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A Nonlinear Structural Model for Volatility Clustering

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Abstract

A simple nonlinear structural model of endogenous belief heterogeneity is proposed. News about fundamentals is an IID random process, but nevertheless volatility clustering occurs as an endogenous phenomenon caused by the interaction between different types of traders, fundamentalists and technical analysts. The belief types are driven by an adaptive, evolutionary dynamics according to the success of the prediction strategies in the recent past conditioned upon price deviations from the rational expectations fundamental price. Asset prices switch irregularly between two different regimes – close to the fundamental price fluctuations with low volatility, and periods of persistent deviations from fundamentals triggered by technical trading – thus, creating time varying volatility similar to that observed in real financial data.

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1 Introduction

Volatility clustering is one of the most important ‘stylized facts’ in financial time series data. Whereas price changes themselves appear to be unpredictable, the magnitude of those changes, as measured e.g. by the absolute or squared returns, appears to be predictable in the sense that large changes tend to be followed by large changes – of either sign – and small changes tend to be followed by small changes. Asset price fluctuations are thus characterized by episodes of low volatility, with small price changes, irregularly interchanged by episodes of high volatility, with large price changes. This phenomenon was first observed by Mandelbrot (1963) in commodity prices.¹ Since the pioneering papers by Engle (1982) and Bollerslev (1986) on autoregressive conditional heteroskedastic (ARCH) models and their generalization to GARCH models, volatility clustering has been shown to be present in a wide variety of financial assets including stocks, market indices, exchange rates, and interest rate securities.²

In empirical finance, volatility clustering is usually modeled by a *statistical* model, for example by a (G)ARCH model or one of its extensions, where the conditional variance of returns follows a low order autoregressive process. Other approaches to modelling volatility clustering and long memory by statistical models are fractionally integrated time series models (Granger (1980), Granger and Joyeux (1980)) and multi-fractal models (Mandelbrot (1997, 1999)). Whereas all these models are extremely useful as a statistical description of the data, they do not offer a structural explanation of why volatility clustering is present in so many financial time series. Rather the statistical models postulate that the phenomenon has an exogenous source and is for example caused by the clustered arrival of random ‘news’ about economic fundamentals.

The volatility of financial assets is a key feature for measuring risk underlying many investment decisions in financial practice. It is therefore important to gain theoretical insight into economic forces that may contribute to or amplify volatility and cause, at least in part, its clustering. In this paper we present a simple nonlinear structural model where price changes are driven by a combination of exogenous random news about fundamentals and evolutionary forces underlying the trading process itself. Volatility clustering becomes an endogenous phenomenon caused by the interaction between *heterogeneous* traders, fundamentalists and technical analysts, having different trading strategies and expectations about future prices and dividends of a risky asset. Fundamentalists believe that prices will move towards its fundamental RE value, as given by the expected discounted sum of future dividends. In contrast, the technical analysts observe past prices and try to extrapolate historical patterns. The chartists are not completely unaware of the fundamental price however, and condition their technical trading rule upon the deviation of the actual price from its fundamental value. The fractions of the two different

¹Mandelbrot (1963, pp. 418–419) notes that Houthakker stressed this fact for daily cotton prices, at several conferences and private conversation.

²See, for example, Pagan (1996) or Brock (1997) for further discussion of ‘stylized facts’ that are observed in financial data.

trader types change over time according to evolutionary fitness, as measured by utility from past realized profits (or equivalently forecasting accuracy in the recent past), conditioned upon price deviations from the rational expectations (RE) fundamental price. The heterogeneous market is characterized by an irregular switching between phases of low volatility, where price changes are small, and phases of high volatility, where small price changes due to random news are reinforced and can become large due to trend following trading rules. Volatility clustering is thus driven by heterogeneity and conditional evolutionary learning. Although our model is very simple, it is able to generate autocorrelation patterns of returns, and absolute and squared returns very similar to those observed in 40 years of daily S&P500 data.

Due to heterogeneity in expectations, our evolutionary model is a *nonlinear* dynamical system exhibiting periodic and even chaotic fluctuations in asset prices and returns. Nonlinear dynamic models can generate a wide variety of irregular patterns. In particular, nonlinear dynamic models can generate any given autocorrelation pattern.³ A nonlinear, chaotic model, buffeted with dynamic noise, with almost no autocorrelations in returns but at the same time persistence in absolute and squared returns, with slowly decaying autocorrelations, may thus provide a structural explanation of the unpredictability of the first moment of asset returns and at the same time the remaining structure of the second moment. In fact, there are two well-known, fundamentally important concepts in nonlinear dynamics which appear to be extremely well suited as a description of the phenomenon of volatility clustering. The first is the phenomenon of *intermittency*, as introduced by Pomeau and Manneville (1980). Intermittency means chaotic asset price fluctuations characterized by phases of almost periodic fluctuations irregularly interrupted by sudden bursts of erratic fluctuations. In our conditional evolutionary model intermittency occurs characterized by close to the RE fundamental steady state fluctuations suddenly interrupted by price deviations from the fundamental triggered by technical trading. The second phenomenon naturally suited to describe volatility clustering is *coexistence of attractors*. In particular, our evolutionary model exhibits coexistence of a stable steady state and a stable limit cycle. When buffeted with dynamic noise, irregular switching occurs between close to the fundamental steady state fluctuations, with small price changes, and periodic fluctuations, triggered by technical trading, with large price changes. It is important to note that both, intermittency and coexistence of attractors, are *persistent* phenomena, which are by no means special for our conditionally evolutionary systems, but occur naturally in nonlinear dynamic models, and moreover are robust with respect to and sometimes even reinforced by dynamic noise.⁴

³Sakai and Tokumaru (1980) already showed that the class of chaotic one-dimensional asymmetric tent maps can generate any AR(1) autocorrelation structure. To see that a higher dimensional chaotic map can generate any desired autocorrelation structure, consider the nonlinear difference equation $(x_t, y_t) = (a_1 x_{t-1} + \dots + a_L x_{t-L} + y_{t-1}, 1 - 2y_{t-1}^2)$. As is well known, the second coordinate y_t follows a chaotic process with zero mean and zero autocorrelations at all lags. Since y_t is generated independently of past values of x_t , the series x_t and y_t are uncorrelated. The first coordinate x_t thus follows a linear AR(L) process driven by a chaotic series with zero autocorrelations at all lags, and thus has the desired autocorrelation structure.

