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A Robust $\bar{X}$ Control Chart

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This article studies alternative standard deviation estimators that serve as a basis to determine the $\bar{X}$ control chart limits used for real-time process monitoring (phase II). Several existing (robust) estimation methods are considered. In addition, we propose a new estimation method based on a phase I analysis, that is, the use of a control chart to identify disturbances in a data set retrospectively. The method constructs a phase I control chart derived from the trimmed mean of the sample interquartile ranges, which is used to identify out-of-control data. An efficient estimator, namely the mean of the sample standard deviations, is used to obtain the final standard deviation estimate from the remaining data. The estimation methods are evaluated in terms of their mean squared errors and their effects on the performance of the $\bar{X}$ phase II control chart. It is shown that the newly proposed estimation method is efficient under normality and performs substantially better than standard methods when disturbances are present in phase I. Copyright © 2012 John Wiley & Sons, Ltd.

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1. Introduction

The $\bar{X}$ control chart is a widely applied technique for effectively monitoring the location of processes. When the parameters of a quality characteristic of the process are unknown, control charts can be applied in a two-stage procedure. In phase I, control charts are used retrospectively to study a historical data set and determine the samples that are out of control. On the basis of the resulting reference sample, the process parameters are estimated and control limits are calculated for phase II. In phase II, control charts are used for real-time process monitoring (cf. Vining[1]). Recent initiatives on phase II control charts address, amongst other things, the joint statistical design of the $\bar{X}$ and the standard deviation control charts (Mukherjee and Chakraborti[2] and Chen and Pao[3]), the variable sampling interval $\bar{X}$ control chart (Zhang *et al.*[4]), the use of different control charting rules (Kim *et al.*[5] and Riaz *et al.*[6]) and the use of alternative estimators (Schoonhoven *et al.*[7] and Schoonhoven and Does[8]).

Let Y_{ij}, $i=1,2,3,\ldots$and $j=1,2,\ldots,n$ denote phase II samples of size n taken in sequence of the process variable to be monitored. We assume the Y_{ij}'s to be independent and $N(\mu+\delta\sigma,\sigma^2)$ distributed, where δ is a constant. When $\delta=0$, the mean of the process is in control; otherwise the process mean has changed. Let $\bar{Y}_i=\frac{1}{n}\sum_{j=1}^{n}Y_{ij}$ be an estimate of $\mu+\delta\sigma$ based on the ith sample Y_{ij}, $j=1,2,\ldots,n$.

In practice, the process parameters μ and σ are usually unknown. Therefore, they must be estimated from samples taken when the process is assumed to be in control (i.e., phase I). The resulting estimates are used to monitor the location of the process in phase II. Define $\hat{\mu}$ and $\hat{\sigma}$ as unbiased estimates of μ and σ, respectively, based on k phase I samples of size n, which are denoted by X_{ij}, $i=1,2,\ldots,k$. The control limits can be estimated by

$$\widehat{UCL}=\hat{\mu}+C_n\hat{\sigma}/\sqrt{n},\ \ \widehat{LCL}=\hat{\mu}-C_n\hat{\sigma}/\sqrt{n}, \tag{1}$$

where C_n is the factor such that the expected probability of having a false alarm p equals the desired type I error probability. Let F_i denote the event that $\bar{Y}_i$ is above $\widehat{UCL}$ or below $\widehat{LCL}$. We define $P(F_i|\hat{\mu},\hat{\sigma})$ as the probability that sample i generates a signal given $\hat{\mu}$ and $\hat{\sigma}$, that is,

$$P\left(F_i|\hat{\mu},\hat{\sigma}\right)=P\left(\hat{\mu}_i<\widehat{LCL}\ \text{or}\ \hat{\mu}_i>\widehat{UCL}|\hat{\mu},\hat{\sigma}\right). \tag{2}$$

Given $\hat{\mu}$ and $\hat{\sigma}$, the distribution of the run length (RL) is geometric with parameter $P(F_i|\hat{\mu},\hat{\sigma})$. Consequently, the conditional average run length (ARL) is given by

$$E(RL|\hat{\mu},\hat{\sigma})=\frac{1}{P(F_i|\hat{\mu},\hat{\sigma}).} \tag{3}$$

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In contrast with the conditional RL distribution, the unconditional RL distribution takes into account the random variability introduced into the charting procedure through parameter estimation. It can be obtained by averaging the conditional RL distribution over all possible values of the parameter estimates. The unconditional p is

$$p = E(P(F_i|\hat{\mu}, \hat{\sigma})), \tag{4}$$

and the unconditional ARL is

$$ARL = E(\frac{1}{P(F_i|\hat{\mu}, \hat{\sigma})}). \tag{5}$$

Quesenberry[9] showed for the $\bar{X}$ control chart that the unconditional ARL is higher than in the case where the process parameters are known. He concluded that, if limits are to behave like known limits, the number of samples in phase I should be at least $400/(n-1)$.

Jensen *et al.*[10] conducted a literature survey of the effects of parameter estimation on control chart properties and identified some issues for future research. One of their main recommendations is to study robust estimators for μ and σ, because most studies have only considered standard estimators. The effect of using these robust estimators on phase II should also be assessed (Jensen *et al.*,[10] p. 360). Schoonhoven *et al.*[11] analyzed several robust location estimation methods, including several methods that use a phase I control chart. In addition, they proposed a new phase I control chart derived from the trimean. Their results indicate that the $\bar{X}$ phase II control chart (with σ known) based on the new estimation method performs well under normality and outperforms the other charts when contaminations are present in phase I. However, the effect of the process location method on the performance of the $\bar{X}$ phase II control chart is more limited than the effect of the standard deviation estimator. The present article therefore looks at the effect of alternative standard deviation estimators under various phase I scenarios.

So far the literature has proposed several robust standard deviation estimators. Rocke[12] considered the interquartile range and the 25% trimmed mean of the interquartile ranges. Rocke[13] gave the practical details for the construction of the charts based on these estimators. Tatum[14] proposed a method, constructed around a variant of the biweight *A* estimator, that is resistant to diffuse disturbances, that are, disturbances that are equally likely to perturb any observation, and localized disturbances, that is, disturbances that affect all observations in a sample. Schoonhoven *et al.*[15] studied several estimators used to construct the standard deviation phase II control chart. They found that Tatum's estimator is robust against diffuse disturbances but less robust against shifts in the process standard deviation in phase I. They proposed an estimator based on the mean deviation to the median supplemented with sample screening in phase I. The advantage of this estimator is that it works well when localized variance disturbances are present but it is less robust when there are asymmetric diffuse disturbances. Finally, Schoonhoven and Does[16] proposed a standard deviation estimation method where the control chart based on the mean deviation to the median is supplemented with screening using an individual control chart in phase I. They investigated the effect of the estimation method on the standard deviation control chart.

In this article, we develop an estimation method to derive the standard deviation for the $\bar{X}$ control chart when both μ and σ are unknown. Apart from the new method, several alternative estimation methods are included in the comparison. The methods are evaluated in terms of their mean squared errors (MSE) and their effect on the $\bar{X}$ phase II control chart performance. We consider the situation where the phase I data are uncontaminated and normally distributed, as well as various types of contaminated phase I situations.

The paper is structured as follows. Subsequently, we present the estimation methods for the standard deviation and assess the MSE of the estimators. In the following sections, we present the design schemes for the $\bar{X}$ phase II control chart and derive the control limits. Next, we describe the simulation procedure and present the effect of the proposed methods on the phase II performance. The final section offers some recommendations and issues for future research.

2. Proposed phase I estimators

To understand the behavior of the estimators, it is useful to distinguish two groups of disturbances, namely diffuse and localized (cf. Tatum[14]). Diffuse disturbances are outliers that are spread over all of the samples, whereas localized disturbances affect all observations in one sample. We analyze various types of standard deviation estimators and compare them under various types of disturbances. The first and second subsections introduce the standard deviation and location estimators, respectively, whereas the second subsection presents the MSE of the standard deviation estimators.

2.1. Standard deviation estimators

Recall that X_{ij}, $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$ denote the phase I data. The X_{ij}'s are assumed to be independent and $N(\mu, \sigma^2)$ distributed. We denote by $X_{i,(v)}$, $v = 1, 2, \ldots, n$ the vth order statistic in sample i. We look at several robust estimators proposed in the existing literature and introduce a new method incorporating a phase I control chart.

The first estimator of σ is based on the mean of the sample standard deviations

$$\bar{S} = \frac{1}{k}\sum_{i=1}^{k} S_i \tag{6}$$

where S_i is the ith sample standard deviation defined by

$$S_i = \left(\frac{1}{n-1} \sum_{j=1}^{n} \left(X_{ij} - \bar{X}_i \right)^2 \right)^{1/2}.$$

An unbiased estimator of σ is given by $\bar{S}/c_4(n)$, where $c_4(n)$ is defined by

$$c_4(n) = \left(\frac{2}{n-1} \right)^{1/2} \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}.$$

Note that this estimator is slightly less efficient under normality than the pooled sample standard deviation. The latter, however, is most sensitive to contaminations (Schoonhoven *et al.*[15]).

