hspace{1cm} (3.22)$$

For comparison, consider the eigenmode expansion of the solution:

$$\phi(x, t) = \phi_K(x) + h(x, t) = \phi_K(x) + g_0(t)f_0(x) + \sum_{a=1} g_a(t)f_a(x) + \cdots$$

$$= \phi_K(x) + (A_0 + B_0t)f_0(x) + \sum_{a=1} g_a(t)f_a(x) + \cdots. \hspace{1cm} (3.23)$$

Comparing Eqs. (3.21) and (3.22) to the mode expansion above, we see that

$$(\Delta x) = -\frac{1}{\sqrt{M}} A_0, \hspace{1cm} (3.24)$$

$$(\Delta v) = -\frac{1}{\sqrt{M}} B_0. \hspace{1cm} (3.25)$$

As promised, the coefficients $A_0$ and $B_0$, which characterize the amplitude and evolution of the zero mode [see Eq. (3.20)], determine the phase shift and velocity change of the solitary wave. Arguably, the phase shift and velocity change are the most important physical outcomes of ultrarelativistic collisions. We will calculate their leading order values [part of a series in $(\gamma v)^{-1}$] in the upcoming sections. The other modes with $E > 0$ are related to oscillating fluctuations on top of the kink.

Note that the zero mode expresses the Lorentz symmetry of the background solution. At leading order, we are free to describe a kink as a translated kink plus a compensating zero mode. However, for the convenience of the perturbative calculation we are about to do, we will choose the initial amplitude of the zero mode to be zero. The zero mode introduces further subtleties at higher order [for us third order in $1/(\gamma v)$], which can possibly be addressed by using collective coordinates [24]. We leave this for future work.

**IV. PERTURBATION THEORY FOR KINK COLLISIONS IN $1 + 1$ DIMENSIONS**

Consider a stationary kink $\phi_K(x)$ centered around $x = 0$, and incoming kink $\phi_K[\gamma(x + vt)]$ approaching the stationary kink from the positive $x$ direction with a speed $v$ (see the top panel in Fig. 8). The kinks collide around $t = 0$. The space-time area occupied by these kinks in the

---

11Note that the dimensionality of the modes $f_w(x)$ and $f_i(x)$ are different.

12We thank Erick Weinberg for alerting us to this possibility.
FIG. 6 (color online). In (a) we show snapshots of the source function \( S(x, t) \) before, during and after the collision. \( S(x, t) \) is nonzero only during the collision and its spatial extent at a fixed time is \( L/\gamma \). In (b) we show \( W(x, t) = W_0(x) + \Delta W(x, t) \). \( \Delta W \) remains nonzero around the incoming kink and its spatial extent is also \( L/\gamma \). We used the periodic \( V(\phi) = (1 - \cos \phi)(1 - \alpha \sin^2 \phi) \) with \( \alpha = 0.7 \) for these plots.

The rest frame of the stationary kink as well as their interaction area is shown in Fig. 7(a). The orange strip represents the fast moving solitary wave, whereas the green strip represents the stationary one. The interaction area \( \mathcal{A}_{\text{int}} \) is denoted by the black parallelogram. Because of Lorentz contraction, the spatial width of the orange strip scales as \( L = \gamma L \). The collision takes place during a time interval \( |t| = t_{\text{int}} = (L/v)(1 + \gamma^{-1}) \) and the space-time interaction area \( \mathcal{A}_{\text{int}} = L^2/(\gamma v) \).

Now let us write down the full solution, before, during and after the collision as follows:\(^{13}\)

\[
\phi(x, t) = \phi_K(x) + \phi_K[\gamma(x + vt)] + h(x, t). \tag{4.2}
\]

Using the equation of motion (3.1) and (3.2), the linearized equation of motion for the perturbation \( h(x, t) \) is

\[
\partial^2_t h - \partial^2_x h + W_0(x)h = -\Delta W(x, t)h - S(x, t), \tag{4.3}
\]

where

\[
W_0(x) = V''[\phi_K(x)], \tag{4.4}
\]

\[
\Delta W(x, t) \equiv V''(\phi_K(x) + \phi_K[\gamma(x + vt)]) - W_0(x). \tag{4.5}
\]

\[
S(x, t) \equiv V'(\phi_K(x) + \phi_K[\gamma(x + vt)]) - V'[\phi_K(x)] - V'[\phi_K[\gamma(x + vt)]]. \tag{4.6}
\]

\( W_0(x) \) is the mass term for the single, stationary kink. \( \Delta W \) is the change in mass due to the fast moving kink and is nonzero only when evaluated on the moving kink, whereas \( S \) is the source which is nonzero only when the two kinks overlap. Typical functional forms of these three quantities are shown in Fig. 6. Notice the Lorentz contraction of the spatial extent of \( S \) and \( \Delta W \).

Now, if we were to only consider perturbations about an isolated kink and ignore the second kink, the right-hand side of Eq. (4.3) would be zero. However for the case under consideration, the right-hand side of Eq. (4.3) gets two additional terms because of the second kink: an \( h \)-dependent term with coupling \( \Delta W(x, t) \) and an \( h \)-independent external source term \( S(x, t) \). Both \( S \) and \( \Delta W h \) only become active once the collision begins. While \( \Delta W \neq 0 \) before the collision, recall that a linear sum of two kinks is a solution before the collision and hence \( h = 0 \) then. In this sense we can think of the incoming kink, via its interaction with the stationary one, as sourcing perturbations of the stationary kink.

We now want to see how the collision excites eigenmodes of the single kink \( \{f_a(x)\} \) defined in Eq. (3.14). First we write the general solution to Eq. (4.3) in terms of these \( \{f_a(x)\} \):

\[
h(x, t) = \sum_{a=0}^{\infty} G_a(t)f_a(x). \tag{4.7}
\]

We have used \( G_a(t) \) instead of the free field \( g_a(t) \) introduced in Eq. (3.18) (Sec. III) as the excitations are now sourced by the interaction terms \( S \) and \( \Delta W \). Plugging Eq. (4.7) into Eq. (4.3) and using the orthonormality of \( f_a(x) \), we get

\[
G_a''(t) + E_a G_a(t) = -\sum_b M_{ab}(t)G_b(t) - S_a(t), \tag{4.8}
\]

where we have defined the transition matrix \( M_{ab}(t) \) and the projected source \( S_a(t) \) as

---

\(^{13}\)If the potential is not periodic, we have to subtract a "reference" \( \Delta \phi \), i.e.

\[
\phi(x, t) = \phi_K(x) + \phi_K[\gamma(x + vt)] - \Delta \phi + h(x, t). \tag{4.1}
\]

For more information on this requirement, see [32].
Recall that in the previous section, Eq. (3.18) characterized the evolution of eigenmodes around an isolated kink. There was neither mixing of modes nor external sources. In Eq. (4.8) above, we see that $M_{ab}(t)$ is sourced by $S_a$, and its evolution is a priori coupled with all $G_b$ via $M_{ab}$.