⁴Coexistence of attractors is a *generic*, structurally stable phenomenon, occurring for an open set of

Whereas the fundamentalists have some ‘rational valuation’ of the risky asset, the technical analysts use a simple extrapolation rule to forecast asset prices. An important critique from ‘rational expectations finance’ upon heterogeneous agent models using simple habitual rule of thumb forecasting rules is that ‘irrational’ traders will *not* survive in the market. For example, Friedman (1953) argues that irrational speculative traders would be driven out of the market by rational traders, who would trade against them by taking infinitely long opposite positions, thus driving prices back to fundamentals. In an efficient market, ‘irrational’ speculators would simply lose money and disappear from the market. However, for example, De Long et al. (1990) have shown that a constant fraction of noise traders may on average earn higher expected returns than rational or smart money traders, and may survive in the market with positive probability. Brock and Hommes (1997a,b, 1998, 1999) have also discussed this point extensively in a series of papers, and stress the fact that in an evolutionary framework technical analysts are not ‘irrational’, but they are in fact *boundedly rational*, since in periods when prices deviate from the RE fundamental price, chartists make better forecasts and earn higher profits than fundamentalists. See also the survey in Hommes (2000) or the interview with Buz Brock in Woodford (2000). In another related recent paper LeBaron (2000) argues that ‘... *in a dynamically evolving market long horizon investors will have to compete with a heterogeneous group of short horizon types who may end up dominating, or at least doing well enough to survive*’. The short horizon traders could be identified with noise traders or technical analysts, whereas the long run horizon traders could be identified with fundamentalists. In a heterogeneous world, on average, technical analysts and fundamentalists may earn approximately equal profits, so that in general fundamentalists *cannot* drive chartists out of the market.

We would like to relate our work to some other recent literature. Agent based evolutionary modeling of financial markets is becoming quite popular and recent contributions include the computational oriented work on the Santa Fe artificial stock market by Arthur et al. (1997) and LeBaron et al. (1999), the stochastic multi-agent models of Lux and Marchesi (1999a,b) and the evolutionary markets based on out-of-equilibrium price formation rules by Farmer (2000).⁵ Malliaris and Stein (1999) discuss price volatility as the outcome of a heterogeneous agents dynamic system with an underlying stochastic foundation. Brock et al. (1992) contains empirical work showing that simple technical trading rules applied to the Dow Jones Index may yield positive returns, suggesting extra structure above and beyond the EMH fundamental. Another recent branch of work concerns adaptive learning in asset markets. For example, Timmerman (1993, 1996) shows that excess volatility in stock returns can arise under learning processes that converge (slowly) to RE. Routledge (1999) investigates adaptive learning in the Grossman-Stiglitz model where traders can choose to acquire a costly signal about dividends, and derives conditions under which

parameter values. Intermittency occurs when e.g. a stable cycle disappears and the system has a strange (chaotic) attractor. Recent mathematical results on homoclinic bifurcations have shown that strange attractors are *persistent* in the sense that they typically occur for a positive Lebesgue measure set of parameter values, see e.g. Palis and Takens (1993) for a mathematical treatment.

⁵An early example of a heterogeneous agent model is Zeeman (1974); other examples include Frankel and Froot (1988), Kirman (1991), Chiarella (1992), Brock (1993), and Lux (1995).

the learning process converges to RE.⁶ An important characteristic that distinguishes our approach is the heterogeneity in expectation rules, with time varying fractions of trader types driven by evolutionary competition. These adaptive, evolutionary forces can lead to endogenous asset price fluctuations around the (stable or unstable) benchmark RE fundamental steady state, thus creating excess volatility and volatility clustering.

In our model, volatility clustering is driven by conditional evolutionary learning of different trader types. Haugen (1999), for example, provides a motivation of this approach by arguing that the most important part of stock volatility is created by the trading process itself. He calls this part ‘price-driven volatility’, in contrast to the ‘event-driven volatility’ (which is consistent with the EMH) and the ‘error-driven volatility’ (due to over- and under reactions of the market to informations coming from the real world).⁷ The price driven volatility is created because *“investors focus their attention on changes in the value of the market index. They respond to signals from the market index, and then simultaneously take action by trading in individual assets.”* This causes a *“complex process by which the market reacts to its own price history, (therefore) a particular sequence, or configuration, of price changes may trigger successive price reactions that eventually build into a volatility increase or decrease.”* (Haugen, 1999, p. 127, 128). Investors watch and learn from the pricing process, which they, as traders, cocreate. Thus, the nature of price-driven volatility are *“price reactions that feed on themselves.”* The theory presented here is in line with these observations, since in our setup an increase in volatility may be triggered by ‘news’ about economic fundamentals and reinforced by evolutionary forces between competing, boundedly rational trading strategies.

The paper is organized as follows. Section 2 presents the conditional evolutionary asset pricing model with fundamentalists and technical analysts. In section 3 we compare the time series properties of the model, in particular the autocorrelation patterns of returns, squared returns, and absolute return with those of 40 years of daily S&P 500 data. Finally, section 4 presents some concluding remarks.

⁶In Routledge (1999), the fraction of informed traders is fixed over time. De Fontnouvelle (2000) investigates a Grossman-Stiglitz model where traders can choose to buy a costly signal about dividends, with fractions of informed and uninformed traders changing over time according to evolutionary fitness. De Fontnouvelle is in fact an application of the Brock and Hommes (1997a) *Adaptive Rational Equilibrium Dynamics (ARED)* framework, which is also underlying our heterogeneous agent asset pricing model, to the Grossman-Stiglitz model and leads to unpredictable (chaotic) fluctuations in asset prices.

⁷The role of quasi-rational overreaction and biased traders in financial markets is emphasized in Thaler (1993, part III) and in Thaler (1994, part V). Recent papers addressing the issue of market under- and overreaction are Daniel et al. (1998) and Veronesi (1999).

2 A Heterogeneous Agents Model

Our nonlinear model for volatility clustering will be a standard discounted value asset pricing model with two types of traders, fundamentalists and technical analysts. The model is closely related to the present discounted value asset pricing model with *heterogeneous* beliefs introduced by Brock and Hommes (1997b, 1998). One important extension is that our technical analysts condition their price forecast upon the deviation of the actual price from the rational expectations fundamental price, similar to the approach taken in the Santa Fe artificial stock market in Arthur et al. (1997) and LeBaron et al. (1999).