The second estimator is based on the mean sample range

$$\bar{R} = \frac{1}{k} \sum_{i=1}^{k} R_i, \tag{7}$$

where R_i is the range of the ith sample

$$R_i = X_{i,(n)} - X_{i,(1)}.$$

An unbiased estimator of σ is $\bar{R}/d_2(n)$, where $d_2(n)$ is the expected range of a random $N(0,1)$ sample of size n. Values of $d_2(n)$ can be found in Duncan[17] (Table M).

Rocke[12] proposed the mean of the sample interquartile ranges

$$\overline{IQR} = \frac{1}{k} \sum_{i=1}^{k} IQR_i, \tag{8}$$

where IQR_i is the interquartile range of sample i defined by

$$IQR_i = Q_{i,3} - Q_{i,1},$$

with $Q_{i,q}$ the qth quartile of sample i (with $q = 1, 2, 3$). We use the following definitions for the quartiles: $Q_{i,1} = X_{i,(a)}$ and $Q_{i,3} = X_{i,(b)}$ with $a = \lceil n/4 \rceil$ and $b = n - a + 1$, where $\lceil z \rceil$ denotes the ceiling function, that is, the smallest integer not less than z. This means that $Q_{i,1}$ and $Q_{i,3}$ are defined as the second smallest and the second largest observations, respectively, for $4 \le n \le 7$ and as the third smallest and the third largest values, respectively, for $8 \le n \le 11$. An unbiased estimator of σ is given by $\overline{IQR}/d_{IQR}$, where d_{IQR} is a normalizing constant. The value of this normalizing constant is 0.990 for $n = 5$ and 1.144 for $n = 9$.

Rocke[12] also proposed the trimmed mean of the sample interquartile ranges

$$\overline{IQR}_\alpha = \frac{1}{k - 2\lceil k\alpha \rceil} \times \left[\sum_{v=\lceil k\alpha \rceil + 1}^{k - \lceil k\alpha \rceil} IQR_{(v)} \right], \tag{9}$$

where $IQR_{(v)}$ denotes the vth ordered value of the sample interquartile ranges. We consider the 20% trimmed mean of the sample interquartile ranges, which trims the 10 smallest and the 10 largest sample interquartile ranges when $k = 50$ and the 20 smallest and the 20 largest sample interquartile ranges when $k = 100$. An unbiased estimator of σ is given by $\overline{IQR}_{20}/d_{\overline{IQR}_{20}}$, where $d_{\overline{IQR}_{20}}$ is a normalizing constant. The value of this normalizing constant is 0.925 for $n = 5$ and 1.108 for $n = 9$.

We also evaluate a robust estimator proposed by Tatum.[14] His method has proven to be robust to both diffuse and localized disturbances. The estimation method is constructed around a variant of the biweight A estimator. The method begins by calculating the residuals in each sample, which involves substracting the sample median M_i from each value: $res_{ij} = X_{ij} - M_i$. If n is odd, then in each sample, one of the residuals will be zero and is dropped. As a result, the total number of residuals is equal to $m' = nk$ when n is even and $m' = (n-1)k$ when n is odd. Tatum's estimator is given by

$$S_c^* = \frac{m'}{(m'-1)^{1/2}} \frac{\left(\sum_{i=1}^{k} \sum_{j:|u_{ij}|<1} res_{ij}^2 \left(1 - u_{ij}^2\right)^4 \right)^{1/2}}{\left| \sum_{i=1}^{k} \sum_{j:|u_{ij}|<1} \left(1 - u_{ij}^2\right)\left(1 - 5u_{ij}^2\right) \right|}, \tag{10}$$

where $u_{ij} = h_i res_{ij}/(cM^*)$, M^* is the median of the absolute values of all residuals,

$$h_i = \begin{cases} 1 & E_i \le 4.5, \\ E_i - 3.5 & 4.5 < E_i \le 7.5, \\ c & E_i > 7.5, \end{cases}$$

and $E_i = IQR_i/M^*$. The constant c is a tuning constant. Each value of c leads to a different estimator. Tatum showed that $c = 7$ gives an estimator that loses some efficiency when no disturbances are present but gains efficiency when disturbances are present. We apply this value of c in our study. Note that we have $h_i = E_i - 3.5$ for $4.5 < E_i \leq 7.5$ in the equations instead of $h_i = E_i - 4.5$ as presented by Tatum (Tatum,[14] p.129). An unbiased estimator of σ is given by $S_c^*/d^*(c,n,k)$, where $d^*(c,n,k)$ is the normalizing constant. In our study, we use the corrected normalizing constants given in Schoonhoven *et al.*[15]

We now present a new estimation method based on the principle of phase I control charting (cf. Jones-Farmer *et al.*[18]). We build a phase I control chart using a robust estimator for the standard deviation, namely $\overline{IQR}_{20}$. A disadvantage of this estimator is that it is not very efficient under normality. To address this, we use $\overline{IQR}_{20}$ to construct the phase I limits with which we screen the estimation data for disturbances, but then use the efficient estimator $\bar{S}$ to obtain a standard deviation estimate from the remaining data. The phase I standard deviation control chart limits are given by $\widehat{UCL}_{I\bar{Q}R_{20}} = U_n\overline{IQR}_{20}/d_{I\bar{Q}R_{20}}$ and $\widehat{LCL}_{I\bar{Q}R_{20}} = L_n\overline{IQR}_{20}/d_{I\bar{Q}R_{20}}$. For simplicity, we derive U_n and L_n from the 0.99865 and 0.00135 quantiles of the distribution of IQR/d_{IQR}. These quantiles are obtained by simulation, and 1,000,000 simulation runs are used. The respective values of U_n and L_n are 3.220 and 0.035 for $n = 5$ and 2.487 and 0.145 for $n = 9$. We then plot the IQR_i/d_{IQR}'s of the phase I samples on the phase I control chart. Charting the IQR instead of the sample standard deviation or the sample range ensures that localized variance disturbances are identified and samples that contain only one single outlier are retained. A standard deviation estimate that is expected to be robust against localized variance disturbances is based on the mean of the sample interquartile ranges of the samples that fall between the control limits

$$\overline{IQR}' = \frac{1}{k'}\sum_{i\varepsilon K} IQR_i \times 1_{\widehat{LCL}_{I\bar{Q}R_{20}} \leq IQR_i/d_{IQR} \leq \widehat{UCL}_{I\bar{Q}R_{20}}}(IQR_i),$$

with K the set of samples which are not excluded and k' the number of non-excluded samples. The resulting estimate $I\bar{Q}R'/d_{IQR}$ is unbiased.

Although the remaining phase I samples are expected to be free from localized variance disturbances, they could still contain diffuse disturbances. To eliminate such disturbances, the next step is to screen the individual observations using a phase I individuals control chart. To screen the individual observations, we determine the residuals in each sample by subtracting the trimean value from each observation in the corresponding sample: $resid_{ij} = X_{ij} - TM_i$ with

$$TM_i = \left(Q_{i,1} + 2Q_{i,2} + Q_{i,3}\right)/4.$$

Note that $Q_{i,2}$ is the median of sample i. Substracting the sample trimeans ensures that the variability is measured within samples and not between samples. According to Tukey,[19] using the trimean instead of the mean or the median gives a more useful assessment of location or centering. The control limits of the individuals chart are given by $\widehat{UCL}_{I\bar{Q}R'} = 3\overline{IQR}'/d_{IQR}$ and $\widehat{LCL}_{I\bar{Q}R'} = -3\overline{IQR}'/d_{IQR}$. The residuals $resid_{ij}$ that fall above $\widehat{UCL}_{I\bar{Q}R'}$ or below $\widehat{LCL}_{I\bar{Q}R'}$ are considered out of control and their corresponding observations are removed from the phase I data set. The final estimate is the mean of the sample standard deviations S_i and is calculated from the observations deemed to be in control

$$\bar{S}' = \frac{1}{k^\wedge}\sum_{i\varepsilon K^\wedge} S_i\left(\left\{X_{ij} \times 1_{\widehat{LCL}_{I\bar{Q}R'} \leq resid_{ij} \leq \widehat{UCL}_{I\bar{Q}R'}}\right\}\right)(X_{ij}), \tag{11}$$

with $K^\wedge$ the set of samples which are not excluded and $k^\wedge$ the number of non-excluded samples. The normalizing constant is 0.980 for $n = 5$ and 0.984 for $n = 9$. This adaptively trimmed standard deviation is denoted by *ATS*.

The proposed standard deviation estimators are summarized in Table I.

2.2. Location estimator

The aforementioned standard deviation estimators are used to construct the $\bar{X}$ phase II control limits. To ensure a fair comparison, we use the same location estimator in each case. The location estimation method uses a procedure similar to *ATS*. This procedure was proposed by Schoonhoven *et al.*[11] and turned out to perform much better than the standard estimating procedures based on, for example, the mean, median, trimmed mean and Hodges–Lehmann estimator. The procedure consists of two steps.

