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A NEW LEARNING TRAJECTORY FOR TRIGONOMETRIC FUNCTIONS

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Educational research has shown that many secondary school students consider the subject of trigonometric functions as difficult and only develop shallow and fragmented understanding. It is unclear which of the two popular approaches to introducing trigonometry, namely the ratio method and the unit circle method, works best. In this study we propose a new framework for trigonometric understanding and a new, dynamic geometry supported trajectory for learning trigonometric functions. We also report on the results of a classroom case study in which the new approach has been implemented and researched. We discuss the task-related difficulties that students faced in their concept development and we describe their trigonometric understanding in terms of our framework.

BACKGROUND

In the triangular geometry, the sine and cosine of an acute angle are defined as ratios of pairs of sides of a right triangle. This is referred to as the ratio method of introducing trigonometric functions. The right triangle is often embedded in the unit circle, but then the notion of angle actually gets the meaning of rotation angle. The sine and cosine of a particular angle are now defined as the horizontal and vertical coordinate of a point obtained by rotating the point (1,0) about the origin over the angle. This is called the unit circle method. It is unclear which approach works best. In practice, a combination of these approaches is often applied. In both approaches, the trigonometric functions are functions of an angle and not of a real number. This is more or less repaired by introducing the radian. It is important to make clear to students that the notion of angle differs in the two approaches: it is in the ratio method an angle of a triangle with values between 0 and 90 degrees, whereas it is in the unit circle method a rotation angle which has both a magnitude and a direction.

The research literature on students' understanding of trigonometric functions is sparse, but most studies conclude that students develop in the aforementioned approaches a shallow and disconnected understanding of trigonometric functions and underlying concepts, and have difficulty using sine and cosine functions defined over the domain of real numbers (Challenger, 2009; Moore, 2010, in press; Weber, 2005, 2008). The sine and cosine functions may have been defined, but the graphs of these real functions remain mysterious or merely diagrams produced by a graphing calculator or mathematics software. The complex nature of trigonometry makes it challenging for students to understand the topic deeply and conceptually.

In this paper we present a model of trigonometric understanding. It played an important role in the design of a new instructional approach to sine and cosine functions and in the analysis of the data regarding students' understanding of these functions that were collected in a classroom case study.

A MODEL OF TRIGONOMETRIC UNDERSTANDING

Our model of trigonometric understanding is based on a conceptual analysis of mathematical ideas within and among three contexts of trigonometry, namely triangle trigonometry, unit circle trigonometry, and trigonometric function graphs (Figure 1). The conceptual analysis of trigonometry based on angle measure by Thompson (2008) was a source of inspiration: we also strive for coherence between mathematical meanings at various levels of trigonometry. But our approach differs in two aspects: (1) we use the functional relationship between arc length and corresponding vertical position for the case of sine in the unit circle prior to angle measure; (2) our model is broader than Thompson's example using angle measure in the sense that it includes trigonometric functions in the domain of real numbers. We presume that it is easier for students to study first covariation of quantities having the same unit for measurement — in our approach a relationship between path length and displacement along coordinate axes — than to start with a function from angle measure (degrees or radians) to length measure (using the length of the hypotenuse of a right triangle or the radius of a circle as a unit).

