The time-variation of volatility and the evolution of expectations

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Chapter 2

GO-GARCH: A new multivariate volatility model

2.1 Introduction

The 'holy grail' in multivariate GARCH modeling is without any doubt a parameterization of the covariance matrix that is feasible in terms of estimation at a minimum loss of generality. The general multivariate GARCH models available parameterize the covariance matrix by a very large number of parameters that are hard to estimate, which often leads to convergence difficulties of estimation algorithms. Therefore, the choice of the multivariate model is often determined by means of practical considerations i.e. the ease of estimation. The strong restrictions are often not believed to reflect the 'truth', but they are imposed to guarantee feasibility.

Some of the best known multi-variate GARCH models available include the VECH model of Bollerslev *et al.* (1988), the constant correlation model of Bollerslev (1990), the factor ARCH model of Engle *et al.* (1990), and the BEKK model studied by Engle and Kroner (1995). For an overview of the multivariate GARCH models, as well as tests for misspecification, see the paper by Kroner and Ng (1998). An extensive survey of empirical applications of time-varying covariance models in finance can be found in Bollerslev *et al.* (1992). In particular the model of Bollerslev (1990) has been a popular choice for modelling high-variate time series. A test for its assumption of a constant correlation is introduced in a recent paper by Tse (2000). Shortly after, both Engle (2002) and Tse and Tsui (2002)

generalized the model to allow for time-varying correlations.

A somewhat different approach is the Orthogonal GARCH (O-GARCH) or principal components GARCH method. The principal components approach has first been applied in a GARCH-type context by Ding (1994). Shortly after, Alexander and Chibumba (1996) introduced the strongly related O-GARCH model. Thereafter, O-GARCH has been a popular choice to model the conditional covariances of financial data (see e.g. Klaassen, 1999), mainly because the model remains feasible for large covariances matrices (see e.g. Alexander, 2002). Recently, the model has been elaborated along with applications by Alexander (1998, 2001).

The O-GARCH model implicitly assumes that the observed data can be linearly transformed into a set of uncorrelated components by means of an orthogonal matrix. These unobserved components can be interpreted as a set of uncorrelated factors that drive the particular economy or market, similar to that in the Factor (G)ARCH approach of Engle *et al.* (1990). The orthogonality assumption, however, appears to be very restrictive. Indeed, if a linkage with a set of uncorrelated economic components exists, why should the associated matrix be orthogonal? The O-GARCH model is also known to suffer from identification problems, mainly because estimation of the matrix is entirely based on unconditional information (the sample covariance matrix). For example, when the data exhibits weak correlation, the model has substantial difficulties to identify a matrix that is truly orthogonal (see e.g. Alexander, 2001).

The multivariate GARCH model proposed in this chapter can best be seen as a natural generalization of the O-GARCH model. Clearly, orthogonal matrices are very special, and they only reflect a very small subset of all possible invertible linear maps. The generalized O-GARCH model (GO-GARCH) allows the linkage to be given by any possible invertible matrix. Estimation of the matrix requires the use of conditional information, which in turn solves possible identification problems[1]. The parameters are relatively easy to estimate, so that a substantial increase in the degrees of freedom is obtained at a very affordable price.

The next section will introduce the generalized Orthogonal GARCH model (GO-GARCH). Estimation is discussed in Section 2.3. Sections 2.4 and 2.5 present some simulation results and an empirical example, respectively. Section 2.6 concludes.

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[1]For example, the data is not required to exhibit strong dependence for the method to work.

2.2 Generalized Orthogonal GARCH

2.2.1 Notation

In a multivariate GARCH setting, the conditional covariance matrix of the m-dimensional zero mean random variable depends on elements of the information set up to time $t-1$, denoted by $\Im_{t-1}$. Assume that x_t is normally distributed and that its conditional covariance matrix V_t is measurable with respect to $\Im_{t-1}$, the multivariate GARCH model is then described by:

$$x_t \,|\Im_{t-1} \sim N\left(0, V_t\right), \tag{2.1}$$

where we have assumed that x_t is second order stationary so that $V = E(V_t)$ exists. The information set $\Im_t$ contains both lagged values of the squares and cross-products of x_t and elements of the conditional covariance matrices up to time t, i.e. lagged values of V_t. The challenge in multivariate GARCH modeling is to find a parameterization of V_t as a function of $\Im_{t-1}$ that is fairly general while feasible in terms of estimation.

In the following we will frequently use the terms conditional information and unconditional information. We specify unconditional information as information that can be extracted from the unconditional covariance matrix. By conditional information we mean the information set $\Im_t$ as introduced above.

