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Turbulence: Large-scale sweeping and the emergence of small-scale Kolmogorov spectra

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The dynamics of fully developed hydrodynamic turbulence still is a basically unsolved theoretical problem, due to the strong-coupling long-range nonlinearities in the Navier-Stokes equations. The present analysis focuses on the small-scale fluctuations in a turbulent boundary layer with one external length scale y_o. After taking a (2+1)D spatiotemporal spectral transform of the fluctuating vorticity fields, care is taken of large-scale sweeping which arises as a collective zero mode from the nonlinear flow terms. The "unswept" small-scale nonlinearities are then shown to be asymptotically locally isotropic (i.e., for wave numbers $k\to\infty$) by internal consistency, which allows to close the nonlinear hierarchy. The Navier-Stokes equations (without external forcing) are integrated to give the spectral response of the fluctuating small-scale velocity fields on the presence of a locally isotropic blob of turbulence while it is being swept around over an arbitrary steady state mean velocity profile, using viscous boundary conditions at y=0. Averaging the response spectrum over all possible orientational configurations and sweep velocities results in a novel self-consistency integral for the 4D energy spectrum function. The distribution of turbulence sweep velocities is modeled by means of Lévy-type densities, having an algebraic tail with power $p > 1$. The generic case (which includes Von Kármán's logarithmic mean velocity profile) is found to correspond to $1 < p < 3$. Asymptotic analysis of the self-consistency integral leads to a differential equation which fixes the scaling exponent λ of the unswept frequency Δ and admits a nonempty, integrable and positive definite Airy-type frequency spectrum $\mathcal{E}'(k,\Delta/k^{\lambda}) \sim k^{\mu}$ with so-called "normal" Kolmogorov scaling, that is, $\mu = -7/3$ and $\lambda = 2/3$. Anomalous scaling is possible for one special mean profile.

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I. INTRODUCTION

Modern mathematical fluid mechanics began in 1755 with Euler's reformulation of Newton's laws (1687) in his famous equations for the motion of a material continuum. However, it took almost another century until Navier and Stokes (in 1823 [1] and 1843 [2], respectively) opened up the way to turbulence theory per se by including the effects of viscosity into Euler's equations, which prompted Reynolds in 1883 to introduce the important number that bears his name. Indeed, the number $\mathrm{Re} = UL/\nu$ is a measure of the relative significance of the nonlinear and viscous terms in the Navier-Stokes equations (ν being the specific viscosity), the fluid motion typically becoming unstable for large enough numbers $\mathrm{Re} \gg 1$. Usually U is taken to be the mean velocity while L is an external length scale.

A next step in turbulence modeling was taken in 1922 by Richardson, who introduced the concept of a cascade of turbulent "eddies" which are breaking down from the largest to the smallest scales. Apart from drawing the attention to the vortical nature of turbulence and to the notion of "sweeping"—of the small eddies (the "whirls") by the large ones (the "gusts")—this lead Kolmogorov in 1941 [3] to give a precise definition of the smallest turbulence scales, which now bear his name (see, e.g., [4]). For example, the Kolmogorov length scale reads $\ell_{\mathrm{K}} = (\nu^3/\varepsilon)^{1/4}$, where ε is the specific energy dissipation rate of the turbulent fluctuations.

While the nonuniversal, geometry dependent large-scale turbulent structures might be handled by numerical computations, for instance using "large-eddy simulations" (see, e.g., Ref. [5]), modern theoretical efforts focus on the important wave number range $1/L \ll k \ll 1/\ell_{\mathrm{K}}$, the so-called "inertial" range where the statistical properties of the turbulent fluctuations are predominantly isotropic. In fact, once it is accepted that there exists a range of wave numbers k where the isotropic energy spectrum is determined solely by the mean energy dissipation rate ε, the famous Kolmogorov K41 spectrum $E(k) \sim k^{-5/3}$ follows at once by a trivial dimensional argument. The main purpose of the present analysis is to give this kinematic result a more fundamental dynamical basis in the Navier-Stokes equations.

In essence, all theoretical efforts concern the nonlinear terms which give rise to an open hierarchy of statistical correlations. One may distinguish between closure models in physical space and in spectral space. To the first category belong the "mixing-length" models, first proposed by Prandtl (1925) to define a turbulent (or eddy) viscosity ν_{T}, and their more sophisticated offspring such as the k-ε model (see, e.g., Refs. [6,7]).

For small scales (i.e., large wave numbers k) writing $\nu_{\mathrm{T}} \approx \ell u_\ell$ (see, e.g., Refs. [5,7]) and interpreting u_ℓ^2 as the kinetic energy of eddies of size $\ell \approx 1/k$, suggests that one may take u_ℓ^2 to be proportional to the second-order longitudinal structure function $D_2^{\|}(\ell)$—which filters away the large scales $L \gg \ell$ so that it can be expressed in terms of the energy spectrum $E(k)$ of the smaller scales only, even for power-law spectra. For example, if $E(k) \sim k^{-q}$ (with $1 < q < 3$) one has $D_2^{\|} \sim \ell^{q-1}$.

Strictly speaking, $D_2^{\|}(\mathbf{r})$ depends on $\ell = |\mathbf{r}|$ only in isotropic turbulence. However, apart from satisfying theoretical arguments, there is by now considerable empirical evidence that the small-scale part of turbulent flows indeed tends to show asymptotic isotropy (i.e., for $\ell\to 0$). In fact, local isotropy is already prominent at scales much larger than the Kolmogorov microscale (see, e.g., Refs. [8,9]).

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For isotropic turbulence with a constant energy dissipation rate ε, the so-called Kármán-Howarth equation (1938) relates $D_2^{\parallel}(\ell)$ to $D_3^{\parallel}(\ell)$ by means of the Navier-Stokes equations (see, e.g., Refs. [7,8,10]). It clearly still involves the closure problem and can only be integrated using, for example, Obukhov's "constant skewness" hypothesis—the skewness being $\mathrm{Sk} = D_3^{\parallel}/(D_2^{\parallel})^{3/2}$, with the empirical value Sk$\approx$−0.4 for all $\ell \ll L$ (see, e.g., Refs. [7,8]). This yields $D_3^{\parallel}(\ell) = -\frac{4}{5}\varepsilon\ell$ in the inertial subrange, which is one of the few basic cornerstones of turbulence theory as it does not depend on the experimental value of Sk. In the same range one then has the K41 result $D_2^{\parallel}(\ell) \sim \ell^{2/3}$. However, the latter can not be considered "fundamental" as its derivation involves an *ad hoc* closure hypothesis having no underlying physical basis.

Kolmogorov's K41 scaling is by-and-large experimentally confirmed (see, e.g., Refs. [5–9,11,12]), but slight deviations in the exponents have been reported—in particular, in laboratory experiments (see, e.g., Refs. [4,7]). This so-called "anomalous" scaling has been attributed to "intermittency" related to nonuniform energy dissipation, which admits the definition of a fractal dimension $D_\lambda \leqslant 3$ (see, e.g., Ref. [4]). Theoretical efforts on anomalous scaling thus go into constructing models for D_λ. It here suffices to mention the β and the lognormal model. To date, however, it is unclear how these cascade models are connected with the actual fluid dynamics.

The 3D spatial Fourier transform of the Navier-Stokes equations yields an equation for $E(k,t)$—which can be considered as the spectral equivalent of the Kármán-Howarth equation—where the energy flux $\Pi(k)$ is defined in terms of three-mode interactions (see, e.g., Ref. [5–7,13]). Many "one-point" models have been proposed to close this equation. For instance, Pao's hypothesis (1965) amounts to putting $\Pi = \alpha^{-1}\varepsilon^{1/3}k^{5/3}E(k)$, which—in contrast to the spectrum obtained from the "constant skewness" hypothesis—yields a positive definite spectrum with an exponential-type cutoff in the viscous subrange. Obviously, it reproduces the K41 spectrum in the inertial subrange by construction. Another oft-quoted closure proposal is due to Heisenberg (1948) which, however, leads to a power-law viscous cutoff.

The earliest "two-point" closure model is known as the quasinormal hypothesis—dating back to Chou (1940) and Millionshtchikov (1941)—which amounts to approximating the nonlinear four-mode interactions in the dynamical equation for $\Pi(k,t)$ itself by products of two-point correlations using Gaussian statistics (see, e.g., Refs. [4,5,7,8]).

However, such an approximation does not account for the skewness of turbulent probability densities. As a consequence, the ensuing energy spectrum is not positive definite. Further progress has been made only by implementing an *ad hoc* eddy-viscosity ν_{T}, assuming a particular time dependence for the response functions and making a Markovian approximation—known as the "eddy-damped quasinormal Markovian" (EDQNM) model (see, e.g., Refs. [7,14]).