However, a closer inspection of $S_a$ and $M_{ab}$ allows us to decouple the evolution of $G_a$ at leading order in $1/\gamma$. With these considerations in mind let us now expand $G_a$ as

$$G_a(t) = \sum_n G_a^{(n)}(t)$$

with the ansatz that $G_a^{(n)} \sim 1/(\gamma \nu)^{-n}$ for $t \sim t_{\text{int}}$. We will see that this ansatz is indeed confirmed at the end of the calculation. Using this expansion in Eq. (4.8), assuming that $S_a \sim M_{ab} \sim 1/(\gamma \nu)$, and collecting terms order by order in $1/(\gamma \nu)$ we get

$$G_a^{(1)}(t) + E_a G_a^{(1)}(t) = -S_a(t),$$

$$G_a^{(n)}(t) + E_a G_a^{(n)}(t) = -\sum_b M_{ab}(t) G_b^{(n-1)}(t), \quad n \geq 2.$$
For $u \to 1$, $\gamma v \to \gamma$. Hence we have taken $O[\gamma^{-1}] = O[(\gamma v)^{-1}]$. As promised, $G^{(1)}_a(t)$ is sourced by the projected source term $S_a$ only; mixing via the transition matrix $M_{ab}$ only occurs at the next order and beyond.

The solution to the above Eqs. (4.13) and (4.14) is given by

$$G^{(1)}_a(t) = -\int_{-t_{\text{int}}}^{t} d\tau \sin[\sqrt{E_a(t-\tau)] S_a(\tau),$$  

$$G^{(n)}_a(t) = -\sum_b \int_{-t_{\text{int}}}^{t} d\tau \sin[\sqrt{E_a(t-\tau)] X_b G^{(n-1)}_b(\tau), \quad n \geq 2,$$

where $\sin[\sqrt{E_a(t-\tau)]/\sqrt{E_a}$ is the Green’s function for the operator $\delta^2_t + E_a$. For the zero mode with $a = 0$ and $E_0 = 0$, the above solutions take the form

$$G^{(1)}_0(t) = -\int_{-t_{\text{int}}}^{t} d\tau (t-\tau) S_0(\tau),$$

$$G^{(n)}_0(t) = -\sum_b \int_{-t_{\text{int}}}^{t} d\tau (t-\tau) M_{0b} G^{(n-1)}_b(\tau). \quad n \geq 2.$$  

Based on our assumptions, the superposition of the two solitary waves is an exact solution for $t < -t_{\text{int}}$. Hence, in writing down the above solutions we have assumed $G^{(a)}_a(t < -t_{\text{int}}) = 0$.

Let us now concentrate on the solution for $G^{(1)}_a(t)$ for $t > t_{\text{int}}$.

$$G^{(1)}_a(t > t_{\text{int}}) = A^{(1)}_a \cos\sqrt{E_a} t + B^{(1)}_a \sin\sqrt{E_a} t,$$

while the zero mode ($E_0 = 0$) satisfies

$$G^{(1)}_0(t > t_{\text{int}}) = A^{(1)}_0 + B^{(1)}_0 t.$$  

For $t > t_{\text{int}}$, the collision is by definition over and $S_a(t > t_{\text{int}}) = 0$. $G^{(a)}_a(t > t_{\text{int}})$ satisfy the “free field” equation (3.18). The nonzero coefficients $A^{(1)}_a$ and $B^{(1)}_a$ (including $A^{(1)}_0$ and $B^{(1)}_0$) are generated by the collision through $S_a(t > t_{\text{int}}) \neq 0$. While the linear $t$ dependence might seem peculiar, it simply reflects the fact that the previously stationary solitary wave can be set into motion by the collision. Furthermore, we will show below that this velocity change is zero at leading order (i.e. $B^{(1)}_0 = 0$).

Also note that for calculating $G^{(a>1)}_a(t)$, we require knowledge of the orthonormal basis $\{f_a(x)\}$—an endeavor which we will postpone to a later publication. In the next subsection, we will focus on the zero mode and the explicit evaluation of $A^{(1)}_0$ and $B^{(1)}_0$. These coefficients are related to the phase shift and velocity change of the stationary solitary wave at leading order in $(\gamma v)^{-1}$ [see Eqs. (3.24) and (3.25)].
Putting everything together in the expression for $B_0^{(1)}$ we get

$$
B_0^{(1)} = \frac{M^{-1/2}}{(\gamma v)} \int_{\mathcal{A}_{int}' \setminus \mathcal{A}_B} dp dq \phi_k'(q) [V'[\phi_k(q)] + V'[\phi_k(p)]
- V'[\phi_k(q) + \phi_k(p)].
$$

(4.30)

Let us calculate each of these three terms separately. Each will be zero. The key step in the manipulations is the following. Based on our localization assumption stated in Eq. (3.6), we assume that $\phi = 0$ for $x \leq -L/2$ and $\phi = \Delta \phi$ for $x \geq L/2$. With that in mind let us look at the first term in Eq. (4.30):

$$
\int_{\mathcal{A}_{int}' \setminus \mathcal{A}_B} dp dq \phi_k'(q) [V'[\phi_k(q)] + V'[\phi_k(p)]
- V'[\phi_k(q) + \phi_k(p)] = 0,
$$

(4.31)

since $V(0) = V(\Delta \phi) = 0$. For the second term in Eq. (4.30), we have

$$
\int_{\mathcal{A}_{int}' \setminus \mathcal{A}_B} dp dq \phi_k'(q) V'[\phi_k(p)]
= \int_{\mathcal{A}_{int}' \setminus \mathcal{A}_B} dp dq \phi_k'(q) \phi_k'(p)
= \int_{-L/2}^{L/2} dq \phi_k'(q) [\phi_k'(L/2) - \phi_k'(-L/2)]
= 0.
$$

(4.32)

Above, we used $\phi_k'(p) = V'[\phi_k(p)]$ and $\phi_k'(L/2) - \phi_k'(-L/2) = 0$. Finally, for the third term in Eq. (4.30) we have

$$
\int_{\mathcal{A}_{int}' \setminus \mathcal{A}_B} dp dq \phi_k'(q) [V[\phi_k(q) + \phi_k(p)]
- V[\phi_k(q) + \phi_k(p)] = 0,
$$

(4.33)

where we used $V(\phi + \Delta \phi) = V(\phi)$.