In the first step, we determine a location estimate that is robust against both localized and diffuse mean disturbances, namely the 20% trimmed mean of the sample trimeans

Table I. Proposed standard deviation estimators

Estimators	Notation
Mean of sample standard deviations	$\bar{S}$
Mean of sample ranges	$\bar{R}$
Mean of sample interquartile ranges	$I\bar{Q}R$
20% trimmed mean of sample interquartile ranges	$I\bar{Q}R_{20}$
Tatum's estimator	$D7$
$I\bar{Q}R'$ control chart with screening	ATS

$$\overline{TM}_{\alpha} = \frac{1}{k - 2\lceil k\alpha \rceil} \times \left[\sum_{v=\lceil k\alpha \rceil+1}^{k-\lceil k\alpha \rceil} TM_{(v)} \right].$$

Note that we start with the entire data set. The respective upper and lower control limits for the sample location are given by $\widehat{UCL}_{\overline{TM}_{20}} = \overline{TM}_{20} + 3\hat{\sigma}/\sqrt{n}$ and $\widehat{LCL}_{\overline{TM}_{20}} = \overline{TM}_{20} - 3\hat{\sigma}/\sqrt{n}$, where σ is estimated by the corresponding standard deviation estimator from Table I. We then plot the TM_i's of the phase I samples on the control chart. Charting the TM_i's instead of the $\bar{X}_i$'s ensures that localized disturbances are identified and samples that contain only one single outlier are retained. A location estimate that is expected to be robust against localized mean disturbances is the mean of the sample trimeans of the samples that fall between the control limits

$$\overline{TM}' = \frac{1}{k^*} \sum_{i\varepsilon K^*} TM_i \times 1_{\widehat{LCL}_{\overline{TM}_{20}} \le TM_i \le \widehat{UCL}_{\overline{TM}_{20}}}(TM_i),$$

with K^* the set of samples which are not excluded and k^* the number of non-excluded samples.

Although the remaining phase I samples are expected to be free from localized mean disturbances, they could still contain diffuse disturbances. To eliminate such disturbances, the next step is to screen the individual observations using a phase I individuals control chart with respective upper and lower control limits given by $\widehat{UCL}_{\overline{TM}'} = \overline{TM}' + 3\hat{\sigma}$ and $\widehat{LCL}_{\overline{TM}'} = \overline{TM}' - 3\hat{\sigma}$, where σ is estimated by the corresponding standard deviation estimator from Table I. The observations X_{ij} that fall above $\widehat{UCL}_{\overline{TM}'}$ or below $\widehat{LCL}_{\overline{TM}'}$ are considered out of control and removed from the phase I data set. The final estimate is the mean of the sample means and is calculated from the observations deemed to be in control

$$\bar{\bar{X}}' = \frac{1}{k''} \sum_{i\varepsilon K''} \frac{1}{n_i'} \sum_{j\varepsilon N_i'} X_{ij} \times 1_{\widehat{LCL}_{\overline{TM}'} \le X_{ij} \le \widehat{UCL}_{\overline{TM}'}}(X_{ij}), \tag{12}$$

with K'' the samples which are not excluded, k'' the number of non-excluded samples, N'_i the observations that are not excluded in sample i and n'_i the number of non-excluded observations in sample i.

2.3. *Efficiency of proposed standard deviation estimators*

For comparison purposes, we assess the MSE of the proposed standard deviation estimators as was performed in Tatum.[14] The MSE will be estimated as

$$MSE = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\sigma}^i - \sigma\right)^2,$$

where $\hat{\sigma}^i$ is the value of the unbiased estimate in the ith simulation run and N is the number of simulation runs. We consider the uncontaminated case, that is, the situation where all X_{ij} are from the N (0,1) distribution as well as four types of disturbances (cf. Tatum[14]):

1. A model for diffuse symmetric variance disturbances in which each observation has a 95% probability of being drawn from the N (0,1) distribution and a 5% probability of being drawn from the $N(0, a)$ distribution, with $a = 1.5, 2.0, \ldots, 5.5, 6.0$.
2. A model for diffuse asymmetric variance disturbances in which each observation is drawn from the $N(0, 1)$ distribution and has a 5% probability of having a multiple of a χ_1^2 variable added to it, with the multiplier equal to $0.5, 1.0, \ldots, 4.5, 5.0$.
3. A model for localized variance disturbances in which observations in five (when $k = 50$) or 10 (when $k = 100$) samples are drawn from the $N(0, a)$ distribution, with $a = 1.5, 2.0, \ldots, 5.5, 6.0$.
4. A model for diffuse mean disturbances in which each observation has a 95% probability of being drawn from the $N(0, 1)$ distribution and a 5% probability of being drawn from the $N(b, 1)$ distribution, with $b = 0.5, 1.0, \ldots, 9.0, 9.5$.

The MSE is obtained for $k = 50$, 100 samples of sizes $n = 5$, 9. The number of simulation runs N is equal to 50,000.

Figures 1–4 show the MSE of the proposed estimators. The following results can be observed. The standard estimators $\bar{S}$ and $\bar{R}$ are not robust against either localized or diffuse disturbances. The $I\bar{Q}R$ is less efficient under normality when there are no contaminations, but performs reasonably well when there are diffuse disturbances. The reason why $I\bar{Q}R$ performs so well in these situations is that it trims the highest and lowest observations in each sample. However, this estimator remains biased when there are asymmetric diffuse disturbances because the trimming method does not take the distribution of the contaminations into account. Furthermore, this estimator is not efficient when there are localized variance disturbances as it trims only the observations within the sample instead of the sample interquartile ranges.

An estimator that combines within-sample and between-sample trimmings, namely $I\bar{Q}R_{20}$, performs reasonably well for all types of contaminations. However, its efficiency is relatively low under normality. $D7$ is efficient under normality as well as for contaminated data but relatively less so when the contamination consists of localized variance disturbances.

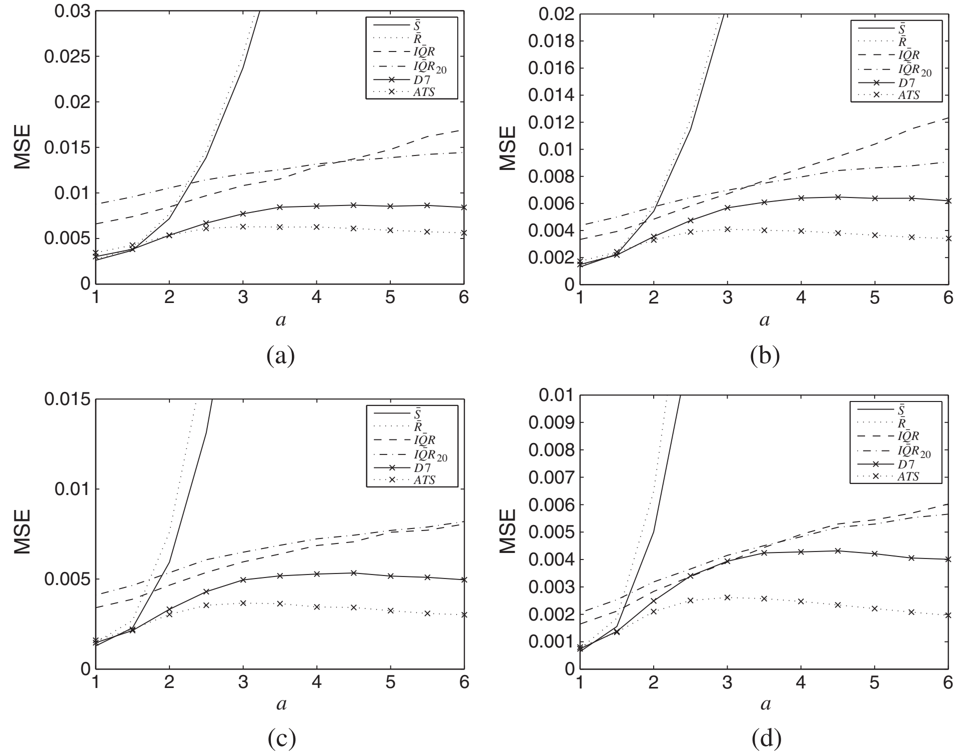

Figure 1. Mean squared errors of estimators when symmetric diffuse variance disturbances are present: (a) $n=5, k=50$; (b) $n=5, k=100$; (c) $n=9, k=50$; and (d) $n=9, k=100$

