Quantum integrable models out of equilibrium
Mossel, J.J.

Citation for published version (APA):
Mossel, J. J. (2012). Quantum integrable models out of equilibrium
Chapter 5

Domain wall quench

5.1 Introduction

So far we discussed integrable models and how correlation functions can be computed on eigenstates. In this chapter we will use the Algebraic Bethe ansatz which we covered in chapter 3 to study an explicit quench problem. The system is prepared in an inhomogeneous initial state and then the subsequent time evolution governed by the XXZ model is studied. It is worth to stress that the XXZ model is a fully interacting system and falls in class of integrable models with linear density character defined in chapter 4. Most exact studies in the literature are done for effectively free models which therefore can be classified as having constant density character. For the specific quench problem we consider, we can study both finite size systems as well as the thermodynamic limit and thereby we obtain a good understanding of the finite size effects. The region where we have full control is when the system is gapped. Universal features of gapped systems will be discussed. The results of this chapter are published in [53].

5.2 Setup

We consider an isolated spin chain of $N$ sites, each occupied by a local spin-1/2 degree of freedom. For definiteness, we put the spins on a ring and impose periodic boundary conditions. The thought experiment we perform consists in preparing the quantum state of the system at $t = 0$ as

$$|\phi\rangle = |\downarrow \ldots \downarrow \uparrow \ldots \uparrow\rangle. \quad (5.1)$$

This state thus contains a magnetic domain wall between sites $M$ and $M + 1$, and another (anti-) domain wall between sites $N$ and 1 (in view of the periodic boundary conditions). This state can be prepared in different ways: we can view it as being created by an Ising model with an appropriate position-dependent field, or as resulting from a sudden polarizing pulse applied on a section of an initially
5.2 Setup

fully polarized chain. For times \( t > 0 \), we let this state evolve unitarily in time under the antiferromagnetic XXZ Hamiltonian

\[
H_{XXZ} = J \sum_{j=1}^{N} \left[ \frac{1}{2\Delta} (S_j^- S_{j+1}^+ + S_j^+ S_{j+1}^-) + S_j^z S_{j+1}^z \right].
\] (5.2)

Note that the Hamiltonian is rescaled by a factor of \( 1/\Delta \) as compared to the previous chapters, in order the have a well-defined Ising limit (\( \Delta \to \infty \)). For \( \Delta > 1 \) the spectrum of this theory is gapped, while for \( -1 < \Delta \leq 1 \) the system is in the quantum critical regime. We consider the case when the system is antiferromagnetic (i.e. \( J > 0 \)). In order to simplify the expressions and without loss of generality we put \( J = 1 \) throughout this chapter. The XXZ chain can now be realized experimentally [155] using cold atomic gases.

The initial domain wall state (5.1) is not an eigenstate of the XXZ Hamiltonian away from the Ising limit \( \Delta \to \infty \). On the other hand, for any fixed value of \( \Delta \), the exact eigenstates of this model form a basis in the Hilbert space on which we can at least in principle decompose the initial domain-wall state to arbitrary accuracy. Given this decomposition, the solution of the Schrödinger equation becomes straightforward, and we obtain the exact time-dependent wavefunction after the quench as the linear decomposition

\[
|\phi(t)\rangle = \sum_n e^{-iE_n t} c_n |\Psi_n\rangle,
\]

\[c_n \equiv \langle \Psi_n | \phi \rangle,\] (5.3)

where the sum is over all the \( \left( \begin{array}{c} N \\ M \end{array} \right) \) eigenstates of \( H_{XXZ} \) at fixed total magnetization, and the complex amplitudes \( c_n \) represent the overlaps between the starting state and the exact eigenfunctions. Since we always work with normalized wavefunctions, the coefficients \( c_n \) should by definition satisfy the constraint

\[
\sum_n |c_n|^2 = 1.
\] (5.4)

This constraint will constitute an important sum rule, allowing to quantify the accuracy of our results by assessing how faithfully the resulting wavefunction is reproduced.

All the complexity of the problem is therefore hidden in two places. First, wavefunctions \( |\Psi_n\rangle \) and their energy \( E_n \) must be known. This is standardly handled by the Bethe Ansatz discussed in chapters 2 and 3. Second, the overlaps \( c_n \) must also be known. This is a problem of much greater complexity, which in the present situation finds its solution in the framework of the Algebraic Bethe Ansatz.

Despite being in possession of these two fundamental building blocks, one major difficulty remains. Since the Hilbert space is exponentially large in system size \( N \), the summation in (5.3) is difficult to handle, and must in practice be truncated in order to reach sizes sufficiently large to allow the discrimination between finite- and infinite-size behaviour. We will show in the next section that this truncation is possible in the situation we consider. In particular, this puts us in position to study the long-time average of observables according to any prescription desired,
a common one being

$$\langle O \rangle \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle O(t) \rangle,$$  \hspace{1cm} (5.5)

and to make reliable observations on the relaxation/thermalization of the prepared state.

The time evolution of this domain wall state has been studied in similar settings, using different techniques. For gapless spin chains this type of quench is studied in using bosonization [156] and CFT’s [157]. An exact analysis has been carried out for the XX-chain\(^1\) in [158]. The short time regime for the XXZ chain is studied using tDMRG in [34]. We here offer a complement to these studies, consisting in results from integrability.

### 5.3 Spectral analysis

Before delving into the computation of the overlap coefficients, we first study the spectrum of the XXZ chain to obtain some intuition of what states are expected to be important. The space of eigenstates of the XXZ chain is spanned by Bethe wavefunctions, each described by a set of rapidity parameters \(\{\lambda\}\) obtained as solution to the Bethe equations. Bethe wavefunctions are generically quite complicated objects, and can contain both unbound and bound magnons. In summary, solutions of the Bethe ansatz equations can be classified using strings (see chapter 2 for a discussion). An \(n\)-string in a set of \(n\) complex rapidities sharing the same real part and invariant under complex conjugation. These strings can be interpreted as bound states of ‘mass’ \(n\). In the Ising-limit it can be shown that an \(n\)-string corresponds to \(n\) adjacent down spins. The dispersion relation for a string an \(n\)-string as a function of its rapidity \(\lambda^n_\alpha\) is

$$\epsilon_n(\lambda^n_\alpha) = -\tanh(\eta) \frac{\sinh n\eta}{\cosh n\eta - \cos 2\lambda^n_\alpha},$$  \hspace{1cm} (5.6)

with \(-\pi/2 < \lambda^n_\alpha < \pi/2\). The total energy of a Bethe wavefunction is simply the sum of the single string energies

$$E = \sum_{\alpha, n} \epsilon_n(\lambda^n_\alpha) \quad n = 1 \ldots M, \quad \alpha = 1 \ldots M_n.$$  \hspace{1cm} (5.7)

We can easily obtain the band structure by considering the thermodynamic limit. For a given string configuration \(\{M_n\}\) one can define: \(E^{max}(\{M_n\})\) by putting all \(\lambda^n_\alpha\) to \(\pi/2\). Similarly one can define \(E^{min}(\{M_n\})\) by putting all \(\lambda^n_\alpha\) to 0. The \(M\)-string is a special case since it has a vanishing bandwidth in the limit \(M \to \infty\). In the Ising limit (\(\eta \to \infty\)) the energy of a state made of \(m\) different strings is \(-m\). This corresponds in the Ising language to a state made of \(m\) distinct blocks of down spins and clarifies the bound states interpretation in this limit. In fig. 5.1, we give the band structure of the dominant classes of excitations as a function of \(\Delta\). The number of particles corresponds to the number of strings a state is made of.