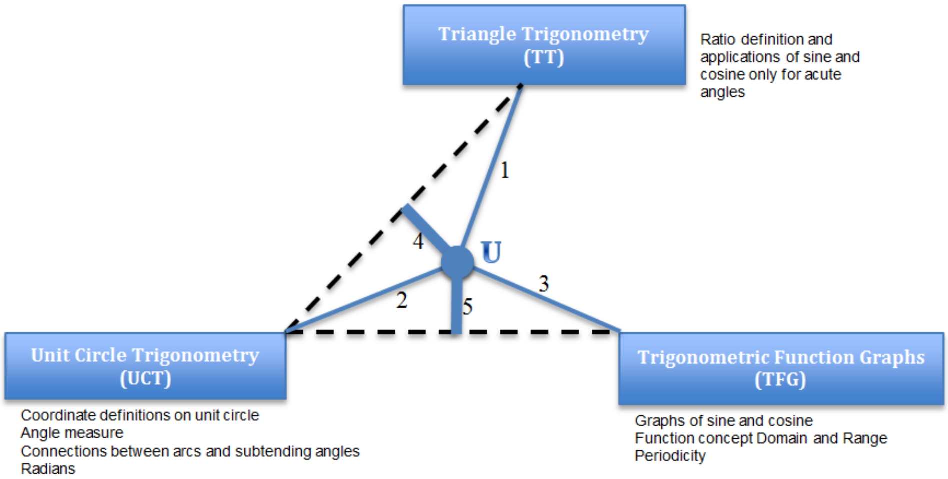

Figure 1: A model of trigonometric understanding

The contexts TT, UCT, and TFG represent three contexts in which trigonometry can be partially understood, while the central point U represents the desired trigonometric understanding of students. The numbered line segments indicate that trigonometric understanding should entail aspects in the three contexts and the connections among them. It is important to keep in mind that point U should not be considered static. It may have different places in between the three contexts with respect to the quality of different students' understanding because different tasks may require different aspects of trigonometric understanding. The line segments labelled 1, 2 and 3 represent understanding of different aspects within three different contexts. These aspects are not only about factual knowledge, but also concern the students' ability to elaborate on them. A deeper level of understanding is represented by the thicker line segments 4 and 5 in Figure 1, which represent understanding the connections among the contexts represented by the dashed lines. Various aspects were hypothesized as important for students' trigonometric understanding: they are listed in Table 1 and formed the basis of the design of a hypothetical learning trajectory of trigonometric functions.

segment	*context or connection*	*Aspects of trigonometric understanding*
1	Triangle (TT)	Ratio definition of sine and cosine in a right triangle, and applications
2	Unit circle (UCT)	Coordinate definitions of sine and cosine, and applications with angles or real numbers
		Application of the coordinate definitions to evaluate sine and cosine for certain inputs
		Trigonometric relationships deducible from the unit circle, e.g. sine is an odd function
		Relationship between arcs and subtended angles through the notion of radian
		Converting between degrees and radians via proportionality
3	Graphs (TFG)	Connection of the sine and cosine graph with real functions
		Functional properties of trigonometric graphs, e.g., domain and range of functions
		Trigonometric relation revealed by the graphs, e.g., even and odd property of functions
4	Connection TT-UCT	Integration of the ratio method with the unit circle method via reference triangles, affording trigonometric values of angles larger than 90 degrees
5	Connection UCT-TFG	Construction and interpretation of trigonometric graphs
		Explanation of properties of trigonometric functions

Table 1: Aspects of the new model of trigonometric understanding

A NEW LEARNING TRAJECTORY FOR TRIGONOMETRY

The basic idea in the new approach is to avoid an early introduction of radians as angle measure, but instead make the winding function of the real line onto the unit circle the principal concept and use the concept of arc and arc length to introduce sine and cosine as real functions. We presume that, when the sine function is introduced, it is easier for secondary students to consider a function with distance as input and height as output than to consider a functional relationship between angle measure and height. The nature and unit of the involved quantities are in this case the same. Also it is important in our approach that students can first practice with pencil and paper and develop understanding of the geometric construction of the sine and cosine function. Our reasoning is similar to that of Weber (2005) and Moore (2010): students better experience the application of a particular process and reflect on it before they mentally apply the same process. To get hands-on experience with the process of applying a winding function onto the unit circle and with coordinate functions, students first explore the winding function defined on a regular polygon that circumscribes the unit circle and is oriented in the Cartesian plane such that the point (1,0) is a midpoint of a vertical edge. A point P moves counter-clockwise along the rim of the regular polygon and the covariation of the travelled distance and the vertical position is studied. In case of an n-gon, this leads to a sine-like function s_n. Students begin with the construction of the function s_4 by pencil and paper. Next they explore the same construction by a dynamic geometry package; see Figure 2 for a screenshot of the GeoGebra activity used in class. The purpose of this applet is to promote student construction of knowledge: it is meant to function as a didactic object (Thompson, 2002), that is, as an object to talk about in a way that enables and supports reflective mathematical discourse.

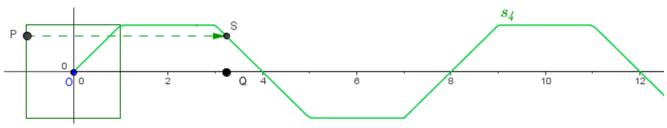

Figure 2: Screenshot of a GeoGebra activity on the graph of the sine-like function s_4.

The next step in the trajectory is to extend the graph over the negative horizontal axis and to make a similar construction of a cosine-like function c_4. Various trigonometric-like properties of the functions s_4 and c_4 can be explored such as $s_4(x+8) = s_4(x)$, $s_4(-x) = -s_4(x)$, and $s_4(2-x) = c_4(x)$, for all x. Also note that by construction $s_4(x) = x$ for small values of x. What can be done for a square can also be done for a pentagon, hexagon, and so on. With great effort, students can explore cases for small values of n by pencil and paper, but for most regular n-gons dynamic geometry software is helpful: see the screenshot of a GeoGebra activity in Figure 3 and 4.