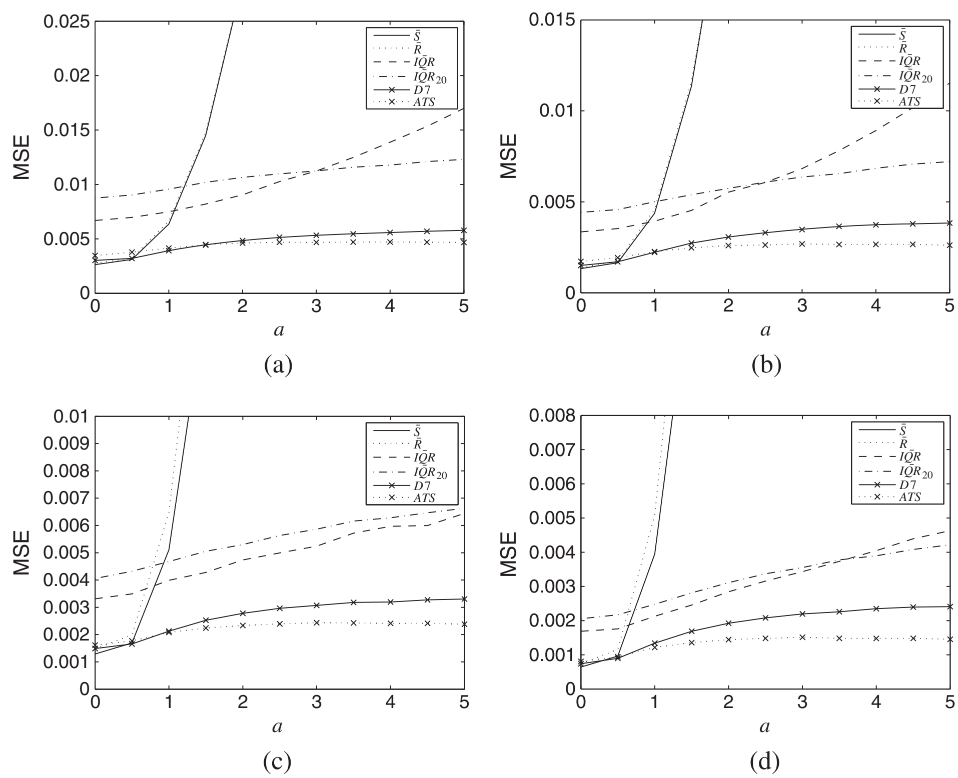

Figure 2. Mean squared errors of estimators when asymmetric diffuse variance disturbances are present: (a) $\boldsymbol{n=5, k=50}$; (b) $\boldsymbol{n=5, k=100}$; (c) $\boldsymbol{n=9, k=50}$; and (d) $\boldsymbol{n=9, k=100}$

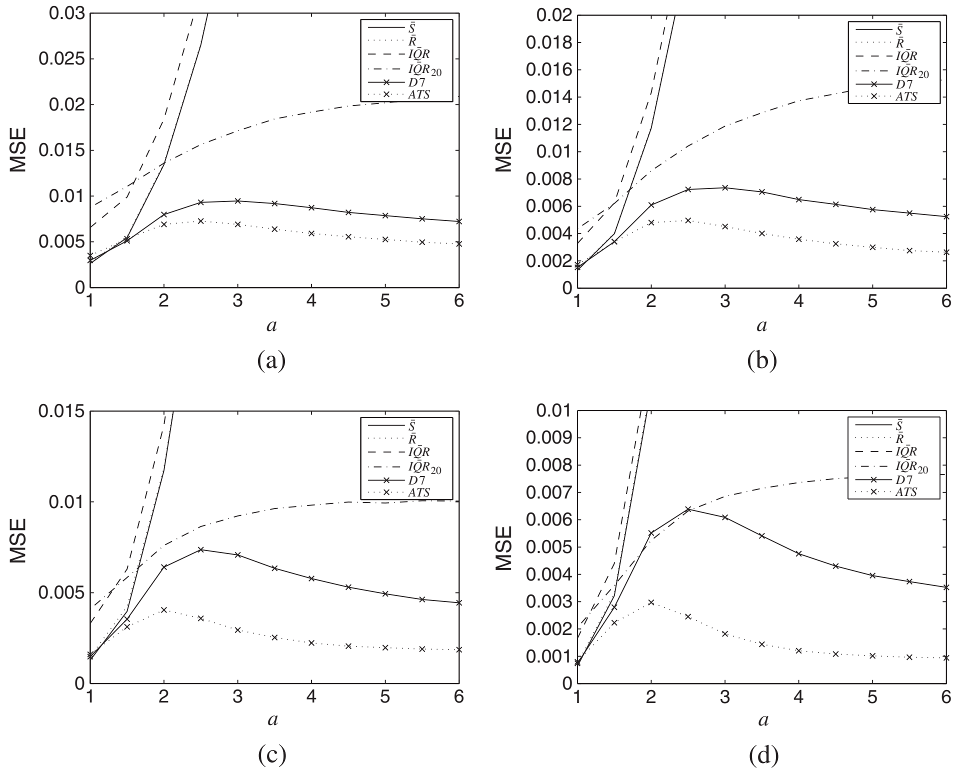

Figure 3. Mean squared errors of estimators when localized variance disturbances are present: (a) $n = 5, k = 50$; (b) $n = 5, k = 100$; (c) $n = 9, k = 50$; and (d) $n = 9, k = 100$

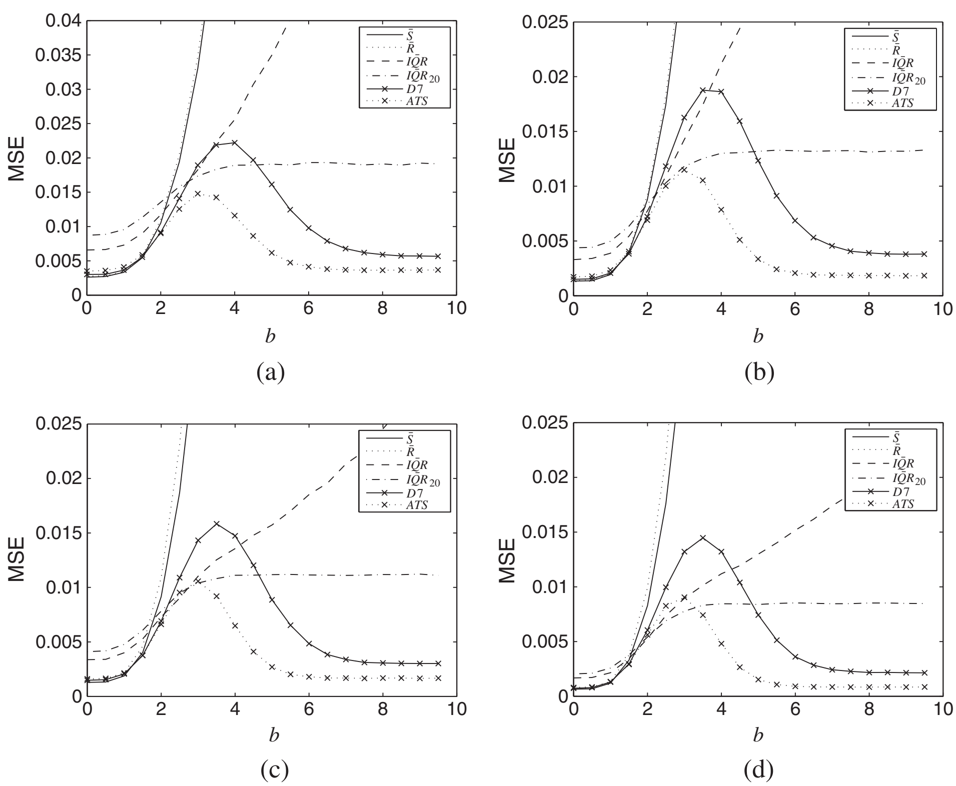

Figure 4. Mean squared errors of estimators when diffuse mean disturbances are present: (a) $n = 5, k = 50$; (b) $n = 5, k = 100$; (c) $n = 9, k = 50$; and (d) $n = 9, k = 100$

The estimator *ATS* is slightly less efficient under normality than the standard estimators, but much more robust than $I\bar{Q}R$ and $I\bar{Q}R_{20}$. Moreover, it shows outstanding performance when contaminations are present. We can therefore conclude that this estimator effectively filters out extreme observations.

3. Derivation of the phase II control limits

We now turn to the effect of the proposed estimators on the performance of the $\bar{X}$ phase II control chart. The formulae for the $\bar{X}$ control limits with estimated limits are given by (1). The factor C_n that is used to obtain accurate control limits when the process parameters are estimated is derived, such that the probability of a false signal equals the chosen type I error probability *p*. The factors cannot be obtained easily in analytic form. Therefore, they are obtained by means of simulation. The chosen type I error probability *p* is 0.0027. 50,000 simulation runs are used. The resulting factors are presented in Table II.

4. Control chart performance

In this section, we evaluate the effect on $\bar{X}$ phase II performance of the proposed standard deviation estimators. We consider the same phase I situations as those used to assess the MSE with *a*, *b* and the multiplier equal to 4 to simulate the contaminated case (cf. the Section on Efficiency of Proposed Standard Deviation Estimators).

The performance of the phase II control charts is assessed in terms of the unconditional *p* and *ARL* as well as the conditional *ARL*. The conditional *ARL* values express the ARL values for the control limits associated with the 2.5% and 95.7% quantiles of simulated *p* in the in-control situation. We consider different shifts of size $\delta\sigma$ in the mean in phase II, namely δ equal to 0, 0.25, 0.5 and 1. The performance characteristics are obtained by simulation. The next section describes the simulation method, followed by the results for the control charts constructed in the uncontaminated situation and various contaminated situations.