\(^1\)The XX-chain is the XXZ chain for \(\Delta = 0\) and can be mapped to a problem of free fermions.
5.3 Spectral analysis

Figure 5.1: Sketch of the energy values covered by the highest three energy bands of the XXZ model as a function of $\Delta$ compared with the energy of the initial state $E_0$. Indicated is the number of particles the states in different bands are made of.

<table>
<thead>
<tr>
<th>Order $(1/\Delta)$</th>
<th>strings</th>
<th># particles</th>
<th>Energy (Ising limit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${M}$</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>${1, M-1}$</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>${1_2, M-2}$</td>
<td>3</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td>${2, M-2}$</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>${1_3, M-3}$</td>
<td>4</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>${1,2, M-3}$</td>
<td>3</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td>${3, M-3}$</td>
<td>2</td>
<td>-2</td>
</tr>
</tbody>
</table>

Table 5.1: The hierarchy of states, in order of importance of contributions to the normalization sum rule (5.4). Each partitioning of $M$ rapidities into strings leads to an independent excitation class, only the few shown here being of relevance to the current setup.

From a perturbative analysis, we expect at large anisotropy $\Delta$ a well-defined hierarchy of states in terms of the importance of their overlap with the domain wall state. This hierarchy is presented in table 5.1. The most important states are those consisting of a single $M$–string. They are followed by successively more complicated partitionings of the $M$ rapidities into more and more individual bound or unbound states, each partitioning representing a whole continuum of excitations. The number of particles in the state is defined here to be the number of elements in the partitioning. From a perturbative analysis it can be easily shown that for sufficiently large $M$ the system is insensitive to small changes of $M$ (i.e., of how distant the domain and anti-domain walls are from each other). This allows us to perform the calculation with an $M$ slightly below $N/2$ without loss of generality. The reason for this choice is that for these values the string hypothesis is better satisfied (i.e. the deviations (2.72) are sufficiently small) for the majority of string states.
5.4 The overlap coefficients

The overlap matrix is derived using the Algebraic Bethe Ansatz which is covered in chapter 3. We first briefly recall the important results for this section. States, respectively dual states can be constructed as

\[ |\psi\rangle = \prod_{j=1}^{M} B(\lambda_j)|0\rangle, \]
\[ \langle \psi| = \langle 0| \prod_{j=1}^{M} C(\lambda_j) \] (5.8)

with the operators \( A(\lambda), B(\lambda), C(\lambda), D(\lambda) \) satisfying the commutation relations (3.31a). The norm of a Bethe state in case of strings is given by [92, 93, 94, 95]:

\[ \langle \psi|\psi\rangle = \varphi(i\eta)^M \prod_{j \neq k, \lambda_j \neq \lambda_k - i\eta} \frac{\varphi(\lambda_j - \lambda_k + i\eta)}{\varphi(\lambda_j - \lambda_k)} \det \Phi^{(r)}(\{\lambda\}) + O(\delta) \] (5.9)

where \( \Phi^{(r)} \) is called the reduced Gaudin matrix

\[ \Phi^{(r)}_{(j,\alpha)(k,\beta)} = \delta_{jk}\delta_{\alpha\beta} \left( N \frac{d}{d\lambda}_\alpha \theta_j(\lambda^j_\alpha) - \sum_{(l,\gamma) \neq (j,\alpha)} \frac{d}{d\lambda}_\alpha \Theta_{jl}(\lambda^j_\alpha - \lambda^l_\gamma) \right) + (1 - \delta_{jk}\delta_{\alpha\beta}) \frac{d}{d\lambda}_\alpha \Theta_{jk}(\lambda^j_\alpha - \lambda^k_\beta). \] (5.10)

We introduce the notation

\[ \varphi(\lambda + i\eta) = \begin{cases} \lambda + i \Delta = 1 \\ \sinh(\lambda + i\eta) |\Delta = \cos(\eta)| < 1 \\ \sin(\lambda + i\eta) |\Delta = \cosh(\eta)| > 1. \end{cases} \] (5.11)

The element for deriving is the overlap formula is based on Slavnov’s expression for the scalar product between an eigenstate \( \langle \psi(\{\lambda\}) | \) (represented by a set of rapidities \( \{\lambda\} \) that satisfy the Bethe equations) and an arbitrary state \( |\psi(\{\mu\})\rangle \) (without restrictions on \( \{\mu\} \)) [90]:

\[ \langle \psi(\{\lambda\})|\psi(\{\mu\})\rangle = \frac{\det H(\{\lambda\}, \{\mu\})}{\prod_{j > k} \varphi(\lambda_j - \lambda_k) \prod_{j < k} \varphi(\mu_j - \mu_k)} \] (5.12)

with

\[ H_{ab} = \frac{\varphi(i\eta)}{\varphi(\lambda_a - \mu_b)\varphi(\lambda_a - \mu_b + i\eta)} \left( \prod_{l=1}^{M} \varphi(\lambda_l - \xi + i\eta) - d(\mu_b) \prod_{l=1}^{M} \varphi(\lambda_l - \mu_b - i\eta) \right). \] (5.13)
5.4 The overlap coefficients

5.4.1 General overlap

The initial state can be written in the Algebraic Bethe Ansatz language using the inverse mapping (3.48):

\[ S_j = \prod_{k=1}^{j-1} (A + D)(\xi_k)B(\xi_j) \prod_{l=j+1}^{N} (A + D)(\xi_l). \]  

(5.14)

Making use of \( \prod_{j=1}^{N} (A + D)(\xi_j) = 1 \) and \( (A + D)(\xi_j)|0\rangle = |0\rangle \) we can write the state as:

\[ |\phi\rangle = \prod_{j=1}^{M} B(\xi_j)|0\rangle \]  

(5.15)

which is a normalized state. The overlap between an eigenstate for finite \( \Delta \) and the initial state \( |\phi\rangle \) can be computed using Slavnov’s theorem for the scalar product.