Agents can either invest their money in a risk free asset, say a T-bill, that pays a fixed rate of return r , or they can invest their money in a risky asset, for example a large stock or a market index traded at price p_t (ex-dividend) at time t , that pays uncertain dividends y_t in future periods t , and therefore has an uncertain return. Let E_{ht} and V_{ht} denote the ‘beliefs’ (forecasts) of trader type h about conditional expectation and conditional variance. The demand z_{ht} of type h for the risky asset is derived from myopic mean-variance maximization, and given by

$$z_{ht} = \frac{E_{ht}[p_{t+1} + y_{t+1} - (1+r)p_t]}{aV_{ht}[p_{t+1} + y_{t+1} - (1+r)p_t]} = \frac{E_{ht}[p_{t+1} + y_{t+1} - (1+r)p_t]}{a\sigma^2}, \quad (1)$$

where a is the risk aversion parameter and the conditional variance $V_{ht}[p_{t+1} + y_{t+1} - (1+r)p_t] = \sigma^2$ is assumed to be constant and equal for all types.⁸ Let z^s denote the supply of outside risky shares per investor, assumed to be constant, and let n_{ht} denote the fraction of type h at date t . Equilibrium of demand and supply yields

$$\sum_{h=1}^H n_{ht} \frac{E_{ht}[p_{t+1} + y_{t+1} - (1+r)p_t]}{a\sigma^2} = z^s, \quad (2)$$

where H is the number of different trader types.

In the case of zero supply of outside risky assets,⁹ i.e. $z^s = 0$, the market equilibrium equation may be rewritten as

$$(1+r)p_t = \sum_{h=1}^H n_{ht} E_{ht}(p_{t+1} + y_{t+1}). \quad (3)$$

In a world where all traders are identical and expectations are *homogeneous* the arbitrage market equilibrium equation (3) for the price p_t of the risky asset reduces to

$$(1+r)p_t = E_t(p_{t+1} + y_{t+1}), \quad (4)$$

⁸Gaunersdorfer (2000) analyzes the case with time varying beliefs about variances and shows that the results are quite similar to those with constant ones. Therefore we concentrate on this simple case.

⁹This assumption is without loss of generality, since one can introduce a risk adjusted dividend $y_{t+1}^\# = y_{t+1} - a\sigma^2 z^s$ to obtain the market equilibrium equation (3), as in Brock (1997).

where E_t denotes the common conditional expectation of all traders at the beginning of period t , based on a publically available information set such as past prices and dividends. The arbitrage equation (4) states that today's price of the risky asset must be equal to the sum of tomorrow's expected price and expected dividend, discounted by the risk free interest rate. It is well known that in a world where expectations are *homogeneous*, where *all* traders are rational, and where it is *common knowledge* that all traders are rational, the fundamental rational expectations equilibrium price, or the *fundamental price* is

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t(y_{t+k})}{(1+r)^k}, \quad (5)$$

given by the discounted sum of expected future dividends. We will focus on the simplest case of a dividend process y_t with a constant mean, i.e. $E_t(y_{t+1}) = \bar{y}$, so that the fundamental price is constant and given by¹⁰

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}. \quad (6)$$

It is important to note that so-called speculative bubble solutions, growing at a constant rate $1+r$, also satisfy the arbitrage equation (4) at each date. In a homogeneous, perfectly rational world the existence of these *speculative bubbles* is excluded by the transversality condition

$$\lim_{t \rightarrow \infty} \frac{E(p_t)}{(1+r)^t} = 0,$$

and the constant fundamental solution (6) is the only solution of (4) satisfying this condition. Along a speculative bubble solution traders would have perfect foresight, but prices would diverge to infinity. In a homogeneous, perfectly rational world traders realize that speculative bubbles cannot last forever and therefore, they will never get started.

In the asset pricing model with heterogeneous beliefs, market equilibrium in (3) states that the price p_t of the risky asset equals the discounted value of tomorrow's expected price plus tomorrow's expected dividend, averaged over all different trader types. In such a *heterogeneous* world, temporary bubbles with prices deviating from the fundamental, may arise, when the fractions of traders believing in those bubbles is large enough. Notice that, within our heterogeneous agents equilibrium model (3) the standard present discounted value model is nested as a special case. In the nested RE benchmark, asset prices are only driven by economic fundamentals. In contrast, the heterogeneous agent model generates excess volatility driven by evolutionary competition between different trading strategies, leading to unpredictability and volatility clustering in asset returns.

In order to complete the model, we have to be more precise about traders' expectations (forecasts) about future prices and dividends. For simplicity we focus on the case where

¹⁰Notice that in our setup, the constant benchmark fundamental $p^* = \bar{y}/r$ could easily be replaced by another, more realistic time varying fundamental price p_t^* , as stressed in Brock and Hommes (1997b).

expectations about future dividends are the same for all traders and given by

$$E_{ht}(y_{t+1}) = E_t(y_{t+1}) = \bar{y}, \quad (7)$$

for each type h . All traders are thus able to derive the fundamental price $p^* = \bar{y}/r$ in (6) that would prevail in a perfectly rational world. Traders nevertheless believe that in a heterogeneous world prices will in general *deviate* from their fundamental value. We focus on a simple case with two types of traders, with expected prices given respectively by¹¹

$$E_{1t}[p_{t+1}] \equiv p_{1,t+1}^e = p^* + v(p_{t-1} - p^*), \quad 0 \leq v \leq 1, \quad (8)$$

$$E_{2t}[p_{t+1}] \equiv p_{2,t+1}^e = p_{t-1} + g(p_{t-1} - p_{t-2}), \quad g \geq 0. \quad (9)$$

Trader type 1 are *fundamentalists*, believing that tomorrow's price will move in the direction of the fundamental price p^* by a factor v . Of special interest is the case $v = 1$, for which

$$E_{1t}[p_{t+1}] \equiv p_{1,t+1}^e = p_{t-1}. \quad (10)$$

We call this type of traders *EMH-believers*, since the naive forecast of the last observed price as prediction for tomorrow's price, is consistent with an efficient market where prices follow a random walk. Trader type 2 are simple *trend followers*, extrapolating the latest observed price change. The market equilibrium equation (3) in a heterogeneous world with fundamentalists and chartists as in (8)–(9), with common expectations on dividends as in (7), becomes

$$(1 + r)p_t = n_{1t}(p^* + v(p_{t-1} - p^*)) + n_{2t}(p_{t-1} + g(p_{t-1} - p_{t-2})) + \bar{y}, \quad (11)$$

where n_{1t} and n_{2t} represent the fraction of fundamentalists and chartists respectively, at date t . At this point we also would like to introduce (additive) dynamic noise into the system, to obtain

$$(1 + r)p_t = n_{1t}(p^* + v(p_{t-1} - p^*)) + n_{2t}(p_{t-1} + g(p_{t-1} - p_{t-2})) + \bar{y} + \varepsilon_t, \quad (12)$$

where ε_t are IID random variables representing the *model approximation error* in that our model can at best be only an approximation of the real world. One can interpret this noise term also as coming from *noise traders*, i.e., traders, whose behavior is not explained by the model but considered as exogenously given (c.f. for example, Kyle, 1985).

