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Cavecchi, Y.

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Flame Propagation on the Surfaces of Rapidly Rotating Neutron Stars during Type I X–ray Bursts

Yuri Cavecchi, Anna L. Watts, Jonathan Braithwaite, Yuri Levin

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Abstract: We present hydrodynamic simulations of a deflagration flame in a thin helium ocean on the surface of an accreting and rapidly rotating neutron star. The flame moves horizontally with velocities of order $10^5$ cm s$^{-1}$, crossing the ocean in few seconds, which is broadly consistent with the rise times of Type 1 X–ray bursts. The heat is transported from the burning to the unburnt fuel by a combination of top-to-bottom conduction and mixing driven by a baroclinic instability. The speed of the flame propagation is therefore a sensitive function of the ocean conductivity and spin, and we explore this dependence for an astrophysically relevant range of both parameters.

Our simulations assume a plane–parallel geometry and use a new hydrodynamics solver developed by Braithwaite & Cavecchi. We resolve the vertical $z$ direction and the horizontal $x$ one, while assuming that the dynamical variables are invariant with respect to the $y$ coordinate. The disparity between the horizontal and vertical length scales is dealt with by introducing grid cells that are strongly elongated in the horizontal direction. Critically, we adopt $\sigma$–pressure coordinates to filter out the irrelevant sound waves in the vertical direction that would otherwise impractically limit our time steps.
5 Flame Propagation during Type I X-ray Bursts

5.1 Introduction

Type I bursts are tremendous thermonuclear explosions on the surface of accreting neutron stars (NSs), with luminosities that can easily reach the Eddington limit. They are characterized by a fast increase in the X-ray luminosity, known as the rise, that lasts from less than few seconds up to ten seconds, and by an exponential, slow decay, the tail, that lasts from tens of seconds to few minutes (for a detailed observational review, see Galloway et al., 2008). There are more than ninety known X-ray sources that have shown Type I bursts (for an up-to-date list see In’t Zand web page http://www.sron.nl/~jeanz/bursterlist.html). All of these sources are low mass X-ray binaries, where the NS is accreting matter from the outer layers of the companion.

A comparison between the accretion luminosity and the time averaged luminosity due to the burst indicates that, while the accretion luminosity source is mainly gravitational energy converted into heat, the extra luminosity comes from nuclear reactions within the accreted fluid (Lewin et al., 1993, and references therein). Type I burst spectra are very well fit by a black body (Swank et al., 1977; Hoffman et al., 1977), which again is a clue pointing to the thermonuclear origin. Moreover, the time evolution of the spectrum indicates an initial increase of the emitting area and subsequent slow cooling of the whole surface (Lewin et al., 1993; Strohmayer et al., 1997). Putting all these facts together has lead to the following broad picture.

First, the matter of the companion is accreted onto the NS surface and then starts spreading around. Whether the accreted fluid can spread freely or is confined to part of the NS surface probably depends on whether the accretion takes place via a boundary layer (the disc directly ‘touching’ the NS, Inogamov & Sunyaev, 2010) or via magnetic channelling and, subsequently, on the strength of the magnetic field itself (Brown & Bildsten, 1998). The majority of bursters do not show persistent pulsations, while those that do feature very weak magnetic fields. It is therefore reasonable to assume that the accreted material spreads over all of the NS surface, forming a thin highly combustible ocean consisting of mostly light elements.

As new fluid is piling up, the deeper layers are compressed until eventually the temperature and density are high enough to trigger nuclear reactions, mainly of H or He or both. Depending on the accretion rate and the composition of the fluid, the burning can be stable or unstable (see Fujimoto et al., 1981): in the latter case Type I bursts occur. It seems unlikely that the whole star ignites at the very same moment (Shara, 1982); instead it is more likely that the ocean ignites locally and that the resulting flame propagates laterally
and engulfs the whole (or a substantial fraction) of the NS surface. It is the physics of the lateral propagation of the thermonuclear flame that is the focus of our study.

During the rise, the X-ray luminosity increases, until the peak (which may be a plateau lasting several seconds) is reached. After that, the energy production decreases and the burst dies out. Some open questions are: what is the vertical structure of the burning atmosphere? And then, what is the horizontal structure of the burning layer? What drives the propagation of the flame? Do the adjacent fluid elements ignite because of heat conduction or because unburnt fluid is compressed by the burning and consequently expanding parts? Does turbulent mixing help the flame propagation, stirring hot, burnt, matter into the cooler, unburnt, zones (discussions in Bildsten, 1995; Spitkovsky et al., 2002)?

Another very intriguing open question involves oscillations that have been detected during many bursts (burst oscillations, BOs). These are nearly-periodic variations of the intensity of the lightcurve that are observed during different parts of the bursts. They have been seen in bursts from both persistent and intermittent accretion powered X-ray pulsars and from non-pulsating sources (see Watts, 2012, for a review). Most strikingly, when BOs can be compared to accretion pulsations in pulsars, it can be seen that BO frequencies are within a few hertz of the spin frequency of the NS.

When they discovered the first case of BOs, Strohmayer et al. (1996) noted that BOs must be the result of some kind of asymmetry of the surface emission. As the star revolves, brighter and dimmer patches alternate through the line of sight and cause the lightcurve to fluctuate. However what causes such asymmetries is a mystery. Some sources show oscillations with constant frequency throughout the entire duration of the bursts, while others have upwards drifting frequency during either rise or tail (Watts, 2012). What causes this variety of behaviour is also unknown.

Strohmayer et al. (1996) suggested a hot-spot without proposing a confinement mechanism. For the slowly-spinning (11Hz) magnetic neutron star in IGR J17480–2446, Cavecchi et al. (2011) showed that a magnetically confined hot-spot is indeed the only plausible explanation. However in the majority of sources with rapidly-spinning, non-magnetic neutron stars this is not a likely explanation and other possibilities must be explored. Spitkovsky et al. (2002, hereafter SLU) suggested that BOs in the rise might be produced by burning hurricanes confined by the Coriolis force (in some respects similar to anti-cyclones on earth), while BOs in the tail might be produced by Jupiter-Red-Spot-like structures in the differentially rotating heated ocean. However,
no predictive model has emerged from these suggestions, and the BO frequency drift of a few Hz has not been be explained.

As an alternative, Heyl (2004) suggested that global $m=1$ surface modes, i.e. giant waves in the ocean, excited by the flame propagation might lead to compression and expansion of the fluid and therefore explain modulation of the surface temperature and brightness. As the ocean cools, the mode frequency would change, and therefore Heyl’s picture could in principle explain the drifts of frequency in the tail. However, detailed calculations showed that the surface-wave speeds were a factor of a few larger than that required by observations, leading to predictions of unacceptably large frequency drifts. Attempts to fix this problem by invoking coupling to a crust-interface mode (Piro & Bildsten, 2005) have been cast into doubt by Berkhout & Levin (2008), who showed that coupling would be ineffective. Therefore, at present, mode models remain discrepant with observations.