In the case when \( \{\mu_j\} = \{\xi_j\} \) the matrix \( H \) simplifies:

\[ H_{ab} = \frac{\varphi'(i\eta)}{\varphi(\lambda_a - \xi_b)\varphi(\lambda_a - \xi_b + i\eta)} \prod_{l=1}^{M} \varphi(\lambda_l - \xi_b + i\eta). \]  

(5.16)

For the XXZ spin chain all inhomogeneity parameters should be set to \( \xi_j = i\eta/2 \forall j \). The limit should be taken carefully, there are \( M(M - 1) \) poles coming from: \( \prod_{j<k} \varphi(\xi_j - \xi_k) \). Since the columns of the matrix \( H \) are also \( M(M - 1) \)-fold degenerate we can apply l'Hôpital’s rule. First we notice that we can easily extract factors \( \prod_{l=1}^{M} \varphi(\lambda_l - \xi_b + i\eta) \) from the determinant, since applying l'Hôpital’s rule to those terms does not remove any degeneracies.

\[ \langle \psi | \phi \rangle = \prod_{k=1}^{M} \prod_{l=1}^{M} \varphi(\lambda_l - \xi_k + i\eta) \frac{\det \tilde{H}(\{\lambda_j\}, \{\xi_j\})}{\prod_{j>k} \varphi(\lambda_j - \lambda_k)\prod_{j<k} \varphi(\xi_j - \xi_k)}, \]  

(5.17)

\[ \tilde{H}_{ab} = \frac{\varphi'(i\eta)}{\varphi(\lambda_a - \xi_b)\varphi(\lambda_a - \xi_b + i\eta)}. \]  

(5.18)

Note that this determinant of \( \tilde{H} \) is the same as that in the partition function of the 6-vertex model with domain wall boundary conditions [159]. However in this case both the sets \( \{\lambda_j\} \) and \( \{\xi_j\} \) play the role of boundary conditions and satisfy no Bethe equations. To be able to apply l'Hôpital we are interested in \( \partial_{\xi_b} \tilde{H}_{ab} \).

First we rewrite \( \tilde{H}_{ab} \):

\[ \tilde{H}_{ab} = \frac{\varphi'(\lambda_a - \xi_b)}{\varphi(\lambda_a - \xi_b)} - \frac{\varphi'(\lambda_a - \xi_b + i\eta)}{\varphi(\lambda_a - \xi_b + i\eta)}. \]  

(5.19)

Next, consider

\[ \partial_{\xi_b} \left( \frac{\varphi'(\lambda_a - \xi_b)}{\varphi(\lambda_a - \xi_b)} \right)^n = n \left( \frac{\varphi'(\lambda_a - \xi_b)}{\varphi(\lambda_a - \xi_b)} \right)^{n+1} - n \left( \frac{\varphi'(\lambda_a - \xi_b)}{\varphi(\lambda_a - \xi_b)} \right)^{n-1}. \]  

(5.20)
Using this result it is easy to derive for \( n \) even

\[
\partial^n_{\xi_b} \tilde{H}_{ab} = \sum_{j=0}^{n/2} d^n_j \left\{ \left( \frac{\varphi'(\lambda_a - \xi_b)}{\varphi(\lambda_a - \xi_b)} \right)^{2j+1} - \left( \frac{\varphi'(\lambda_a - \xi_b + i\eta)}{\varphi(\lambda_a - \xi_b + i\eta)} \right)^{2j+1} \right\},
\tag{5.21}
\]

and for \( n \) odd we get a similar result. In general the coefficients \( d^n_j \) are complicated expressions. However it is straightforward to derive that \( d^n_{n/2} = n! \) for \( n \) even. Since in this case there is no mixing between the left and right terms in (5.20). It will turn out that theses coefficients are the only ones we need, since all terms with coefficient \( d^n_j \) for \( j < n/2 \) can be removed using row manipulations while leaving the determinant invariant. Differentiating the denominator:

\[
\prod_{j<k} \varphi(\xi_j - \xi_k)
\]

is much simpler. First we put \( \xi_1 = i\eta/2 \). If we now consider \( \xi_2 \) we get a single zero in the limit \( \xi_2 = i\eta/2 \). We can continue this logic and see that \( \xi_j \) gives rises to \( j-1 \) zero’s, hence we get a factor \((j-1)!\). We also notice that the factors \( n! \) from the numerator cancel against the factors from the denominator so we are left with

\[
\langle \psi | \phi \rangle = \prod_{l=1}^M \varphi(\lambda_l + i\eta/2)^M \frac{\det \tilde{H}}{\prod_{j<k} \varphi(\lambda_j - \lambda_k)} \tag{5.22}
\]

\[
\tilde{H}_{ab} = \left( \frac{\varphi'(\lambda_a - i\eta/2)}{\varphi(\lambda_a - i\eta/2)} \right)^b - \left( \frac{\varphi'(\lambda_a + i\eta/2)}{\varphi(\lambda_a + i\eta/2)} \right)^b. \tag{5.23}
\]

The matrix \( \tilde{H} \) has a similar structure to a Vandermonde determinant. This makes the matrix ill-conditioned and is therefore not suitable for a numerical evaluation. However, for a given string structure the determinant can be written in terms of sums of Vandermonde determinants. In the following sections this is done explicitly for the most important cases. The normalized overlap is

\[
\frac{\langle \psi | \phi \rangle}{\sqrt{\langle \psi | \psi \rangle}} = \frac{\prod_{l=1}^M \varphi(\lambda_l + i\eta/2)^M \det \tilde{H}}{\sqrt{-1^{M(M-1)/2} \varphi(i\eta)^M \prod_{j<k} \varphi(\lambda_j - \lambda_k + i\eta) \det \Phi}} \tag{5.24}
\]

These expressions hold for all three parameterizations (5.11).

### 5.4.2 Overlap in case of an \( M \)-string

In the case where the solution is an \( M \)-string: \( \lambda_{a,j} = \lambda_a + i\eta(M + 1 - 2j)/2 \) the determinant of \( \tilde{H} \) is exactly a Vandermonde determinant. A Vandermonde matrix is defined as: \( V_{jk} = x_j^{k-1} \) with all \( x_j \) distinct. Its determinant is given by

\[
\det V = \prod_{j<k} (x_j - x_k). \tag{5.25}
\]