The EDQNM model has been used to cure the "small-parameter problem" in the renormalization group (RNG) analysis of turbulence [15]. Indeed, the RNG analysis of critical phenomena reminds one of Richardson's cascade. Therefore, in Ref. [16] Forster, Nelson, and Stephen (see also, e.g., Refs. [4,5]) added a fictitious random forcing term in the 3D Fourier transformed Navier-Stokes equations, taken to be homogeneous isotropic Gaussian white noise with a power-law energy spectrum $\Phi(k) = D^{\mathrm{ext}}k^{3-\epsilon}$, where $0 < \epsilon \ll 1$. The nonlinear flow terms in the Navier-Stokes equations are given a coupling parameter λ_ϵ, which must also be small. The fields are then expanded in powers of λ_ϵ and the highest wave numbers are repeatedly eliminated. The parameter-flow equations for the renormalized viscosity and coupling parameter have a nontrivial fixed point at $\lambda_\epsilon^* = (\epsilon/3A)^{1/2}$, where $A \approx 1/10\pi^2$. The turbulence energy spectrum becomes $E(k) \approx (D^{\mathrm{ext}}/\nu)\,\ell_{\mathrm{K}}^{\epsilon/3}k^{1-2\epsilon/3}$, which agrees with the K41 spectrum if $\epsilon = 4$, implying $\lambda_\epsilon^* \approx 4\pi$, which violates the basic presumptions of the method. The problem has also been eased by an intricate analysis of the large-scale stirring (sweeping), using "short-distance operator-product expansions" in the RNG analysis, and accounting for Galilean invariance (see, e.g., Ref. [17]).

The EDQNM closure is connected to a Markovian version of Kraichnan's "random-coupling" model (see, e.g., Refs. [4,5,14]), which on its turn is related to his so-called direct-interaction (DI) closure approximation. However, the ensuing DI inertial-range energy spectrum has the form $E(k) \sim k^{-3/2}$ (see, e.g., Refs. [4,14]), which is strongly at variance with Kolmogorov's K41 spectrum. This defect of the DI approximation has been shown to be due to its noninvariance under (random) Galilean transformations (RGTs) so that the large eddies have a direct action on the small eddies. To quote Leslie [14] (p. 186): "The particular defect of DI [..] was its failure to properly represent the sweeping around of small eddies by big ones." It has been proposed to remedy these sweeping complications by means of quasi-Lagrangian transformations. For example, Kraichnan's Lagrangian-history direct-interaction approximation indeed yields the K41 spectrum. However, the significance of the various steps involved is unclear (see, e.g., Refs. [5,8,14]).

Another quasi-Lagrangian theory—eliminating sweeping by subtracting a single Lagrangian displacement from all points in the needed vicinity of a reference point $\mathbf{r}_o$—is due to Belinicher and L'vov (BL) [18]. To quote from Ref. [19] (p. 7–8): "The theory [of hydrodynamic turbulence] exhibits two types of nonlinear interactions. The larger of the two is known to any person who has watched how a small floating object is entrained in the eddies of a river and swept along [..] with the turbulent flow. [..] However, sweeping is just a kinematic effect that does not lead to energy redistribution between scales. [..] The redistribution of energy results from the second type of interaction, that stems from the shear and torsion effects that are sizable only if they couple fluid motions of comparable scales. This second type of nonlinearity is smaller in size but crucial in consequence." In the present work, this pivotal notion of two types of nonlinear interactions will be implemented from the outset.

Closure of the BL-transformed Navier-Stokes equations—including a stirring force—is obtained by so-called fusion rules and bridges, such that the normal K41 result follows at each level of a field-theoretical expansion. In that context anomalous scaling can only result from nonperturbative effects (see, e.g., Ref. [20]) and requires an *ad hoc* outer renormalization scale $L[\gg \ell \gg \ell_{\mathrm{K}}]$.

In this article we take a fresh look at the problem by including what is known by now—and what has been outlined in this Introduction—concerning such crucial concepts as, for example, (i) the closure problem, (ii) especially for small-scale locally isotropic fields, while (iii) accounting for large-scale sweeping effects, (iv) including an external length scale, and (v) excluding fictitious external stirring. This amounts to studying the small-scale motion of turbulent fluid blobs—as dictated by the Navier-Stokes equations—under the influence of externally given background fields. The latter comprise both the mean velocity profile and the large-scale sweeps.

While using the concept of turbulence as the dynamics of a nonlinear self-driven stochastic vorticity field as originally outlined in Ref. [21], the present analysis differs in several aspects from that work. For example, although Ref. [21] does contain the idea of isotropic closure at the small scales, the irreducible part of the dynamics only involved linear mean shear and large-scale sweeping was not considered. The importance of the latter was recognized in, for example, Refs. [22,23], albeit in an *ad hoc* fashion. That problem has been cured in Ref. [24], but the irreducible dynamics remained restricted to linear shear. The present analysis removes these restrictions. A detailed account will be available in Ref. [25].

This leads to a picture of turbulence as a statistically steady state of small blobs of locally isotropic fluid fluctuations, which are being swept all over the place before arriving at a particular point in space—the Eulerian point of view. The large-scale sweeps are uniform, which guarantees Galilean invariance, but against a nonuniform mean background velocity profile the blobs will loose their original unswept isotropy. After averaging over both all orientational configurations and all possible sweeps, however, the isotropic state can be recovered and its properties determined by self-consistency.

II. NAVIER-STOKES DYNAMICS

Consider incompressible turbulent flow over an infinitely extended flat wall (in the $y=0$ plane). The mean flow is taken in the x direction, that is, $\langle U_1\rangle = \overline{U}(y)$, and $\langle U_2\rangle = \langle U_3\rangle = 0$. Using Reynolds decomposition for U_i and the vorticity $\Omega_i = \epsilon_{ijk}\partial U_k/\partial x_j$ [i.e., $U_i = \langle U_i\rangle + u_i$, and $\Omega_i = \langle\Omega_i\rangle + \omega_i$ where $\langle\Omega_i\rangle = \delta_{i3}\overline{\Omega}$ with $\overline{\Omega} = -\overline{U}'(y)$], one has

$$\frac{\partial\omega_i}{\partial t} + \overline{U}\frac{\partial\omega_i}{\partial x} = -\omega_2\delta_{i1}\overline{\Omega} - v\delta_{i3}\overline{\Omega}' + \overline{\Omega}\frac{\partial u_i}{\partial z} + \nu\nabla^2\omega_i + \mathcal{F}_i \tag{2.1}$$

for the vorticity fluctuations, with

$$\mathcal{F}_i = \omega_j s_{ij} - u_j\frac{\partial\omega_i}{\partial x_j}. \tag{2.2}$$

The strain-rate fluctuations $s_{ij} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ cause both rotation and stretching ($s_{ij} > 0$) [or squeezing ($s_{ij} < 0$)] of the vorticity vector. Note that $\mathbf{u} \equiv (u,v,w)$.

Now introduce the (2 + 1)D spectral transformation

$$\tilde{\omega}_i(\mathbf{k}_\parallel,\omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\omega_i(\mathbf{r},t)\,e^{-i\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel + i\omega t}d\mathbf{r}_\parallel dt, \tag{2.3}$$

where $\mathbf{r}_\parallel = (x,z)$ and $\mathbf{k}_\parallel = (k_1,k_3)$. Equation (2.1) then yields

$$\left(-if_\nu + u_2^{\mathrm{s}}\frac{\partial}{\partial y} - \nu\frac{\partial^2}{\partial y^2}\right)\tilde{\omega}_i = -\tilde{\omega}_2\delta_{i1}\overline{\Omega} - \tilde{v}\delta_{i3}\overline{\Omega}' + ik_3\overline{\Omega}\tilde{u}_i + \tilde{\mathcal{F}}_i, \tag{2.4}$$

with

$$f_\nu = \omega - k_1\overline{U} - \mathbf{k}_\parallel\cdot\mathbf{u}_\parallel^{\mathrm{s}} + i\nu k_\parallel^2, \tag{2.5}$$

where ($\mathbf{k}''_\parallel = \mathbf{k}_\parallel - \mathbf{k}'_\parallel$)

$$u_j^{\mathrm{s}}\tilde{\omega}_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{u}_j(\mathbf{k}'_\parallel,\omega')\mathsf{P}\tilde{\omega}_i(\mathbf{k}''_\parallel,\omega'')d\mathbf{k}'_\parallel d\omega'. \tag{2.6}$$

The unswept nonlinear interactions are given by

$$\tilde{\mathcal{F}}_i = -i\mathbf{k}(y)\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\tilde{\mathbf{u}}(\mathbf{k}'_\parallel,\omega')\mathsf{Q}\tilde{\omega}_i(\mathbf{k}''_\parallel,\omega'') - \tilde{u}_i(\mathbf{k}'_\parallel,\omega')\mathsf{Q}\tilde{\omega}(\mathbf{k}''_\parallel,\omega'')]d\mathbf{k}'_\parallel d\omega', \tag{2.7}$$

where $\mathbf{k}(y) = (k_1, -i\frac{\partial}{\partial y}, k_3)$ and $\mathsf{Q} = 1 - \mathsf{P}$, with P being defined by $\mathsf{P}\tilde{\omega}_i(\mathbf{k}_\parallel - \mathbf{k}'_\parallel, \omega - \omega') \equiv \tilde{\omega}_i(\mathbf{k}_\parallel,\omega)$ for $|\mathbf{k}'_\parallel| < k_o^{\mathrm{s}}O(1/y_o)$, and zero elsewhere. For $k_o^{\mathrm{s}} \to 0$, the u_j^{s} will be y independent.

The spectral projector P extracts the collective large-scale sweeping contributions from the nonlinear terms in the original Eulerian Navier-Stokes equations so that the unswept nonlinearities only involve small-scale fields. In the present work, this is the formal implementation of the pivotal notion of "two types of nonlinear interactions" in hydrodynamic turbulence—as emphasized by the quote from Ref. [19] in the Introduction—separating the large-scale gusts from the small-scale whirls.