The market equilibrium equation (12) represents the first part of the model. The second, *conditionally evolutionary* part of the model describes how the fractions of fundamentalists and technical analysts change over time. The basic idea is that fractions are updated according to past performance, conditioned upon the deviation of actual prices from the fundamental price. The evolutionary competitive part of the updating scheme follows Brock and Hommes (1997a,b, 1998a) and Gaunersdorfer (2000); the additional conditioning upon deviations from the fundamental is similar to the approach taken in the Santa

¹¹For example, Frankel and Froot (1986) and Kirman (1998) have been using exactly the same fundamental and chartist trader types.

Fe artificial stock market by Arthur et al. (1997) and LeBaron et al. (1999). We emphasize that agents are not irrational, but in fact *boundedly rational* in the sense that most of them will choose the forecasting rule that performed best in the recent past, conditioned upon deviations from the fundamental. Performance will be measured by risk adjusted past profits, that is, utilities derived from realized profits.

The first, evolutionary part of the updating of fractions of fundamentalists and technical analysts is described by the discrete choice probabilities

$$\tilde{n}_{ht} = \exp[\beta U_{h,t-1}] / Z_t, \quad h = 1, 2 \quad (13)$$

where $Z_t = \sum_{h=1}^2 \exp[\beta U_{h,t-1}]$ is just a normalization factor such that the fractions add up to one. $U_{h,t-1}$ measures the evolutionary fitness of predictor h in period $t-1$, given by utilities of realized past profits as discussed below. The parameter β is called the intensity of choice, measuring how fast the mass of traders will switch to the optimal prediction strategy. In the special case $\beta = 0$, both fractions will be constant and equal to $1/2$. In the other extreme case $\beta = \infty$, in each period *all* traders will use the same, optimal strategy.¹²

In order to define the performance measure U_{ht} , consider realized excess returns per share over period t to period $t+1$, which can be computed as

$$R_{t+1} = p_{t+1} + y_{t+1} - (1+r)p_t = p_{t+1} - p^* - (1+r)(p_t - p^*) + \delta_{t+1}, \quad (14)$$

where $\delta_{t+1} = y_{t+1} - \bar{y}$ is a martingale difference sequence. The term δ_t represents *intrinsic uncertainty* about economic fundamentals in a financial market, in our case unexpected random news about future dividends. Thus, realized excess returns (14) can be decomposed in an EMH-term δ_t and an endogenous dynamic term explained by the theory represented here, as stressed in BH (1998). Utilities of realized profits in period t of investortype h are given by

$$\pi_{ht} := \pi(R_{t+1}, E_{ht}[R_{t+1}]) = R_{t+1} z(E_{ht}[R_{t+1}]) - \frac{a}{2} \sigma^2 z^2(E_{ht}[R_{t+1}]), \quad (15)$$

where¹³

$$z(E_{ht}[R_{t+1}]) = \arg \max \pi_{ht} = \frac{E_{ht}[R_{t+1}]}{a\sigma^2} = z_{ht}.$$

Since the fractions (13) are independent of the utility level, that is, the fractions do not change if the same term is subtracted off the exponents, without loss of generality we can subtract the same term π_t from all utilities π_{ht} to obtain

$$\begin{aligned} \pi_{ht} - \pi_t &= \pi(R_{t+1}, E_{ht}[R_{t+1}]) - \pi(R_{t+1}, R_{t+1}) \\ &= -\frac{1}{2a\sigma^2} (E_{ht}[R_{t+1}] - R_{t+1})^2 \\ &= -\frac{1}{2a\sigma^2} (p_{t+1} - p_{h,t+1}^e + \delta_{t+1})^2, \end{aligned}$$

¹²Brock and Hommes (1997a, 1998) show that for a high intensity of choice, the evolutionary adaptive system is close to having a *homoclinic orbit*, Poincaré's classical notion and key feature of chaotic systems.

¹³Note that in our mean-variance framework maximizing expected utilities of profits is equivalent to maximizing expected utilities of wealth, hence $z(E_{ht}[R_{t+1}])$ coincides with z_{ht} in equation (1).

where $\pi_t = \pi(R_{t+1}, R_{t+1})$ is the utility of profits of perfect foresight investors. Note that in the absence of random shocks, $\delta_t \equiv 0$, these differences simply reduce to a negative constant times the squared prediction errors. Taking also previous periods into account, we define the fitness measure as

$$U_{ht} = \pi_{h,t-1} - \pi_{t-1} + \eta U_{h,t-1} = -\frac{1}{2a\sigma^2}(p_t - p_{ht}^e + \delta_{t+1})^2 + \eta U_{h,t-1}, \quad (16)$$

where the parameter η , $0 \leq \eta \leq 1$, represents ‘memory strength’. In the special case $\eta = 0$, fitness is given by utilities of realized profits in the most recently observed period, whereas for positive η it is an exponentially moving average of past utilities of profits.¹⁴ The timing of the coupling between the market equilibrium equation (12) and the evolutionary selection of strategies (13) is important. The market equilibrium price p_t in (12) depends upon the fractions n_{ht} . The notation in (13) stresses the fact that these fractions n_{ht} depend upon *past* fitnesses $U_{h,t-1}$, which in turn depend upon past prices p_{t-1} and dividends y_{t-1} in periods $t - 1$ and further in the past. After the equilibrium price p_t has been revealed by the market, it will be used in evolutionary updating of beliefs and determining the new fractions $n_{h,t+1}$. These new fractions $n_{h,t+1}$ will then determine a new equilibrium price p_{t+1} , etc. In the adaptive belief system, market equilibrium prices and fractions of different trading strategies thus co-evolve over time.

In the second step of updating of fractions, the conditioning on deviations from the fundamental by the technical traders is modeled as

$$n_{2t} = \tilde{n}_{2t} \exp[-(p_{t-1} - p^*)^2/\alpha], \quad \alpha > 0 \quad (17)$$

$$n_{1t} = 1 - n_{2t}. \quad (18)$$

According to (17) the fraction of technical traders decreases more, the further prices deviate from their fundamental value p^* . As long as prices are close to the fundamental, updating of fractions will almost completely be determined by evolutionary fitness, that is, by (13), but when prices move far away from the fundamental, the correction term $\exp[-(p_{t-1} - p^*)^2/\alpha]$ in (17) becomes small, representing the fact that more and more chartists start believing that a price correction towards the fundamental price is about to occur. Our conditional evolutionary framework thus models the fact that technical traders are conditioning their charts upon information about fundamentals, as is common practice in real markets.

The noisy conditional evolutionary asset pricing model with fundamentalists versus chartists is given by (12), (13), (16), and (17–18). A detailed mathematical analysis of the (*deterministic*) *skeleton*,¹⁵ where the noise terms ε_t and δ_t are set equal to zero, is

¹⁴Brock and Hommes (1997b, 1998, 1999) use past realized profits as performance measure with $\pi_{ht} = R_{t+1}z(E_{ht}[R_{t+1}])$. For $\eta = 1$ this fitness function coincides with accumulated wealth. In this non-risk adjusted case the fitness function is however inconsistent with the traders being myopic mean-variance maximizers of wealth.