4.1. Simulation procedure

The performance characteristics *p* and ARL for estimated control limits are determined by averaging the conditional characteristics, that is, the characteristics for a given set of estimated control limits, over all possible values of the control limits. Recall the definitions of $p(F_i|\hat{\mu},\hat{\sigma})$ from (2), $E(RL|\hat{\mu},\hat{\sigma})$ from (3), $p = E(p(F_i|\hat{\mu},\hat{\sigma}))$ from (4) and $ARL = E\left(\frac{1}{p(F_i|\hat{\mu},\hat{\sigma})}\right)$ from (5). These expectations will be obtained by simulation: numerous data sets are generated, and for each dataset, $p(F_i|\hat{\mu},\hat{\sigma})$ and $E(RL|\hat{\mu},\hat{\sigma})$ are computed. By averaging these values we obtain the unconditional values.

Enough replications of the aforementioned procedure were performed to obtain sufficiently small relative estimated standard errors for *p* and ARL. The relative estimated standard error is the estimated standard error of the estimate relative to the estimate. The relative standard error of the estimates is never higher than 0.80%.

4.2. Simulation results

The performance metrics are obtained in the in-control situation ($\delta = 0$) as well as in the out-of-control situation ($\delta \neq 0$). When $\delta = 0$, the process is in control, so we want *p* to be as low as possible and ARL to be as high as possible. When $\delta \neq 0$, that is, in the out-of-control situation, we want to achieve the opposite.

Table III shows the performance metrics for the $\bar{X}$ phase II charts under normality. In this case, we have estimated both the in-control μ and σ in phase I. Compared with the $\bar{X}$ phase II performance presented in Schoonhoven *et al.*,[11] where only the mean was estimated to isolate the effect of estimating the location parameter, the ARL values are much higher than the desired 370. Thus,

Table II. Factors C_n to determine Phase II control limits

Chart	Factors for control limits			
	$n = 5$		$n = 9$	
	$k = 50$	$k = 100$	$k = 50$	$k = 100$
$\bar{S}$	3.065	3.030	3.050	3.025
$\bar{R}$	3.070	3.035	3.055	3.025
$I\bar{Q}R$	3.125	3.060	3.080	3.040
$I\bar{Q}R_{20}$	3.155	3.080	3.090	3.045
D7	3.070	3.035	3.050	3.025
ATS	3.085	3.040	3.055	3.025

Table III. Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values under normality

			p				ARL			
n	k	Chart	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$
5	50	$\bar{S}$	0.0027	0.0073	0.029	0.21	489	193	44.9	5.24
							(155; 1256)	(64.1; 418)	(18.6; 85.7)	(3.21; 7.94)
		$\bar{R}$	0.0027	0.0073	0.028	0.21	500	196	46.2	5.30
							(163; 1318)	(63.3; 489)	(18.4; 98.1)	(3.19; 8.58)
		$I\bar{Q}R$	0.0027	0.0071	0.027	0.20	770	289	60.7	6.09
							(110; 3127)	(75.7; 1838)	(22.6; 307)	(3.48; 17.4)
		$I\bar{Q}R_{20}$	0.0027	0.0069	0.026	0.19	1066	375	73.2	6.74
							(96.4; 5089)	(42.8; 1332)	(13.5; 227)	(2.69; 14.9)
		$D7$	0.0027	0.0073	0.028	0.21	507	201	46.2	5.30
							(145; 1374)	(81.2; 493)	(23.1; 98.6)	(3.56; 8.63)
		ATS	0.0027	0.0072	0.028	0.20	543	211	48.4	5.45
							(135; 1536)	(57.8; 451)	(17.1; 91.2)	(3.06; 8.32)
	100	$\bar{S}$	0.0027	0.0075	0.029	0.22	419	159	38.5	4.83
							(194; 818)	(77.1; 398)	(21.5; 83.7)	(3.48; 7.66)
		$\bar{R}$	0.0027	0.0074	0.029	0.22	428	161	38.9	4.85
							(193; 857)	(76.7; 282)	(21.4; 61.8)	(3.48; 6.50)
		$I\bar{Q}R$	0.0027	0.0074	0.029	0.21	521	189	43.9	5.18
							(146; 1487)	(61.1; 522)	(17.9; 103)	(3.14; 8.89)
		$I\bar{Q}R_{20}$	0.0027	0.0072	0.028	0.21	598	213	47.9	5.41
							(133; 1929)	(56.6; 553)	(16.8; 108)	(3.03; 9.26)
		$D7$	0.0027	0.0074	0.029	0.21	431	163	39.1	4.88
							(189; 877)	(81.6; 290)	(22.6; 63.2)	(3.57; 6.59)
		ATS	0.0027	0.0073	0.029	0.21	446	168	40.1	4.94
							(179; 940)	(80.0; 296)	(22.3; 64.3)	(3.53; 6.68)
9	50	$\bar{S}$	0.0027	0.012	0.063	0.48	427	106	18.0	2.13
							(189; 846)	(148; 157)	(23.8; 24.6)	(2.31; 2.48)
		$\bar{R}$	0.0027	0.012	0.063	0.48	440	107	18.2	2.14
							(186; 923)	(53.1; 171)	(10.9; 26.2)	(1.75; 2.54)
		$I\bar{Q}R$	0.0027	0.012	0.062	0.47	537	124	20.1	2.22
							(142; 1512)	(38.2; 257)	(8.61; 35.8)	(1.61; 2.94)
		$I\bar{Q}R_{20}$	0.0027	0.012	0.061	0.46	588	135	21.1	2.26
							(135; 1858)	(36.7; 388)	(8.37; 48.7)	(1.60; 3.36)
		$D7$	0.0027	0.012	0.064	0.48	432	106	18.0	2.14
							(182; 885)	(65.8; 184)	(12.7; 27.6)	(1.85; 2.59)
		ATS	0.0027	0.012	0.063	0.48	441	107	18.2	2.15
							(177; 931)	(127; 171)	(21.1; 26.3)	(2.21; 2.55)
	100	$\bar{S}$	0.0027	0.012	0.065	0.49	397	95.5	16.4	2.06
							(232; 643)	(56.4; 132)	(11.4; 21.6)	(1.79; 2.33)
		$\bar{R}$	0.0027	0.012	0.063	0.48	398	92.8	16.4	2.06
							(223; 668)	(61.2; 135)	(12.1; 21.9)	(1.83; 2.35)

(*Continues*)

Table III. *Continued.*

			p				*ARL*			
n	*k*	Chart	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$
		$I\bar{Q}R$	0.0027	0.012	0.064	0.49	440 (182; 941)	99.8 (47.2; 226)	17.3 (10.0; 32.1)	2.10 (1.71; 2.75)
		$I\bar{Q}R_{20}$	0.0027	0.012	0.064	0.48	462 (174; 1066)	103 (46.6; 193)	17.6 (9.94; 28.7)	2.12 (1.70; 2.65)
		D7	0.0027	0.012	0.065	0.49	403 (225; 670)	93.1 (61.1; 130)	16.4 (12.1; 21.4)	2.07 (1.83; 2.33)
		ATS	0.0027	0.012	0.065	0.49	401 (219; 681)	92.9 (75.8; 143)	16.4 (14.1; 22.7)	2.07 (1.92; 2.38)

Table IV. Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when symmetric variance disturbances are present in phase I