Introducing the notation

\[
a_j = \frac{\varphi'(\lambda_j + i\eta/2)}{\varphi(\lambda_j + i\eta/2)}, \quad j = 1 \ldots M + 1, \tag{5.26}
\]
The reduced Gaudin determinant for an \( M \)-string takes the simple form:

\[
\text{det} \, \tilde{H} = \text{det} \begin{pmatrix}
a_2 - a_1 & \ldots & a_2^M - a_1^M \\
\vdots & \ddots & \vdots \\
a_{M+1} - a_M & \ldots & a_{M+1}^M - a_M^M
\end{pmatrix} = \text{det} \begin{pmatrix} 1 & a_1 & \ldots & a_1^M \\
\vdots & \vdots & \ddots & \vdots \\
1 & a_{M+1} & \ldots & a_{M+1}^M
\end{pmatrix}
\]

\[
= \prod_{j<k}^{M+1} \left( \frac{\varphi'(\lambda_{\alpha}^M + i\eta(M + 2 - 2j)/2)}{\varphi'(\lambda_{\alpha}^M + i\eta(M + 2 - 2k)/2)} - \frac{\varphi'(\lambda_{\alpha}^M + i\eta(M + 2 - 2j)/2)}{\varphi'(\lambda_{\alpha}^M + i\eta(M + 2 - 2k)/2)} \right)
\]

The reduced Gaudin determinant for an \( M \)-string takes the simple form:

\[
\text{det} \Phi^{(r)}_M = N \frac{\varphi(i\eta M)}{\varphi(\lambda_{\alpha}^M - i\eta/2)\varphi(\lambda_{\alpha}^M + i\eta/2)}.
\]

This results in the normalized overlap for an \( M \)-string:

\[
\frac{\langle \psi | \phi \rangle}{\sqrt{\langle \psi | \psi \rangle}} = \prod_{n=1}^{M-1} \varphi(i\eta n) \varphi(\lambda_{\alpha} - i\eta n / 2)^M \left[ \frac{\varphi(\lambda_{\alpha} - i\eta M / 2)\varphi(\lambda_{\alpha} + i\eta M / 2)}{-1^{M(M-1)/2} N} \right]^{1/2}.
\]

The absolute value squared of the overlap is

\[
|\langle \psi | \phi \rangle|^2 = \frac{N \varphi(\lambda_{\alpha} - i\eta M / 2)\varphi(\lambda_{\alpha} + i\eta M / 2)}{-1^{M(M-1)/2} N}.
\]

### 5.4.3 Overlap for two \((m, \bar{m})\)-strings

We consider states constructed from two strings with lengths \( m \) and \( \bar{m} \) such that \( m + \bar{m} = M \), and \( m \leq \bar{m} \): \( \lambda_{\alpha}^m = \lambda_{\alpha}^m + i\eta(m + 1 - 2j)/2 \) and \( \lambda_{\alpha}^\bar{m} = \lambda_{\alpha}^\bar{m} + i\eta(\bar{m} + 1 - 2k)/2 \). Some convenient notations are:

\[
a_j \equiv \frac{\varphi'(\lambda_{\alpha}^m + i\eta/2)}{\varphi(\lambda_{\alpha}^m + i\eta/2)} \quad j = 1 \ldots m + 1
\]

\[
b_j \equiv \frac{\varphi'(\lambda_{\alpha}^\bar{m} + i\eta/2)}{\varphi(\lambda_{\alpha}^\bar{m} + i\eta/2)} \quad j = 1 \ldots \bar{m} + 1.
\]

Now the overlap matrix takes the form:

\[
\text{det} \begin{pmatrix}
a_2 - a_1 & \ldots & a_2^M - a_1^M \\
\vdots & \ddots & \vdots \\
a_{m+1} - a_m & \ldots & a_{m+1}^M - a_m^M \\
b_2 - b_1 & \ldots & b_2^M - b_1^M \\
\vdots & \ddots & \vdots \\
b_{\bar{m}+1} - b_{\bar{m}} & \ldots & b_{\bar{m}+1}^M - b_{\bar{m}}^M
\end{pmatrix} = \text{det} \begin{pmatrix} 1 & a_1 & \ldots & a_1^M \\
\vdots & \vdots & \ddots & \vdots \\
1 & a_{m+1} & \ldots & a_{m+1}^M \\
0 & 1 & b_1 & \ldots & b_1^M \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & b_{\bar{m}+1} & \ldots & b_{\bar{m}+1}^M
\end{pmatrix}
\]

96
Expanding this determinant in the first column results in a sum over \(m + 1\) Vandermonde determinants. After factoring out common factors we obtain:

\[
\det \tilde{H} = \sum_{j=1}^{m+1} \left( \prod_{k \neq j}^{m+1} (a_k - b_l) \prod_{l=1}^{m+1} (a_p - a_q) \right) \prod_{r<s}^{m+1} (b_r - b_s). \tag{5.34}
\]

### 5.4.4 Overlap for a general string solution

Consider a state constructed of \(n\) strings with lengths: \(l_1 \leq l_2 \ldots \leq l_n\). Introducing the notation

\[
a^\alpha_j \equiv \frac{\varphi^\prime(\lambda^\alpha_{\alpha,j} + i\eta/2)}{\varphi(\lambda^\alpha_{\alpha,j} + i\eta/2)} \quad j = 1 \ldots l_\alpha + 1 \quad \alpha = 1 \ldots n, \tag{5.35}
\]

the overlaps can be written as a sum over Vandermonde determinants:

\[
\det \tilde{H} = \sum_{j_1=1}^{l_1+1} \ldots \sum_{j_{n-1}=1}^{l_{n-1}+1} \left( \prod_{\alpha<\beta}^{n} \prod_{j_\alpha \neq j_\beta}^{l_\alpha+1} (a^\alpha_j - a^\beta_k) \prod_{\gamma=1}^{n} \prod_{j<k \neq j_\gamma}^{l_\gamma+1} (a^\gamma_j - a^\gamma_k) \right). \tag{5.36}
\]

### 5.4.5 Connection with the emptiness formation probability

It is worth to mention that the overlap coefficients are special a case of the emptiness formation probability. For a state |\(\Psi\rangle\) the emptiness formation probability \(\tau(m)\) is the probability to detect a contiguous block of \(m\) down-spins 

\[
\tau(m) = \langle \Psi | \prod_{j=1}^{m} (1/2 - S_z^j) | \Psi \rangle.
\]

If we now consider the special case where \(m = M\) equals to number of down-spins, this is precisely \(|\langle \Psi | \phi \rangle|^2\)

\[
\tau(M) = \langle \Psi | \prod_{j=1}^{M} (1/2 - S_z^j) | \Psi \rangle
\]

\[
= \sum_{\{j_n\}} \langle \Psi | \prod_{j=1}^{M} (1/2 - S_z^j) | \{j_n\} \rangle \langle \{j_n\} | \Psi \rangle
\]

\[
= \langle \Psi | \prod_{j=1}^{M} (1/2 - S_z^j) | \phi \rangle \langle \phi | \Psi \rangle = |\langle \Psi | \phi \rangle|^2. \tag{5.37}
\]

In the first step we inserted a resolution of the identity in terms |\(\{j_n\}\rangle\) which are states with down-spins at positions \(\{j_n\}\). Since this sum can be restricted to states with only \(M\) down-spins it is clear that the only state that is projected out by \(\prod_{j}^{M} (1/2 - S_z^j)\) is |\(\phi\rangle\).