¹⁵This terminology is used, for example, by Tong (1990).

given in Gaunersdorfer, Hommes, and Wagener (2000). This analysis is important to understand time series properties of the noisy model. The fundamental value p^* is a unique steady state, which is locally stable for $g < 2(1 + r)$. As g is increased, the steady state is destabilized by a *Hopf bifurcation*¹⁶ at $g = 2(1 + r)$ and a stable invariant circle with periodic or quasi-periodic dynamics emerges. This invariant circle may undergo bifurcations as well, turning into a strange (chaotic) attractor. Thus, our nonlinear evolutionary system exhibits (quasi)periodic as well as chaotic fluctuations of asset prices and returns. We also find parameter regions, for which two attractors, a stable steady state and a stable (quasi)periodic cycle, coexist. When buffeted with dynamic noise, in such a case irregular switching occurs between close to the fundamental steady state fluctuations and (quasi)periodic fluctuations triggered by technical trading.¹⁷

In the next section we analyze time series properties of the noisy model and present an example where the endogenous fluctuations in returns is characterized by volatility clustering.

3 Time Series Properties

We are interested in the statistical properties of time series generated by our model and how they compare with those of real data. In particular, we are interested in the autocorrelation structure of the returns, and absolute and squared returns generated from the heterogeneous agents market equilibrium model (12–13), (16–18). Returns are defined as relative price changes,

$$r_t = \frac{p_{t+1} - p_t}{p_t}. \quad (19)$$

We focus on a typical example in which strong volatility clustering occurs, with ‘EMH-believers’ ($v = 1$ in (8)) and technical traders. In the absence of random shocks ($\varepsilon_t \equiv \delta_t \equiv 0$), there are two coexisting attractors in the example, a locally stable fundamental steady state and an attracting quasiperiodic cycle, as illustrated in figure 1.¹⁸ Depending upon the initial state, the system will settle down either to the stable fundamental steady state or to the stable cycle.

The corresponding time series of the deterministic skeleton of prices, returns and fractions of EMH believers along the cycle, as shown in the bottom pannel of figure 1, yield important insight into the economic mechanism driving the price movements. Prices start just

¹⁶A bifurcation is a qualitative change in the dynamics when parameters change. See, for example, Kuznetsov (1998), for an extensive mathematical treatment of bifurcation theory.

¹⁷As a technical remark, Gaunersdorfer, Hommes and Wagener (2000) show that the mathematical generating mechanism for these coexisting attractors is a so-called Chenciner or degenerate Hopf bifurcation (see Kuznetsov (1998, pp. 404–408)). Any (noisy) model with two coexisting attractors produces some form of volatility clustering. We emphasize that the Chenciner bifurcation is not special, but it is a generic phenomenon in nonlinear dynamic models with at least two parameters.

¹⁸In this example we have set $\eta = 0$. We also have run simulations with non-zero η , however this did not alter the results much.

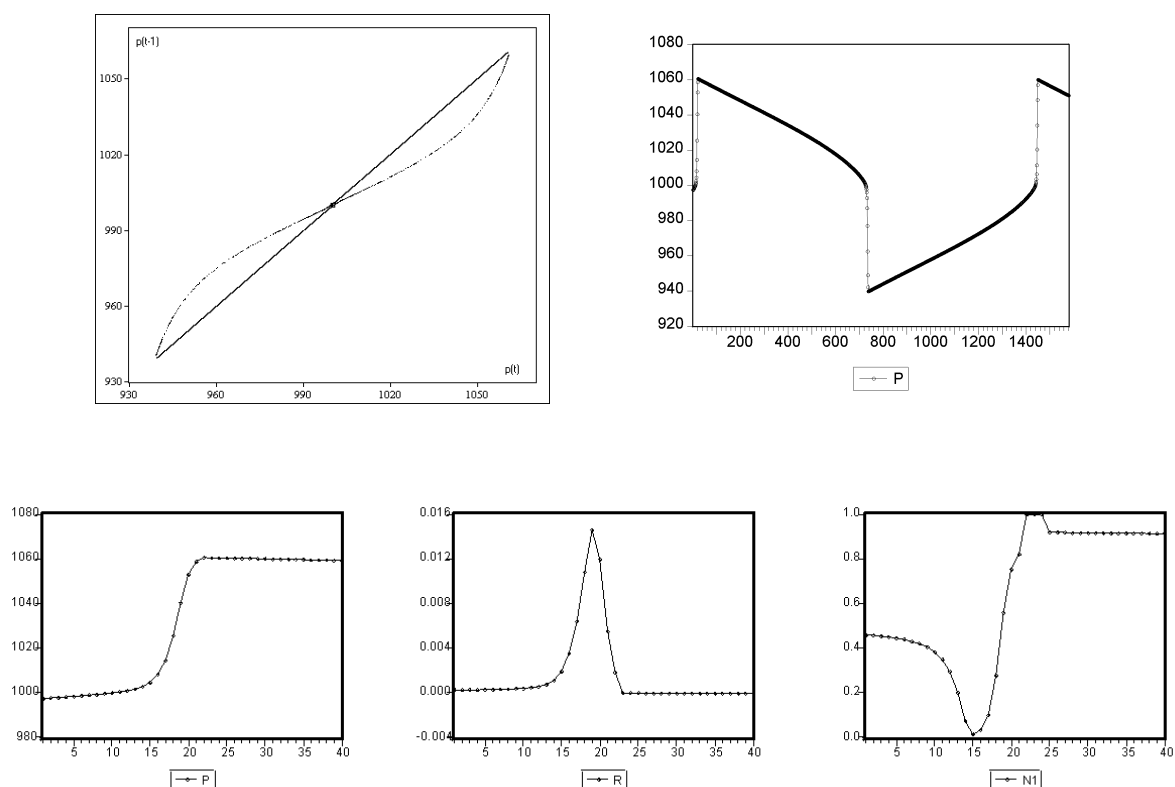


Figure 1: *Top panel: Left figure: phase space projection of prices p_t for deterministic skeleton without noise, where p_t is plotted against p_{t-1} : coexisting limit cycle and stable fundamental steady state $p^* = 1000$ (marked as a square). Right figure: corresponding time series along the limit cycle. Bottom panel: Time series of prices, returns, and fractions of EMH-believers. Parameter values: $\beta = 2$, $r = 0.001$, $v = 1$, $g = 1.89$, $\bar{y} = 1$, $\alpha = 2000$, $\eta = 0$, $a\sigma^2 = 1$ and $\varepsilon_t \equiv \delta_t \equiv 0$.*