In summary, we have just shown that

$$
B_0^{(1)} = 0,
$$

(4.34)

$$
\Delta v = - \frac{1}{\sqrt{M}} p_0^{(1)} + \mathcal{O}[(\gamma v)^{-2}] = 0 + \mathcal{O}[(\gamma v)^{-2}].
$$

(4.35)

2. Phase shift

Let us now turn our attention to $A_0^{(1)}$:

$$
A_0^{(1)} = \int_{-\tau_0}^{\tau} d\tau S_0(\tau)
= -M^{-1/2} \int_{\mathcal{A}_{int}} d\tau d\chi \tau \phi_k'(\chi) [V'[\phi_k(\chi)]
+ V'[\phi_k(\chi + v\tau)] - V'[\phi_k(\chi)]
+ \phi_k'(\chi + v\tau)]
$$

$$
+ \int_{\mathcal{A}_{int}} d\tau d\chi \frac{P}{\gamma v} \phi_k'(q) [V'[\phi_k(q)]
+ V'[\phi_k(q) + \phi_k(p)]
$$

$$
- M^{-1/2} \int_{\mathcal{A}_{int}} d\tau \int_{\mathcal{A}_{int}} dp dq \phi_k'(q) V'[\phi_k(q)]
+ \int_{\mathcal{A}_{int}} d\tau \int_{\mathcal{A}_{int}} dp dq \phi_k'(q) V'[\phi_k(q)]
+ V'[\phi_k(p)] - V'[\phi_k(q) + \phi_k(p)]
$$

$$
= \int_{\mathcal{A}_{int}} d\tau \int_{\mathcal{A}_{int}} dp dq \phi_k'(q) V'[\phi_k(q)]
+ \int_{\mathcal{A}_{int}} d\tau \int_{\mathcal{A}_{int}} dp dq \phi_k'(q) V'[\phi_k(q)]
+ V'[\phi_k(p)] - V'[\phi_k(q) + \phi_k(p)].
$$

(4.36)

Notice the extra $\tau$ factor in the integrand of $A_0^{(1)}$ becomes $\tau = p/(\gamma v) - q/v$. As in the case of $B_0^{(1)}$, we will integrate each of the above terms separately.

For the first term in Eq. (4.36), the integral is

$$
\int_{\mathcal{A}_{int}} d\tau \int_{\mathcal{A}_{int}} dp dq \phi_k'(q) V'[\phi_k(q)]
= \int_{-L/2}^{L/2} dp \int_{-L/2}^{\Delta \phi} d\phi \phi_1 \phi_1\phi_1 V'(\phi_1)
$$

$$
= - \int_{-L/2}^{L/2} dp \int_{0}^{\Delta \phi} d\phi \phi_1 \phi_1\phi_1 V'(\phi_1)
$$

$$
= - \int_{-L/2}^{L/2} dp \int_{0}^{\Delta \phi} d\phi \phi_1 \phi_2 \phi_2\phi_2 V'(\phi_1)
$$

$$
= - \frac{1}{2} \int_{0}^{\Delta \phi} \frac{d\phi_1 d\phi_2}{\sqrt{V(\phi_1)V(\phi_2)}} V(\phi_1).
$$

(4.37)

In the third line we integrated by parts with boundary terms giving no contribution. The fourth line converts the coordinate space variables to the field space variables. In the fifth line we used $\phi'(x) = \sqrt{2V(\phi_k(x))}$ which can be obtained by integrating the equation of motion $\phi''(x) = V[\phi_k(x)]$ by parts. Note that $p(\phi)$ and $q(\phi)$ are invertible in the range of interest.

The second term in Eq. (4.36) can be converted to a boundary term of the $p$ coordinate and evaluates to zero:
For the third term in Eq. (4.36) we have

\[
\int_{\mathcal{A}_{\phi K}^n} dp dq q' \phi_K(q) V[\phi_K(p)] = \int_{\mathcal{A}_{\phi K}^n} dp dq q' \phi_K(q) \phi_K(p) = \int_{-L/2}^{L/2} dq q' \phi_K(q) [\phi_K(L/2) - \phi_K(-L/2)] = 0. \tag{4.38}
\]

In line above equation, we used

\[
A_{\phi_0 K} = \int_{0}^{\Delta \phi} d\phi_0 \int_{0}^{\Delta \phi} d\phi_1 \int_{0}^{\Delta \phi} d\phi_2 [V(\phi_1 + \phi_2) - V(\phi_1) - V(\phi_2)] - \frac{2}{(\gamma v)} \Delta \phi^2.
\]

With similar manipulations, one can show that the fourth and sixth terms of Eq. (4.36) are zero whereas the fifth evaluates to \(-M^{-1/2} \Delta \phi^2/(\gamma v)^2\). Combining all of these results, we can finally write down a closed form expression for \(A_{\phi_0}^{(1)}\) in Eq. (4.36):

\[
A_{\phi_0}^{(1)} = \frac{M^{-1/2}}{2(\gamma v)} \frac{1}{\gamma v} \int_{0}^{\Delta \phi} d\phi_1 \int_{0}^{\Delta \phi} d\phi_2 \left[ V(\phi_1 + \phi_2) - V(\phi_1) - V(\phi_2) \right] - \frac{2}{(\gamma v)} \Delta \phi^2.
\]

where in the last line we have kept the leading order term in \((\gamma v)^{-1}\). With this \(A_{\phi_0}^{(1)}\), the phase shift is

\[
\Delta x = -\frac{A_{\phi_0}^{(1)}}{\sqrt{M}} + O[(\gamma v)^{-2}] = \frac{1}{2(\gamma v)M} \int_{0}^{\Delta \phi} d\phi_1 \int_{0}^{\Delta \phi} d\phi_2 \left[ V(\phi_1) + V(\phi_2) - V(\phi_1 + \phi_2) \right] + O[(\gamma v)^{-2}]. \tag{4.42}
\]

Note that in the second line above, \(O[(\gamma v)^{-2}]\) contains higher order corrections to \(A_{\phi_0}^{(1)}\) calculated in this subsection, as well as corrections from \(n > 1\) terms in Eq. (4.18).

Being able to write the kink-kink interaction as a simple integral of \(V\) in field space is a very powerful tool. For example, since the field profile of an antikink is given by \(\phi_K(-x)\), it is straightforward to verify that Eq. (4.42) also gives the phase shift between a pair of kink-antikink. We can hence conclude that the leading order result of a kink-kink collision and a kink-antikink collision are identical. The sign of the phase shift depends on the details of the potential. As we will show in the next section, it is possible to get positive and negative phase shifts corresponding to an attractive and repulsive interaction, respectively.