### 5.4.6 Results

The right panel of the same figure shows the normalization sum rule saturation coming from these excitations. For sufficiently large values of anisotropy, the saturation is essentially complete using these states only. Approaching the gapless
5.5 Work probability distribution

regime however requires increasingly large numbers of classes of excitations, since all overlaps scale to zero as can be expected from Anderson’s orthogonality catastrophe scenario [160]. A further issue with the gapless limit $\Delta \rightarrow 1$ is that in its vicinity, solutions to Bethe equations involving complex rapidities become more and more difficult to find, and often require considering deviated strings in detail [83]. We will not burden ourselves with these issues here, and only consider systems sufficiently deep in the gapped regime where such deviations are negligible.

![Figure 5.2: Normalization sum rule saturation coming from summing over all states of each basic excitation family, as a function of $\Delta$. As the gapless regime is approached, all overlaps scale to zero.](image)

### 5.5 Work probability distribution

One of the most straightforward measurable quantities which can be obtained from the knowledge of the overlaps $c_n$ is the work probability distribution [161] defined as:

$$P(W) = \sum_n |\langle \phi | \Psi_n \rangle|^2 \delta(W - E_n + E_0).$$

(5.38)

The work expectation value can easily be computed analytically: $\langle W \rangle = \langle \phi | H_{XXZ} - H_0 | \phi \rangle = 0$ and $\langle W^2 \rangle = \frac{1}{2\Delta^2}$. $\langle W \rangle$ can thus be considered intensive here, which for a generic quantum quench is not the case. In figure 5.3 we plot $P(W)$ for two different values of $\Delta$. The calculations are done for a finite system size ($N = 250, M = 100$), therefore the results are binned in energy (using bins larger than the interlevel spacing) in order to generate smooth curves. The contribution of the $M$-strings is a narrow peak with a vanishing bandwidth in the thermodynamic limit, see section 5.3. The $1, M - 1$ states have two van Hove singularities\(^2\) [162] at the end edge of the energy band, resulting in two sharp peaks in the work probability distribution. We will verify this by computing exactly the density of states in the thermodynamic limit (section 5.7.1). Other states made up of two strings lie within the energy band of the $1, M - 1$-strings and display a similar structure.

\(^2\)A van Hove singularity is a divergence in the density of states as a result of a flat dispersion relation: $\partial_\lambda \epsilon(\lambda) = 0$. 

98
Figure 5.3: Work probability distribution for $N = 250$ and $M = 100$ and anisotropies $\Delta = 1.5$ (left) and 3 (right). Indicated are the contributions from the important string states. The vertical scale of the full peak of the $M-$string is cut in the plot for clarity.

Since the energies are intensive in this energy domain it is expected that the results for $P(W)$ we obtain closely mimics the one in the thermodynamic limit. This is motivated by considering the difference in $P(W)$ for two different system sizes, see figure 5.4, showing the fact that finite-size effects rapidly disappear.

Figure 5.4: For $\Delta = 1.5$ we plot the difference of $P(W)$ obtained from different system sizes. Left: the difference between $N = 200$ and $N = 150$. Right: the difference between $N = 250$ and $N = 200$. The differences fall off with increasing system size, and for the sizes presented are smaller by about two orders of magnitude than the value of $P(W)$ obtained (see figure 5.3).

5.6 Relaxation dynamics: Loschmidt echo

We now turn to the question of whether the system effectively displays some form of relaxation. Of course here, since we consider a single realization and not an ensemble in the presence of a bath, relaxation can only come from the relative dephasing of the various terms in the right-hand side of (5.3).

It is not a priori easy to guess the outcome of the time evolution, since the
5.6 Relaxation dynamics: Loschmidt echo

states occupy coherent modes followed by overlapping continua, and each state contributes according to its relative overlap. Moreover, this outcome can essentially depend on which observable is considered, via the form factor values of the operator concerned, each operator favouring specific sets of state combinations. As a concrete example, we will focus on studying the Loschmidt echo \cite{163, 161} which is the overlap between the initial state and the time evolved state,

\[
\mathcal{L}(t) = |\langle \phi | e^{iH_0 t} e^{-iHt} | \phi \rangle|^2 \\
= \sum_{m,n} e^{i(E_m - E_n)t} |\langle \phi | \Psi_n \rangle|^2 |\langle \phi | \Psi_m \rangle|^2 \\
= \left| \int P(W) e^{iWt} dW \right|^2.
\]

This will allow us to quantify if the system dephases, if so how quickly and to what point, and whether initial state (partial) revival can take place. Plots of the Loschmidt echo for short and long times are presented in figure 5.5. We can separate different regimes in the time dependence. First and foremost, the small time, transient regime plotted in the left panel can be easily understood using simple perturbation theory, which predicts an initial decay quadratic in time with an anisotropy-dependent coefficient:

\[
\mathcal{L}(t) = 1 + (\langle H \rangle^2 - \langle H^2 \rangle) t^2 + O(t^4) \\
= 1 - \frac{1}{2\Delta^2} t^2 + O(t^4).
\]

More complex behaviour becomes evident when considering longer times, as in Figure 5.5: Loschmidt echo as a function of time for the domain wall quench for \( \Delta = 1.5 \) and \( N = 200, M = 80 \). In the left panel, the short time behaviour is shown. The dashed line is the result from perturbation theory. At large times, plotted in the right panel, one can distinguish effective relaxation towards the long-time average, but also finite-size induced small-scale revivals after a certain point (see main text). Note the change of scale in the horizontal axis in going from one figure to the other.

the right panel of figure 5.5. One point worth emphasizing is that our method,
Domain wall quench

unlike e.g. real-time numerics, is (in view of the exact wavefunctions and energies we use) directly applicable irrespective of the time \( t \) considered, and thus valid also at large times. There however, finite-size effects show up, originating from the discrete nature of the energy levels. Numerical inaccuracies also prevent us from giving numerical values at arbitrarily large times. Returning to the figure, a number of interesting features are displayed. First, interferences between the various terms lead to oscillations around the long-time average

\[
\mathcal{Z} = \sum_{m,n} \delta(E_m - E_n) \langle \phi | \Psi_n \rangle^2 / \langle \psi | \psi_m \rangle^2. \tag{5.41}
\]

These oscillations are accompanied by a decaying envelope which can be interpreted as effective relaxation. This behaviour carries on until a new regime is entered, where finite-size effects take over and lead to accidental partial revivals where the Loschmidt echo shows rapid, larger-scale variations. More generally, the echo looks chaotic after the onset of these revivals, and the numerical calculations lose their meaningfulness. The revival time is observed to grow with system size, first approximately linearly with system size, but more generally in an unspecifiable and nontrivial fashion.