Clearly, a better understanding of flame propagation could shed some light on the origin of BOs: determining, for example, the degree of asymmetry during the rise (thereby explaining BOs in the early stages), or establishing whether it is possible to excite global modes in the burst tail. A better understanding of burst oscillations is also relevant to efforts to measure NS parameters, such as mass and radius, using burst oscillation pulse profile modelling (Strohmayer, 1992; Poutanen & Beloborodov, 2006; Lamb et al., 2009; Morsink & Leahy, 2011; Altamirano et al., 2011). Knowing the emission pattern from the star, a natural result of understanding flame spreading, would reduce parameter degeneracy in pulse profile fitting, assisting efforts to make indirect measurements of mass and radius\(^1\). As outlined in the introductory chapter of this thesis, this is of primary importance to constraining the NS equation of state, testing gravitational wave theories and studying binary evolution (Strohmayer & Bildsten, 2006).

To answer the many outstanding questions about burst physics, researchers have turned primarily to numerical simulations. Whilst some groups have probed the details of reaction chains and rates (i.e. Fujimoto et al., 1981; Hanawa & Fujimoto, 1984; Wallace & Woosley, 1984; Woosley et al., 2004), others have focused on the structure and dynamical evolution of the bursts. Some studies have simulated the vertical structure of the atmosphere with one or two zone models (i.e. Paczynski, 1983; Narayan & Heyl, 2003; Cooper & Narayan, 2006), whilst others have used more resolved hydrodynamical simulations, highlighting for example the importance and extension of convection

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\(^1\)Even though one must remember that many other obstacles remain that hinder such determinations, especially in connection with the uncertainties in the spectral modeling.
(Woosley et al., 2004; Weinberg et al., 2006). However all of these models are one-dimensional. The need for multi-dimensional simulations arises as soon as one wants to study the structure of convection without parameterizations. For example Malone et al. (2011) have analyzed in great detail how convective cells develop, using the MAESTRO code which assumes the low Mach number approximation (Nonaka et al., 2010, see also the results of Lin et al. 2006).

Early attempts at multi-dimensional simulations of the flame propagation mechanism simulations include Nozakura et al. (1984), who followed the zonal approach, and Fryxell & Woosley (1982a), who hydrodynamically simulated the first few milliseconds of a detonation in a thick layer. Zingale et al. (2001) continued this line of research, following the detonating flame for up to hundreds of milliseconds. The problem with simulations of detonations is that the thick layers they require are unlikely to accumulate between bursts, so that in reality the flame probably develops via deflagration (Malone et al., 2011). The most recent studies of flame propagation, which last again only a few milliseconds (far shorter than the timescales of real X-ray bursts), are those by Simonenko et al. (2012a,b). In the first of these papers, the flame propagates via a detonation wave, in a manner similar to that in Zingale et al. (2001). The authors find that propagation is due to the hot fluid expanding and spilling over the top of the cold fluid, which then compresses and ignites. In the second paper, the authors explore regimes with densities $\sim 2 \times 10^7$ g/cm$^3$ at the base of the ocean, conditions where one might expect deflagration rather than detonation. However, in this case the simulations did not develop a steadily propagating flame.

None of these previous studies take into account the rotation of the neutron star. However SLU studied the role of the Coriolis force on flame dynamics and concluded that the Rossby adjustment radius may determine the horizontal scale of the burning front. They also pointed out the importance of the ageostrophic flow at the hot-cold fluid interface as part of the mechanism for the flame propagation itself. That said, SLU used a two layer, shallow water, method to simulate the propagation of the flame and therefore had to make phenomenological assumptions on the heat and momentum transport in the vertical direction. In this paper our simulations do not involve such assumptions, and are fully resolved in the vertical direction, allowing us to make a detailed study of the flame-propagation physics, taking full account of rotation.

In this paper, we simulate flame propagation on a domain that has horizontal extent that is a substantial fraction of the surface of the NS. At the heart of our simulations is the hydrodynamical code described in Braithwaite
& Cavecchi (2012, hereafter BC). It is a multidimensional code, which by construction enforces hydrostatic equilibrium in the vertical direction (this assumption is justified since the timescale for sound propagation in the vertical direction in the burning layer is much shorter than the nuclear reaction timescale). Hydrostatic equilibrium allows us to use a longer time step, which would be otherwise limited by sound wave propagation in the vertical direction. It also allows us to use pressure as the vertical coordinate. This is a great advantage, because we can follow the inflation of the fluid without the need for extra grid points which would lie unused for most of the simulation, consuming extra memory and increasing calculation time (for more details see BC).

In Sec. 5.2 we briefly review the numerical code described in BC and introduce the additions and modifications made to the code in order to study thermonuclear flame propagation. We then report our results on flame spreading in Sec. 5.3. We conclude with a brief summary in Sec. 5.4.

### 5.2 Numerical implementation

In this section we describe the modifications made to the code reported by BC in order to make it suitable for study of flame propagation during Type I X-ray bursts. First however we briefly review the most salient features of the code as outlined in BC. It is a 3D magnetohydrodynamical code, which uses the $\sigma$-coordinate system (a pressure coordinate system, see below) on a staggered grid: thermodynamical variables like temperature, pressure, density and heat sources are evaluated at the centres of the grid cells, while velocities are evaluated on the “faces” of the cells. The code is 3D, but for the scope of this paper we will be using a 2D version by assuming variables are independent of one of the horizontal dimensions. We also neglect magnetic fields, postponing this for future research.

The assumption of vertical hydrostatic equilibrium, justified by the short vertical sound crossing time (much shorter than any time scale of interest in burst simulations), allows us to discard vertically propagating sound waves, the numerical resolution of which consumed the lion’s share of the cpu time in previous numerical experiments. In our simulations we can therefore employ much longer timesteps than previous studies. Vertical equilibrium leads naturally to the introduction of a vertical pressure coordinate. This in turn makes it possible to follow the fluid as it expands, without the need of extra grid cells.

The code evolves the two horizontal components of the fluid velocity, the
5.2 Numerical implementation

pressure (which acts as a pseudo density) and the temperature using a three step Runge-Kutta scheme, while the spatial derivatives are calculated with sixth-order finite differences. Pressure is defined as

\[ P = \sigma P_* + P_T \quad (5.1) \]

where \( P_* = P_B - P_T \) and \( P_B \) and \( P_T \) are the pressure at the bottom and at the top of the simulation. \( P_T \) is a constant parameter in the simulations and \( \sigma \in [0, 1] \) becomes the vertical coordinate (0 corresponding with the top, 1 with the bottom). \( P_* \) can be shown to become a pseudo density and the continuity equation becomes

\[ \frac{\partial P_*}{\partial t} = -I_{\sigma=1} \quad (5.2) \]

where

\[ I \equiv \int_{0}^{\sigma} \nabla_{\sigma} \cdot (P_* u) \, d\sigma' , \quad (5.3) \]

\( u \) is the horizontal velocity and \( \nabla_{\sigma} = \hat{x} \partial_{x} + \hat{y} \partial_{y} \), taken at constant \( \sigma \).