2.2.2 Representation

The key assumption of the GO-GARCH model is the following:

Assumption 2.1 *The observed economic process $\{x_t\}$ is governed by a linear combination of uncorrelated economic components[2] $\{y_t\}$:*

$$x_t = Zy_t. \tag{2.2}$$

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[2]Note that there might be more components than the number of variables observed, so that exposing a set of reliable components could be troublesome. However, as the components are assumed to be described by independent GARCH-type models, a new set of uncorrelated components can be constructed by aggregating the 'original' components. Under certain conditions, the (extracted) aggregated components are also described by GARCH-type processes, see for example Drost and Nijman (1993) in which temporal aggregation of GARCH processes is considered. However, it is known that the GARCH-type 'features' typically become weaker under aggregation. As a consequence, the accuracy with which the components are described by GARCH-type models increases as more components can be extracted, which will result in better fits.

The linear map Z that links the unobserved components with the observed variables is assumed to be constant over time, and invertible.

Without loss of generality[3], we normalize the unobserved components to have unit variance, so that:

$$V = Ex_t x_t^T = ZZ^T. \tag{2.3}$$

An explicit example, which we will denote the GO-GARCH(1,1) model, would be:

$$x_t = Zy_t \quad y_t \sim N(0, H_t), \tag{2.4}$$

where each component is described by a GARCH(1,1) process:

$$H_t = diag(h_{1,t}, ..., h_{m,t}) \tag{2.5}$$

$$h_{i,t} = (1 - \alpha_i - \beta_i) + \alpha_i y_{i,t-1}^2 + \beta_i h_{i,t-1} \quad i = 1, ..., m, \tag{2.6}$$

where $H_0 = I$ equals the unconditional covariance matrix of the components[4]. The conditional covariances of $\{x_t\}$ are given by:

$$V_t = ZH_tZ^T. \tag{2.7}$$

2.2.3   Identification

Let P and Λ denote the matrices with, respectively, the orthonormal eigenvectors and the eigenvalues of the unconditional covariance matrix $V = ZZ^T$.

Let us assume that an orthogonal linear linkage Z indeed exists, so that $x_t = Zy_t$. The unconditional covariance matrix V is then given by: $V = ZHZ^T$, where H is diagonal. Then the orthogonal matrix P, the O-GARCH estimator for Z, is only guaranteed to coincide with Z, when the diagonal elements of H are all distinct. Identification problems thus arise when some of the uncorrelated components have similar unconditional variance.

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[3]Note that the unconditional variances of the components and the matrix Z are directly related. Let $\{y_t\}$ denote the components with original scaling, and let the normalized set of components be denoted by $\{\tilde{y}_t\}$, so that $\{\tilde{y}_t\} = \{Dy_t\}$, where D represents the diagonal normalization matrix. The observed process is then given by $\{x_t\} = \{Zy_t\} = \{\tilde{Z}\tilde{y}_t\}$, where $\tilde{Z} = ZD^{-1}$.

[4]Ling and McAleer (2003) provide a method for treating the initial value when it comes to asymptotic theory for multivariate GARCH.

To see this, suppose that all components have unit variance, so that $V = ZIZ^T = I$. Clearly, the matrix Z is no longer identified by the eigenvector matrix of V, as for every orthogonal matrix Q, we have $(ZQ)(ZQ)^T = I$. Note that the eigenvalues of V reflect the variances of the components when the model is well identified. The estimations should therefore be interpreted with caution when some of the eigenvalues are almost identical. Problems of this type are known to occur when, for example, the data exhibits weak dependence[5]. The next lemma states that the linkage Z is well identified when conditional information is taken into account.

Lemma 2.1 *Let Z be the map that links the uncorrelated components $\{y_t\}$ with the observed process $\{x_t\}$. Then there exists an orthogonal matrix U_0 such that:*

$$P\Lambda^{\frac{1}{2}}U_0 = Z. \tag{2.8}$$

Proof. The result follows directly from Singular Value Decomposition, see e.g. Horn and Johnson (1999). ∎

Let the estimator for U_0 be denoted by U. Without loss of generality, we restrict the determinant of U to be 1[6].

It can be verified that the orthogonal matrices P and Λ have $\frac{m(m-1)}{2}$ and m degrees of freedom, respectively. Together with the $\frac{m(m-1)}{2}$ degrees of freedom for U, we have $m + m(m-1) = m^2$ degrees of freedom for the invertible matrix Z. The matrices P and Λ will be estimated by means of unconditional information, as they will be extracted from the sample covariance matrix V. Conditional information is required to estimate U_0.