This separation opens up the way for the occurrence of small parameters. For example, with an unswept frequency $\Delta \sim k^\lambda$ (and $1/2 < \lambda < 1$), both $\Delta/k \sim k^{\lambda-1}$ and, for example, $k/\Delta^2 \sim k^{1-2\lambda}$ become small for $k \to \infty$. Clearly, if both types of terms contribute equally—as will be seen, they actually do—one at once gets $\lambda = \frac{2}{3}$, which amounts to normal Kolmogorov time scaling (see, e.g., Ref. [11]). In the present theory, Δ will emerge naturally as the frequency ω (in the laboratory coordinate frame) after subtraction of both mean Taylor advection and sweeping at y_o, that is, $\Delta = \omega - k_1\overline{U}_o - \mathbf{k}\cdot\mathbf{u}^{\mathrm{s}}$.

Now take ik_3 times Eq. (2.4) for $i=1$ minus ik_1 times Eq. (2.4) for $i=3$. Using $\tilde{\omega}_i = i\epsilon_{ijl}k_j(y)\tilde{u}_l$ and transversality, this yields

$$\left[-if_\nu(y) + u_2^{\mathrm{s}}\frac{\partial}{\partial y} - \nu\frac{\partial^2}{\partial y^2}\right]\left(k_\parallel^2 - \frac{\partial^2}{\partial y^2}\right)\tilde{v} - ik_1\overline{\Omega}'\tilde{v} = \tilde{\mathcal{F}}, \tag{2.8}$$

where $\tilde{\mathcal{F}} = ik_3\tilde{\mathcal{F}}_1 - ik_1\tilde{\mathcal{F}}_3$. Equation (2.8) describes the response of the normal-to-the-surface velocity component $\tilde{v}$ on the source field $\tilde{\mathcal{F}}$. The other components of the fluctuating velocity are most easily obtained by first using the solution $\tilde{v}$ from Eq. (2.8) to solve Eq. (2.4) for $i=2$, and finally to determine the response of the parallel-to-the-surface velocity components $\tilde{u}$ and $\tilde{w}$ by means of transversality, that is, $\tilde{u} = i(k_1\frac{\partial\tilde{v}}{\partial y} - k_3\tilde{\omega}_2)/k_\parallel^2$, and $\tilde{w} = i(k_3\frac{\partial\tilde{v}}{\partial y} + k_1\tilde{\omega}_2)/k_\parallel^2$.

For $\nu \to 0$, Eq. (2.8) typically involves three modes $\tilde{v} \approx e^{-ky}$ with k being of the order of $\nu^{-1/2}$ in the Taylor range (see, e.g., Refs. [4,11]) and one very short-range mode $\tilde{v} \approx e^{-|u_2^s|y/\nu}$ (see Ref. [25]). Since the latter yields response contributions which are a factor $1/k$ smaller than those from the slow modes, it will not be significant for the ensuing small-scale turbulence spectra at $y_o O(\nu^0)$. This is tantamount to neglecting both the viscous operator $\nu(\partial/\partial y)^2$ and the term $ik_1\overline{\Omega}'\tilde{v}$ in Eq. (2.8), which is then solved by

$$\tilde{v} = \frac{1}{2k_\parallel}\int_0^\infty e^{-k_\parallel|y-y'|}\tilde{\varphi}(y')dy', \tag{2.9}$$

with

$$\tilde{\varphi} = C_o e^{i\sigma(y)} + \int_0^y e^{i[\sigma(y)-\sigma(y')]}\tilde{\mathcal{F}}(y')\,dy'/u_2^s, \tag{2.10}$$

where $\sigma(y) = \int_0^y f_{\nu=0}(y')dy'/u_2^s$, satisfying both $\tilde{v}|_{y=0} = 0$ and the viscous no-slip condition $(\partial\tilde{v}/\partial y)|_{y=0} = 0$ at the boundary if

$$\int_0^\infty e^{-k_\parallel y}\tilde{\varphi}(y)dy = 0. \tag{2.11}$$

This yields $C_o(\mathbf{k}_\parallel) = -K_o/I_o$, with

$$I_o = \int_0^\infty e^{-k_\parallel y + i\sigma(y)}dy, \tag{2.12}$$

$$K_o = \int_0^\infty dy\int_0^y dy' e^{-k_\parallel y + i[\sigma(y)-\sigma(y')]}\tilde{\mathcal{F}}(y')/u_2^s. \tag{2.13}$$

For the 4D spectral transform $[\tilde{v}(\mathbf{k}_\parallel,y) \Rightarrow \hat{v}(\mathbf{k})]$, one thus obtains

$$\hat{v}(\mathbf{k}) = \left[\frac{\mathcal{K}(k_2)}{K_o} - \frac{\mathcal{I}(k_2)}{I_o}\right]\frac{K_o}{2\pi k^2}, \tag{2.14}$$

with

$$\mathcal{I}(k_2) = \int_0^\infty e^{-ik_2 y + i\sigma(y)}dy, \tag{2.15}$$

$$\mathcal{K}(k_2) = \int_0^\infty dy\int_0^y dy' e^{-ik_2 y + i[\sigma(y)-\sigma(y')]}\tilde{\mathcal{F}}(y')/u_2^s, \tag{2.16}$$

where the dependence on $\mathbf{k}_\parallel$ has been suppressed.

III. 4D TURBULENCE SPECTRA

The 4D spectra (of second order) are given by the space-time Fourier transform of the 4D (local) correlation functions $\mathcal{R}_{ij}(\rho,\tau)|_o = \langle u_i(\mathbf{r}_o+\rho,t_o+\tau)u_j(\mathbf{r}_o,t_o)\rangle$, where $\mathbf{r}_o = (x_o,y_o,z_o)$. Using the spectral representation of Sec. II for the fields, one gets

$$\mathcal{S}_{ij}(\mathbf{k},\omega) = \int_{-\infty}^\infty\int_{-\infty}^\infty e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_o - i(\omega+\omega')t_o}\hat{v}(\mathbf{k}',\omega) \times \langle\hat{u}_i(\mathbf{k},\omega)\hat{u}_j(\mathbf{k}',\omega')\rangle d\mathbf{k}'d\omega'. \tag{3.1}$$

Therefore, multiply Eq. (2.14) for $\hat{v}(\mathbf{k},\omega)$ by $e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_o - i(\omega+\omega')t_o}\hat{v}(\mathbf{k}',\omega')$, and integrate over $\mathbf{k}'$ and ω', as in Eq. (3.1). This leads to $\mathcal{S}_{vv} = \sum_{n=0}^1 \mathcal{S}_{vv}^{[n]}$. Only the bulk term $\mathcal{S}_{vv}^{[0]}$ will be relevant in what follows, because the boundary term $\mathcal{S}_{vv}^{[1]}$ makes no contribution to the leading orders in the analysis, due to the factor $e^{-k_\parallel y}$ in I_o and K_o. Upon using Eq. (2.16) for $\mathcal{K}$, introducing the Fourier transform $\tilde{\mathcal{F}}(\mathbf{k}_\parallel,y) \Rightarrow \hat{\mathcal{F}}(\mathbf{k})$, and at once taking advantage of both time-translation invariance to do the ω' integration and of $\mathbf{r}_\parallel$-translation invariance (for steady flow over an infinitely extended flat wall in the $y = 0$ plane) in doing the $\mathbf{k}'_\parallel$ integrations [i.e., $\langle\hat{u}_i\hat{u}_j\rangle = \langle\hat{u}_i\hat{u}_j\rangle_o\delta(\mathbf{k}_\parallel+\mathbf{k}'_\parallel)\delta(\omega+\omega')$], one obtains

$$\mathcal{S}_{vv}^{[0]} = \frac{1}{2\pi k^2}\int_0^\infty dy\int_0^y dy'\int_{-\infty}^\infty d\bar{k}_2\int_{-\infty}^\infty dk'_2 \times e^{i\tau(y,y')/u_2^s}\langle\hat{\mathcal{F}}(\bar{k}_2)\hat{v}(k'_2)\rangle_o/u_2^s, \tag{3.2}$$

with

$$\tau(y,y') = (k_2+k'_2)u_2^s y_o + y\Delta - y'\bar{\Delta} + k_1\int_y^{y'}[\overline{U}(y'') - \overline{U}_o]dy'', \tag{3.3}$$

where

$$\Delta = \omega - k_1\overline{U}_o - \mathbf{k}\cdot\mathbf{u}^s \tag{3.4}$$

represents the unswept turbulence frequency at $y = y_o$. The source frequency $\bar{\Delta}$ follows from Δ by letting $k_2 \Rightarrow \bar{k}_2$. Note that $\Delta - \bar{\Delta} = -(k_2 - \bar{k}_2)u_2^s$.