In the next section we check our results for the phase shift and velocity change for a number of examples. For the Sine-Gordon case, these quantities are compared to the exact results. For other cases, we compare our answers to those obtained by full numerical integrations of the equations of motion. We will find that our order by order results agree exceptionally well with both exact results (when available) and numerical simulations of collisions.

V. EXAMPLES

A. Sine-Gordon

Consider the normalized Sine-Gordon potential

\[
V(\phi) = 1 - \cos \phi. \tag{5.1}
\]

Let us calculate the leading order phase shift in this model based on our result (4.42). For the Sine-Gordon case, the kink solution is given by \(\phi_K(x) = 4\tan^{-1}[e^x]\). From Eq. (3.7), we get \(M = 8\) and using (4.42), the phase shift is

\[
\Delta x = \frac{2}{(\gamma v)} + O[(\gamma v)^{-2}]. \tag{5.2}
\]

In the Sine-Gordon case, the two kink solution can be obtained analytically, based on which one can calculate the phase shift exactly [2,3,19]:
\[ \Delta x = \ln \left(\frac{\gamma + 1}{\gamma - 1}\right) = \frac{2}{(\gamma v)} + O[(\gamma v)^{-3}] \]  

(5.3)

Thus, the phase shift calculated based on Eq. (4.42) at leading order agrees exactly with the phase shift (again at leading order) based on the exact Sine-Gordon kink-kink (or kink-antikink or antikink-antikink) solution. Also note that for the Sine-Gordon case, the solution shows that there is no velocity change due to the collision. This is again consistent with our leading order result \( \Delta v = 0 + O[(\gamma v)^{-2}] \).\(^{15}\)

**B. Away from Sine-Gordon**

We now consider models for which analytic solutions are not known. For concreteness we consider models of the form

\[ V(\phi) = (1 - \cos \phi)(1 - \alpha \sin^2 \phi), \]  

(5.4)

where \(-1 < \alpha < 1\). Note that these potentials are periodic with a period \( \Delta \phi = 2\pi \). Importantly, these models are not necessarily small deformations of the Sine-Gordon case (\( \alpha = 0 \)).

We numerically simulate the collision of two kinks in this model. Initially, one of the kinks is stationary and another is moving towards the stationary one from the positive to negative direction. The stationary kink profile is obtained numerically by a relaxation technique.\(^{16}\) A Lorentz transformation of the stationary kink profile is then used to obtain the profile of the incoming kink. We take a superposition of the two profiles to obtain initial conditions for the collision. The initial conditions used are shown in Fig. 8 (top). The equation of motion for the field is evolved using a fourth order Runge Kutta method (with rigid boundary conditions).

After the collision, we calculate the phase and velocity shift of the stationary solitary wave from the numerically evolved \( \phi(x, t) \) as follows. First, for \( t_{\text{int}} < t \leq 10 t_{\text{int}} \) we carry out the following projection:

\[ I(t) = \int_{-L}^{L} dx f_0(x) \left[ \phi(x, t) - \phi_K(x) - \phi_K(\gamma(x + vt)) \right] \]

(5.5)

\[ = \int_{-L}^{L} dx f_0(x) \left[ \phi(x, t) - \phi_K(x) - 2\pi \right]. \]

(5.6)

Note the \( 2\pi \) arises from the asymptotic value of the second kink after it has moved away from the stationary kink. We then fit this numerically calculated \( I(t) \) to a line \( \tilde{A}_0 + \tilde{B}_0 t \) and determine the coefficients \( A_0 \) and \( B_0 \). These coefficients are directly related to the phase and velocity shift of the stationary kink. To understand why we fit \( I(t) \) to a straight line, and why these numerically calculated coefficients provide a measure of the phase and velocity shift, let us express \( I(t) \) in terms of our eigenmodes \( f_a(x) \) and their coefficients \( G_a(t) \). For \( t > t_{\text{int}} \):

\[ I(t) = \int_{-L}^{L} dx f_0(x) h(x, t) = \int_{-L}^{L} dx f_0(x) \sum_a G_a(t) f_a(x) \]

\[ = G_0(t) = G_0^{(1)}(t) + \sum_{n>1} G_0^{(n)}(t) \]

\[ = A_0^{(1)} + B_0^{(1)} t + \sum_{n>1} G_0^{(n)}(t), \]  

(5.7)

where in the second step we expanded \( h(x, t) \) in terms of eigenmodes, in the third step we used the orthonormality of \( f_a(x) \).\(^{17}\) in the fourth step we expanded \( G_0(t) \) as a series and in the fifth step we explicitly write down the leading order term in the form of its eigenmode (3.20). As we have shown earlier, \( G_0^{(1)}(t \sim t_{\text{int}}) \sim 1/(\gamma v)^{-n} \) with \((\gamma v) \gg 1\). Hence, when we fit the numerically calculated \( I(t) \) to a straight line \( \tilde{A}_0 + \tilde{B}_0 t \), we are estimating the coefficients \( A_0^{(1)} \) and \( B_0^{(1)} \). Once these coefficients have been estimated numerically, it is easy to find the phase shift and the velocity change of the stationary kink via Eqs. (3.24) and (3.25).

We carried out a large number of high resolution simulations of the collisions, varying both the velocity of the incoming kink \( v \) and parameter \( \alpha \) in the potential (5.4). Using the projection \( I(t) \) discussed above, we calculated the phase shift and the velocity change of the stationary kink as a function \( (\gamma v) \) and \( \alpha \). We summarize our results below.

Figure 9 shows the comparison of the phase shift calculated based on Eq. (4.42) and that from the numerical simulations. For this figure, we used \( \alpha = 0.5 \) and varied the speed \( v \) of the incoming kink. Note that the \( (\gamma v) \) dependence is wonderfully captured by our leading order result, even when \( (\gamma v) = 3 \). As expected, the difference between the numerically calculated phase shift and the one based on Eq. (4.42) diminishes as \( (\gamma v) \) increases.