below the fundamental price $p^* = 1000$ and slowly increase with small positive return. The fraction of EMH-believers slowly decreases, or equivalently, the fraction of trend followers slowly increases. This is due to the fact that when prices slowly increase trend followers perform slightly better than EMH-believers, and evolutionary forces thus increase the fraction of trend followers. As the fraction of trend followers increases, the increase in prices is reinforced, which in turn causes the fraction of trend followers to increase even more, etc. At some critical phase from periods 15 – 22 prices rapidly move to a higher level. During this phase returns increase and volatility jumps to a high value, with a peak around period 19. As the price level moves to a high level of 1060 in period 22, the fraction of EMH believers also increases to a value close to 1 in period 22, since the fraction of trend followers decreases when prices move too far away from the fundamental. When the market is dominated by EMH-believers prices decrease and move slowly in the direction

of the fundamental price again¹⁹ with small negative returns close to zero and with low volatility. Thereafter, the fraction of trend followers slowly increases again, finally causing a rapid decrease in prices to a value of 940, far below the fundamental, in period 740. Prices slowly move into the direction of the fundamental again to complete a full price (quasi-)cycle of about 1400 periods. The price cycle is thus characterized by a period of small changes and low volatility when EMH-believers dominate the market, and periods of rapid increase or decrease of prices with high volatility. The periods of rapid change and high volatility are triggered by technical trading; the conditioning of their charts upon the fundamental prevents the price to move too far away from the fundamental and leads to a new period of low volatility.

Adding dynamic noise to the system destroys the regularity of prices and returns along the cycle and leads to an irregular switching between phases of low volatility, with returns close to zero, and phases of high volatility, initiated by technical trading. Figure 2 compares 10000 time series observations of the same example buffeted with dynamic noise²⁰ with daily S&P 500 data from the last 40 years. In the S&P 500 returns series the October 1987 crash and the two days thereafter have been excluded.²¹ It should come as no surprise that the price series in the top panels are quite different, since S&P 500 is non-stationary and strongly increasing, whereas our model is stationary.²² Prices in our evolutionary model are highly persistent and *close* to having a *unit root*.²³ The model price series clearly exhibits sudden large movements, which are triggered by random shocks and amplified by technical trading. When prices move too far away from the fundamental $p^* = 1000$ technical traders condition their rule upon the fundamental and switch to the EMH-belief. With many EMH believers in the market, prices have a (weak) tendency to return to the fundamental value. As prices get closer to the fundamental, trend following behavior may become dominating again and trigger another fast price movement.

Table 1 shows some descriptive statistics for both return series. The means and medians of

¹⁹In the case where all agents are EMH believers, the market equilibrium equation without noise (11) reduces to $p_t = (p_{t-1} + rp^*)/(1+r)$, which is a linear difference equation with fixed point p^* and stable eigenvalue $1/(1+r)$, so that prices always move slowly into the direction of the fundamental. Notice also that when all agents are EMH believers, the market equilibrium equation with noise (12) becomes $p_t = (p_{t-1} + rp^* + \varepsilon_t)/(1+r)$, which is a stationary AR(1) process with mean p^* and root $1/(1+r)$ close to 1, for r small. In case all traders believe in a random walk, the implied actual law of motion is thus very close to a random walk and EMH-believers only make small forecasting errors which may be hard to detect in the presence of noise.

²⁰In the simulation the noise term $\delta_t \equiv 0$, whereas $\varepsilon_t \sim N(0,10)$. We also have run simulations with non-zero δ_t , however this did not alter the results much.

²¹The returns for these days were about -0.20 , $+0.05$, and $+0.09$. In particular, the crash affects the autocorrelations of squared S&P 500 returns, which drop to small values of 0.03 or less for all lags $k \geq 10$ when the crash is included.

²²By replacing our IID dividend process by a non-stationary dividend process, e.g. by a geometric random walk, prices in our heterogeneous agent model will also rapidly increase, similar to the S&P 500 series. We intend to study such non-stationary evolutionary systems in future work.

²³For $v = 1$ and $r = 0$ the characteristic polynomial of the Jacobian at the steady state has an eigenvalue equal to 1. Note that the Jacobian of a linear difference equation $y_t = \alpha_0 + \sum_{k=1}^L \alpha_k y_{t-k}$ has an eigenvalue 1 if and only if the time series $y_t = \alpha_0 + \sum_{k=1}^L \alpha_k y_{t-k} + \varepsilon_t$ has a unit root equal to 1.