Note that there is a continuum of matrices Q for which a set of linearly independent components $u_t = Qx_t$ can be obtained. For every choice of orthogonal matrix U, the linear transformation $Q = U^T\Lambda^{-\frac{1}{2}}P^T$ induces an uncorrelated series with unit variance: $Eu_tu_t^T = QVQ^T = U^TU = I$. Clearly, these components often still exhibit a form of nonlinear correlation. Therefore, linear independence can be very deceiving, as it might give the impression that the linkage between the observed variables and the uncorrelated components is uncovered, when more often it is not. The original components can only[7] be restored by means of the inverse of Z.

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[5]Given that the observed data is normalized to have unit variance, which is common practice.

[6]More precisely, U is considered an element of $SO(m)$, which denotes the set of all m-dimensional orthogonal matrices with positive determinant.

[7]Equivalent matrices, in the sense that they only exchange variables for example, are included.

According to Lemma 2.1, the model is well identified as there exists a U_0 that is associated with the original Z. Indeed, the additional $\frac{m(m-1)}{2}$ degrees of freedom induced by the extra term U extends the representation to full generality, in the sense that any invertible linkage Z can in principle be estimated from the data, instead of orthogonal matrices only.

One way to parameterize the estimator for the orthogonal matrix U_0 would be by means of rotation matrices:

Lemma 2.2 *Every m-dimensional orthogonal matrix U with $\det(U) = 1$ can be represented as a product of $\binom{m}{2} = \frac{m(m-1)}{2}$ rotation matrices:*

$$U = \prod_{i<j} R_{ij}\left(\theta_{ij}\right) \quad -\pi \leq \theta_{ij} \leq \pi, \tag{2.9}$$

where $R_{ij}\left(\theta_{ij}\right)$ performs a rotation in the plane spanned by e_i and e_j over an angle θ_{ij}.

Proof. See Vilenkin (1968). ■

The rotation angles[8] $\{\theta_{ij}\}$ are commonly referred to as the Euler angles, which can be estimated by means of maximum likelihood.

We have noted earlier already that the O-GARCH model suffers from identification problems, for example when the data exhibits weak dependence. These problems should not arise when conditional information is exploited, as proposed in the GO-GARCH model. For example, when the independent components appear to be observed directly, we expect the estimator for U_0 to be close to P^T, since $\hat{Z} = P\Lambda^{\frac{1}{2}}P^T = V^{\frac{1}{2}}$ is approximately diagonal when the data is virtually independent.

2.2.4   Time-varying correlations

The implied conditional correlations $\{R_t\}$ of the observed process $\{x_t\}$ can be computed as:

$$R_t = D_t^{-1}V_t D_t^{-1}, \quad D_t = (V_t \circ I)^{\frac{1}{2}}, \tag{2.10}$$

where $\{V_t\} = \{ZH_t Z^T\}$ denotes the conditional covariances of $\{x_t\} = \{Zy_t\}$, and where $\circ$ denotes the Hadamard product.

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[8]Note that the values for the angles will depend on the ordering of the rotation matrices. The ordering should not affect the estimation results.

This theoretical example illustrates how possible lower and upper bounds for the correlation depend on the type of linear map Z_θ. Let Z_θ be the following two dimensional map:

$$Z_\theta = \begin{pmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{pmatrix}, \tag{2.11}$$

where θ measures the extent to which the uncorrelated components are mapped in the same direction. For $\theta = 0$ the map is not invertible yielding perfect correlation between the observed variables, whereas for $\theta = \frac{1}{2}\pi$ we have the identity map, so that the observed variables are completely uncorrelated. Let the conditional variances of the uncorrelated components be denoted by (h_{1t}, h_{2t}). It can be verified that the conditional correlation between the observed variables, denoted by ρ_t, is given by:

$$\rho_t = \frac{h_{1t}\cos\theta}{\sqrt{h_{1t}}\sqrt{h_{1t}\cos^2\theta + h_{2t}\sin^2\theta}}. \tag{2.12}$$

If we assume that $h_{it} > 0$, we can define $z_t = \frac{h_{2t}}{h_{1t}}$, so that ρ_t can be expressed as:

$$\rho_t = \frac{1}{\sqrt{1 + z_t\tan^2\theta}}. \tag{2.13}$$

For finite samples, the variable z_t will have finite lower and upper bounds. As a consequence, the conditional correlation ρ_t is also bounded.

Note that a constant linkage Z_θ gives rise to time-varying correlations between the observed variables. These correlations rise on average when the components are mapped more in the same direction. We can not exclude the possibility that the 'economic mechanism' Z evolves over time. If so, endogenizing Z and make it time-varying, might improve the fit of the time-varying correlations. Extending the GO-GARCH model to allow for a non-constant Z, however, is left for further research. A first step would be to test for a constant linkage, for example by means of test on structural change such as the Chow test.