Defining the Lagrangian derivative as \( \frac{D}{Dt} = \partial_t + u_x \partial_x + u_y \partial_y + \dot{\sigma} \partial_{\sigma} \), the momentum and energy equations are

\[ \frac{Du}{Dt} = -\nabla_{\sigma} \phi - \sigma \frac{\nabla P_*}{\rho} - 2\Omega \times u + F_{\text{visc}} . \quad (5.4) \]
\[ c_p \frac{DT}{Dt} = \frac{1}{\rho} \frac{DP}{Dt} + Q , \quad (5.5) \]

where

\[ \phi = gz = P_* \int_{\sigma}^{1} \frac{d\sigma'}{\rho} ; \quad (5.6) \]

\( \rho \) is the density, \( \Omega \) is the rotation vector of the star, parallel to the \( z \) direction, \( F_{\text{visc}} \) are the viscous forces, \( c_p \) is the heat capacity at constant pressure and \( Q \) is the heat per unit mass per unit time (see BC for more details); the equation of state in BC is a perfect monoatomic gas.

We now move on to discuss the modifications to the code that were implemented to render it suitable for Type I burst simulations.

Equation of state

The first relevant change is to the equation of state (EOS) for the fluid in our simulations, which has to be able to describe the physics of the NS ocean. We take into account the composition of the fluid by expressing it in terms of
the mass fraction: $X$ is the fractional mass of H, $Y$ that of He and $Z=1-X-Y$ the fraction of all heavier elements. For a fully ionized perfect gas, the perfect gas EOS used in BC becomes (assuming full ionization):

$$P = \frac{\rho RT}{\mu}$$

with

$$\mu = \frac{12}{7 + 17X + 2Y} \text{g mol}^{-1}.$$  \hspace{1cm} (5.8)

However, since conditions in the oceans of NSs will easily lead to electron degeneracy, which plays an important role in the vertical support of the ocean against the gravitational field, we must take this into account in our simulations. We still consider the atoms to be fully ionized; however whilst the nuclei are assumed to be a perfect gas, the electrons can be (partially) degenerate and (partially) relativistic. We also need to include radiation pressure.

For this purpose we adapted the publicly available routine helmeos$^2$ of Timmes & Swesty (2000). It uses the density $\rho$, temperature, $X$ and $Y$ to derive pressure, energy, the thermodynamic potentials and their derivatives with respect to $\rho$, $T$, $X$ and $Y$. In our code structure $\rho$ is a derived quantity, while pressure is a main one (see BC, sec. 2). To circumvent the problem of passing density as input parameter to the routine, we interface the original helmeos with a zero-finding routine that calls it repeatedly with different values of $\rho$ until convergence in pressure is achieved. Subsequent calls use the previous value of $\rho$ as an initial guess. Given the Courant conditions we impose, the information in each grid point does not change much in one time step, and convergence is achieved within two or three calls. This is done in parallel for each grid point.

The choice for the EOS is important for the evolution equation for the temperature $T$. We still derive it from the first law of thermodynamics:

$$\frac{DE}{Dt} = \frac{1}{\rho^2} P \frac{DP}{Dt} + Q$$

(where $E$ and $Q$ are the energy and heating rate per unit mass), but we have two different results depending on the choice of the equation of state. If we use the perfect gas equation of state, the evolution equation for temperature has the form (compare to BC, eq. 20):

$$\frac{c_P}{\mu} \frac{DT}{Dt} = \frac{1}{\rho} \frac{DP}{Dt} + \frac{c_P T}{12} \left(17 \frac{DX}{Dt} + 2 \frac{DY}{Dt}\right) + Q,$$  \hspace{1cm} (5.10)

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where \( c_P = R(\gamma - 1)/\gamma \) and \( \gamma \) is the adiabatic index and \( R = 8.3144621 \times 10^7 \) erg K\(^{-1}\) mol\(^{-1}\) is the gas constant\(^3\).

When we include electron degeneracy and radiation pressure, by contrast, we have

\[
\frac{\dot{c}_P}{D_t} = \frac{DP}{D_t} + B \frac{DX}{D_t} + C \frac{DY}{D_t} + Q, \quad (5.11)
\]

with

\[
D = \left( \frac{P}{\rho^2} - \frac{\partial E}{\partial \rho_{T,X,Y}} \right) \quad (5.12)
\]

\[
A = D \frac{\partial \rho}{\partial P_{T,X,Y}} \quad (5.13)
\]

\[
\dot{c}_P = \left( \frac{\partial E}{\partial T_{\rho,X,Y}} - D \frac{\partial \rho}{\partial T_{P,X,Y}} \right) \quad (5.14)
\]

\[
B = \left( \frac{\partial E}{\partial X_{\rho,T,Y}} - D \frac{\partial \rho}{\partial X_{P,T,Y}} \right) \quad (5.15)
\]

\[
C = \left( \frac{\partial E}{\partial Y_{\rho,T,X}} - D \frac{\partial \rho}{\partial Y_{P,T,X}} \right), \quad (5.16)
\]

where we make use of the relations

\[
\frac{\partial \rho}{\partial P_{T,X,Y}} = \frac{1}{\partial P/\partial \rho_{T,X,Y}} \quad (5.17)
\]

\[
\frac{\partial \rho}{\partial T_{P,X,Y}} = - \frac{\partial P/\partial T_{\rho,X,Y}}{\partial P/\partial \rho_{T,X,Y}} \quad (5.18)
\]

\[
\frac{\partial \rho}{\partial X_{P,T,Y}} = \frac{\partial P/\partial X_{\rho,T,Y}}{\partial P/\partial \rho_{T,X,Y}} \quad (5.19)
\]

\[
\frac{\partial \rho}{\partial Y_{P,T,X}} = - \frac{\partial P/\partial Y_{\rho,T,X}}{\partial P/\partial \rho_{T,X,Y}} \quad (5.20)
\]

since the routine returns variables as functions of \( \rho, T, X \) and \( Y \).

A further complication comes from the fact that HELMEOS actually uses \( \bar{A} = 12/(1 + 11X + 2Y) \) and \( \bar{Z} = \bar{A}(1 + X)/2 \), (instead of \( X \) and \( Y \) directly) for \( P \), and that the derivatives of \( E \) and \( P \) are evaluated with respect to \( T, \bar{A}, \bar{Z} \) and \( \rho \). Therefore, for equations (5.15), (5.16), (5.19) and (5.20) we also

---

\(^3\)Note that in BC we include \( \mu \) in the definition of \( R \).
need

\[
\frac{\partial}{\partial X} = - \frac{\bar{A}^2}{12} \left( \frac{11}{\bar{A}} \frac{\partial}{\partial A} + (5 - Y) \frac{\partial}{\partial Z} \right)
\] (5.21)

\[
\frac{\partial}{\partial Y} = - \frac{\bar{A}^2}{12} \left( 2 \frac{\partial}{\partial A} + (1 + X) \frac{\partial}{\partial Z} \right)
\] (5.22)