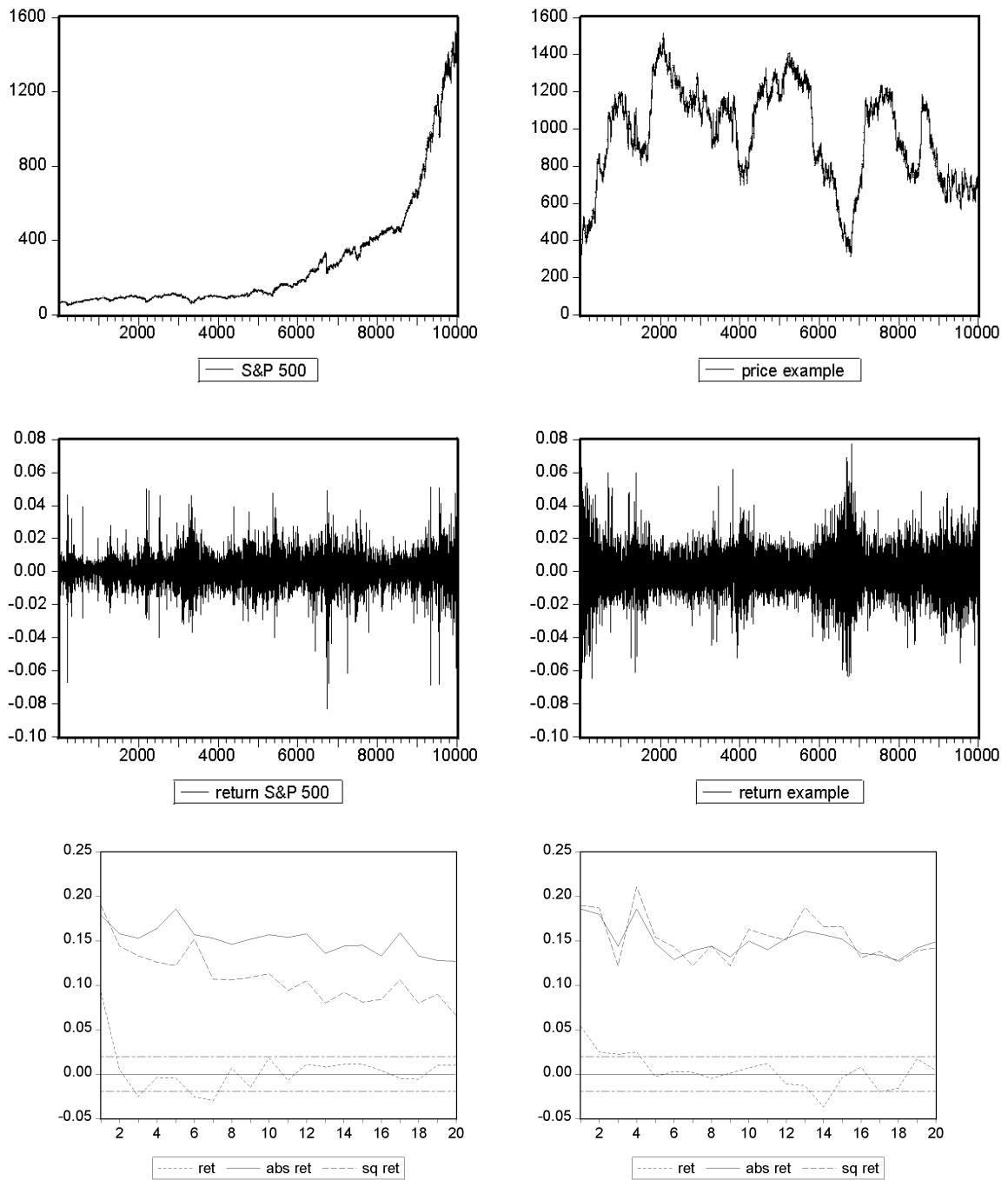


Figure 2: Daily S&P 500 data, 08/17/1961–05/10/2000 (left panel) compared with data generated by our model (right panel), with dynamic noise $\varepsilon_t \sim N(0, 10)$ and other parameters as in Figure 1: price series (top panel), returns series (middle panel), and auto-correlation functions of returns, absolute returns, and squared returns (bottom panel).

both returns series are close to 0 and the range and standard deviations are comparable in size, although slightly larger for our example. The S&P 500 returns have negative skewness, which is not the case in our example.²⁴ This should not come as a surprise, because our simple stylized model is in fact symmetric around the fundamental steady state equilibrium, since both type of traders behave symmetric with respect to high or low prices and with respect to positive or negative changes in prices. Finally, both returns series show excess kurtosis, though the kurtosis coefficient of our example is smaller than the coefficients for the S&P 500 returns. This may be due to the fact that in our simple evolutionary system chartists' price expectations are always conditioned upon the same distance function of price deviations from the fundamental price, i.e. upon the weighted distance $(p_{t-1}^2 - p^*)^2/\alpha$ as described by (17). Nevertheless, our simple stylized evolutionary model clearly exhibits excess kurtosis.

	S&P 500	Example
Mean	0.000348	0.00015
Median	0.000214	$-1.02E-05$
Maximum	0.051152	0.077428
Minimum	-0.082789	-0.064346
Std. Dev.	0.008658	0.012514
Skewness	-0.187095 (**)	0.046455
Kurtosis	8.512094 (**)	5.389321 (**)

Table 1: *Descriptive statistics for returns shown in figure 2.*
 (**) *null hypothesis of normality rejected at the 1% level*

We next turn to the time series patterns of returns fluctuations and the phenomenon of volatility clustering. In real financial data autocorrelation functions (ACF) of returns are roughly zero at all lags. For high frequencies they are slightly negative for individual securities and slightly positive for stock indices. Autocorrelations functions of volatility measures such as absolute or squared returns are positive for all lags with slow decay for stock indices and a faster decay for individual stocks. This is the well-known stylized fact known as volatility clustering.

Figure 2 (bottom panel) shows autocorrelation plots of the first 20 lags of the returns series and the series of absolute and squared returns. Both returns series have significant, but small, autocorrelations at the first lag ($\rho_1 = 0.092$ for the S&P 500 and $\rho_1 = 0.054$ for our example). For the S&P 500 the autocorrelation coefficient at the second lag is insignificant and at the third lag slightly negative significant ($\rho_2 = 0.005$, $\rho_3 = -0.025$), whereas in our model example autocorrelation coefficients are only slightly significant

²⁴Skewness statistics are not significant nor of the same sign for all markets. Nevertheless, some authors examine the skewness in addition to excess kurtosis. In a recent paper Harvey and Siddique (2000) argue that skewness may be important in investment decisions because of induced asymmetries in realized returns.

for the three following lags ($\rho_2 = 0.025$, $\rho_3 = 0.022$, $\rho_4 = 0.025$). For all higher order lags autocorrelations coefficients are close to zero and almost always insignificant. Our noisy conditional evolutionary model thus has almost no linear dependence in the returns series, and may therefore be seen as a reasonable first order approximation of real financial returns series.²⁵

The bottom panel in figure 2 also shows that for the absolute and squared returns the autocorrelations coefficients of the first 20 lags are strongly significant and positive. Table 2 reports the numerical values of the autocorrelation coefficients at the first 5 lags. They are comparable in size for both series, though the autocorrelations coefficients of the squared returns of the S&P 500 decay faster than those of our example.

lag n	S&P 500		Example	
	$ r_t $	r_t^2	$ r_t $	r_t^2
1	0.179	0.190	0.186	0.190
2	0.158	0.144	0.180	0.187
3	0.153	0.133	0.144	0.122
4	0.164	0.126	0.186	0.211
5	0.186	0.122	0.147	0.154

Table 2: *Autocorrelations of the absolute and squared returns shown in figure 2.*

Finally, we estimate a simple GARCH(1,1) model on the returns series. The GARCH(1,1) model is specified as

$$r_t = c_1 + e_t, \quad e_t \sim N(0, \sigma_t) \quad (20)$$

$$\sigma_t^2 = c_2 + \gamma_1 \delta_{t-1}^2 + \gamma_2 \sigma_{t-1}^2. \quad (21)$$

Using the variance equation (21) and the error in squared returns $\nu_t = e_t^2 - \sigma_t^2$, it follows that the squared error e_t^2 of the mean equation (20) satisfies

$$e_t^2 = c_2 + (\gamma_1 + \gamma_2)e_{t-1}^2 + \nu_t - \gamma_2\nu_{t-1}. \quad (22)$$

As is well known, for many financial returns series the sum $\gamma_1 + \gamma_2$ of the ARCH(1) and GARCH(1) coefficients is smaller than but close to unity, representing the fact that the squared error term e_t^2 in (22) follows a stationary, but highly persistent process. The estimated parameters are $\gamma_1 = 0.068529$ and $\gamma_2 = 0.929095$ with $\gamma_1 + \gamma_2 = 0.997624$ for the S&P 500 returns and $\gamma_1 = 0.026086$ and $\gamma_2 = 0.971687$ with $\gamma_1 + \gamma_2 = 0.997773$

²⁵Brock and Hommes (1997b) calibrate their evolutionary asset pricing model to ten years of monthly IBM prices and returns. They present (noisy) chaotic time series with autocorrelations of prices and returns similar to the autocorrelation structure in IBM prices and returns. In particular, the noisy chaotic returns series have (almost) no significant autocorrelations. However, these series do not exhibit volatility clustering, since there are no significant autocorrelations in squared returns.

for the returns series of our example.²⁶ Our conditional evolutionary model thus exhibits long memory with long range autocorrelations and captures the phenomenon of volatility clustering. It may thus be seen as a reasonable second order approximation of asset returns.