			p				*ARL*			
n	k	Chart	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$
5	50	$\bar{S}$	4.3×10^{-4}	0.0014	0.0070	0.081	4.3×10^{4} (546; 6.7×10^{4})	3609 (343; 1.4×10^{4})	441 (76.1; 1675)	20.2 (7.12; 58.7)
		$\bar{R}$	4.1×10^{-4}	0.0013	0.0067	0.079	1.4×10^{4} (562; 7.8×10^{4})	364 (275; 1.5×10^{4})	475 (61.7; 1758)	21.1 (6.35; 61.1)
		$I\bar{Q}R$	0.0015	0.0042	0.018	0.15	2041 (166; 9948)	657 (109; 2979)	120 (30.1; 449)	9.08 (4.10; 33.3)
		$I\bar{Q}R_{20}$	0.0016	0.0046	0.018	0.15	2288 (144; 1.2×10^{4})	746 (70.4; 2700)	128 (20.3; 413)	9.28 (3.34; 22.4)
		$D7$	0.0015	0.0044	0.019	0.16	1107 (221; 3820)	401 (102; 1209)	82.6 (27.1; 209)	7.38 (3.94; 14.0)
		ATS	0.0019	0.0053	0.022	0.17	898 (170; 3064)	335 (69.9; 860)	70.8 (19.9; 157)	6.73 (3.33; 11.7)
	100	$\bar{S}$	3.5×10^{-4}	0.0012	0.0065	0.082	5922 (880; 2.4×10^{4})	1599 (309; 5101)	255 (66.9; 713)	15.2 (6.80; 32.5)
		$\bar{R}$	3.3×10^{-4}	0.0012	0.0062	0.079	6506 (912; 2.7×10^{4})	174 (302; 6155)	271 (65.4; 837)	15.9 (6.73; 36.1)
		$I\bar{Q}R$	0.0014	0.0042	0.018	0.16	1116 (240; 3632)	375 (99.8; 952)	77.1 (26.6; 171)	7.17 (3.91; 12.4)
		$I\bar{Q}R_{20}$	0.0016	0.0046	0.019	0.16	1131 (204; 4062)	370 (83.7; 1046)	76.5 (23.0; 185)	7.10 (3.61; 13.1)
		$D7$	0.0015	0.0044	0.019	0.16	874 (303; 2126)	304 (126; 785)	64.8 (32.1; 146)	6.57 (4.36; 11.0)
		ATS	0.0019	0.0053	0.022	0.18	693 (235; 1696)	247 (190; 507)	55.1 (49.3; 101)	5.96 (5.40; 8.83)
9	50	$\bar{S}$	2.9×10^{-4}	0.0019	0.015	0.23	1.0×10^{4} (934; 4.8×10^{4})	1445 (185; 1.2×10^{4})	127 (67.7; 732)	5.14 (2.60; 14.0)
		$\bar{R}$	2.0×10^{-4}	0.0014	0.011	0.20	2.2×10^{4} (1162; 1.2×10^{5})	2702 (230; 2.2×10^{4})	206 (32.7; 1246)	6.42 (2.80; 19.3)
		$I\bar{Q}R$	0.0016	0.0078	0.045	0.40	983 (218; 3166)	209 (62.1; 459)	29.4 (12.2; 56.3)	2.62 (1.83; 3.66)
		$I\bar{Q}R_{20}$	0.0017	0.0079	0.046	0.41	1047 (196; 3690)	219 (109; 533)	30.3 (18.5; 63.2)	2.64 (2.11; 3.87)
		$D7$	0.0016	0.0078	0.046	0.41	799 (271; 1945)	180 (85.7; 348)	26.7 (15.4; 45.0)	2.52 (2.00; 3.26)
		ATS	0.0019	0.0091	0.051	0.43	667 (225; 1618)	154 (64.3; 274)	23.7 (12.5; 37.6)	2.39 (1.85; 3.00)
	100	$\bar{S}$	2.5×10^{-4}	0.0017	0.014	0.23	6623 (1446; 2.2×10^{4})	934 (244; 2308)	95.2 (34.4; 203)	4.61 (2.89; 7.15)
		$\bar{R}$	1.7×10^{-4}	0.0012	0.011	0.20	1.2×10^{4} (1851; 4.6×10^{4})	1492 (305; 5238)	135 (40.9; 390)	5.49 (3.13; 10.2)

(*Continues*)

Table IV. *Continued.*

n	k	Chart	p $\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	*ARL* $\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$
		$I\bar{Q}R$	0.0016	0.0079	0.047	0.42	785	160	24.6	2.44
							(287; 1831)	(67.6; 416)	(13.1; 51.2)	(1.89; 3.42)
		$I\bar{Q}R_{20}$	0.0017	0.0081	0.048	0.42	785	160	24.4	2.44
							(263; 1953)	(61.9; 317)	(12.2; 42.1)	(1.84; 3.17)
		$D7$	0.0016	0.0077	0.046	0.42	724	152	23.7	2.41
							(354; 1367)	(91.8; 240)	(16.3; 33.9)	(2.06; 2.86)
		ATS	0.0019	0.0091	0.053	0.45	591	129	20.9	2.28
							(287; 1106)	(71.4; 195)	(13.6; 29.0)	(1.92; 2.67)

Table V. Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when asymmetric variance disturbances are present in phase I

			p				ARL			
n	k	Chart	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$
5	50	$\bar{S}$	2.9×10^{-4}	9.2×10^{-4}	0.0045	0.054	2.4×10^{10} ($549; 2.2\times10^{7}$)	4.1×10^{14} ($277; 2.1\times10^{7}$)	1.5×10^{8} ($62.3; 1.1\times10^{6}$)	6.2×10^{4} (6.38; 7196)
		$\bar{R}$	2.8×10^{-4}	9.2×10^{-4}	0.0046	0.055	4.2×10^{9} ($575; 2.2\times10^{7}$)	2.9×10^{7} ($194; 1.4\times10^{6}$)	2.5×10^{7} ($95.4; 9.6\times10^{4}$)	3532 (5.40; 1172)
		$I\bar{Q}R$	0.0016	0.0045	0.018	0.15	4219 (158; 1324)	1376 (72.3; 2951)	158 (20.6; 446)	9.88 (3.38; 23.6)
		$I\bar{Q}R_{20}$	0.0018	0.0050	0.020	0.16	1944 ($132; 1.0\times10^{4}$)	670 (62.8; 8338)	118 (18.4; 1152)	8.74 (3.17; 42.0)
		$D7$	0.0019	0.0053	0.022	0.17	820 (189; 2565)	318 (76.4; 702)	67.7 (21.4; 132)	6.62 (3.47; 10.5)
		ATS	0.0022	0.0061	0.024	0.19	723 (155; 2292)	283 (77.7; 647)	60.7 (22.0; 123)	6.23 (3.49; 10.1)
	100	$\bar{S}$	1.7×10^{-4}	6.07×10^{-4}	0.0033	0.048	8.7×10^{5} ($1107; 1.5\times10^{6}$)	6.7×10^{4} ($467; 2.9\times10^{5}$)	1.6×10^{4} ($94.7; 2.4\times10^{4}$)	123 (8.34; 402)
		$\bar{R}$	1.7×10^{-4}	6.3×10^{-4}	0.0034	0.049	6.8×10^{5} ($1097; 1.0\times10^{6}$)	7.0×10^{6} ($402; 1.4\times10^{5}$)	5843 ($83.1; 1.3\times10^{4}$)	69.0 (7.75; 260)
		$I\bar{Q}R$	0.0016	0.0045	0.019	0.16	1204 (223; 4523)	400 (109; 1148)	78.8 (29.0; 200)	7.23 (4.07; 13.8)
		$I\bar{Q}R_{20}$	0.0018	0.0051	0.021	0.17	988 (188; 3510)	341 (80.2; 1006)	69.9 (22.3; 179)	6.75 (3.54; 12.7)
		$D7$	0.0018	0.0053	0.022	0.18	669 (254; 1525)	246 (99.4; 450)	55.1 (26.4; 91.1)	5.94 (3.91; 8.30)
		ATS	0.0022	0.0061	0.025	0.19	571 (209; 1312)	213 (83.8; 397)	49.0 (23.0; 82.0)	5.54 (3.62; 7.77)
9	50	$\bar{S}$	1.3×10^{-4}	8.5×10^{-4}	0.0071	0.14	1.4×10^{8} ($1173; 1.7\times10^{7}$)	1.1×10^{7} ($208; 1.1\times10^{6}$)	1.9×10^{5} ($30.5; 3.5\times10^{4}$)	190 (2.72; 166)
		$\bar{R}$	1.0×10^{-4}	6.7×10^{-4}	0.0058	0.12	2.1×10^{9} ($1408; 4.1\times10^{7}$)	2.1×10^{8} ($260; 3.1\times10^{6}$)	9.0×10^{10} ($36.0; 9.2\times10^{4}$)	322 (2.92; 314)
		$I\bar{Q}R$	0.0018	0.0084	0.048	0.42	880 (194; 2921)	200 (49.7; 444)	28.5 (10.4; 54.7)	2.56 (1.73; 4.60)
		$I\bar{Q}R_{20}$	0.0019	0.0087	0.049	0.42	905 (177; 3084)	203 (61.5; 768)	28.9 (12.1; 82.3)	2.57 (1.81; 4.29)
		$D7$	0.0020	0.0092	0.052	0.44	628 (232; 1423)	154 (131; 243)	23.6 (21.3; 34.4)	2.38 (2.23; 2.88)
		ATS	0.0023	0.010	0.056	0.45	551 (198; 1248)	137 (63.0; 216)	21.8 (12.3; 31.3)	2.30 (1.83; 2.76)
	100	$\bar{S}$	7.4×10^{-5}	5.7×10^{-4}	0.0054	0.13	4.8×10^{5} ($2429; 1.9\times10^{6}$)	3.6×10^{4} ($577; 1.0\times10^{5}$)	1405 (66.0; 4594)	15.1 (3.86; 45.8)
		$\bar{R}$	5.5×10^{-5}	4.4×10^{-4}	0.0043	0.11	1.2×10^{6} ($3124; 3.6\times10^{6}$)	1.0×10^{5} ($532; 2.9\times10^{5}$)	2.5×10^{3} ($62.6; 1.1\times10^{4}$)	19.9 (3.82; 77.3)

(Continues)

Table V. *Continued.*

			p				ARL			
n	*k*	Chart	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$
		$I\bar{Q}R$	0.0018	0.0086	0.050	0.43	699 (254; 1592)	152 (60.7; 443)	23.6 (12.1; 52.6)	2.40 (1.83; 3.44)
		$I\bar{Q}R_{20}$	0.0019	0.0089	0.051	0.43	685 (235; 1674)	150 (58.4; 354)	23.4 (11.7; 45.4)	2.38 (1.81; 3.24)
		D7	0.0019	0.0092	0.053	0.45	572 (293; 1026)	131 (123; 192)	21.2 (20.1; 28.6)	2.29 (2.21; 2.64)
		ATS	0.0022	0.010	0.058	0.46	496 (251; 894)	116 (81.4; 178)	19.4 (14.9; 27.0)	2.20 (1.97; 2.57)