5.7 Thermodynamic limit

The Bethe-Gaudin-Takahashi equations for the \( M \)-strings reduce to the simple equation:

\[
\theta_M(\lambda^M) = \frac{2\pi}{N} I_M^M. \tag{5.42}
\]

For even \( M \) and \( \Delta > 1 \) the \( N \) possible quantum numbers \( I_M^M \) are \( \{-\frac{N-1}{2}, -\frac{N-3}{2}, \ldots, \frac{N-1}{2}\} \), which follows from (2.92). In the thermodynamic limit, we can define a density function \( \rho_M(\lambda) \) for an \( M \)-string solution centered around \( \lambda \):

\[
\rho_M(\lambda) = \frac{N}{2\pi} \frac{d}{d\lambda} \theta_M(\lambda) = \frac{N}{2\pi} \frac{-i\varphi(\varphi_M)}{\varphi(\lambda^M - i\varphi_M/2)\varphi(\lambda^M + i\varphi_M/2)} \tag{5.43}
\]

such that \( \int_{-\pi/2}^{\pi/2} \rho_M(\lambda) d\lambda = N \). This function \( \rho_M(\lambda) \) should be interpreted as the number of solutions there are with a rapidity in the domain \( \lambda, \lambda + d\lambda \). The total contribution of all \( N \) \( M \)-string solutions is:

\[
\lim_{N \to \infty} \sum_{l=1}^{N} \frac{|\langle \psi_l^M | \phi \rangle|^2}{\langle \psi_l^M | \psi_l^M \rangle} = \int_{-\pi/2}^{\pi/2} \frac{\prod_{n=1}^{M-1} |\varphi(i\eta)^2|}{N(\varphi(\lambda - i\varphi_M/2)\varphi(\lambda + i\varphi_M/2))^{M-1}} \rho_M(\lambda) d\lambda
\]

\[
= \frac{-i\varphi(i\eta) \prod_{n=1}^{M-1} |\varphi(i\eta)|^2}{2\varphi'(iM\eta/2)^{2M}} 2F_1 \left( \frac{1}{2}, M, 1, \varphi'(i\eta M/2)^{-2} \right) \tag{5.44}
\]

where \( 2F_1(a,b,c,z) \) is the hypergeometric function:

\[
2F_1(a,b,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} \quad (x)_n = x(x+1)(x+2)\ldots(x+n-1). \tag{5.45}
\]
In the limit when both $M$ and $N$ are sent to infinity a simpler expression can be obtained:

$$
\lim_{M,N \to \infty} \sum_{l=1}^{N} \left| \langle \psi_M^l | \phi \rangle \right|^2 = \lim_{M \to \infty} \int_{-\pi/2}^{\pi/2} \prod_{n=1}^{M-1} \frac{|\varphi(i\eta)|^2 - i\varphi(i\eta M)}{(\varphi(i\eta M)/2)^{M-1}} \pi \varphi'(i\eta M) d\lambda + O\left(\frac{1}{\varphi'(i\eta M)}\right)
$$

$$
= \prod_{n=1}^{M-1} \frac{|\varphi(i\eta)|^2}{\varphi'(i\eta M)/2} + O\left(\frac{1}{\varphi'(i\eta M)}\right)
$$

$$
= \prod_{n=1}^{\infty} \left(1 - e^{-2n\eta}\right)^2.
$$

(5.46)

**Thermodynamic limit: XXX**

The isotropic limit $\Delta \to 1$ should be taken with care, since some M-string solutions correspond to different string configurations. For the XXX case we have the following restriction for the quantum number of an M-string:

$$
|I_M| \leq \frac{(N-M)}{2}.
$$

From the Bethe-Gaudin-Takahashi equation:

$$
\theta_M(\lambda^M) = 2\pi I^M/N
$$

(5.47)

we see that in the limit $N \to \infty$ and finite $M$ allowed values for $\lambda$ go from minus to plus infinity. In this limit we can derive the weight function:

$$
\rho_M(\lambda) = \frac{N}{2\pi} \frac{d}{d\lambda} \theta_M(\lambda) = \frac{N}{2\pi} \frac{M}{(\lambda^M - iM/2)(\lambda^M + iM/2)},
$$

(5.48)

so the contribution of the $M$-string for finite $M$ becomes:

$$
\lim_{N \to \infty} \sum_{l=1}^{N} \sum_{m=1}^{2M} \left| \langle \psi_M^l | \phi \rangle \right|^2 = \int_{-\infty}^{\infty} d\lambda \frac{(M-1)!^2}{N((\lambda - iM/2)(\lambda + iM/2))^{M-1}} \rho_M(\lambda)
$$

$$
= \frac{(2M - 2)!}{M^{2M-2}}.
$$

(5.49)

**Overlap for several strings**

The analysis given for the $M$-string contributions in the thermodynamic limit can be generalized to several strings. In case of a finite number of strings and in the limit $N \to \infty$ the scattering between strings becomes negligible and the Bethe-Takahashi equations decouple

$$
\theta_n(\lambda_n) = \frac{2\pi}{N} \frac{I_n^N}{I_n^0} + O(1/N).
$$

(5.50)

Note that, although the density of strings vanishes, the density of down spins can still be finite. For simplicity, we consider the case with two strings of length $m$ and $\bar{m} = M - m$ with rapidities $\lambda$ and $\bar{\lambda}$. Analogous to the case of a single $M$-string, we can introduce a density function $\rho_{m,\bar{m}}(\lambda, \bar{\lambda})$, which factorizes in the limit $N \to \infty$

$$
\rho_{m,\bar{m}}(\lambda, \bar{\lambda}) = \rho_{m}(\lambda) \rho_{\bar{m}}(\bar{\lambda}) + \text{vanishing corrections},
$$

(5.51)
with $\rho_m(\lambda)$ defined in (5.43). For the overlap coefficients we have to study equations (5.34) and (5.24) in the limit $N \to \infty$. The only explicit factors of $N$ appear in the Gaudin determinant (5.10) for the norm, which in this limit simply takes the form

$$\det(\Phi_m^{(r)}) = \left( N d\theta_m(\lambda) \right) \left( N d\theta_m(\bar{\lambda}) \right) + O(N).$$

(5.52)

Furthermore, one can show that all remaining factors of the overlap coefficients remain of order one in the limit $N \to \infty$, both for finite $M$ and infinite $M$. From this we conclude

$$|\langle \psi^{1,M-1} | \phi \rangle|^2 = O(1/N^2).$$

(5.53)

The same logic can be applied to show that for a state made of $m$ strings the corresponding overlaps are of order $|\langle \psi^{(m)} | \phi \rangle|^2 = O(1/N^m)$.