IV. LOCAL ISOTROPY

Since for $k \to \infty$ small-scale turbulence will be approximately locally homogeneous and isotropic—indeed, the experimental data confirm this expectation (see, e.g., Ref. [9])—from here on the nonlinear source fields will be taken to represent such turbulence in a first-order approximation, as the starting point for a systematic asymptotic analysis. For homogeneous isotropic turbulence (i.e., with $\overline{U} \equiv \overline{U}_o$) Eq. (2.14) reduces to $\hat{v}^\iota = \mathcal{K}^\iota/2\pi k^2$, with

$$\mathcal{K}^\iota(k_2) = \int_{-\infty}^\infty d\bar{k}_2\int_{-\infty}^\infty dy\int_{-\infty}^y dy' e^{i(y\Delta - y'\bar{\Delta})/u_2^s}\hat{\mathcal{F}}(\bar{k}_2)/u_2^s. \tag{4.1}$$

Putting $y' = y - \xi$, the y integral leads to $2\pi|u_2^s|\delta(\Delta - \bar{\Delta})$ while the ξ integral yields $iu_2^s\mathcal{P}(1/\bar{\Delta})$—where $\mathcal{P}$ stands for "the principal value of." Hence, using the relation $\Delta - \bar{\Delta} = -(k_2 - \bar{k}_2)u_2^s$ one obtains $\mathcal{K}^\iota = 2\pi i\hat{\mathcal{F}}/\Delta$ so that $\hat{v}^\iota = i\hat{\mathcal{F}}/(k^2\Delta)$. More general, with $\overline{\Omega}^{(\prime)} \equiv 0$—that is, eliminating any explicit dynamical reference to an external scale—the Navier-Stokes Eq. (2.4) yields $-i\Delta_\nu\hat{\omega}_i^\iota = \hat{\mathcal{F}}_i$, where $\Delta_\nu = \Delta + i\nu k^2$. For the small-scale velocity fluctuations $\hat{u}_i = i\epsilon_{ijl}k_j\hat{\omega}_l/k^2$, this leads to $\hat{u}_i = i\hat{\mathcal{V}}_i/\Delta$ with $\hat{\mathcal{V}}_i = i\epsilon_{ijl}k_j\hat{\mathcal{F}}_l/k^2$. Note that $\hat{\mathcal{F}}$ is identical to $k^2\hat{\mathcal{V}}_2$.

Using the detailed properties of the nonlinear forces [see Eq. (2.7)], it has been shown that isotropy is a self-consistent property, at least up to third-order correlations (see Ref. [24]). The equation $-i\Delta_\nu\hat{\omega}_i^\iota = \hat{\mathcal{F}}_i$ is also the starting point for deriving the 4D spectral version of the Kármán-Howarth equation, which leads to Kolmogorov's "four-fifth" law for the third-order (static) longitudinal structure function—although for homogeneus turbulence this requires an *ad hoc* external stirring force (see, e.g., Refs. [4,10,24]). For the 4D isotropic energy spectrum it yields $[\Delta = \Delta_{\nu=0}]$

$$\mathcal{E}^\iota(k,\Delta) = \varepsilon_\mu^{1/3}k^\mu\mathsf{F}(\Delta/\varepsilon_\lambda^{1/3}k^\lambda), \tag{4.2}$$

where $\mu = -(1+2\lambda)$ [with $\lambda < 1$ being the (as yet arbitrary) time scaling exponent] and $\varepsilon_\mu^{1/3} = \varepsilon\, \varepsilon_\lambda^{-2/3}$, where ε is the turbulence energy dissipation rate; $\mathsf{F}(\varpi)$ is related to the (as yet unknown) third-order temporal correlation function, that is, $\mathcal{R}_3^{\parallel}(r,\tau) = -\frac{2}{15}\varepsilon r F(\tau/r^\lambda)$ [with $F(0)=1$] [24].

Generalizing Eq. (4.2) to higher (i.e., nth) order structure functions leads to the exponents $\mu_n = 1 - n + (n-4)\lambda$, which allows to make a connection with the lognormal random cascade model (see, e.g., Refs. [4,24] and Appendix A).

Before proceeding with Eq. (3.2), it is worth noticing that the response spectra $\mathcal{S}_{ij}$ for $\{i,j\}\neq\{v,v\}$—with locally isotropic input fields—can be shown to satisfy

$$\mathcal{S}_{ij}^{[0]} = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\frac{\mathcal{E}^{[0]}}{4\pi k^2}, \tag{4.3}$$

with the energy response spectrum $\mathcal{E}^{[0]}$ being defined by, for example, $\mathcal{E}^{[0]} = (4\pi k^4/k_\parallel^2)\mathcal{S}_{vv}^{[0]}$ [25]. This shows that the spectrum function $\mathcal{E}^{[0]}$ is unambiguously defined, so that the ensuing asymptotic result for the energy spectrum (in Sec. V) will indeed be generic, that is, does not depend on the particular pick of $i=j=2$. Using $\hat{\mathcal{F}} = -ik^2\Delta\hat{v}'$ on the right-hand side of Eq. (3.2), and invoking the locally homogeneous isotropic expression for the correlation function in the response integral, one thus arrives at

$$\mathcal{E}^{[0]} = -i\frac{k^2}{2\pi u_2^{\mathrm{s}}}\int_{-\infty}^{\infty}\frac{d\bar{k}_2}{\bar{k}^2}\int_0^\infty dy\int_0^y dy' e^{i\bar{\tau}(y,y')/u_2^{\mathrm{s}}} \times \bar{\Delta}\,\mathcal{E}^{\iota}(\bar{k},\bar{\Delta}), \tag{4.4}$$

where $\bar{\tau} = \tau(y,y')|_{k_2'=-\bar{k}_2}$ [see Eq. (3.3)].

V. THE 4D ENERGY SPECTRUM

The spectral response embodied in Eq. (4.4) describes the sweeping of a blob of turbulent fluid with an input isotropic energy spectrum $\mathcal{E}^{\iota}$ across, and the distortion by, the mean velocity profile $\overline{U}(y)$, to-and-fro a surface (at $y=0$) where the fluctuating fields satisfy viscous boundary conditions. Of course, the resulting output spectrum $\mathcal{E}^{[0]}$ is not fully isotropic anymore, but to observe statistically steady turbulence (at y_o) it should reproduce the input spectrum after averaging over all possible orientations during all possible sweeps. This yields the basic self-consistency equation

$$\mathcal{E}^{\iota}(k,\Delta) = -i\left\langle\frac{k^2}{2\pi u_2^{\mathrm{s}}}\int_{-\infty}^{\infty}\frac{d\bar{k}_2}{\bar{k}^2}\int_0^\infty dy\int_0^y dy' \times e^{i\bar{\tau}(y,y')/u_2^{\mathrm{s}}}\bar{\Delta}\,\mathcal{E}^{\iota}(\bar{k},\bar{\Delta})\right\rangle. \tag{5.1}$$

In this form the equation is far too complicated to be solved for $\mathcal{E}^{\iota}$. In fact, there is no reason to do so because in its derivation various contributions of relative order $1/k$ have been neglected. Therefore, Eq. (5.1) will be analyzed asymptotically for $k\to\infty$.

First of all, it suffices to expand $\mathcal{E}^{\iota}(\bar{k},\bar{\Delta})$ about $(k,\bar{\Delta})$ and only keep the leading term. This considerably simplifies the directional average, which—since only the component k_1 remains—may now be taken as $\langle\,\rangle_{\mathrm{sph}} = (1/2k)\int_{-k}^{k}dk_1$. With $\eta = y - y_o$, expand $\overline{U}(y_o+\eta)$ in powers of η [see Eq. (3.3)]. Next, let $\eta = \sigma+\zeta$ and $\eta' = \sigma-\zeta$ so that the integration domain in the new variables is the halfspace $\zeta\geqslant 0$. Using $d\bar{k}_2 = d\bar{\Delta}/|u_2^{\mathrm{s}}|$, this leads to

$$\mathcal{E}^{\iota}(k,\Delta) = -\frac{i}{\pi}\int_{-\infty}^{\infty}d\bar{\Delta}\int_{-\infty}^{\infty}d\sigma\int_0^\infty d\zeta \times\left\langle\frac{e^{i[(\Delta-\bar{\Delta})\sigma+(\Delta+\bar{\Delta})\zeta]/u_2^{\mathrm{s}}}j_o(k\phi/u_2^{\mathrm{s}})}{|u_2^{\mathrm{s}}|u_2^{\mathrm{s}}}\right\rangle\bar{\Delta}\,\mathcal{E}^{\iota}(k,\bar{\Delta}), \tag{5.2}$$

with

$$\phi = \sum_{n=2}^{\infty}\sum_{m=0}^{n}\frac{1}{n!}\binom{n}{m}\sigma^{n-m}[\zeta^m-(-\zeta)^m]\overline{U}_o^{(n-1)}, \tag{5.3}$$

where $j_o(z) = \sin(z)/z$ is the spherical Bessel function of order zero and $\overline{U}_o^{(n)} = (\partial^n\overline{U}/\partial y^n)|_{y_o}$. The remaining brackets in Eq. (5.2) stand for averaging over all sweeps.

With the unswept frequency scaling as k^λ $(1/2<\lambda<1)$, $\mathcal{E}^{\iota}(k,\Delta)$ may be written as a function of k and Δ/k^λ, like in Eq. (4.2), and Eq. (5.2) shows that near $\bar{\Delta}=\Delta\sim k^\lambda$ one has $\zeta\sim k^{-\lambda}$. Since the leading term from Eq. (5.3) is $\phi\sim 2\sigma\zeta$, a functional relation between input and output spectra thus exists only for $k\sigma\zeta\ O(1)$, that is, $\sigma\sim k^{\lambda-1}$ so that $|\Delta-\bar{\Delta}|\sim k^{1-\lambda}$. Both the spectrum function and the 'linear' response exponent in the integral in Eq. (5.2) expand in powers of $k^{1-2\lambda}$, and the leading nontrivial contribution will be of order $k^{2-4\lambda}$.