Let us finally briefly discuss the generality of the presented example. In order to get strong volatility clustering, the parameter $v = 1$ (or v very close to 1) is important, but the results are fairly robust with respect to the choices of the other parameter values; see Gaunersdorfer, Hommes and Wagener (2000) for a detailed analysis of the underlying dynamics. In general, when $0 \leq v < 1$ volatility clustering becomes weaker, and sometimes also significant autocorrelations in returns may arise. The fact that $v = 1$, in which case type 1 are EMH-believers (or v very close to 1, in which case the fundamentalists adapt only slowly into the direction of the fundamental), yields the strongest volatility clustering results may be understood as follows. When EMH-believers dominate the market asset prices are highly persistent and mean reversion is weak, since the evolutionary system is close to having a unit root (see footnote 19). Apparently, the interaction between unit root behavior far from the fundamental steady state with relatively small price changes driven only by exogenous news, and larger price changes due to amplification by trend following rules in some neighborhood around the fundamental price yields the strongest form of volatility clustering. We emphasize that all these results have been obtained for an IID dividend process and a corresponding constant fundamental price p^* . Including a non-stationary dividend process and accordingly a non-stationary time varying fundamental process p_t^* may lead to stronger volatility clustering also in the case $0 \leq v < 1$. We leave this conjecture for future work.

4 Concluding Remarks

We have presented a nonlinear structural model for volatility clustering. Fluctuations in asset prices and returns are caused by a combination of random news about economic fundamentals and evolutionary forces. Two typical trader types have been distinguished. The first type are *fundamentalists* (*‘smart money’* traders), believing that the price of an asset returns to its fundamental value given by the discounted sum of future dividends or *‘EMH-believers’*, believing that prices follow a random walk. The second trader type are *chartists* or *technical analysts*, believing that asset prices are not solely determined by fundamentals, but that they may be predicted in the short run by simple technical trading rules based upon patterns in past prices, such as trends or cycles. The fraction of each of the two types is determined by past success of the two strategies, conditioned upon how far prices deviate from their fundamental value. This leads to a highly nonlinear, conditionally evolutionary model buffeted with noise. The time series properties of our model are similar to those observed in real financial series. In particular, the autocorrelation structure of the returns and absolute and squared returns series of our noisy nonlinear evolutionary system

²⁶We checked that the squared residuals from the estimated GARCH(1,1) model has almost no significant autocorrelations.

are similar to those observed in real financial data, with little or no linear dependence in returns and high persistence in absolute and squared returns. Although the model is simple, it may be viewed as a second order approximation to real data in the sense that it captures the first two moments of the distribution of asset returns. Our model thus might serve as a good starting point for a structural explanation – by a tractable model – of further stylized facts in finance, such as cross correlation between volatility and volume.

The *persistent* phenomenon of coexistence of a stable fundamental steady state and a stable (quasi)periodic cycle plays an important role in generating volatility clustering. But there is also a strikingly simple economic intuition of why the phenomenon of volatility clustering should in fact be expected in our conditionally evolutionary system. When EMH-believers dominate the market prices are highly persistent, changes in asset prices are small and only driven by news, returns are close to zero and volatility is low. As prices move towards the fundamental, trend followers perform better than EMH-believers and due to evolutionary forces, their influence in the market gradually increases. When trend followers start dominating the market, a rapid change in asset prices occurs with large (positive or negative) returns and high volatility. The price trend cannot persist forever, since prices cannot move away too far from the fundamental because technical traders condition their charts upon the fundamental. In the noisy conditionally evolutionary system both, the low and the high volatility phases, are persistent and the interaction between the two phases is highly irregular. The nonlinear interaction between heterogeneous trading rules in a noisy environment thus causes unpredictable asset returns and at the same time volatility clustering and the associated predictability in absolute and squared returns.

Our model is also able to explain empirical facts like ‘fat tails’, i.e. it generates excess kurtosis in the returns. This is due to the fact that it implies a decomposition of returns into two terms, one martingale difference sequence part according to the conventional EMH theory, and an extra *speculative* term added by the evolutionary theory. The heterogeneity in the model thus creates *excess volatility*. However, because of the simplicity of the model there are also some shortcomings compared to real financial data, which we would like to discuss briefly. Our model does not generate returns series which exhibit strong skewness. This is due to the fact that our agents use trading rules which are exactly symmetric with respect to the constant fundamental value of the risky asset. As a consequence, the evolutionary model is also symmetric with respect to the fundamental price. Our model is therefore also not able to explain leverage effects, i.e. the effect that volatility increases as the market goes down and decreases as the market goes up. To capture such asymmetric properties one could introduce a more realistic time depending fundamental price, extend the number of trader types, for example by introducing groups of ‘optimistic’ and ‘pessimistic’ technical trader types or introduce heterogeneity in the expectations about the fundamental to capture the effect that traders do not have perfect information and/or have asymmetric information about economic fundamentals. Another shortcoming is that our model is a stationary model and generates unrealistic price series. By replacing our IID dividend process by a non-stationary dividend process, e.g. by a geometric random

walk, prices will also rapidly increase, similar to the S&P 500 series. We intend to study such non-stationary models within the presented framework in future work.

In our model excess volatility and volatility clustering are created or reinforced by the trading process itself, which seems to be in line with common financial practice. If the evolutionary interaction of boundedly rational, speculative trading strategies amplifies volatility, this has important consequences for risk management and regulatory policy issues in real financial markets. Our model predicts that ‘good’ or ‘bad’ news about economic fundamentals are *amplified* by evolutionary forces. Small fundamental causes may thus occasionally have big consequences and trigger large changes in asset prices. In the time of globalization of international financial markets, small shocks in fundamentals in one part of the world may thus cause large changes of asset prices in another part of the world. Our simple structural model shows that a stylized version of this theory already fits real financial data surprisingly well. Our results thus call for more financial research in this area to build more realistic models to assess investors’ risk to speculative trading and evolutionary amplification of changes in underlying fundamentals.

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