The final evolution equation for the temperature is therefore \( (D/Dt = \partial/\partial t + u \cdot \nabla \sigma + \dot{\sigma} \partial/\partial \sigma) \):

\[
\frac{\partial T}{\partial t} = T_{t, \text{adv}} + T_{t, \text{thermodyn}} + Q/\bar{c}_P,
\] (5.23)

where the contributions to \( \partial T/\partial t \) are separated into

\[
T_{t, \text{adv}} = - (u \cdot \nabla \sigma T + \dot{\sigma} \partial T/\partial \sigma)
\] (5.24)

\[
T_{t, \text{thermodyn}} = \frac{A}{\bar{c}_P} \frac{DP}{Dt} + T_{t, \mu}
\] (5.25)

\[
T_{t, \mu} = \frac{B}{\bar{c}_P} \frac{DX}{Dt} + \frac{C}{\bar{c}_P} \frac{DY}{Dt}
\] (5.26)

and \( Q/\bar{c}_P \), so that we can test the relative importance of the contributions of the different terms (see Sec. 5.3). We also include an artificial diffusive term (with a small coefficient, see Sec. 3.4 of BC) to ensure numerical stability. In the case of the perfect gas EOS the evolution equation is very similar to equation (5.23).

Finally, the term \( DP/Dt \) is evaluated according to equation (19) of BC, which does not depend on the EOS, but on the choice of the \( \sigma \)-coordinate system. \( DX/Dt \) and \( DY/Dt \) have to be treated carefully: in case of reactions, or any change in composition, they have to be evaluated explicitly (see Sec. 5.2).

**Conduction**

Since conduction may play an important role in flame propagation, we include a physical conduction term in \( Q \) of the form:

\[
Q_{\text{cond}} = \frac{1}{\rho} \nabla \left( \frac{16 \sigma_B T^3}{3 \rho \kappa_c} \nabla T \right)
\] (5.27)
where $\kappa_c$ is the effective opacity due both to radiative and conductive processes. In the $\sigma$-coordinate system $Q_{\text{cond}}$ takes the form:

$$\frac{1}{P_*} \left\{ \nabla_\sigma \left[ \frac{16 \sigma_B T^3}{3 \kappa_c \rho} \left( \frac{P_*}{\rho} \nabla_\sigma T + \nabla_\sigma \phi \frac{\partial T}{\partial \sigma} \right) \right] + \frac{\partial}{\partial \sigma} \left[ \frac{16 \sigma_B T^3}{3 \kappa_c P_*} \left( \nabla_\sigma \phi \frac{3}{2} \frac{\partial T}{\partial \sigma} + \frac{P_*}{\rho} \nabla_\sigma \phi \nabla_\sigma T + g \frac{\partial T}{\partial \sigma} \right) \right] \right\}$$

(5.28)

where $\phi = g z = P_\star \int_{\sigma}^{1} \frac{1}{\rho} \rho d\sigma'$. The terms that include $\phi$ are due to the fact that the transformation matrix between the Cartesian coordinate system and the $\sigma$ coordinate system is not orthogonal everywhere. These contributions turn out to be small and can be neglected.

We have implemented two possibilities for evaluating $\kappa_c$: either it has a fixed value set at the beginning of the simulation or it is calculated for each and every grid point taking into account the composition and the thermodynamical variables at that position. For this second option we adapted the publicly available routines of Timmes: sig99\(^4\). The opacities calculated in this way take into account radiation, scattering and the degree of degeneracy (see Timmes, 2000, and references therein). Based on the values of density, temperature and composition in the simulations that we wanted to perform, we decided to use an average constant value of $\kappa_c = 0.07 \text{ cm}^2 \text{ g}^{-1}$ in our reference simulation, which speeds up the calculations whilst still preserving the critical physics.

As anticipated in section 5.2, we include in equation (5.23) an hyperdiffusive term (see BC, section 3.4.2), which mimics conduction. This term is unphysical and only used to ensure numerical stability. In the horizontal direction, in particular, it will be unphysically high, and may partly limit the conclusions we can draw from our simulations. However test runs involving much lower hyperdiffusivity yielded flame velocities (see section 5.3) which differ by only a few percent from the values reported in this paper.

**Sources and sinks of heat: nuclear burning and cooling**

Since we are interested in simulating Type I bursts, we implement helium burning via the triple-\(\alpha\) reaction according to (see Clayton, 1984):

$$Q_n = 5.3 \times 10^{18} \rho_5^2 \left( \frac{Y}{T_9} \right)^3 e^{-4.4/T_9} \text{ erg g}^{-1} \text{ s}^{-1},$$

(5.29)

where $T_9$ the temperature in units of $10^9$ K, $Y$ is the mass fraction of He and $\rho_5$ is the density in units of $10^5$ g cm$^{-3}$. Including only the triple-$\alpha$ process is of course a simplification, since there are many other reaction chains that should be taken into account (this model would not be correct even for a pure He accretor). However we leave this refinement for a later investigation.

During burning the composition is evolved according to

$$\frac{DY}{Dt} = -\frac{Q_n}{\epsilon_\alpha} \tag{5.30}$$

which corresponds to

$$\frac{\partial Y}{\partial t} = -\mathbf{u} \cdot \nabla Y - \dot{\sigma} \frac{\partial Y}{\partial \sigma} - \frac{Q_n}{\epsilon_\alpha} \tag{5.31}$$

where the first two terms come from advection and the third is the consumption of He due to nuclear reactions ($\epsilon_\alpha = 5.84 \times 10^{17}$ erg g$^{-1}$ is the energy production per gram per nucleon). We also include a form of artificial diffusion as described in Sec. 3.4 of BC to ensure numerical stability.

In terms of sinks of entropy, we include the possibility of cooling from the uppermost layers. We use a simple formula, derived under the assumption that energy is only transported through the layers above the computational domain that we do not simulate, without additional sinks or sources within the atmosphere. This is a somewhat coarse approximation, particularly since expansion of the upper layers may occur. We use the temperature of the top grid cell in order to evaluate the flux due to radiation and conduction:

$$F = \frac{16\sigma_B T^3}{3\rho \kappa_c} \frac{dT}{dz} \tag{5.32}$$

where $F$ is the flux, which we assume to be constant in our plane parallel approximation and $\sigma_B$ and $\kappa_c$ are as defined in Sec. 5.2; we further assume that $\kappa_c$ is constant across the layers above the simulation. Rearranging and integrating in the vertical, $z$, direction, from the top of the simulation ($T$) to the top of the NS atmosphere ($atm$), we have

$$F \int_{atm}^{T} \rho \, dz = \frac{16\sigma_B}{3\kappa_c} \int_{atm}^{T} T^3 \, dT. \tag{5.33}$$

In hydrostatic equilibrium, the integral on the left hand side reduces to $P_T/g$, so that

$$F \frac{P_T}{g} = \left. \frac{4\sigma_B T^4}{3\kappa_c} \right|_{atm} \tag{5.34}$$
Then, assuming that the temperature at the top of the atmosphere is negligible with respect to the one at the top of the simulation, we obtain

\[ F = \frac{4\sigma_B}{3\kappa_c P_T/g} T_4^4 \]  

(5.35)

This is the flux from the surface of a grid cell at the top of the simulation. In order to derive the entropy loss per unit mass, we multiply the flux by the surface area \( S \) of the cell and divide by the mass within it:

\[ Q_{\text{cool}} = F \frac{S}{\rho T S \Delta z_T} \]  

(5.36)

so that \( \Delta z \approx H \approx P_T/g\rho_T \)

\[ Q_{\text{cool}} = \frac{4g^2 \sigma_B}{3\kappa_c P_T^2} T_4^4 \]  

(5.37)

This is the sink term we use in our simulations; it could also be used as a first approximation to calculate the bolometric lightcurve of the bursts.