The first nonzero 'nonlinear' response contribution arising from the Bessel function is of order $k^{2\lambda-2}$. Since only both response terms together yield a nonempty energy spectrum, one must have $k^{2-4\lambda} = k^{2\lambda-2}$, which at once yields $\lambda=\frac{2}{3}$, that is, normal Kolmogorov K41 scaling. In the following analysis, the time-scaling exponent will be set at this value from the outset. So-called anomalous scaling (where $\lambda\neq\frac{2}{3}$) is discussed in Appendix B. The above scaling arguments are now implemented in Eq. (5.2) by defining $\varpi = \Delta/k^{2/3}$, putting $\bar{\Delta}-\Delta = \vartheta k^{1/3}$, and letting $\sigma\Rightarrow\sigma k^{-1/3}$ and $\zeta\Rightarrow\zeta k^{-2/3}$.

Although the notions of 'linear' and nonlinear response contributions are somewhat arbitrary, they are useful in organizing the mathematics. The term linear is reserved for those contributions to the self-consistency equation which would survive in the case of a perfectly 'linear' mean velocity profile. On the right hand side of Eq. (5.2)—apart from expanding $\mathcal{E}^{\iota}$—this amounts to merely evaluating the exponential in powers of ϑ. Through second order there will be one contribution of $O(1)$, two of $O(k^{-1/3})$, and three of $O(k^{-2/3})$. They may be listed as $\mathcal{E}^{\iota}|_{\mathsf{L},nm}O(k^{-n/3})$, with $m=0,1,\ldots,n$. One finds that $\mathcal{E}^{\iota}|_{\mathsf{L},00} = \mathcal{E}^{\iota}$ is exactly canceled by the $\mathcal{E}^{\iota}$ on the left-hand side of the equation, while all $\mathcal{E}^{\iota}|_{\mathsf{L},1m}=0$ by ϑ symmetry. The second-order terms may be written as

$$\mathcal{E}^{\iota}|_{\mathsf{L},20} = \frac{\partial^2\varpi\mathcal{E}^{\iota}}{2\,\partial\varpi^2}\mathsf{L}_{20},\quad \mathcal{E}^{\iota}|_{\mathsf{L},21} = \frac{\partial\varpi\mathcal{E}^{\iota}}{\partial\varpi}\mathsf{L}_{21},$$
$$\mathcal{E}^{\iota}|_{\mathsf{L},22} = \frac{\varpi\mathcal{E}^{\iota}}{2}\mathsf{L}_{22}. \tag{5.4}$$

The coefficients $\mathsf{L}_{nm}(\varpi)$ are obtained from Eq. (5.2) by making use of the harmonic property $(i\xi)^m e^{ix\xi} = \partial^m e^{ix\xi}/\partial x^m$ (with $\xi = \zeta/u_2^{\mathrm{s}}$, and $x = 2\varpi$) to evaluate the ζ integral, and noticing that since the Bessel function $j_o(z)$ is even in z only $\cos(\vartheta\sigma/u_2^{\rm s})$ contributes in the σ integral, which yields

$$\int_{-\infty}^{\infty} d\sigma \cos\left(\frac{\vartheta\sigma}{u_2^{\rm s}}\right) j_o\left(\frac{2\sigma\zeta\overline{U}_o'}{u_2^{\rm s}}\right) = \frac{\pi}{2}\left|\frac{u_2^{\rm s}}{\zeta\overline{U}_o'}\right|\Theta\left(|\zeta| - \left|\frac{\vartheta}{2\overline{U}_o'}\right|\right), \quad (5.5)$$

where $\Theta(x)$ is the unit step function. Now noticing that only $\sin(2\omega\zeta/u_2^{\rm s})$ contributes to the real part of the spectrum function, defining the dimensionless sweeping velocity $s \equiv u_2^{\rm s}/u_*^{\rm s}$ and putting $\vartheta = (u_*^{\rm s}\overline{U}_o'/\varpi)\theta$, one obtains $\mathsf{L}_{20} = -2\beta_{\mathsf{L}}k^{-2/3}/\varpi^3$, $\mathsf{L}_{21} = 3\beta_{\mathsf{L}}k^{-2/3}/\varpi^4$, and $\mathsf{L}_{22} = -6\beta_{\mathsf{L}}k^{-2/3}/\varpi^5$ with $\beta_{\mathsf{L}} \equiv -\frac{1}{4}|\overline{U}_o'|^2 u_*^{\rm s\,2}\beta_2^{\mathsf{L}}$, where

$$\beta_2^{\mathsf{L}} = -2\int_0^{\infty} d\theta\,\theta^2\left\langle\frac{\mathrm{si}(\theta/|s|)}{|s|}\right\rangle, \quad (5.6)$$

$\mathrm{si}(z)$ being the sine integral.

To evaluate the turbulence sweep average in Eq. (5.6), consider

$$P_{\rm even}(s) = \frac{c_p}{(1+2s^2/p)^{p/2}}, \quad (5.7)$$

with $c_p = \sqrt{\frac{2}{p}}/B(\frac{1}{2},\frac{p-1}{2})$ [where $B(x,y)$ is the β function]. This distribution contains the Lorentzian for $p = 2$ while it leads to the Gaussian for $p \to \infty$. However, it also covers in a unified manner the general case of so-called Lévy (or Pareto) sweeps, that is, involving a power-tailed probability distribution $P(s) \sim s^{-p}$ as $|s| \to \infty$ for all values of $1 < p < 3$. Such densities are also known, for example, in cosmology [for the distribution of galaxies (see, e.g., Ref. [26]), where $p \approx 2.25$] and in economy [for the distribution of income (see, e.g., Ref. [27]), where $p \approx 2.5$].

Using the distribution (5.7) in Eq. (5.6)—rescaling both $s \Rightarrow s\sqrt{\frac{p}{2}}$ and $\theta \Rightarrow \theta\sqrt{\frac{p}{2}}$ and letting $\tau = 1/s$—it is worth writing the sine integral as a Meijer's G function (see, e.g., Ref. [28]). Letting $\tau = \sqrt{x}$, the ensuing x integral has been tabulated (e.g., in Ref. [28] p. 898 formula 5). Finally, putting $\theta = \sqrt{x}$ and again using a tabulated formula (e.g., Ref. [28] p. 897 formula 4), the upshot for the coefficient β_{L} reads

$$\beta_{\mathsf{L}} = \frac{1}{6}\frac{p}{p-3}|\overline{U}_o'|^2 u_*^{\rm s\,2}. \quad (5.8)$$

Now consider the 'nonlinear' response contributions in the self-consistency equation (5.2), arising from expanding the Bessel function $j_o(k\phi/u_2^{\rm s})$ in powers of $k^{-1/3}$, which requires a 'nonlinear' mean velocity profile, that is, $\overline{U}_o^{(n)} \neq 0$ for $n \geqslant 2$. Through second order there will be one contribution of $O(k^{-1/3})$ and two of $O(k^{-2/3})$. As for the linear response [see above Eq. (5.4)], one has $\mathcal{E}'|_{\mathsf{N},1m} = 0$ by ϑ symmetry while the coefficients $\mathsf{N}_{2m}(\varpi)$ in the second-order contributions

$$\mathcal{E}'|_{\mathsf{N},21} = \tfrac{1}{3}\varpi\mathcal{E}'\overline{U}_o'''\mathsf{N}_{21}, \quad \mathcal{E}'|_{\mathsf{N},22} = \tfrac{1}{2}\varpi\mathcal{E}'|\overline{U}_o''|^2\mathsf{N}_{22}, \quad (5.9)$$

follow from Eq. (5.2) by noticing that $j_o^{(m)}(2\sigma\zeta\overline{U}_o'/u_2^{\rm s}) = (u_2^{\rm s}/2\sigma\overline{U}_o')^m\partial^m j_o(z)/\partial\zeta^m$, doing m partial ζ integrations, and once more invoking the harmonic properties of $e^{ix\xi}$. Also using $(i\sigma)^n e^{ix\sigma} = \partial^n e^{ix\sigma}/\partial x^n$ (with $x = \vartheta/u_2^{\rm s}$), again applying Eq. (5.5) for the σ integral and once more noticing that only $\sin(2\omega\zeta/u_2^{\rm s})$ contributes to the real part of the spectrum function, one obtains $\mathsf{N}_{21}(\varpi) = k^{-2/3}\varpi/\overline{U}_o'^{\,3}$ and $\mathsf{N}_{22}(\varpi) = -3k^{-2/3}\varpi/2\overline{U}_o'^{\,4}$. In contrast to the linear response terms, these nonlinear response contributions do not depend on the moments of the sweeps distribution, that is, they hold for any normalizable density $P(s)$.

It remains to consider "mixed" response contributions, that is, terms arising from Eq (5.2) as products of expanding both the exponential and the Bessel function. Through $O(k^{-2/3})$ there will be two such contributions which, however, are both identically zero (see Ref. [25]).