Table VI. Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when localized variance disturbances are present in phase I

			p				*ARL*			
n	k	Chart	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$
5	50	$\bar{S}$	1.3×10^{-4}	5.2×10^{-4}	0.0030	0.047	2.3×10^{4} ($1843; 1.1\times10^{5}$)	6410 ($1054; 2.1\times10^{4}$)	828 (192; 2442)	32.8 (12.8; 76.9)
		$\bar{R}$	1.3×10^{-4}	5.1×10^{-4}	0.0030	0.046	2.5×10^{4} ($1805; 1.3\times10^{5}$)	6946 ($767; 2.3\times10^{4}$)	868 (144; 2606)	33.7 (10.8; 80.7)
		$I\bar{Q}R$	1.8×10^{-4}	6.3×10^{-4}	0.0033	0.047	1.5×10^{5} ($918; 7.2\times10^{5}$)	3.0×10^{4} ($743; 1.2\times10^{5}$)	2.7×10^{3} ($150; 1.1\times10^{5}$)	60.7 (10.8; 231)
		$I\bar{Q}R_{20}$	0.0011	0.0033	0.014	0.13	3703 ($193; 2.1\times10^{4}$)	1171 (76.8; 4467)	186 (41.4; 636)	11.8 (3.48; 30.0)
		$D7$	0.0015	0.0044	0.018	0.16	1112 (224; 3717)	408 (90.4; 1536)	83.5 (24.5; 257)	7.48 (3.74; 15.8)
		ATS	0.0021	0.0057	0.023	0.18	843 (155; 2899)	321 (216; 780)	68.0 (70.2; 144)	6.58 (6.50; 11.2)
	100	$\bar{S}$	1.2×10^{-4}	4.7×10^{-4}	0.0029	0.048	1.5×10^{4} ($2865; 5.0\times10^{4}$)	3766 (780; 9525)	534 (144; 1222)	25.4 (11.1; 47.3)
		$\bar{R}$	1.1×10^{-4}	4.6×10^{-4}	0.0029	0.047	1.6×10^{4} ($2870; 5.3\times10^{4}$)	3904 ($1034; 1.2\times10^{4}$)	549 (184; 1456)	26.1 (12.8; 53.0)
		$I\bar{Q}R$	1.4×10^{-4}	5.3×10^{-4}	0.0031	0.048	2.9×10^{4} ($1630; 1.6\times10^{5}$)	6682 ($264; 2.7\times10^{4}$)	827 (144; 3022)	32.4 (10.8; 90.0)
		$I\bar{Q}R_{20}$	0.0011	0.0033	0.015	0.14	1775 (285; 6672)	554 (107; 1654)	105 (28.1; 272)	8.68 (4.06; 16.9)
		$D7$	0.0014	0.0043	0.019	0.16	886 (309; 2132)	309 (114; 629)	66.0 (29.6; 120)	6.65 (4.18; 9.89)
		ATS	0.0020	0.0056	0.023	0.19	652 (218; 1603)	236 (89.9; 468)	52.9 (24.4; 94.4)	5.80 (3.73; 8.47)
9	50	$\bar{S}$	1.2×10^{-4}	8.9×10^{-4}	0.0083	0.17	1.5×10^{4} ($2724; 5.1\times10^{4}$)	2.3×10^{3} (505; 6959)	193 (59.9; 487)	6.54 (3.72; 11.4)
		$\bar{R}$	1.2×10^{-4}	9.0×10^{-4}	0.0083	0.17	1.6×10^{4} ($2608; 6.0\times10^{4}$)	2432 ($773; 1.6\times10^{4}$)	201 (82.4; 930)	6.69 (4.26; 16.0)
		$I\bar{Q}R$	1.4×10^{-4}	9.9×10^{-4}	0.0086	0.17	3.0×10^{4} ($1671; 1.7\times10^{5}$)	4082 ($277; 1.4\times10^{4}$)	292 (38.0; 899)	7.66 (3.02; 16.7)
		$I\bar{Q}R_{20}$	0.0014	0.0067	0.040	0.38	1281 (229; 4581)	273 (87.2; 621)	35.7 (15.6; 71.4)	2.82 (2.00; 4.12)
		$D7$	0.0016	0.0076	0.045	0.41	823 (273; 2013)	190 (118; 348)	27.6 (19.6; 45.1)	2.55 (2.18; 3.26)
		ATS	0.0025	0.011	0.060	0.46	509 (176; 1211)	126 (96.9; 225)	20.4 (17.0; 32.2)	2.24 (2.04; 2.79)
	100	$\bar{S}$	1.1×10^{-4}	8.6×10^{-4}	0.0082	0.18	1.2×10^{4} ($3975; 3.0\times10^{4}$)	1650 (690; 2965)	152 (56.6; 249)	5.95 (4.21; 8.00)
		$\bar{R}$	1.1×10^{-4}	8.6×10^{-4}	0.0083	0.18	1.3×10^{4} ($3776; 3.3\times10^{4}$)	1691 (1534; 3841)	154 (142; 303)	5.98 (5.53; 8.84)

(*Continues*)

Table VI. *Continued.*

			p				*ARL*			
n	*k*	Chart	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$
		$I\bar{Q}R$	1.2×10^{-4}	9.0×10^{-4}	0.0084	0.18	1686 (2599; 6.5×10^{4})	2130 (404; 6258)	180 (50.8; 453)	6.38 (3.47; 11.2)
		$I\bar{Q}R_{20}$	0.0014	0.0069	0.042	0.40	971 (310; 2447)	194 (86.61 387)	28.3 (45.6; 49.1)	2.60 (2.02; 3.41)
		D7	0.0015	0.0076	0.045	0.42	748 (359; 1422)	158 (79.6; 249)	24.5 (14.8; 34.9)	2.44 (1.99; 2.90)
		ATS	0.0025	0.011	0.061	0.47	455 (226; 842)	105 (57.4; 184)	18.0 (11.6; 27.5)	2.14 (1.80; 2.58)

Table VII. Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when diffuse mean disturbances are present in phase I

			p				ARL			
n	k	Chart	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$	$\delta=0$	$\delta=0.25$	$\delta=0.5$	$\delta=1$
5	50	$\bar{S}$	2.5×10^{-4}	8.8×10^{-4}	0.0046	0.062	1.2×10^{4} (991; 5.4×10^{4})	6444 (341; 1.5×10^{4})	976 (72.5; 1754)	32.3 (7.15; 60.0)
		$\bar{R}$	2.3×10^{-4}	8.3×10^{-4}	0.0045	0.060	1.3×10^{4} (1030; 6.1×10^{4})	7379 (1068; 4.4×10^{4})	1117 (216; 4737)	35.3 (13.4; 115)
		$I\bar{Q}R$	9.4×10^{-4}	0.0028	0.012	0.12	4141 (245; 2.2×10^{4})	2120 (108; 6419)	306 (28.4; 866)	15.4 (4.05; 36.4)
		$I\bar{Q}R_{20}$	0.0013	0.0036	0.015	0.13	3177 (176; 1.7×10^{4})	1530 (83.2; 9626)	264 (23.1; 1253)	12.4 (3.60; 45.4)
		$D7$	9.6×10^{-4}	0.0029	0.013	0.12	2269 (299; 9368)	1006 (146; 3666)	182 (36.7; 359)	11.5 (4.67; 25.9)
		ATS	0.0017	0.0047	0.019	0.16	1343 (175; 5726)	598 (108; 1457)	115 (29.6; 244)	8.57 (4.07; 15.7)
	100	$\bar{S}$	2.1×10^{-4}	8.1×10^{-4}	0.0045	0.063	7771 (1554; 2.5×10^{4})	3188 (823; 2.2×10^{3})	484 (155; 2638)	22.4 (11.2; 75.0)
		$\bar{R}$	2.0×10^{-4}	7.6×10^{-4}	0.0042	0.061	8521 (1651; 2.8×10^{4})	3520 (951; 1.5×10^{4})	521 (176; 1756)	23.6 (12.1; 57.6)
		$I\bar{Q}R$	8.7×10^{-4}	0.0027	0.012	0.12	2173 (366; 8109)	790 (142; 1924)	143 (35.4; 310)	10.2 (4.63; 18.4)
		$I\bar{Q}R_{20}$	0.0012	0.0036	0.016	0.14	1564 (262; 5767)	568 (99.9; 1570)	106 (26.5; 260)	8.65 (3.93; 16.2)
		$D7$	8.8×10^{-4}	0.0028	0.013	0.13	1662 (433; 4800)	618 (172; 2467)	117 (421.3; 389)	9.27 (5.07; 20.5)
		ATS	0.0016	0.0045	0.019	0.16	934 (247; 2762)	350 (94.9; 1218)	73.2 (25.4; 213)	6.94 (3.83; 13.9)
9	50	$\bar{S}$	2.2×10^{4}	0.0014	0.011	0.20	9178 (1508; 3.2×10^{4})	3394 (2351; 5964)	272 (216; 425)	7.00 (6.50; 10.4)
		$\bar{R}$	1.4×10^{-4}	0.0010	0.0089	0.17	1.5×10^{4} (1923; 5.8×10^{4})	6628 (512; 7.8×10^{4})	462 (60.0; 3614)	9.05 (3.67; 33.9)
		$I\bar{Q}R$	0.0012	0.0060	0.037	0.36	1479 (278; 5152)	373 (64.8; 693)	44.8 (12.7; 77.7)	3.10 (1.87; 4.29)
		$I\bar{Q}R_{20}$	0.0014	0.0066	0.040	0.38	1305 (235; 4557)	332 (391; 964)	41.0 (56.5; 98.8)	2.95 (3.29; 4.74)
		$D7$	0.0011	0.0055	0.035	0.36	1336 (361; 3751)	350 (100; 533)	43.5 (17.4; 63.2)	3.06 (2.11; 3.87)
		ATS	0.0017	0.0081	0.047	0.41	819 (228; 2339)	215 (159; 496)	30.0 (24.8; 58.8)	2.59 (2.37; 3.66)
	100	$\bar{S}$	1.8×10^{-4}	0.0013	0.011	0.20	7369 (2201; 1.9×10^{4})	2127 (450; 1.5×10^{4})	178 (54.6; 893)	6.07 (3.55; 14.9)
		$\bar{R}$	1.3×10^{-4}	9.7×10^{-4}	0.0087	0.17	1.1×10^{4} (2834; 3.1×10^{4})	3640 (999; 7126)	273 (101; 488)	7.51 (4.67; 11.1)