### 5.7.1 van Hove singularities

To address the van Hove singularities observed in the work probability spectrum, we need to derive the density of states. In the thermodynamic limit the density of states for an $n$-string in rapidity space is simply $\rho_n(\lambda)$. Using (5.6), the density of states as a function of energy is

$$\tilde{\rho}(\epsilon_n) = 2\rho_n(\lambda) \left( \frac{d\epsilon_n(\lambda)}{d\lambda} \right)^{-1}$$

$$= \frac{1}{\pi} \frac{\sinh(n\eta)}{\sqrt{\epsilon_n^2 - (\epsilon_n \cosh(n\eta) + \sinh(n\eta) \tanh(\eta))^2}},$$

defined on the interval $\epsilon_n^{(-)} \leq \epsilon_n < \epsilon_n^{(+)}$ with $\epsilon_n^{(\pm)} = -\tanh(\eta) (\tanh(n\eta/2))^\pm$. The factor 2 in the top line comes from the fact that $\epsilon(\lambda)$ is symmetric in $\lambda$. At the edge of the energy bands ($\epsilon_n^{(\pm)}$) the dispersion relation $\epsilon_n(\lambda)$ is flat, resulting in a divergence of $\tilde{\rho}(\epsilon_n)$. One can easily verify that these are square-root divergences by expanding of $\tilde{\rho}(\epsilon_n)$ up to leading order in $\delta_n^{(\pm)} \equiv \pm(\epsilon_n^{(\pm)} - \epsilon)$

$$\tilde{\rho}(\delta_n^{(\pm)}) = \frac{\sqrt{\sinh(n\eta)}}{\pi \sqrt{2 \tanh(\eta) \delta_n^{(\pm)}}} + O(\sqrt{\delta_n^{(\pm)}}).$$

(5.56)

### 5.7.2 Loschmidt echo

We now have all the ingredients to formally write the long-time average of the Loschmidt echo (5.41) in the thermodynamic limit as

$$\mathcal{Z} = \left( \int d\lambda \rho_M(\lambda) ||\phi|\langle \Psi^M(\lambda) ||^2 \right)^2$$

$$+ \int d\lambda \rho_1(\lambda) \left( \int d\lambda \rho_{M-1}(\bar{\lambda}) ||\phi|\langle \Psi^{1,M-1}(\lambda, \bar{\lambda}) ||^2 \right)^2 + \ldots.$$
5.7 Thermodynamic limit

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$\mathcal{L}_\infty$</th>
<th>$\mathcal{L}_{200}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.481212</td>
<td>0.481402</td>
</tr>
<tr>
<td>2</td>
<td>0.725941</td>
<td>0.726285</td>
</tr>
<tr>
<td>3</td>
<td>0.884184</td>
<td>0.884241</td>
</tr>
<tr>
<td>4</td>
<td>0.936021</td>
<td>0.936077</td>
</tr>
</tbody>
</table>

Table 5.2: The long-time average $\mathcal{L}$ in the thermodynamic limit compared with long time average for a finite size system ($N = 200$ and $M = 80$).

order $O(1/N)$. Therefore, only the $M$-strings contribute to the long-time average of the Loschmidt echo in the thermodynamic limit. In (5.46) we derived:

$$
\int d\lambda \rho_M(\lambda) |\langle \phi | \Psi^M(\lambda) \rangle|^2 = \prod_{n=1}^{\infty} (1 - e^{-2n\eta})^2, \quad \Delta > 1,
$$

(5.58)

in which we have used the definition $\cosh \eta \equiv \Delta$. Therefore, the Loschmidt echo is simply

$$
\mathcal{L} = \prod_{n=1}^{\infty} (1 - e^{-2n\eta})^4, \quad \Delta > 1.
$$

(5.59)

This result can be interpreted as the overlap between the initial state and the asymptotic state. The non-zero overlap indicates that the asymptotic state keeps most of the spatial anisotropy from the initial state. We conclude that the system cannot be described by a statistical ensemble (e.g. the generalized Gibbs ensemble) constructed of only macroscopic quantities. To gain some insight in how fast the thermodynamic limit is approached, we compare the result for $\mathcal{L}$ with the one for a finite system with $N = 200$. The finite time average is over intervals before the finite size effects show up. The finite size result $\mathcal{L}_{200}$ exceeds the thermodynamic limit value $\mathcal{L}_\infty$ by a small value, which can be attributed to the transient regime, which can be neglected in the infinite time average.

5.7.3 Expectation values

When considering expectation values, it is useful to introduce a new set of eigenstates $|\tilde{\Psi}\rangle$, by taking a superposition of states with equal energies and equal string lengths, weighted by the corresponding overlaps. For the $M$-string and $1,M-1$ string states these are, respectively,

$$
|\tilde{\Psi}^M\rangle \equiv \int d\lambda \rho_M(\lambda) \langle \Psi^M(\lambda) | \phi \rangle |\Psi^M(\lambda)\rangle
$$

(5.60)

and

$$
|\tilde{\Psi}^{1,M-1}(\lambda)\rangle \equiv \int d\tilde{\lambda} \rho_{M-1}(\tilde{\lambda}) \langle \Psi^{1,M-1}(\lambda, \tilde{\lambda}) | \phi \rangle |\Psi^{1,M-1}(\lambda, \tilde{\lambda})\rangle.
$$

(5.61)
As a result, these states are not translationally invariant. In this basis we can express the long-time average of a local observable $\mathcal{O}$ efficiently as

$$
\langle \mathcal{O} \rangle = \langle \tilde{\Psi}^M | \mathcal{O} | \tilde{\Psi}^M \rangle + \int d\lambda \rho_1(\lambda) \langle \tilde{\Psi}^{1,M-1}(\lambda) | \mathcal{O} | \tilde{\Psi}^{1,M-1}(\lambda) \rangle + \ldots
$$

(5.62)

Counting the powers of $N$ shows that all terms are of the same order and therefore all states with non-negligible overlaps are expected to contribute to the long-time average of local observables $\mathcal{O}$.