Next we check the \( \alpha \) dependence of the phase shift at a fixed \( \gamma v = 100 \). As seen in Fig. 10, the phase shift calculated from the numerical simulations matches well with the leading order result. The reason we fixed \( (\gamma v) = 100 \) is interesting in its own right. When computing \( I(t) \) numerically, we found that when \( \alpha \) was not too close to 1, the deviation of \( I(t) \) from a straight line fit (used to determine the phase shift) was oscillatory and quite small, scaling\(^{17}\). Note that for orthonormality we need \( L \to \infty \); however, in practice the localization of \( f_0(x) \) to \( \sim L \) allows us to cut off the integral at a finite \( L \).
with inverse powers of $\gamma v$. However, for the same $\gamma v$, when $\alpha \rightarrow 1$, the deviations from a straight line fit were quite large. We found that for $\alpha > 0.9$, one has to go to sufficiently high $\gamma v$ to see a good match between the analytically and numerically calculated phase shifts. It is also worth noting that this is the region where the interaction starts becoming repulsive ($\Delta x$ changes sign). Thus, the $\gamma v$ at which our leading order results provide a good approximation can vary with model parameters. In the future, we will explore the relationship between this rate of convergence and the functional form of the potential.

Finally, while we do not show the result here, we also checked that the velocity shift $\Delta v = 0 + O[(\gamma v)^{-2}]$.

In summary, the calculated phase and velocity shift based on our $(\gamma v)$ expansion are in excellent agreement with the results from numerical simulations of ultrarelativistic kinks.

**VI. DISCUSSION**

In this paper, we established a general framework to calculate the outcome of ultrarelativistic collisions between solitary waves in relativistic scalar field theories. We showed that the colliding solitary waves pass through each other, and the perturbations to this free passage behavior are small due to the suppression of the spacetime area of interaction $A_{\text{int}} \propto 1/(\gamma v)$ where the two solitary waves overlap. We present an order by order prescription to calculate the full result of the collision.

We considered collisions of localized, quasistable, scalar-field solitary waves in periodic potentials or potentials with a single minimum. In our setup, an ultrarelativistic solitary wave ($\gamma v \gg 1$) collides with a stationary solitary wave. We showed that for linearized perturbations, the stationary solitary wave’s perturbations can be organized as a power series in $1/(\gamma v)$ for small amplitude perturbations, the corrections from nonlinear effects can also be expressed as a power series, as shown in Appendix B. In such cases, there exists a $\gamma v$ high enough that the full result is under analytical control and well approximated by the leading order effects.

We applied our formalism to a specific example: $(1 + 1)$ dimensional kinks with periodic potentials. We calculated two leading order effects with important physical

**FIG. 9 (color online).** In (a), we plot the numerically calculated phase shift undergone by a stationary kink colliding with an incoming kink as a function of $\gamma v$. For this plot, the scalar field potential $V(\phi) = (1 - \cos \phi)(1 - 0.5\sin^2 \phi)$. The orange curve is the theoretical prediction at leading order in $1/(\gamma v)$ and the black dots are the simulation results. They are in excellent agreement. In (b), we multiply the phase shift by $(\gamma v)$ to show how the numerical and analytic calculations approach each other as $(\gamma v)$ increases.

**FIG. 10 (color online).** The phase shift [multiplied by $(\gamma v)$] undergone by a stationary kink colliding with an incoming kink as a function of the $\alpha$ parameter in the potential $V(\phi) = (1 - \cos \phi)(1 - \alpha \sin^2 \phi)$. For this plot $(\gamma v) = 100$. The orange curve (and orange dots) is the theoretical prediction at leading order in $(\gamma v)^{-1}$ and the black dots are the simulation results. The agreement between the two is excellent.
meanings: phase shift and velocity change. We showed that the leading order results can be expressed in closed forms in terms of the potential. In other words we can know the leading order result before they collide—making analytical predictions which can be checked with experiments (simulations).

We showed that the leading order phase shift for ultra-relativistic collisions is independent of whether the collision is between kinks or between a kink and an antikink. Although there is no direct contradiction, we note that the long-range interaction between kinks is repulsive, whereas between a kink and antikink is attractive [44]. We were also able to construct examples with a zero or a negative phase shift at leading order.

Collisions of solitary waves have been investigated analytically before in a very special and limited subset—integrable (e.g. Sine-Gordon) or approximately integrable cases. Our results agree with these cases. More importantly, we showed that for a potential that was arbitrarily far from being integrable, our prediction still agreed extremely well with numerical simulations.

In summary, understanding soliton interactions has been an active area of research for more than 50 years. Many interesting physical phenomena involve solitons such as fluxons in Josephson junctions [20], nonlinear optical solitons [11], reheating after inflation [15] and domain wall collisions in cosmology [31]. Apart from numerical techniques, there are two standard approaches. One is to model them as being perturbatively close to the integrable Sine-Gordon system. Another approach is to carry out a dynamical systems analysis of the collective-coordinates ordinary differential equations [45,46]. Here, we demonstrated a novel third method—a kinematics based scattering theory at relativistic velocities. Our method works well for collisions at ultrarelativistic velocities, which is exactly where numerical techniques become inefficient. For these collisions, we do not rely on a small deformation from Sine-Gordon; thus, our analytical framework is applicable to a wider range of phenomena.

A. Future directions

(i) Testing the framework with examples.—Perhaps the most natural next step is to test our framework with different examples. Collisions of oscillons, Q-balls and bubbles in 1 + 1 and higher dimensions can be simulated and compared with the $1/(\gamma v)$ behavior (of the leading order effects) predicted by our framework. It would also be interesting to see if collisions of localized objects composed of multiple fields still respect our framework.

(ii) The localization condition.—We have focused our attention on solitary waves with exponentially suppressed tails. That is clearly sufficient but not necessary. In particular, the earliest observation of free passage occurs in strings (vortices) [29]. For them the tails are not only power law, but also lead to infinite integrated energy. However it still has a clean relativistic collision and it might be described by a method similar to our framework. It will be interesting to figure out the most general class of objects that their relativistic collisions allow full analytical descriptions.

(iii) Non-Lorentz invariant theories.—We have focused our attention on Lorentz invariant scalar field theories. However, certain classic systems such shallow water waves described by the Korteg deVries equation [47] are not Lorentz invariant. Nevertheless, they contain two solitary wave solutions where the phase shift does decrease with velocity in a manner reminiscent of our results in this paper. It would be interesting to see whether such systems can still be described within our framework.

(iv) Higher order effects.—We provided the recursive equation to calculate higher order effects in the linearized theory. It is already tedious to do a calculation beyond the leading order with them. Going beyond the linearized equations makes it even more so. Although still doable, the required computation resources may exceed a direct simulation. However, in this paper we naively wrote down all terms without considering symmetries which could have simplified our analysis. This is akin to drawing all Feynman diagrams without recognizing that some (maybe the majority) of them can cancel with others. To investigate such cancellations, in Appendix C we wrote down energy conservation equations order by order. These equations play the role of the optical theorem in perturbation theory. In Appendix C we show that these equations are already quite powerful at leading order. They allow us to conclude that the velocity change is zero at leading order, without any detailed calculations. It is possible that similar techniques can be used to simplify the expressions for higher order effects.