Using the above results for $\mathsf{L}_{nm}(\varpi)$ and $\mathsf{N}_{nm}(\varpi)$ in Eqs. (5.4) and (5.9) for the $\mathcal{E}'|_{nm}$, the differential form of the self-consistency equation (5.1) becomes

$$\beta_{\mathsf{L}}\left(\frac{1}{\varpi^3}\frac{\partial^2\varpi\mathcal{E}'}{\partial\varpi^2} - 3\frac{1}{\varpi^4}\frac{\partial\varpi\mathcal{E}'}{\partial\varpi} + 3\frac{1}{\varpi^4}\mathcal{E}'\right) = \beta_{\mathsf{N}}\varpi^2\mathcal{E}', \quad (5.10)$$

where

$$\beta_{\mathsf{N}} = \frac{1}{3}\frac{\overline{U}_o'''/\overline{U}_o' - \frac{9}{4}|\overline{U}_o''/\overline{U}_o'|^2}{|\overline{U}_o'|^2}. \quad (5.11)$$

Define the nondimensional frequency $\tilde{\omega} = (\beta_{\mathsf{N}}/\beta_{\mathsf{L}})^{1/6}\varpi$, go to the variable $\xi = \tilde{\omega}^3/3$, and put $\mathcal{E}' = k^{\mu}\xi^{1/3}\mathcal{G}'(\xi)$ [with $\mu = -(1+2\lambda)$, where $\lambda = \frac{2}{3}$ is the time scaling exponent as defined in Eq. (4.2)]. Equation (5.10) then yields the modified Bessel equation of order $\frac{1}{3}$ for $\mathcal{G}'(\xi)$. The physical solution is $\mathcal{G}' = \mathcal{C}_o K_{1/3}(\xi)$, where $\mathcal{C}_o$ is an as yet undetermined constant. Hence the ensuing isotropic 4D energy spectrum reads

$$\mathcal{E}'(k,\tilde{\omega}) = \mathcal{C}_o\,k^{-7/3}\xi^{1/3}K_{1/3}(\xi), \quad (5.12)$$

with $\xi = \tilde{\omega}^3/3$ and

$$\tilde{\omega} = \left(\frac{\beta_{\mathsf{N}}}{\beta_{\mathsf{L}}}\right)^{1/6}\frac{\Delta}{k^{2/3}}, \quad (5.13)$$

where Δ is the unswept Eulerian frequency defined in Eq. (3.4). The modified Bessel (or Airy) spectrum function according to Eq. (5.12) is plotted in Fig. 1.

VI. FINAL REMARKS

The self-consistency equation (5.10)—leading to the energy spectrum (5.12)—has emerged as a balance between linear and nonlinear response.

The linear response describes those changes in the small-scale Euclidean energy spectrum $\mathcal{E}'(k,\Delta)$ which arise from the variation in the unswept frequency Δ due to the random large-scale sweeping of a blob of turbulent fluid across a mean velocity profile $\overline{U}(y)$. For such changes to occur, a nonzero value of the linear mean shear is both necessary and sufficient—as seen in Eq. (5.8) for the coefficient β_{L}.

On their own these linear response changes can not be undone so as to retrieve the original spectrum function upon averaging over all possible sweeps and orientations. In fact, for a perfectly linear mean velocity profile a self-consistent nonzero turbulence energy spectrum does not exist. However,

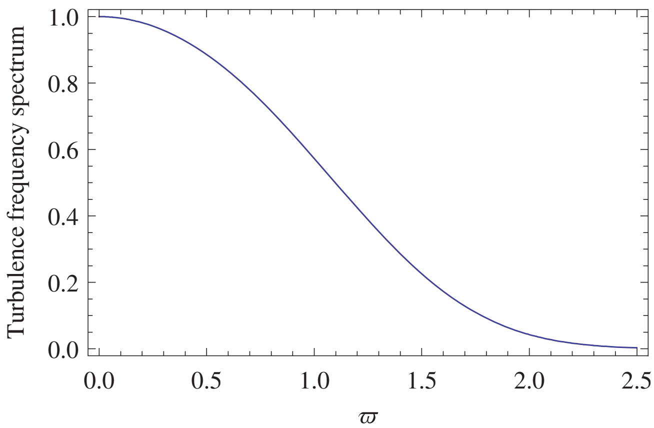

FIG. 1. (Color online) Plot of the self-consistent turbulence frequency spectrum in the scaling range according to Eq. (5.12), normalized at zero frequency, as a function of the nondimensional unswept frequency $\tilde{\omega}$ defined in Eq. (5.13).

the linear response variation of the spectrum is balanced by nonlinear response effects whenever the mean velocity profile is nonlinear—as seen in Eq. (5.11) for the coefficient β_{N}.

While a nonzero value of both β_{L} and β_{N} is thus found to be necessary for the emergence of a nontrivial normal scaling solution of the self-consistency equation, this is not sufficient for that solution to represent an energy spectrum. Namely the ensuing spectrum Eq. (5.12) is positive definite only if the frequency variable $\xi = \varpi^3/3$ is real, which by Eq. (5.13) requires that $\beta_{\mathsf{N}}/\beta_{\mathsf{L}} > 0$.

A nondimensional determinant for the character of the mean velocity profile is defined by $\mathsf{D} = -u_{\mathrm{ref}}^2\beta_{\mathsf{N}}$, with u_{ref} being a reference velocity. The case $\mathsf{D}(y) \equiv 0$ is satisfied for either $\overline{U} = y$ or for the special power law profile $\overline{U} = y^{1/5}$ (in dimensionless units). For incompressible, fully developed turbulent flow over a single infinitely extended flat wall—which is the flow described by the Navier-Stokes Eqs. (2.1)—the mean velocity field is generally taken to be Von Kármán's famous logarithmic profile (see, e.g., Refs. [11,29]).

In view of the ongoing debate about the possibility of a power law profile (see, e.g., Refs. [6,7,30]) putting $\overline{U}'(y_+) = (u_*/\kappa_q)y_+^{q-1}$ (where $y_+ = u_* y/\nu$) one finds that $\mathsf{D} < 0$ for $\frac{1}{5} < q < 1$ while $\mathsf{D} > 0$ for all other values of q. Since there is by now general consensus that q is likely to be in the range $0 < q < \frac{1}{5}$, the generic case will be $\mathsf{D} > 0$—which corresponds to Lévy sweeps with $1 < p < 3$. Taking $u_{\mathrm{ref}} = u_*/\kappa_q$, one gets $\mathsf{D}(y) = \frac{1}{4}(1-q)(1-5q)y_+^{-2q}$. For the logarithmic profile $\mathsf{D} = 1/4$.

The above result for $\mathsf{D}(y)$ needs qualification when the boundary layer has a finite thickness, for example, for pipe or channel flow. In that case a so-called wake region exists in the core of the flow. This is usually modeled by an empirical wake function $\mathcal{W}(y)$ (see, e.g., Refs. [6,7,11]), which is reasonably well studied only for the logarithmic profile and tends to strongly suppress the value of $\mathsf{D}(y)$ in the region where the wake effects set in (i.e., adjacent to but not in the core) [25]. For $D = 0$, an anomalous scaling spectrum emerges (see Appendix B).

Using $\varepsilon = u_*^2|\overline{U}_o'|$ for the local energy production rate (see, e.g., Ref. [11]), the frequency-integrated Kolmogorov energy spectrum $E(k) = \int_{-\infty}^{\infty}\mathcal{E}^\iota(k,\Delta)d\Delta$ becomes $E(k) = \alpha\,\varepsilon^{2/3}k^{-5/3}$, with $\alpha \approx \mathcal{C}_o\sqrt{\pi}(\frac{12}{\varepsilon})^{1/3}\Gamma(\frac{7}{6})$ for $\kappa_q \approx 0.4$, $p \approx 2$, $\mathsf{D} \approx 1$, and $u_*^{\mathrm{s}} \approx 10u_*$. Taking $\alpha \approx 1.5$ from the experimental data, this yields $\mathcal{C}_o \approx 0.5\,\varepsilon^{1/3}$ in agreement with the value obtained from the Reynolds stress in Appendix C. Of course, an *ab initio* calculation of $\mathcal{C}_o$ can only be done after including bulk viscosity effects into the theory.

Finally, it is worth noticing that higher order corrections to the irreducible differential form of the self-consistency equation (5.10) rapidly increase in complexity. Such contributions arise from (i) the terms neglected in the Navier-Stokes equations (2.4) [see below Eq. (2.8)], (ii) the disregarded boundary term $\mathcal{S}_{vv}^{[1]}$ in the response spectrum [see below Eq. (3.1)], (iii) the further iteration of the full nonlinear source terms (2.7) before closing the hierarchy at the level of isotropic fields (see Sec. IV), and (iv) higher order terms in the asymptotic evaluation of the self-consistency integral equation (5.1) (see Sec. V, and Ref. [25]).