In the thermodynamic limit all local expectation values on the state $| \tilde{\Psi}^M \rangle$ can be easily obtained. We first study the ground states of the ferromagnetic XXZ chain in the thermodynamic limit. It was shown [164] that the complete set are the fully polarized states and the so-called kink- and anti-kink ground states. A kink-ground state is a state that interpolates between spin-up and spin-down on opposite ends of the chain; the position of the kink determines the total magnetization. It has been proven that all these ground states are frustration free, i.e. not only the total energy is minimal but also the energy of the local Hamiltonian; which allows for analytic expressions of all local observables [165, 166]. The state $| \tilde{\Psi}^M \rangle$ which has both a kink and an anti-kink can be thought of the product of a kink and anti-kink ground state of the ferromagnetic XXZ chain. We can confirm this by comparing the energy and the overlap with the initial state $| \phi \rangle$ in both cases. Furthermore, we can also check that both states have the same spectral gap. Therefore, for the computation of local observables on the state $| \tilde{\Psi}^M \rangle$ near the kink we can use the results of [165, 166]. For example the magnetization $\langle \tilde{\Psi}^M | S^z_j | \tilde{\Psi}^M \rangle$ is given by

$$
\frac{\langle \tilde{\Psi}^M | S^z_j | \tilde{\Psi}^M \rangle}{\langle \tilde{\Psi}^M | \tilde{\Psi}^M \rangle} = \frac{1}{2} - \sum_{k=0}^{\infty} (-1)^k e^{-\eta k^2} e^{-\eta k(2j+1)},
$$

(5.63)

see for a plot fig. 5.6. It is clear from (5.63) that the magnetization profile becomes flat in the limit $\Delta \to 1$. The methods presented in [165, 166] do not allow for an easy generalization to compute expectation values for excited states. The computation of the long time average $\langle S^z_j \rangle$ remains therefore an open question, but one can expect it to be very similar to fig. 5.6.

### 5.8 Discussion and Conclusions

In this chapter we studied the relaxation dynamics of the gapped XXZ chain after a quench from an initial domain wall state. The techniques that were used are based on the Algebraic Bethe Ansatz for a finite chain. Some of the results were extended to the thermodynamic limit. The key ingredient for studying this quench was deriving a numerically efficient expression for the overlap between the initial state and eigenstates of the XXZ Hamiltonian. Overlaps for different Ising states as initial state could be computed following the same logic, however in the general case of a finite density of domain walls this will become an intractable combinatorial problem.

Since the energies of the relevant part of the spectrum are intensive, a good comparison with the thermodynamic limit is possible. One important result is
that the bandwidth of the $M-$strings is nonzero for finite systems allowing the long time average of the Loschmidt echo going to zero, something that does not happen in the thermodynamic limit. Hence we conclude that the limits $\lim_{T \to \infty}$ and $\lim_{N \to \infty}$ do not commute.

In the thermodynamic limit we derived the infinite time average of the Loschmidt echo, which is non-zero when the system is gapped. This led us to the most important conclusion of this chapter, namely that the system for this quench will not thermalize, i.e. the long time average of observables cannot be predicted by one of the canonical ensembles. Moreover, the generalized Gibbs ensemble also fails, because the only input is macroscopic data (i.e. expectation values of local conserved charges) and can therefore not predict the observed spatial inhomogeneity. A way to cure this, is to include for instance the expectation value of the local magnetization, but since this is a time-dependent quantity this approach does not give any advantage.

The reason why the system does retain most of its spatial inhomogeneity can be explained from its spectrum, and does not directly rely on the fact that the system is integrable. Imagine for instance a gapped system with lowest (or highest) band having zero bandwidth in the thermodynamic limit, such as a system with macroscopically-degenerate ground states. Consider a quench such that $\langle W \rangle$ lies right between the lowest (or highest) energy band and the next one. A lower bound for the overlap $|\langle \phi | \Psi_0 \rangle|^2$ of the lowest band can then be obtained by simple reasoning. Let $E_0$ be the energy of the lowest energy band of zero bandwidth. We denote the band gap by $E_1 - E_0$. The expectation value of the initial Hamiltonian is $\langle H_0 \rangle = E_0 + \Delta E$. We then choose $0 < \Delta E < E_1 - E_0$. In this case we can obtain a lower bound for the total overlap of the $E_0$ states.

$$E_0 + \Delta E = \sum_n |\langle \phi | \Psi_n \rangle|^2 E_n$$

$$\geq |\langle \phi | \Psi_0 \rangle|^2 E_0 + (1 - |\langle \phi | \Psi_0 \rangle|^2) E_1,$$

(5.64)
from which follows the lower bound

$$|\langle \phi | \Psi_0 \rangle|^2 \geq 1 - \frac{\Delta E}{E_1 - E_0}. \quad (5.65)$$

In other words, such a quench must occupy the lowest (resp. highest) dispersionless band with $O(1)$ amplitude. Since these states are dispersionless and separated from other states by a band gap, they cannot dephase, and therefore cannot effectively relax. The large-time asymptotic state will therefore always maintain memory of the ground state occupation amplitude of this band, which is a quench-dependent quantity. Note that such situations trivially cannot show eigenstate thermalization according to the arguments of [12, 13], in which such circumstances were argued to be non-generic and therefore not treated.

The condition used for deriving this lower bound is quite strong. In figure 5.2 one can see that for $1 < \Delta < 1.5$ that $\langle W \rangle$ does not lie in the gap anymore, but the overlaps of the $M$–string are still substantially big. One can also question whether the condition of an isolated peak is essential. From the work probability distribution it is clear that the contributions of states made up of different strings can be distinguished. Simply looking at energy levels is therefore insufficient to assess the presence or absence of relaxation, and one must also take into account the values of the wavefunction overlaps. If we consider for example $n$ strings with a diverging length in the thermodynamic limit, they all will become coherent. Although they are embedded in a continuum, we can reasonably expect a non-zero height peak of zero width in $P(W)$, which again will prevent thermalization.

In the case of evolution under a gapless Hamiltonian, our method loses its efficiency, since all overlaps scale to zero and the number of states to take into account grows accordingly, preventing an efficient truncation of the sum in 5.3. The absence of a gap thus opens the door to dephasing involving arbitrarily complex excitation continua, in correspondence with Anderson’s orthogonality catastrophe principle. Therefore $\mathfrak{L} = 0$ in this case, and the possibility of thermalization remains, although it cannot be quantified here. The presence of a gap is thus determinative to the large time asymptotics of dynamics after the quench release of the domain wall state, but is not very sensitive to system size for large enough systems. In the gapless phase only approximate results are known. Using bosonization approach in was shown [156] that the magnetization profile becomes flat, this seems in agreement with our findings. A more surprising results is that in [156] the two-point functions retain an inhomogeneous structure. Therefore, also in the gapless regime the generalized Gibbs ensemble seems inapplicable.

To summarize, we considered a quench starting from a domain wall state under the time evolution of the gapped XXZ chain. We observed that the large time behavior cannot be described by a statistical ensemble. This can be understood by the fact that the spectrum is gapped, and does not directly rely on the fact that the system is integrable. Furthermore, we were able to assess finite size effects by considering both results for the finite and infinite systems. General conclusions for the large time behavior of the XXZ model after a quench cannot however be drawn from the results presented here. For instance, the two quasi-degenerate ground states of the XXZ chain we are dealing with are separated from all other states by the gap, so one might expect a similar effect if one would perform a
quench starting from the Néel state. However, the low energies are extensive in contrast with the high energy excitations which are intensive. The effect of the gap is much less pronounced and one might at least expect that the staggered magnetization will vanish for all $\Delta > 1$, in correspondence with the results of [36].