(v) An inverse search for integrable systems.—One property of integrable systems is the lack of velocity change after collisions (to all orders). Since we have an analytical expression for the velocity change, setting it to zero order by order in principle provides the set of analytical conditions for integrable systems. This of course relies on the previous point that the full recursive series needs to be simplified to make the condition useful.

(vi) Gravitational effects.—We have completely ignored gravity in our framework. In [48], the authors explore the gravitational effects in bubble

---

We thank Adam Brown for suggesting this possibility.
collisions (in the context of classical transitions). Gravity can have dramatic effects in certain ultrarelativistic collisions. It has been shown by [49] that one can form black holes by colliding ultrarelativistic solitary waves, which of course cannot be seen in our framework. It would be interesting to see whether one can appropriately incorporate gravity into our framework.

ACKNOWLEDGMENTS

M. A. A. is supported by a Kavli Fellowship. I.-S. Y. is supported by the research program of the Foundation for Fundamental Research on Matter (FOM), which is part of the Netherlands Organization for Scientific Research (NWO). M. A. A. and E. A. L. also thank the Department of Mathematics and Theoretical Physics (DAMTP) at Cambridge where part of this work was done. We thank Daniel Baumann, Roger Blandford, Adam Brown, Cliff Burgess, George Efstathiou, Tom Giblin, Lam Hui, Nick Manton, Boris Malomed, Theodorus Nieuwenhuizen, Ignacy Sawicki, Bob Wagoner and Erick Weinberg for useful conversations. We especially thank Adam Brown, Tom Giblin and Erick Weinberg for their insightful comments and suggestions for improvement of this manuscript. M. A. A. thanks the organizers of the Primordial Cosmology workshop at KITP, Santa Barbara, 2013. E. A. L. thanks the organizers of Peyresq Physics 18, 2013, and acknowledges the support of OLAM, Association pour la Recherche Fondamentale, Bruxelles and an FQXi mitgrant for work on “Cosmological Bubble Collisions.” Some of the high resolution numerical simulations were performed on the COSMOS supercomputer, part of the DiRAC HPC, a facility which is funded by STFC and BIS.

APPENDIX A: POSTCOLLISION EVOLUTION

In the main body of the text we were able to show that soon after the collision $t \sim t_{\text{int}}$, the perturbations of a stationary soliton generated by the collision can be organized as a convergent series with the ratio of consecutive terms scaling as $(\gamma v)^{-1}$. However, we alluded to the fact that this scaling might be broken when evaluating the perturbations long after the collision. In this Appendix we discuss this issue in detail and provide a prescription to calculate the perturbation for all time.

To obtain the result of solitary wave collisions valid for all time, we need to carry out two parallel calculations, one in the rest frame of each solitary wave. The full result of the collision is then an appropriate combination from the two parallel calculations. As we will see, the perturbations induced by the collision are small. Hence this combination is no more than a linear superposition of the perturbations from both calculations (in most, though not all, of the space-time regions of interest).

We suggest that the reader refer to Sec. II for definitions and some background for what is discussed below. We begin by writing the solution to the linearized equation of motion (2.2) as a series $h(x, t) = \sum h^{(n)}(x, t)$. Each term is then formally given by

$$h^{(1)}(x, t) = \int dt' dx' G(x, t; x', t') S(x', t'), \quad (A1)$$

$$h^{(n>1)}(x, t) = \int dt' dx' G(x, t; x', t') \Delta W(x', t') h^{(n-1)}, \quad (A2)$$

where $G = (\Box - W_0)^{-1}$. If evaluated at $t \sim t_{\text{int}}$ and $|x| < L_A/2$, we argued in Sec. II that $h^{(n)} \propto A_{\text{int}} h^{(n-1)}$ where the overlap area $A_{\text{int}} \propto 1/(\gamma v)$. However, for $t \gg t_{\text{int}}$ this need not be the case. This is primarily because $\Delta W(x, t) \gg t_{\text{int}}$ does not. Physically, at late times, the integral over $\Delta W h^{(n-1)}$ is related to how perturbations from the outgoing kink influence perturbations on the stationary one.

Instead of the general hierarchy in Eq. (A2), a better approach is to first split the perturbation $h$ into two parts:

$$h(x, t) = h_A(x, t) + h_B(x, t), \quad (A3)$$

where

1. $h_A(x, t) = \text{perturbations on } \phi_A \text{ generated by the collision. } h_A \text{ includes all effects generated from the interaction area } A_{\text{int}}, \text{ including localized perturbations as well as outgoing radiation on and from } \phi_A$.

2. $h_B(x, t) = \text{incoming radiation generated (or reflected) from } \phi_B$. More precisely, $h_B$ includes possible contributions to $h$ at the point $(x, t)$ from the dashed orange box in Fig. 3(b). Note that the dashed orange box represents the relevant space-time occupied by $\phi_B$ beyond $A_{\text{int}}$.

The calculation of $h_A$ is surprisingly simple. It is just the calculation of $h$ in Sec. II, restricting the integration range to $A_{\text{int}}$, but allowing it to be generally valid even for $t > t_{\text{int}}$.

$$h_A(x, t) = \sum_{n=1}^{\infty} h_A^{(n)}(x, t), \quad (A4)$$

where

$$h_A^{(1)}(x, t) = \int_{A_{\text{int}}} dt' dx' G(x, t; x', t') S(x', t') \sim \frac{1}{(\gamma v)}, \quad (A5)$$

$$h_A^{(n)}(x, t) = \int_{A_{\text{int}}} dt' dx' G(x, t; x', t') \Delta W(x', t') h_A^{(n-1)} \sim \frac{1}{(\gamma v)^n}, \quad n \geq 2. \quad (A6)$$

This is because the limit of integration ignores the dashed orange box in Fig. 3(b), which will be given by $h_B$. The convenient way to describe $h_B$ is in the rest frame of the
leaving solitary wave $\phi_B$. The calculation for that is identical to the above, just switching the role of the two solitary waves. That is what we meant by our earlier statement that the full result of the collision is to be described by the combination of two such calculations, one in the rest frame of each solitary wave.