**Tracer particles**

Finally, we add the capability to follow tracer particles. They are assigned initial positions uniformly distributed in the integration domain and are evolved according to

\[ \frac{dx}{dt} = U_x(x, y, \sigma) \]  

(5.38)

\[ \frac{dy}{dt} = U_y(x, y, \sigma) \]  

(5.39)

\[ \frac{d\sigma}{dt} = \dot{\sigma}(x, y, \sigma) \]  

(5.40)

where \( d/dt \) is the rate of change of the particle’s position in the \( \sigma \) coordinates. Time evolution is the same as for all of the other variables (see BC, section 3.3), and the values of the three components of the velocity are derived by means of trilinear interpolation of the fluid velocity.

**5.3 Flame propagation simulations**

First we describe the numerical setup used for all the simulations, then we provide a description of what we see in the runs. Finally we will describe our interpretation of what drives the flame propagation.
Numerical setup

We ran a series of simulations resolving both the horizontal $x$ and vertical $z$ directions, assuming that the dynamical variables are independent of the $y$ coordinate (making the simulations effectively 2D). The fixed initial conditions, common to all of our simulations, are:

\[ P_T = 10^{22} \text{erg cm}^{-3} \quad \quad P_* = (e^{1.7} - 1) \times 10^{22} \text{erg cm}^{-3} \]
\[ X = 0 \quad \quad Y = 1 \]
\[ \nu_1 = 0.03 \quad \quad \nu_2 = 0.5 \]

and

\[ T_0 = 10^8 \text{K} \quad \quad \delta T = 3.81 \times 10^8 \text{K} \]

where $P_T$ and $P_*$ are the pressure at the top and the difference between the bottom and top pressure (see BC, section 2). The choice of $P_*$ means that we simulate 1.7 scale heights. $\nu_1$ and $\nu_2$ are the kinetic diffusive coefficients (see section 3.4 of BC). The corresponding coefficients for the temperature and the composition fractions $X$ and $Y$ are taken to be 1% of these values.

We also use a common initial temperature perturbation in all simulations. We use a $z$-independent temperature profile, which varies in the horizontal direction according to:

\[
T = T_0 + \frac{\delta T}{1 + \exp[(x - 8.7 \text{ km})/0.36 \text{ km}]} \quad (5.41)
\]

This function ensures that the temperature profile is smooth enough that it does not cause numerical issues; 8.7 km corresponds to the position where the temperature perturbation to the background $T_0$ is half of its maximum, while 0.36 km is approximately half the width between where the perturbation is asymptotic to its maximum and where it is asymptotic to its minimum (0 K).

We want to simulate a physical domain of 15 km, with a perturbation centred in the middle at 7.5 km. Since we assume symmetrical conditions around the centre, we actually only simulate the right half of it, from 7.5 km to 15 km: that means that the half width of the perturbation is $\sim 1.6$ km. In this way we simulate an ignition site (the analogue of a general initial inhomogeneity), where we ‘kick’ the initial flame, which is at the centre of our domain. Boundary conditions are symmetric in the vertical direction and reflective in the horizontal one. Our resolution is 240x90, for a horizontal extent of 7.5 km. Gravitational acceleration $g = 2 \times 10^{14} \text{ cm s}^{-2}$ is assumed.
5.3 Flame propagation simulations

<table>
<thead>
<tr>
<th>Run</th>
<th>$\nu$ (Hz)</th>
<th>$\kappa_c$ (g cm$^{-2}$)</th>
<th>$v_f$ (cm s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>450</td>
<td>$1 \times 10^{+0}$</td>
<td>$1.33 \pm 0.03 \times 10^{5}$</td>
</tr>
<tr>
<td>2</td>
<td>450</td>
<td>$7 \times 10^{-1}$</td>
<td>$1.43 \pm 0.02 \times 10^{5}$</td>
</tr>
<tr>
<td>3</td>
<td>450</td>
<td>$5 \times 10^{-1}$</td>
<td>$1.52 \pm 0.02 \times 10^{5}$</td>
</tr>
<tr>
<td>4</td>
<td>450</td>
<td>$3 \times 10^{-1}$</td>
<td>$1.67 \pm 0.03 \times 10^{5}$</td>
</tr>
<tr>
<td>5</td>
<td>450</td>
<td>$1 \times 10^{-1}$</td>
<td>$1.91 \pm 0.04 \times 10^{5}$</td>
</tr>
<tr>
<td>6</td>
<td>450</td>
<td>$7 \times 10^{-2}$</td>
<td>$2.01 \pm 0.05 \times 10^{5}$</td>
</tr>
<tr>
<td>7</td>
<td>450</td>
<td>$5 \times 10^{-2}$</td>
<td>$2.03 \pm 0.05 \times 10^{5}$</td>
</tr>
<tr>
<td>8</td>
<td>450</td>
<td>$1 \times 10^{-2}$</td>
<td>$1.99 \pm 0.02 \times 10^{5}$</td>
</tr>
<tr>
<td>9</td>
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<td>$1.98 \pm 0.03 \times 10^{5}$</td>
</tr>
<tr>
<td>10</td>
<td>50</td>
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<td>$1.11 \pm 0.18 \times 10^{6}$</td>
</tr>
<tr>
<td>11</td>
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<td>$5.30 \pm 0.31 \times 10^{5}$</td>
</tr>
<tr>
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<td>225</td>
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<td>$3.04 \pm 0.10 \times 10^{5}$</td>
</tr>
<tr>
<td>13</td>
<td>900</td>
<td>$7 \times 10^{-2}$</td>
<td>$1.39 \pm 0.02 \times 10^{5}$</td>
</tr>
</tbody>
</table>

Table 5.1: Values of the spin frequency $\nu = \Omega/2\pi$ and the opacity $\kappa_c$ used in the different simulations. In the third column we report the velocity of the flame as measured from the simulations, with errors derived from the least squares fit (see Section 5.3).

to be constant and the geometry is plane parallel. The fluid is at rest at the beginning $U_x = 0$ cm s$^{-1}$ and quickly adjusts to the Rossby solution (see BC, sec. 4.2) before the flame spreads.

Since we want to study the effects of different rotation frequencies and the influence of conduction, we run a series of models employing different values of the spin frequency $\Omega$ and the opacity $\kappa_c$. The parameters for the simulations that we run are given in Table 5.1.