APPENDIX A: THE LOGNORMAL MODEL

Equation (4.2) connects the unswept second-order isotropic 4D spectrum $\mathcal{E}_2^\iota \equiv \mathcal{E}^\iota$ with the third-order spectrum function $\mathcal{S}^\iota$ as $\mathcal{E}_3^\iota \equiv 4\pi k^5\mathcal{S}^\iota$ (see, e.g., Ref. [13] and [24] Sec. 3.3). Its basic structure, that is, $\mathcal{E}_n^\iota \sim (\Delta/k)\,\mathcal{E}_{n-1}^\iota$ with $n = 3$, leads to $\mathcal{E}_n^\iota \sim (\Delta/k)^{n-3}\mathcal{E}_3^\iota$ for arbitrary n. Hence, one gets $\mathcal{E}_n^\iota(k,\delta) \sim k^{\mu_n}\mathsf{F}_n(\delta)$, where $\mathsf{F}_n(\delta) = \delta^{n-2}\mathsf{F}(\delta)$ with $\delta = \Delta/\varepsilon_\lambda^{1/3}k^\lambda$, and $\mu_n = 1 - n + \lambda(n-4)$. Integrating over all frequencies then yields $E_n^\iota(k) \sim k^{\lambda+\mu_n}$, and using $\mathcal{D}_n^\iota(\ell) \sim [kE_n^\iota(k)]|_{k\Rightarrow 1/\ell}$ for the nth-order static structure function $\mathcal{D}_n^\iota(\ell) = \langle[u(\ell)-u]_\parallel^n\rangle$ yields $\mathcal{D}_n^\iota(\ell) \sim \ell^{\zeta_n}$, with $\zeta_n = 1 + (1-\lambda)(n-3)$, which is equivalent to the so-called β model (with fractal dimension $D_\beta = 5 - 3\lambda$; see, e.g., Ref. [4]).

Going beyond single-time scaling by letting $\lambda \Rightarrow \lambda_n$, one has $\mathcal{D}_n(\ell) \sim \ell^{\zeta_n}$ with

$$\zeta_n = 1 + (1-\lambda_n)(n-3), \tag{A1}$$

which implies a characteristic velocity $u_{\mathrm{a}} \sim \ell^{\zeta_n/n}$ of the turbulent eddies which are active in the nth-order correlations. On the other hand, there also exists the eddy spectral group velocity $u_{\mathrm{g}} = \partial\Delta/\partial k|_{k\to 1/\ell}$, that is, $u_{\mathrm{g}} \sim \ell^{1-\lambda_n}$. Obviously putting $u_{\mathrm{a}} \equiv u_{\mathrm{g}}$ at once yields the K41-result $\lambda_n = 2/3$. However, in general $u_{\mathrm{a}} < u_{\mathrm{g}}$ for $\lambda_n > 2/3$. Consequently, during a time lapse τ the turbulent eddies typically sample a volume $V_{\mathrm{g}}(\ell) \sim (u_{\mathrm{g}}\tau)^d \sim \ell^{(1-\lambda_n)d}$ (where d is the dimension of the embedding space, that is., $d = 3$ for 3D turbulence), while they are active only in a volume $V_{\mathrm{a}}(\ell) \sim (u_{\mathrm{a}}\tau)^d \sim \ell^{\zeta_n d/n}$, with $V_{\mathrm{a}} < V_{\mathrm{g}}$ (for $\lambda_n > \frac{2}{3}$). So active eddies of size ℓ fill only a fraction $p_\ell = (\ell/\ell_o)^{(d-D_\lambda)}$ of the volume (where ℓ_o is an external reference scale and where D_λ can be interpreted as a fractal dimension; see, e.g., Refs. [4,26]), that is, $V_{\mathrm{a}}/V_{\mathrm{g}} = (\ell/\ell_o)^{d-D_\lambda}$. Equating powers of ℓ gives $\zeta_n/n + (\lambda_n - 1) = 1 - D_\lambda/d$. Using Eq. (A1) for ζ_n then gives

$$\lambda_n = \frac{2}{3} + \frac{n}{3}\left(1 - \frac{D_\lambda}{d}\right), \tag{A2}$$

which yields

$$\zeta_n = \frac{n}{3}\left[1-(n-3)\left(1-\frac{D_\lambda}{d}\right)\right]. \tag{A3}$$

Writing Eq. (A3) as $\zeta_n = \frac{n}{3}[1-\frac{1}{6}(n-3)\mu_{\ln}]$ shows that this result is equivalent to that for the so-called lognormal model, with parameter $\mu_{\ln} \equiv 6(1-D_\lambda/d)$; see, e.g., Ref. [4].

APPENDIX B: ANOMALOUS SCALING

With $\mathsf{D}\equiv 0$, the linear response terms in the self-consistency equation (5.10) can be balanced only by higher order nonlinear terms. Let us therefore reconsider the expansion in Eq. (5.3) for the argument in the Bessel function $j_o(k\phi/u_2^{\rm s})$, which may be rewritten as $\phi = 2\sigma\zeta\overline{U}_o' + \delta\phi$, with

$$\delta\phi = \sum_{n=3}^{\infty}\sum_{m=0}^{n}\frac{1}{n!}\binom{n}{m}\sigma^{n-m}[\zeta^m-(-\zeta)^m]\overline{U}_o^{(n-1)}. \tag{B1}$$

With the scalings $\sigma \sim k^{\lambda-1}$ and $\zeta \sim k^{-\lambda}$ (see Sec. V), the general term in $k\delta\phi$ becomes of the order of $k^{(n-m-1)(\lambda-1)-(m-1)\lambda}$, which is easily rewritten as $k^{(n-2)(\lambda-1)-(m-1)(2\lambda-1)}$. So the expansion is in terms of increasing powers of $1/k$ for $\lambda > 1/2$, and the leading term is asymptotically small for all $\lambda < 1$. For $n=$ odd, the odd index m runs from $m=1$ through $m=n$. For $n=$ even, one has $m=1$ through $m=n-1$. Putting for the purpose of these preliminary considerations $u_2^{\rm s}\equiv 1$, $j_o(k\phi)$ can now be expanded as

$$j_o(k\phi) = j_o(z) + \sum_{l=1}^{\infty}\frac{1}{l!}(k\delta\phi)^l j_o^{(l)}(z), \tag{B2}$$

where $z = 2k\sigma\zeta\overline{U}_o'$.

A. $l=$ odd. : Here only odd powers of σ contribute to the response integrals, because $j_o^{(l)}(z)$ is odd in σ for odd values of l (and since there are no ϑ powers in the nonlinear response terms per se). Hence, keeping in mind that always $k\sigma\zeta\ O(1)$, a general term in $(\delta\phi)^l$ that is, $k^l\sigma^{N-M}\zeta^M$, will be of the order $\sigma^{N-M-l}\zeta^{M-l}$, where $N=\sum_{i=1}^{l} n_i$ and $M=\sum_{i=1}^{l} m_i$, with $n_i\in(3,\infty)$ and $m_i=$ odd$\in(1,n_i)$, and with $N-M$ being odd. Therefore, $M=l,l+2,\ldots,M_{\max}$ runs here through odd values only and the relevant N values will be even, that is, $N=3l+1,3l+3,\ldots$. Possible time scaling exponents $\lambda^{[l]}$ then follow from the balancing equality $2-4\lambda=(N-M-l)(\lambda-1)-(M-l)\lambda$, which leads to $\lambda^{[l]}=(N-M-l+2)/(N-2M+4)$. This implies that $\partial\lambda^{[l]}/\partial M>0$ for all values of M (and $N\geqslant 3l+1$), so that $\lambda^{[l]}$ has its minimum value for $M=l$. Moreover, one easily verifies that any $M\geqslant l+2$ yields a $\lambda^{[l]}\geqslant 1$, which is an invalid scaling exponent. Therefore, one is left with $\lambda^{[l]}=(N-2l+2)/(N-2l+4)$, that is,

$$\lambda^{[l=\text{odd}]} = \frac{l+3}{l+5},\frac{l+5}{l+7},\ldots. \tag{B3}$$

B. $l=$ even. : Because $j_o^{(l)}(z)$ is even in σ for even values of l, only even σ powers need to be considered in this case. Clearly, $M=l,l+2,\ldots,M_{\max}$ now runs through even numbers only, and with $N-M$ being even one finds that the relevant N values are here given by $3l,3l+2,\ldots$. Again (i.e., as for $l=$ odd) one has $\lambda^{[l]}=(N-M-l+2)/(N-2M+4)$ and $\partial\lambda^{[l]}/\partial M>0$ for all values of M (and $N\geqslant 3l$), while once more any value of $M\geqslant l+2$ produces an invalid $\lambda^{[l]}\geqslant 1$. Hence (i.e., with $M=l$), one again arrives at $\lambda^{[l]}=(N-2l+2)/(N-2l+4)$, which now yields

$$\lambda^{[l=\text{even}]} = \frac{l+2}{l+4},\frac{l+4}{l+6},\ldots. \tag{B4}$$

Hence an energy spectrum with anomalous scaling exponent $\lambda=\frac{3}{4}$ arises (if $D\equiv 0$) from nonlinear response contributions with $l=1$ (i.e., for $M=1$, $N=6$); $l=2$ (for $M=2,N=8$); $l=3$ (for $M=3,N=10$); and $l=4$ (for $M=4,N=12$).