Let us be a bit more precise about the technical combination of $h_A$ and $h_B$ using the idea of initial data. We begin by taking the time slice at 

$$ t = t_{\text{int}} = v^{-1}(L_A + \gamma^{-1}L_B), $$

(A7)

and considering everything to the right-hand side of the solitary wave $\phi_A$ including itself,

$$ x > -L_A/2. $$

(A8)

The initial data on this semi-infinite slice are conveniently given by $h_A(x, t_{\text{int}})$. The remaining half is supplemented by similar initial conditions in the rest frame of $\phi_B$. If we can correctly calculate both, then combining the two provides sufficient initial data to determine the results in the future. The future evolution happens in the background where the two solitary waves are far apart. Hence we can extend the validity of Eqs. (2.14) and (2.15) to $t > t_{\text{int}}$ for $h = h_A$ and $h = h_B$ separately.

Let us once again emphasize the physical meanings of $h_A$ and $h_B$. $h_A$ does not represent the all-time result in the half space-time region $\phi_A$ belongs to. It describes the perturbations originated from the half space-time region at the time of collision. At later times, some of these perturbations will propagate outside this region, and this half also receives perturbations propagating from the other half where $\phi_B$ is.

The perturbations localized on separate solitary waves of course can be added linearly, since they never actually overlap. The region between two solitary waves is in empty space, and the propagating waves from both solitary waves are of small amplitudes. So they also add up linearly. The only exception is when a propagating wave from one solitary wave catches up with the other. That is of little concern to us, since the interaction between a small amplitude incoming wave and a solitary wave is part of the linearized dynamics of one solitary wave—it is not part of the collision dynamics that we want to deal with here.\footnote{One can even further consider the reflection of a mode with high enough momentum such that it comes back and catches up with the original solitary wave it was emitted from. That again can be treated by the intrinsic dynamics of one solitary wave and is separate from the short term effects of the collision.}

**APPENDIX B: NONLINEAR EVOLUTION OF PERTURBATIONS**

In the main text, we only dealt with linearized equations of motion for the perturbations. Here we include a discussion that does not assume linearized perturbations. For simplicity, we will limit ourselves to evaluating the perturbations at $t \sim t_{\text{int}}$. We will find that when we drop the linearization assumption, we have to correct the expressions for $h^{(n=3)}$ provided in Eq. (2.15). We recommend that the reader refer to Sec. II for definitions and background on what is discussed below.

For a solution of the form

$$ \phi(x, t) = \phi_A(x, t) + \phi_B(x, t) + h(x, t), $$

(B1)

the full, nonlinear equation of motion for $h$ is

$$ \Box h = V'(\phi_A + \phi_B + h) - V'(\phi_A) - V'(\phi_B) = W_0 h + S(x, t) + \Delta W(x, t) h + \sum_{n=2}^{\infty} \frac{1}{n!} \left( \frac{d^{n+1} V}{d\phi^{n+1}} \right) h^n. $$

(B2)

Expanding the perturbations $h$ as a series

$$ h(x, t) = \sum_{n=1}^{\infty} h^{(n)}(x, t), $$

(B3)

the order by order solutions can be written as

$$ h^{(1)}(x, t) = \int_{\mathcal{A}_{\text{int}}} dt' dx' G(x, t; x', t') S(x', t'), $$

$$ h^{(2)}(x, t) = \int_{\mathcal{A}_{\text{int}}} dt' dx' G(x, t; x', t') V''(\phi_A + \phi_B) h^{(1)}, $$

$$ h^{(3)}(x, t) = \int_{\mathcal{A}_{\text{int}}} dt' dx' G(x, t; x', t') \left[ V''(\phi_A + \phi_B) h^{(2)} + \frac{1}{2} V'''(\phi_A + \phi_B) (h^{(1)})^2 \right] \ldots $$

(B4)

where $G = (\Box - W_0)^{-1}$ and $\mathcal{A}_{\text{int}} \sim \mathcal{A}_{\text{int}}$ (see discussion in Sec. II). For $n \leq 2$, the expressions for $h^{(n)}$ above agree with those in Eq. (2.15). However, for $n \geq 3$, additional terms appear in the nonlinear case compared to the linearized one. For example, note that the term $(h^{(1)})^2$ in the expression of $h^{(3)}$ above is absent in the corresponding expression for $h^{(3)}$ in Eq. (2.15). Similar terms appear at higher orders as well. Note that these extra terms do not spoil our $h^{(n)} \propto \mathcal{A}_{\text{int}} h^{(n-1)}$ scaling. While not relevant for $n \leq 2$, the nonlinear terms are important for $n \geq 3$ and can be included in the calculation.

**APPENDIX C: ENERGY CONSERVATION**

In scattering theory, the optical theorem often simplifies calculations considerably. In a classical field theory, the optical theorem is a direct consequence of energy conservation. In this Appendix we show that the optical theorem is already useful at the leading order in $(\gamma v)^{-1}$. It shows that $B^{(1)}_0 = 0$ independently from the explicit evaluation given in Sec. IV. The general expression of the theorem
requires analysis of higher order in the perturbation theory, which we postpone for future work.

It is straightforward to compute the total energy of a single stationary kink plus small perturbations:

\[
E_{\text{total stationary}} = \int dx \left\{ \frac{1}{2} \left( \partial_x \phi(x) + h(x, t) \right)^2 + \frac{1}{2} \left( \partial_x \left[ \phi(x) + h(x, t) \right] \right)^2 + V[\phi(x)] + h(x, t) \right\}
\]

\[
= \int dx \left\{ \frac{1}{2} \left[ \phi(x) \right]^2 + V[\phi(x)] \right\}
+ \int dx \left\{ \frac{1}{2} \left[ \sum_a \left[ f_a(x) g_a(t) \right]^2 + [f_a(x) g_a(t)]^2 + V''[\phi(x)] \left[ f_a(x) g_a(t) \right]^2 \right] \right\}
= M + \sum_a \frac{1}{2} E_a (A_a^2 + B_a^2). \tag{C1}
\]

The equation of motion (3.2) ensures that the term linear in \( h \) vanishes. We used the mode expansion \( h = \sum_a f_a g_a \), with \( g_a(t) = A_a \cos (\sqrt{E_a} t) + B_a \sin (\sqrt{E_a} t) \) in the absence of sources. We will keep using the notation that all the \( \phi \) are usual derivatives acting on functions of a single variable.