To help us in finding out how the flame propagates, we use test particles and follow what happens to these fluid elements before, during, and after ignition conditions are met. We place the test particles homogeneously in our grid (note that they are not homogeneous in space since we use a pressure coordinate system) such that $x_{i,j} = i \times \delta x/200$ and $\sigma_{i,j} = j \times 1/200$, $i, j \in [1, 200]$ . We also track what happens at three different points in the atmosphere with fixed horizontal position. These points rise and descend with time, having fixed values of $\sigma$ not $z$, so that this approach is not strictly speaking Eulerian. However it still allows us to see what happens when the flame reaches a determined distance from the ignition point.
General description of the propagating flame

In this section, we give a qualitative description of the burning fluid as a whole. We begin by using one particular run as an example, since the qualitative behaviour is general. Figures 5.1 - 5.3 show the fluid in reference run (6) in its initial conditions and at \( t = 1.14 \) s, when the flame is propagating steadily. Figure 5.1 shows the temperature distribution, Figure 5.2 the burning rate. On these panels we superimpose our tracer particles. Finally, Figure 5.3 shows the density, with isobars superimposed.

In the top panel of Figure 5.1 the fluid is hotter on the left of the image: this is the initial perturbation, where the temperature is \( T = 4.81 \times 10^8 \) K, while at the other side the temperature is \( T = 10^8 \) K. Moving to the lower panel, we see that the flame front has moved to the right. Where the fluid has already burnt, it is hotter (\( T \sim 10^9 \) K) and has expanded by a factor of \( \sim 4^5 \). Looking at Figure 5.2, we can see that the tracer particles have been scattered by the passage of the front, while Figure 5.3 shows a drop in density. Eventually, after the flame has passed (not shown in the figure), the burning diminishes, the temperature decreases and the fluid contracts.

Looking more closely at the propagating front, we see that it is characterized by a slanted interface between the hot burning fluid on the left, and the cold unburnt on the right (in Figures 5.1 - 5.3 lower panels, the interface lies roughly between \( x = 3.7 \times 10^5 \) and \( x = 4.7 \times 10^5 \) cm). We see a decrease in pressure behind the interface and an increase immediately in front of it (see the lowest contours of pressure in the bottom panel of Figure 5.3). They reflect a change in \( P_\star \) (equation 5.1). Because of the hydrostatic approximation, \( P_\star \) is a measure of the column density at each point. A change in \( P_\star \) means mass motion. The decrease before the front and the increase after it therefore show that there has been a motion of the matter from behind the front forward.

The peak of the burning is concentrated in a thin stripe along the interface (Figure 5.2, lower panel) where the density is still high (undiminished by the increase in temperature) and the fuel is still almost pure Helium. Fluid is moving at the interface, primarily along it.

In order to understand what is driving the flame forward, we measure the different terms in the energy equation (5.23): conduction \( Q_{\text{cond}} / \tilde{c}_P \), advection \( T_{t,\text{adv}} \) (motion of the fluid) and thermodynamic compression \( T_{t,\text{thermodyn}} \). For each term, we plot the contributions in Figure 5.4. In the four panels the black line approximates the interface between hot and cold fluid. It is drawn below the region of significant burning in order to clearly demarcate regions where

\[ \text{(In general, the maximum expansion factor can be up to } \sim 4 - 5 \text{ depending on the effective opacity } \kappa_c \text{ that sets the cooling rate.)} \]
Figure 5.1: Initial conditions (above) and conditions at $t = 1.14$ s (below), when the flame is steadily propagating, for reference simulation (6). Temperature distribution. Note the different horizontal and vertical scales.
Figure 5.2: Initial conditions (above) and conditions at \( t = 1.14 \) s (below), when the flame is steadily propagating, for reference simulation (6). Burning rate with the tracer particles superimposed. Note the different horizontal and vertical scales.
Figure 5.3: Initial conditions (above) and conditions at $t = 1.14$ s (below), when the flame is steadily propagating, for reference simulation (6). Density with isobars superimposed (ten levels from $P = 10^{22}$ erg cm$^{-3}$ to $P = 6.2 \times 10^{22}$ erg cm$^{-3}$). Note the different horizontal and vertical scales.
Figure 5.4: Burning rate, and heating rate associated with advection $T_{t,adv}$ for reference simulation (6). The black line indicates the hot-cold fluid interface. The colour scale has been restricted to highlight more details (white regions indicate values above the maximum of the scale, the black ones below the minimum).
Figure 5.4: Continued. Heating rate associated with conduction $Q_{\text{cond}}/\tilde{c}_P$ and thermodynamical compression $T_{t,\text{thermodyn}}$ for reference simulation (6).
burning has started from those where burning is about to start.

It is clear from Figure 5.4 that in the region immediately below the peak of the burning, heat conduction is much more important in increasing the temperature of the unburnt fuel than both advection or thermodynamic compression. It is this process that drives flame propagation, since the main burning occurs in this zone. In the upper part of the interface, advection and thermodynamic compression dominate heat transfer to the unburnt region. Directly above the flame, heat conduction is not effective, while the advective and thermodynamic compressive terms show opposite signs. This is a clear signature of convection, which is expected above the burning regions. We also note that the cells near the topmost part of the hot-cold interface are mostly parallel to it (i.e. almost horizontal, recall the extreme aspect ratio of the interface), while the ones behind the interface are vertical.

To compute flame propagation speed from our simulations, we define the front position as the location with the maximum burning rate. In Figure 5.5, we follow the position of the front for simulation (6) and plot it versus time. At the beginning there is a transitional stage when the flame is started by the initial perturbation of the temperature and then the front adjusts to a steadily spreading configuration (in $\lesssim 0.1$ s). This steady propagation is well fit by a straight line, and the gradient gives us the speed of the flame front. We repeat the fit procedure for all the various runs: the resulting front speeds $v_f$ are reported in Table 5.1.

Having measured front velocities, we can determine the effects of the rotational spin $\Omega$ and the effective opacity $\kappa_c$ (a proxy for the heat conductivity). Overall, the gradient of the inflated fluid is steeper for higher $\Omega$, and the baroclinicity along the interface tends to increase with $\Omega$. Flame propagation speed decreases as rotation rate increases (see next sections and Figure 5.7). Changing $\kappa_c$ also has an effect on flame velocity: the flame is faster for lower $\kappa_c$, but the velocity saturates when $\kappa_c \lesssim 10^{-2}$ cm$^2$ g$^{-1}$ (see Figure 5.8).

A first set of conclusions

Although the fluid moves ageostrophically from behind the interface forward, this motion does not go past the interface (Figures 5.3 bottom panel). We interpret that as follow: the fluid has expanded behind the front because of its higher temperature and the resulting horizontal pressure gradient pushed the hot burning fluid to spill over the unburnt one. The Coriolis force, however, diverts the horizontal $x$ motion in the horizontal $y$ direction and thus creates

---

We want to stress that also these vertical cells are actually elongated in the horizontal direction due to the aspect ratio of our underlying grid cells.
5.3 Flame propagation simulations

Figure 5.5: Flame front position for run (6). The symbols indicate the positions, while the line is the best linear least squares fit. After an initial stage, the flame adjusts to steady propagation. Eventually, the flame reaches the opposite boundary (in this case in \( \sim 3 \) s).