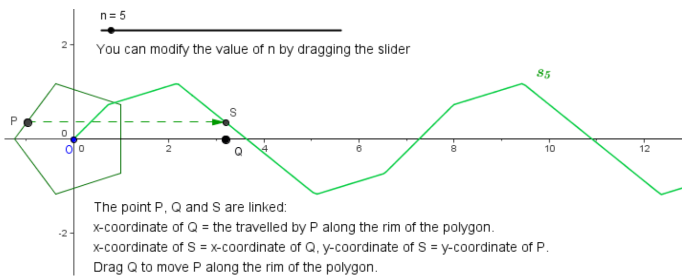

Figure 3: Screenshot of a GeoGebra activity on the graph of the sine-like function s_5.

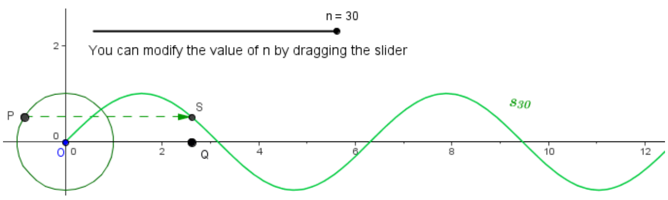

Figure 4: Screenshot of a GeoGebra activity on the graph of the sine-like function s_{30}.

The graph of s_{30} is very smooth and it can hardly be distinguished with the naked eye from the sine graph. Functions s and c can now be introduced by taking limits: $s = \lim_{n\to\infty} s_n$ and $c = \lim_{n\to\infty} c_n$. Of course, this is not done in a formal way at secondary school level. It is only important that the students realize that the winding function for the unit circle can be defined in a way similar to the construction using regular n-gons and that the graphs of s_n and c_n look for large n almost the same as the graphs in case the unit circle is used instead of a regular polygon with many edges. We hope and expect that students develop in this way a process view of the sine- and cosine-like mathematical functions that helps them to understand the mental construction of the sine and cosine functions.

Until this point on the learning trajectory no attention has been paid to rotation angles and no link has been laid with the geometric definition of sine and cosine. Presuming that the students already know that the circumference of the unit circle equals 2π and that a point on the rim of the unit circle is mapped by a rotation of 360 degrees about the origin onto itself, they can find out that a counter-clockwise walk along the rim of the unit circle starting from (1,0) over a distance of x units corresponds with a rotation of $x\cdot 180°/\pi$ about the origin. By drawing a right triangle for a point on the unit circle and using the ratio definition of sine, students can find out that $s(x) = \sin(x\cdot 180°/\pi)$. This formula allows them to compute function values such as $s(\pi)$ and $s(\pi/3)$. The introduction of radian finally boils down to the understanding that an arc of length 1 corresponds with a rotation angle of $180°/\pi$ and that this leads to $s(x) = \sin(x \text{ rad})$. We are close to calling s the sine function and denoting it by sin, too. The introduction of the cosine function can similarly be realized. This finalizes the linking of the unit circle geometry with the triangle geometry.

CLASSROOM CASE STUDY

Based on the model of trigonometric understanding (Figure 1 and Table 1) and the new approach outlined in the previous section, we examined students' concept development and understanding of sine and cosine functions through a classroom case study. Demir (2012) has written a detailed report about this research study. Here we only outline the work and some of the results.

The classroom study was conducted at a secondary school in Amsterdam with a class of 24 pre-university students (17 female and 7 male; age 16-17 yr.), who were classified by their teacher as a high achievement group in mathematics. The first author designed and taught five lessons in class. The cooperating teacher wrote observational notes, translated English mathematical terminology into Dutch when needed, and also helped students during their work. Students' readiness for the instruction was evaluated one week before the start of the instructional sequence through a half-hour diagnostic test. In the lessons, students were encouraged to construct their knowledge through interactions with peers, the researcher (acting as teacher) and the regular teacher. Working in pairs and whole classroom discussions were important elements of the lessons. The grouping of the students in dyads was done on the basis of advice of the cooperating teacher. GeoGebra applets were used as didactic objects in combination with tasks given to students through worksheets.

The research was conducted to find answers to two descriptive research questions: (1) What task-related difficulties do students face in their concept development within the designed instructional sequence based on the new hypothetical learning trajectory of trigonometric functions? (2) What characteristics relating to students' understanding of sine and cosine can be found in the data resulting from the intervention based on the new model of trigonometric understanding

We applied classical research methods for data collection. Participatory classroom observation was an important data source. We held interviews with the cooperating teacher after each lesson to record her impressions of the activities and of how the instructional materials and the ICT tools had functioned. Four semi-structured audio-recorded interviews were held with students after the instructional sequence for the purpose of getting an impression of their understanding of the key mathematical points underlying the instructional sequence. Audio recordings of discussions of students on worksheet tasks gave an impression of their concept development. In each lesson, four dyads were recorded. We collected all completed worksheets of pairs of students. We administered a 50-minutes trigonometry test after the instructional sequence. It was based on our model of trigonometric understanding and designed to assess students' understanding.