For the collision, it is convenient to go to the center-of-mass frame where the two colliding kinks are on equal footing. In the center-of-mass frame, we have

\[
\phi(x, t < 0) = \phi_K[\gamma_c(x - v_c t)] + \phi_K[\gamma_c(x + v_c t)] \tag{C2}
\]

before the collision. The center-of-mass boost factor \( \gamma_c \) and velocity \( v_c \) are related to those of the incoming kink in the stationary frame by

\[
\gamma = 2 \gamma_c^2 - 1 = 2 \gamma_c^2. \tag{C3}
\]

After collision, we have

\[
\phi(x, t > 0) = \phi_K[\gamma_c(x - v t)] + h[\gamma_c(x - v t), \gamma_c(x + v t)] + \phi_K[\gamma_c(x + v t)] + h[\gamma_c(x + v t), \gamma_c(x + v t)]
\]

\[
= \phi_K[\gamma_c(x - v t)] + \sum_a \left[ f_a[\gamma_c(x - v t)] \right] \times \sum_a \left[ f_a[\gamma_c(x + v t)] \right]
+ \sum_a \left[ f_a[\gamma_c(x + v t)] \right] \times \sum_a \left[ f_a[\gamma_c(x + v t)] \right]. \tag{C4}
\]

Symmetry also guarantees that they each carry half of the total energy:\textsuperscript{20}

\[
\frac{1}{2} E_{\text{total CoM}} = \gamma_c M + E_{1st} + E_{2nd}. \tag{C5}
\]

Here \( E_{1st} \) and \( E_{2nd} \) are the corrections to the energy at the first and second order \( h \), respectively. Before we move on to analyze them further, note that conservation of total energy means \( E_{1st} + E_{2nd} = 0 \), since the total energy has always been \( 2 \gamma_c M \). Recall that \( h \) will be given by a power series of \( (\gamma v) \text{ }^{-1} \); conservation of energy provides a cross-order relation at every order. This is commonly known as the optical theorem in perturbation theory.

We will postpone the full scope of the optical theorem to future work. Here we will demonstrate its power at the leading order. Recall that in Sec. IV we explicitly evaluated \( B_0^{(1)} = 0 \) which means no leading order velocity change during the collision. Conservation of energy can give us that answer without an explicit evaluation. The process is a bit technical but the logic is quite simple. \( B_0^{(1)} \) is going to be the only term contributing at the leading order in the energy conservation equation \( E_{1st} + E_{2nd} = 0 \). Therefore it must be zero.

In order to show that we focus on \( E_{1st} \), since \( E_{2nd} \) is automatically a higher order term:

\[
E_{1st} = \int dx \left( \gamma_c \phi_K[\gamma_c(x + v t)] \frac{\partial h}{\partial x} + \gamma_c v_c \phi_K'[\gamma_c(x + v t)] \frac{\partial h}{\partial t} + \frac{dV}{d\phi} h \right). \tag{C6}
\]

Unlike a stationary kink, the equation of motion (3.2) does not make \( E_{1st} \) zero. Before we figure out what it is, let us clarify a few technical details.

We focus on the left moving kink, for which

\[
h = h[\gamma_c(x + v t), \gamma_c(x + v x)]
= \sum_a f_a[\gamma_c(x + v t)] G_a[\gamma_c(x + v x)]. \tag{C7}
\]

From this point on we will omit all arguments to make equations shorter. Since all the functions have one argument only, it should be clear what has been omitted. Again \( \phi \) always means the usual derivative with respect to the single nontrivial argument of the corresponding function. Below, we provide a complete list of how such notation is related to the \( \partial_x \) and \( \partial_t \):

\[
\partial_x \phi_K = \gamma_c \phi_K' \quad \partial_t \phi_K = \gamma_c v_c \phi_K'. \tag{C8}
\]

\[
\partial_x f_a = \gamma_c f_a' \quad \partial_t f_a = \gamma_c v_c f_a'. \tag{C9}
\]

\[
\partial_x G_a = \gamma_c v_c G_a' \quad \partial_t G_a = \gamma_c G_a'. \tag{C10}
\]

\textsuperscript{20}Even for kink-antikink, the energy density is still symmetric.
A SCATTERING THEORY OF ULTRARELATIVISTIC SOLITONS

The equation of motion (3.2) does not make $E_{1st}$ manifestly zero, but it does provide a useful equation:

$$\int dx V f_a G_a = \int dx \phi_k^0 f_a G_a$$

$$= - \int dx \phi_k^0 (f_a G_a + v_c f_a G_a'), \quad \text{(C11)}$$

which results in a more compact expression:

$$E_{1st} = \int dx \sum_a \left[ 2 \gamma_c^2 v_c^2 \phi_k^0 f_a G_a + \gamma_c^2 v_c (1 + v_c^2) \phi_k^0 f_a G_a' \right]. \quad \text{(C12)}$$

Although the integral is nonzero, the contribution from all nonzero modes is zero. To show that, we need to use the property of the modes (3.14):

$$f_a \phi_k^0 G_a' = E_a^{-1} (-f_a'' + V'' f_a) \phi_k^0 G_a'. \quad \text{(C13)}$$

Integrating both sides and noting that both terms on the rhs can be integrated by parts,

$$\int dx V'' \phi_k^0 f_a G_a' = \int dx \frac{1}{\gamma_c^2} \partial_x V f_a G_a'$$

$$= \int dx (-V f_a G_a' - v_c V f_a G_a')$$

$$= \int dx (-\phi_k^0 f_a G_a' + v_c E_a \phi_k^0 f_a G_a). \quad \text{(C14)}$$

Similarly,

$$- \int dx f_a'' \phi_k^0 G_a' = \int dx (f_a'' \phi_k^0 G_a' + v_c f_a'' \phi_k^0 G_a')$$

$$= \int dx (f_a'' \phi_k^0 G_a' - v_c E_a f_a'' \phi_k^0 G_a). \quad \text{(C15)}$$

In the last step of both equations above, we used the fact that after the collision $G_a$ obeys Eq. (3.18), again in line with our discussion in Appendix A. By combining them and plugging then back into Eq. (C13), we get

$$\int dx \phi_k^0 f_a G_a = \int dx \phi_k^0 (v_c^{-1} f_a G_a' + f_a G_a'). \quad \text{(C16)}$$

Combining Eqs. (C16) and (C11), we see that $E_{1st}$ in Eq. (C12) gets exactly zero contributions from the nonzero modes.

For the zero mode, we first use the definition $\phi_k^0 = \sqrt{M v_c} B_0$ and then integrate by parts to get

$$\int dx 2 f_a G_a G_0 = \int dx v_c f_a^3 G_0'. \quad \text{(C17)}$$

Combining Eqs. (C17) and (C12), we get

$$E_{1st} = - \gamma_c \sqrt{M v_c} B_0. \quad \text{(C18)}$$

This proves that $B_0^{(1)}$ is the only term at the leading order of $E_{1st}$, consequently the only term at the leading order of the conservation of energy equation. Therefore it must be zero.

---