A geostrophic current that compensates the horizontal pressure gradient; the configuration is the one of the Rossby adjustment problem (see BC), as anticipated by SLU. In this case the inclination angle \( \alpha \) of the interface should be \( \alpha \sim H/R_{Ro} \), where \( H \) is the scale height of the fluid and \( R_{Ro} \) is the Rossby radius \( R_{Ro} = \sqrt{gH/2\Omega} \), where \( \Omega = 2\pi \nu \) and \( \nu \) is the spin frequency of the neutron star. Measuring the slope of the black line in Figure 5.4, we find that the slope is \( \alpha \sim 3.5 \times 10^{-3} \), so that its horizontal extent is \( \sim 2 - 3R_{Ro} \). This is in accordance with expectations.

Regarding the motion that we observe along the interface, we note that here the fluid is much more baroclinic, that is to say, the iso–surfaces of density and pressure are much more misaligned than elsewhere, as can be seen in Figures 5.3 and 5.6. It is well known from geophysical studies that geostrophic balance is unstable in the presence of baroclinicity. The resulting instability is similar in nature to convection, but with motion, which is no longer vertical, lying within the “wedge of instability” between the iso–pressure and iso–density contours (Pedlosky, 1987). Fujimoto (1988, 1993) and Cumming & Bildsten (2000) in fact already studied the possibility of baroclinic instability in the context of Type I bursts, but their baroclinicity was very mild since
they considered the effects of shear induced by the differential rotation due to expansion of the burning layer, and not the effects of the huge horizontal temperature gradients that develop during flame propagation.

In our case, the source of baroclinicity is the horizontally-inhomogeneous nuclear burning\(^7\) which affects the temperature profile. Its steady propagation is maintained by the Coriolis force, which reinforces the near-geostrophic configuration on time scales of order \(1/\nu\). Following the tracer particle motion, we can see advection along and in front of the interface, which we attribute to the development of baroclinic instability triggered by the burning flame. In the previous section we noted the presence of cells that are highly elongated in the horizontal direction at the upper end of the hot-cold fluid interface: we identify these cells with baroclinicity-induced motion.

\(^7\)Compare Figure 5.6 to the lower panel of Figure 5.2.
Front propagation mechanism

Summarizing the results from Section 5.3, we see that at the top of the interface the fluid is heated up by the spilling of the hot fluid, by advection and thermodynamics, but in the most relevant regions for the flame propagation, heat is brought across the interface mostly by conduction, mainly vertically given the small inclination angle\(^8\), with a contribution by baroclinic instability mixing.

The contribution to \(\partial T/\partial t\) in equation (5.23) from conduction can be written by means of equation (5.27) as

\[
T_{t,\text{cond}} = \frac{Q_{\text{cond}}}{\tilde{c}_p} = \frac{1}{\tilde{c}_p \rho} \nabla \left( \frac{16 \sigma_B T^3}{3 \rho \kappa_c} \nabla T \right). \tag{5.42}
\]

This, in turn, can be approximated as a diffusion process with diffusion coefficient \(D_{\text{cond}}\) given by

\[
D_{\text{cond}} \sim \frac{16 \sigma_B T^3}{3 \tilde{c}_p \rho^2 \kappa_c} \tag{5.43}
\]

and then the timescale for heat diffusion across the vertical scale height \(H\) is

\[
\tau_{\text{cond}} \sim \frac{3 \rho^2 H^2 \tilde{c}_p}{16 \sigma_B T^3 \kappa_c} \tag{5.44}
\]

Equation (5.44) gives

\[
\tau_{\text{cond}} \sim 2.2 \times 10^{-2} \text{ s} \left( \frac{\kappa_c}{0.07 \text{ cm}^2 \text{ g}^{-1}} \right) \left( \frac{\rho}{10^5 \text{ g cm}^{-3}} \right)^2 \left( \frac{T}{10^9 \text{ K}} \right)^{-3} \left( \frac{H}{3 \times 10^2 \text{ cm}} \right)^2 \left( \frac{\tilde{c}_p}{10^8 \text{ erg K}^{-1}} \right) \tag{5.45}
\]

Once the lower fluid has been heated up and starts burning it expands again, and then Coriolis reinforces the Rossby adjustment in a time scale of order \(\nu^{-1} \ll \tau_{\text{cond}}\). This translates a small vertical shift into a long horizontal displacement, where the proportionality is given by the inclination of the interface: \(1/\alpha \sim (2 - 3R_{Ro})/H\), as we will see in the next section.

The effective, advective conduction brought about by baroclinic mixing would act on a time scale given by

\[
\tau_{\text{bar}} \sim H^2/D_{\perp\text{bar}}, \tag{5.46}
\]
\[\text{Heat conduction in the horizontal direction can be neglected since the horizontal length scale is larger by a factor } \sim 10^3 \text{ than the vertical length scale. Indeed, two runs where in one case full conduction was implemented and in the other only the vertical one was used gave identical results.}\]
with

$$D_{\perp \text{bar}} \sim w_{\perp \text{bar}} \lambda_{\perp \text{bar}}, \quad (5.47)$$

where $w_{\perp \text{bar}}$ and $\lambda_{\perp \text{bar}}$ are the physical velocity of the fluid and its length scale perpendicular to the hot-cold fluid interface. As long as $\tau_{\text{bar}} \gg \tau_{\text{conf}}$, conduction will be the most effective mechanism. We expect this condition to hold except in the case of very low heat diffusivity (high $\kappa_c$).

**Front propagation speed**

Following e.g. Landau & Lifshitz (1959); Fryxell & Woosley (1982b), the velocity of flame propagation across the interface should be given, in a deflagration regime, by

$$v_{f\perp} \sim \sqrt{\frac{D_{\text{cond}}}{\tau_n}} \quad (5.48)$$

where $\tau_n$ is the burning time scale, given by

$$\tau_n \sim 1.1 \times 10^{-1} Y^{-3} \exp(4.4 \times 10^9 K/T) \left(\frac{T}{10^9 \text{K}}\right)^3 \left(\frac{\rho}{10^5 \text{g cm}^{-3}}\right)^{-2}. \quad (5.49)$$

In order to estimate the horizontal propagation velocity across the NS, this velocity has to be multiplied by the factor $(2 - 3R_{\text{Ro}})/H$ which expresses the ratio of the area of the burning front to the area of the vertical section of the ocean (see Landau & Lifshitz, 1959). The horizontal velocity becomes

$$v_f \sim \sqrt{\frac{16\sigma_B T^3 g}{3\tilde{c}_p \rho^2 \tau_n H^2 \pi \nu \sqrt{\kappa_c}}} \quad (5.50)$$

or

$$v_f \sim 1.8 \times 10^6 Y^{3/2} \exp(-2.2 \times 10^9 K/T) \text{ cm s}^{-1} \left(\frac{\nu}{450 \text{ Hz}}\right)^{-1} \left(\frac{\kappa_c}{0.07 \text{ cm}^2 \text{g}^{-1}}\right)^{-1/2} \left(\frac{g}{2 \times 10^{14} \text{ cm s}^{-2}}\right)^{1/2} \left(\frac{H}{3 \times 10^2 \text{ cm}}\right)^{-1/2} \left(\frac{\tilde{c}_p}{10^8 \text{ erg K}^{-1}}\right)^{-1/2} \quad (5.51)$$

Again, if $\tau_{\text{bar}} \sim \tau_{\text{cond}}$, then the actual $v_{f\perp}$ should be given by a combination of conduction and advection, with an extra term of order

$$\sqrt{\frac{D_{\perp \text{bar}}}{\tau_n}}. \quad (5.52)$$
to be included in equation (5.48). The result should then be multiplied by the same factor, \((2 - 3R_{Ro})/H\).