Concerning the first research question, the analysis of student's responses to worksheet tasks and the audio recordings of the related group discussions revealed that in general the students were quite successful when working on most tasks. They only faced difficulties in the following tasks regarding their concept development within the instructional sequence: (1) drawing the graph of the vertical position of a moving point along the rim of the unit square against the travelled distance; (2) deriving the formula $s(x) = \sin(x \cdot 180°/\pi)$ of the graph of the vertical position of a moving point on the unit circle plotted against arc length; (3) converting $180°/\pi$ to radians and using it in in the transition to function on real numbers; and (4) calculating the sine and cosine of 210°.

It was clear that the first task was very uncommon to the students. They were only familiar with drawing the graph of a function for which a formula has been given. Once they understood the task, they knew what to do and could determine points on the graph. But then it was difficult to decide how to connect these points: by line segments or curved segments? Having seen many smooth graphs in their school career, students tended to do the same in this task. The difficulties in deriving the formula of $s(x)$ had to do with proportional reasoning to connect arcs and subtended angles, and with the required movement from specific cases toward a general case expressed via a mathematical formula that involves variables. Group discussions revealed that few students could link $180°/\pi$ with the notion of radian as angle measure. The radian concept was probably at this stage in the instructional sequence not fully understood yet and the task was therefore too challenging for the students. Later on in the learning sequence, many obstacles with the conversion between degrees and radians disappeared. The difficulty of calculating sin(210°) and cos(210°) had two sources: (1) some students did not recall the related values of sin(30°) or the method how to compute them; and (2) many students could not figure out how to link triangle trigonometry with unit circle trigonometry. Whole classroom discussed helped students overcome these difficulties.

Concerning the second research question about the students' integrated understanding of trigonometry after the instructional sequence, several findings could be derived from the interviews with students and the trigonometry test. Some of them are discussed below.

In general, the students developed a good level of understanding of aspects in the unit circle context. They were able to evaluate trigonometric functions of real numbers by associating them to an arc on the unit circle and they understood the notion of radian well enough for the construction of knowledge of trigonometric functions based on the fact that the value of the angle measured in radians is equal to the arc length on the unit circle. Although many students understood trigonometric relationship for specific angles, most students were not able not prove a trigonometric equality when expressed in algebraic format.

For the context of trigonometric function graphs, we found that the students conceptualized sine and cosine as functions of real numbers, and that they grasped how to interpret the graphs in terms of domain, range, periodicity, and symmetry properties.

From the trigonometry test and the interviews with students, we concluded that the students showed good understanding of how to integrate right triangle definitions of sine and cosine with the unit circle method in order to calculate trigonometric values of angles larger than 90°. However, most students could not calculate trigonometric values of well-known angles like 30°, 45°, and 60° because they had not memorized these special values, nor the methods to find the values. Nevertheless, they seemed to understand the connections between triangle and unit circle contexts

Students also developed a deep understanding between the unit circle context and the graph context. The most remarkable finding was that the students continued to base their understanding of such connections on arcs. This was found on many occasions. Concerning the construction and comprehension of trigonometric graphs, it was found that students conceptualized trigonometric graphs through the arc length on the unit circle. They explained coordinates of points on the graph with a journey metaphor based on arc length as travelled distances and they related the direction of the movement with the sign of an angle or the sign of a real number. Furthermore, students could explain the shape of trigonometric graphs.

CONCLUSION

We proposed the use of arcs of a unit circle to serve as glue between the unit circle trigonometry and trigonometric function graphs, and the use of the metaphor of travelling along the rim of a geometric object like a regular polygon or a circle to help students develop coherent meanings based on arcs of a unit circle. Angle measure was addressed in our learning trajectory only after most connections between the trigonometric function graphs and the unit circle trigonometry have been set. We examined our new approach in a classroom case study. It provided evidence of the effectiveness in promoting: (1) integrated understanding of trigonometric functions in such way that students do not have as many difficulties and misconceptions as reported before in the research literature on trigonometry; and (2) connected understanding of trigonometric functions as functions defined for angles and as functions defined for the domain of real numbers. Students developed good understanding of trigonometric graphs and related properties of the trigonometric functions.

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