The corresponding response contributions read

$$\mathcal{E}'|_{\mathsf{N},m}^{\mathsf{D}=0} = \frac{\alpha_m}{m!}\varpi\mathcal{E}'\mathsf{N}_m^{\mathsf{D}=0}, \tag{B5}$$

with $\alpha_1=\frac{1}{60}\overline{U}_o^{(5)}$, $\alpha_2=\frac{1}{6}\overline{U}_o^{(2)}\overline{U}_o^{(4)}+\frac{1}{9}\overline{U}_o^{(3)\,2}$, $\alpha_3=\overline{U}_o^{(2)\,2}\,\overline{U}_o^{(3)}$, $\alpha_4=\overline{U}_o^{(2)\,4}$, $\mathsf{N}_1^{\mathsf{D}=0}=\frac{2}{3}k^{-1}\varpi^3/\overline{U}_o^{\prime\,5}$, $\mathsf{N}_2^{\mathsf{D}=0}=-\frac{5}{3}k^{-1}\varpi^3/\overline{U}_o^{\prime\,6}$, $\mathsf{N}_3^{\mathsf{D}=0}=5k^{-1}\varpi^3/\overline{U}_o^{\prime\,7}$, $\mathsf{N}_4^{\mathsf{D}=0}=-\frac{35}{2}k^{-1}\varpi^3/\overline{U}_o^{\prime\,8}$.

Using the above results for $\mathsf{N}_m^{\mathsf{D}=0}(\varpi)$ in Eq. (B5) for the $\mathcal{E}'|_{\mathsf{N},m}^{\mathsf{D}=0}$ and summing over m, the ensuing anomalous self-consistency equation reads

$$\beta_{\mathsf{L}}\left(\frac{1}{\varpi^3}\frac{\partial^2\varpi\mathcal{E}'}{\partial\varpi^2}-3\frac{1}{\varpi^4}\frac{\partial\varpi\mathcal{E}'}{\partial\varpi}+3\frac{1}{\varpi^4}\mathcal{E}'\right)=\beta_{\mathsf{N}}^{\mathsf{D}=0}\varpi^4\mathcal{E}', \tag{B6}$$

where $\beta_{\mathsf{N}}^{\mathsf{D}=0}=\sum_{m=1}^{4}\frac{1}{m!}\alpha_m[k\mathsf{N}_m^{\mathsf{D}=0}/\varpi^3]$. A nondimensional frequency is defined by $\tilde{\omega}=(\beta_{\mathsf{N}}^{\mathsf{D}=0}/\beta_{\mathsf{L}})^{1/8}\varpi$. Introducing the variable $\xi=\tilde{\omega}^4/4$, and again (i.e., mutatis mutandis, as in Sec. V) putting $\mathcal{E}'=k^{\mu}\xi^{1/4}\mathcal{G}'(\xi)$ [with $\mu=-(1+2\lambda)$, where now $\lambda=\frac{3}{4}$], Eq. (B6) leads to the modified Bessel equation of order $\frac{1}{4}$ for $\mathcal{G}'(\xi)$. The physical solution is $\mathcal{G}'=\mathcal{C}_o^{\mathsf{D}=0}K_{1/4}(\xi)$, and the ensuing anomalous scaling isotropic 4D energy spectrum reads

$$\mathcal{E}'(k,\tilde{\omega})=\mathcal{C}_o^{\mathsf{D}=0}k^{-5/2}\xi^{1/4}K_{1/4}(\xi), \tag{B7}$$

with $\xi=\tilde{\omega}^4/4$, and

$$\tilde{\omega}=\left(\frac{\beta_{\mathsf{N}}^{\mathsf{D}=0}}{\beta_{\mathsf{L}}}\right)^{1/8}\frac{\Delta}{k^{3/4}}, \tag{B8}$$

Strictly speaking, $\mathsf{D}(y)\equiv 0$ only for the nontrivial special profile $\overline{U}=y^{1/5}$ (see Sec. VI). This leads to the value $\beta_{\mathsf{N}}^{\mathsf{D}=0}=-4/9375$, which is of the same sign as the normal coefficient β_{N} [defined in Eq. (5.11)] for all $q<\frac{1}{5}$. That is, both the logarithmic profile and the $\mathsf{D}\equiv 0$ profile $\overline{U}=y^{1/5}$ coexist for Lévy sweeps with $1<p<3$.

Finally, it worth noticing that there are no mixed response contributions that would spoil Eq. (B6) (see Ref. [25]).

APPENDIX C: REYNOLDS STRESS

The off-diagonal spectra have been studied in Sec. IV [see, e.g., Eqs. (4.3) and (4.4)]. Using the expansion of $\mathcal{E}'(\bar{k},\bar{\Delta})$ about (k,Δ), first of all notice that the term due to $\mathcal{E}'(k,\Delta)$ itself vanishes under orientational averaging [because

$\langle k_{i\neq j}k_j f(k_1)\rangle_{\rm sph}=0$]. Indeed, for perfectly isotropic turbulence $\langle \mathcal{S}_{i\neq j}\rangle_{\rm sph}=0$. The same holds for the terms involving $\partial/\partial\Delta$ only. Nonzero contributions will come from terms containing a $\partial/\partial k_2$ [arising since $\{\bar{k}_i\}=(k_1,\bar{k}_2,k_3)$]. One readily verifies that $\langle k_{i\neq j}k_j k_2 f(k_1)\rangle_{\rm sph}=0$ unless $\{i,j\}=\{1,2\}$. Defining the Reynolds stress spectrum function $\mathcal{T}(k,\Delta)$ by $\langle \mathcal{S}_{uv}\rangle=-\mathcal{T}/4\pi k^2$, and noticing that the spherical average only involves k_1 and k_2 (so that it simplifies, mutatis mutandis, as in Sec. V), one obtains $\mathcal{T}=\sum_{n=1}^{2}\mathcal{T}_n$, with

$$\mathcal{T}_n=-\frac{k^2\overline{U}_o'}{\pi}\int_{-\infty}^{\infty}d\bar{\Delta}\int_0^{\infty}d\sigma\int_0^{\infty}d\zeta \times\left\langle\frac{\cos[(\bar{\Delta}-\Delta)\sigma/u_2^{\rm s}]\cos[(\bar{\Delta}+\Delta)\zeta/u_2^{\rm s}]}{|u_2^{\rm s}|u_2^{\rm s}}j_o^{(2n)}(z)\right\rangle \times\frac{\partial\bar{\Delta}\mathcal{D}_n\bar{\mathcal{E}}^{\iota}}{\partial\bar{\Delta}}. \tag{C1}$$

where $j_o^{(n)}(z)$ is the nth derivative of the spherical Bessel function (with $z=2k\sigma\zeta\overline{U}_o'/u_2^{\rm s}$) while the operator $\mathcal{D}_n$ is defined by $\mathcal{D}_n\bar{\mathcal{E}}^{\iota}=(\delta_{n1}+\frac{1}{2}k\partial/\partial k)[\mathcal{E}^{\iota}(k,\bar{\Delta})/k^2]$. The remaining brackets in Eq. (C1) stand for averaging over sweeps only.

For very weak mean shearing, that is, $\overline{U}_o'\to 0$ so that $z\approx 0$, both σ and ζ integral yield a δ function, and one gets ($s\equiv u_2^{\rm s}/u_*^{\rm s}$)

$$\mathcal{T}_n=\pi k^2\overline{U}_o'\frac{(-1)^{n+1}}{2n+1}\left\langle\frac{s}{|s|}\right\rangle\delta(\Delta)\mathcal{D}_n\mathcal{E}^{\iota}. \tag{C2}$$

Summing up the contributions for $n=1,2$ and using the energy spectrum found in Sec. V, the basic upshot reads

$$\mathcal{T}(k,\Delta)=\mathcal{C}_o'\overline{U}_o'k^{-7/3}\delta(\Delta), \tag{C3}$$

where $\mathcal{C}_o'=\frac{2\pi}{45}[\Gamma(\frac{1}{3})/2^{2/3}]\langle{\rm sign}\, s\rangle\mathcal{C}_o$. The usually observed frequency-integrated stress spectrum thus becomes $T(k)=\mathcal{C}_o'\overline{U}_o'k^{-7/3}$, which confirms Lumley's conjecture (see, e.g., Refs. [12,31]) based on kinematical considerations and dimensional analysis à la Kolmogorov K41.

For z $O(1)$, the $\delta(\Delta)$ in Eq. (C3) becomes a function of the nondimensional frequency $\hat{\omega}=\Delta/|ku_*^{\rm s}\overline{U}_o'|^{1/2}$—which is indeed a sharply peaked function on the frequency scale $\Delta\sim k^{2/3}$ of the energy spectrum. To study the exact form of this function, various skewed sweep densities are discussed in Ref. [25]. In particular, applying Lévy's stable distributions (see, e.g., Refs. [27,32]) expression for $P_{\rm odd}(s)$ as a Hilbert-type transform of $P_{\rm even}(s)$ and using Eq. (5.7) for the latter, all integrations can be done analytically in terms of Meijer G functions. It is worth noticing that the ensuing stress spectrum changes sign near $\hat{\omega}\approx 2$.

Taking tentatively $\langle{\rm sign}\, s\rangle\approx\frac{3}{5}$ for the skewness (see, e.g., Ref. [6] p. 666) and comparing the present $T(k)$ with the Reynolds stress spectrum measured in the atmospheric surface layer (see, e.g., Ref. [12]), one obtains $\mathcal{C}_o\approx\frac{75}{2\pi}[2^{2/3}/\Gamma(\frac{1}{3})]\zeta_1(0)\varepsilon^{1/3}$ for the coefficient in the isotropic energy spectrum (5.12), where $\zeta_1(0)\approx 0.1$ is an experimentally determined coefficient. This amounts to $\mathcal{C}_o\approx 0.5\varepsilon^{1/3}$, as in Sec. VI.

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