The expected values from equation (5.51) are comparable to what we see in our simulations within half a order of magnitude. We verify the dependence of \(v_f\) on spin rate \(\nu\) using runs 6-10. In Figure 5.7, we can see that increasing the rotation frequency slows down the flame, as expected, with a \(1/\nu\) dependence.

By contrast Figure 5.8, where we plot \(v_f\) against the inverse heat conduction (our effective opacity \(\kappa_c\)), shows more complex behaviour. For opacities \(\kappa_c \gtrsim 0.05\) g cm\(^{-2}\), the flame speed increases approximately with \(1/\sqrt{\kappa_c}\) as expected from equation (5.51). Below it, the velocity seems to asymptote with \(v_f \sim 1.99 \times 10^5\) cm s\(^{-1}\). In our simulations we see a change in the morphology of the flame: indeed all simulations showed a flame leaning on the hot-cold fluid interface similar to the lower panel of Figure 5.2, apart from simulation (9) \((\kappa_c = 0.001\) g cm\(^{-2}\)), which does not show such a leaning flame and has a much more vertical structure of the temperature profile\(^9\), see Figure 5.9, and simulation (8) \((\kappa_c = 0.01\) g cm\(^{-2}\)) where this trend is beginning to become apparent.

We interpret this point as marking the transition where the conduction time scale \(\tau_{\text{cond}}\) becomes comparable to the burning time scale \(\tau_n\) and the

Figure 5.7: Velocity of flame propagation versus \(1/\nu\). All these runs have \(\kappa_c = 0.07\) cm\(^2\) g\(^{-1}\).
5.4 Discussion and conclusions

In this paper, we have been able to simulate for the first time the lateral propagation of a deflagrating vertically resolved flame on the surface of a neutron star. We find that after an initial post-ignition adjustment, the front propagates steadily with constant velocity, until it reaches the opposite side of the simulation box. The fact that the flame velocity is constant, Figure 5.5, gives us confidence that, regardless of the physics of localised ignition, steady flame propagation depends only on the physics acting in the ocean layer and the conditions therein. After all the surface has been traversed...
Figure 5.9: Temperature and reaction rate for simulation (9) after the flame is steadily propagating. The morphology of the flame is different to that shown in Figure 5.2. This simulation shows the asymptotic behaviour seen in Figure 5.8.
by the flame, the fluid cools down slowly, in a time which depends on the
opacity, whilst still burning the residual fuel. We note that in 2003, Anatoly
Spitkovsky (unpublished) obtained somewhat similar flames using the Pencil
code. Due to computational constraints, however, he assumed unphysically
large neutron-star spins, so that the Rossby radius was comparable to the
ocean scale height. As far as we are aware, in Spitkovsky’s simulations the
micro-physics of the flame propagation mechanism was not identified, and
full exploration of parameter range was not carried out (Spitkovsky, private
communication).

We have explored the dependence of the flame speed on the spin frequency
of the neutron star \( \nu \) and the heat conductivity of the fluid (expressed as
an inverse of the effective opacity \( \kappa_c \)). We measured velocities in the range
\( 1.33 \times 10^5 - 1.11 \times 10^6 \text{ cm s}^{-1} \), which cross the entire domain of 7.5 km in each
direction in 0.7 - 5.6 s. These numbers are in good agreement with the rise
times observed from Type I burst sources, suggesting that we have included
all the relevant physics in our simulations and that we are now in a position
to explore in more detail the behaviour of flame propagation during Type I
bursts.

The flame propagates through a combination of the ageostrophic forward
flow of the burning fluid on top of the as-yet unburnt fluid (as argued previously
in SLU), and top-to-bottom heat transport across the large-area strongly-
inclined interface between burning and cold fluid. Heat transport leading
to ignition is effected primarily by microscopic heat conduction and, in runs
where conductivity was set to lower values, by baroclinic motions.

In Section 5.3 we derived an order of magnitude estimate for the velocity
that the front would have if it were driven by conduction. We calculated a
dependence of the speed on \( \kappa_c \) and \( \nu \) of the form \( 1/\nu \sqrt{\kappa_c} \) (Figures 5.7 and
5.8) and confirmed these expectations with the results of our simulations.
The breakdown of the \( \kappa_c \) dependence and the existence of asymptotes on
the limits of both small and large \( \kappa_c \) can be understood qualitatively. In
particular, we observe the existence of a possible asymptote in the velocity
when the effective opacity is too small, which we explain as follows. When the
opacity decreases sufficiently, the conduction time scale becomes shorter than
the nuclear burning time scale. The latter becomes the bottleneck, the burning
front width becomes comparable to the scale height, and the nuclear burning
time scale becomes the time scale of vertical expansion. This translates into
a horizontal velocity of \( \sim R_{Ro}/\tau_n \), as already anticipated by SLU.

There are a number of hydrodynamical issues that now have to be explored
further. Firstly, the effect of the baroclinic instability at the hot–cold fluid in-
terface has not been explored thoroughly. Secondly, the flow in the $y$-direction has a velocity comparable to the sound speed, and Kelvin-Helmholtz instabilities that might be generated by this flow need to be investigated. Other aspects of the flame propagation will be explored in future work, including the effects of a better burning prescription that might include other elements apart from Helium. We also aim to investigate the possibility of exciting large-scale waves in the ocean and include the effect of magnetic fields.

Finally, some of our simulations suggest that in the absence of a sufficiently strong Coriolis force the flame will die out. This leads to an important question: can the flame cross the equator? Near the equatorial belt the effective Coriolis force is much weaker and this could lead to rapid lateral spreading of the burning front, and subsequent quenching of the burning by enhanced cooling. This would have important consequences on the efforts to determine NS radius from observations of type-I bursts (see, e.g., Steiner et al., 2010), since it is usually assumed that the whole star is burning at the peak, and the derived radius of the burning area is used as measure of the NS radius. If the flame cannot cross the equator, this fact has to be taken into account when dealing with those estimates. This would also have important implications for burst recurrence times, and may help to explain the properties of multi-peak bursts (Bhattacharyya & Strohmayer, 2006). We plan to investigate this possibility by introducing a variable Coriolis parameter in our code for a future work, to simulate properly the changes that would occur as a flame approaches the equatorial belt.

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