(*Continues*)

Table VII. *Continued.*

n	k	Chart	p $\delta=0$	p $\delta=0.25$	p $\delta=0.5$	p $\delta=1$	ARL $\delta=0$	ARL $\delta=0.25$	ARL $\delta=0.5$	ARL $\delta=1$
		$I\bar{Q}R$	0.0011	0.0059	0.037	0.38	1137 (376; 2822)	259 (82.2; 592)	34.2 (15.1; 67.5)	2.82 (2.01; 3.92)
		$I\bar{Q}R_{20}$	0.0013	0.0068	0.041	0.39	979 (316; 2457)	220 (71.6; 373)	30.8 (13.6; 47.9)	2.67 (1.92; 3.38)
		$D7$	0.0010	0.0054	0.035	0.37	1166 (484; 2483)	265 (101; 521)	35.8 (17.6; 61.1)	2.87 (2.14; 3.74)
		ATS	0.0017	0.0082	0.048	0.42	702 (303; 1479)	163 (93.3; 259)	24.7 (16.4; 36.0)	2.44 (2.05; 2.94)

also estimating the process standard deviation has substantially more impact on the $\bar{X}$ phase II control chart than only estimating the process mean.

The conditional ARL values are presented in parentheses. The first value in parentheses represents the ARL for the control limits associated with the 97.5% quantile of the simulated p in the in-control situation, whereas the second value represents the ARL for the control limit associated with the 2.5% quantile of the simulated p in the in-control situation. The results show that the conditional ARL values vary quite strongly, even when k equals 100.

In the absence of any contamination, the charts based on $\bar{S}$, $\bar{R}$, $D7$ and ATS show comparable performance. The charts based on $I\bar{Q}R$ and $I\bar{Q}R_{20}$ are less powerful under normality.

The analysis shows that, when there are disturbances in the phase I data, the performance of all charts changes considerably: p decreases and the ARL values increase. Thus, when the phase I data are contaminated, shifts in the process mean are less quickly detected. When symmetric disturbances are present (Table IV), their impact is the smallest for the charts based on $I\bar{Q}R$, $I\bar{Q}R_{20}$, $D7$ and ATS. These charts are also least affected when there are asymmetric disturbances (Table V). Both tables show that the chart based on ATS outperforms the others.

When there are localized disturbances (Table VI), the charts based on the estimators $D7$ and ATS perform best, the reason being that these charts trim extreme samples. Finally, in the case of diffuse mean disturbances (Table VII), the charts based on $I\bar{Q}R_{20}$, $D7$ and ATS perform better than the other charts.

Overall, the ATS chart performs best. Under normality, the chart essentially matches the performance of the standard charts based on $\bar{S}$ and $\bar{R}$ and, in the presence of any contamination, the chart outperforms the alternatives.

5. Concluding remarks and future research

In this article, we have considered several estimation methods for the standard deviation parameter. The MSE of the estimators has been assessed under various circumstances: the uncontaminated situation and various situations contaminated with diffuse symmetric and asymmetric variance disturbances, localized variance disturbances and diffuse mean disturbances. Moreover, we have investigated the effect of estimating the standard deviation estimator on the $\bar{X}$ phase II control chart performance when the methods are used to determine the phase II limits.

The standard methods suffer from a number of problems. Estimators that are based on the principle of trimming observations (e.g., $I\bar{Q}R$) perform reasonably well when there are diffuse disturbances but not when there are localized disturbances. In the latter situation, estimators that include a method to trim sample statistics (e.g., $I\bar{Q}R_{20}$) are efficient. All of these methods are biased when there are asymmetric disturbances, as the trimming principle does not take into account the asymmetry of the disturbance.

A phase I analysis—using a control chart to study a historical dataset retrospectively and trim the data adaptively—does take into account the distribution of the data and is therefore very suitable for use during the estimation of σ. In this article, we have proposed a new type of phase I analysis. The initial estimate of σ for the phase I control chart is given by an estimator that is robust against both diffuse and localized disturbances, namely $I\bar{Q}R_{20}$. We have shown that this estimator is not very efficient under normality. However, when $I\bar{Q}R_{20}$ is only used to construct the phase I control chart limits, and when the standard estimation method $\bar{S}$ is used to determine the final estimate of σ after screening, the resulting estimator (ATS) is efficient under normality. Moreover, ATS outperforms the other estimation methods when there are contaminations. It is therefore a suitable method for determining the value of σ in the $\bar{X}$ phase II control chart limits.

References

1. Vining G. Technical advice: phase I and phase II control charts. *Quality Engineering* 2009; **21**:478–479.
2. Mukherjee A, Chakraborti S. A distribution-free control chart for the joint monitoring for location and scale. *Quality and Reliability Engineering International* 2012; **28**:335–352.
3. Chen H, Pao Y. The joint economical statistical design of $\bar{X}$ and R charts for nonnormal data. *Quality and Reliability Engineering International* 2011; **27**:269–280.
4. Zhang Y, Castagliola P, Wu Z, Khoo MBC. The variable sampling interval $\bar{X}$ chart with estimated parameters. *Quality and Reliability Engineering International* 2012; **28**:19–34.
5. Kim YB, Hong JS, Lie CH. Economic-statistical design of 2-of-2 and 2-of-3 runs rule scheme. *Quality and Reliability Engineering International* 2009; **25**:215–228.
6. Riaz M, Mehmood R, Does RJMM. On the performance of different control charting rules. *Quality and Reliability Engineering International* 2011; **27**:1059–1067.
7. Schoonhoven M, Riaz M, Does RJMM. Design schemes for the $\bar{X}$ control chart. *Quality and Reliability Engineering International* 2009; **25**:581–594.
8. Schoonhoven M, Does RJMM. The $\bar{X}$ control chart under non-normality. *Quality and Reliability Engineering International* 2010; **26**:167–176.
9. Quesenberry CP. The effect of sample size on estimated limits for $\bar{X}$ and X control chart. *Journal of Quality Technology* 1993; **25**:237–247.
10. Jensen WA, Jones-Farmer LA, Champ CW, Woodall WH. Effects of parameter estimation on control chart properties: a literature review. *Journal of Quality Technology* 2006; **38**:349–364.
11. Schoonhoven M, Nazir ZN, Riaz M, Does RJMM. Robust location estimators for the $\bar{X}$ control chart. *Journal of Quality Technology* 2011; **43**:363–379.
12. Rocke DM. Robust Control Charts. *Technometrics* 1989; **31**:173–184.
13. Rocke DM. $\bar{X}_Q$ and R_Q charts: robust control charts. *The Statistician* 1992; **41**:97–104.
14. Tatum LG. Robust estimation of the process standard deviation for control charts. *Technometrics* 1997; **39**:127–141.

15. Schoonhoven M, Riaz M, Does RJMM. Design and analysis of the standard deviation control chart with estimated parameters. *Journal of Quality Technology* 2011; **43**:307–333.
16. Schoonhoven M, Does RJMM. A robust standard deviation control chart. *Technometrics* 2012; **54**:73–82.
17. Duncan AJ. Quality Control and Industrial Statistics. R.D. Irwin Inc.: Homewood, 1986; 5.
18. Jones-Farmer LA, Jordan V, Champ CW. Distribution-free phase I control charts for subgroup location. *Journal of Quality Technology* 2009; **41**:304–316.
19. Tukey JW. Exploratory Data Analysis. Addison-Wesley: Boston, 1997.

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