2.3 Estimation

The parameters that need to be estimated by means of conditional information, include the vector θ of rotation coefficients that will identify the invertible matrix Z (see lemma 3), and the parameters (α, β) for the m uni-variate GARCH(1,1) specifications. The log

likelihood $L_{\theta,\alpha,\beta}$ for the GO-GARCH model can be represented as:

$$
\begin{aligned}
L_{\theta,\alpha,\beta} &= -\frac{1}{2} \sum_t m \log\left(2\pi\right) + \log|V_t| + x_t^T V_t^{-1} x_t & (2.14) \\
&= -\frac{1}{2} \sum_t m \log\left(2\pi\right) + \log\left|Z_\theta H_t Z_\theta^T\right| + y_t^T Z_\theta^T (Z_\theta H_t Z_\theta^T)^{-1} Z_\theta y_t & (2.15) \\
&= -\frac{1}{2} \sum_t m \log\left(2\pi\right) + \log\left|Z_\theta Z_\theta^T\right| + \log|H_t| + y_t^T H_t^{-1} y_t, & (2.16)
\end{aligned}
$$

where $Z_\theta Z_\theta^T = P \Lambda P^T$ is independent of θ. For the initial value of H_t we take the identity matrix, which equals the implied unconditional covariance of $\{y_t\}$. Even in high-variate cases, when the covariance matrices are very large, it should not be a problem to maximize the log likelihood over the $\frac{m(m-1)}{2} + 2m$ parameters. Note that in order to avoid convergence difficulties of estimation algorithms, we propose a kind two-step estimation. We exploit unconditional information first, so that the number of parameters for Z that are estimated through maximum likelihood is $\frac{m(m-1)}{2}$ instead of m^2 (see lemma 2).

Conditions for strong consistency of the maximum likelihood estimator for general multivariate GARCH are derived by Jeantheau (1998). These conditions are verified by Comte and Lieberman (2003) for the general BEKK model, in which a result of Boussama (1998), concerning the existence of a stationary and ergodic solution to the multivariate GARCH(p,q) process, is used.

It can be verified that the more general BEKK model has the GO-GARCH model nested as a special case. Strong consistency of the quasi MLE for GO-GARCH can therefore be established by appealing to Jeantheau's conditions, following Comte and Lieberman. To keep it simple, we focus on the GO-GARCH(1,1) model as in (2.4), but it can be verified that the results also hold for the more general GO-GARCH(p,q) model. To apply the results of Jeantheau (1998), we assume that starting value of the process is drawn from its stationary distribution P_{θ_0}, although Comte and Lieberman (2003) indicate that consistency holds for an arbitrary starting value. We refer to Ling and McAleer (2003) for a more extensive discussion of the treatment of the initial value and its implication for asymptotic properties.

Proposition 2.1 *Consider the GO-GARCH(1,1) model, where α_i and β_i correspond to the GARCH(1,1) parameters of the independent components. Assume that B0-B2 holds,*

and that the components are stationary, i.e.

$$\alpha_i + \beta_i < 1 \quad for\ i=1,...,m. \tag{2.17}$$

Then the MLE is consistent.

Proof. The result follows directly from the derivation of Comte and Lieberman (2003).
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How to conduct inference is beyond the scope of this chapter. However, as Comte and Lieberman (2003) have proven asymptotic normality of the quasi-MLE for the BEKK model, having GO-GARCH nested as a special case, we conjecture that this property is also inherited by GO-GARCH. Some caution will be in place though, since we proposed a kind of two-step estimation which will affect the distribution of the estimator. For example, the standard errors might be underestimated by the Fisher Information matrix. We leave a precise study of the asymptotic distribution for further work. For tests on possible misspecification of the multivariate GARCH model see Kroner and Ng (1998), and the more recent paper by Tse (2002).

2.4 Simulation results

This section aims to illustrate the behavior of the GO-GARCH model by experimenting with artificial data.

2.4.1 Orthogonal linkage

We constructed the independent components by generating from 4 univariate GARCH(1,1) models to build a 4-variate time series. The conditional variance of each component is described by:

$$h_{i,t} = c_i + \alpha_i y_{i,t-1}^2 + \beta_i h_{i,t-1} \quad i = 1,...4. \tag{2.18}$$

The values that are assigned to the parameters (c, α, β) are summarized in Table 2.1.

The parameters are chosen so that variances are nearly integrated, which is commonly observed in financial data. Also note that the parameters are chosen in such a way that

component	c	α	β
1	0.08	0.10	0.88
2	0.03	0.08	0.90
3	0.05	0.15	0.80
4	0.10	0.20	0.70

Table 2.1: *GARCH parameters*

some of the unconditional variances are identical. The length of the artificial data set is 3000 observations, which is equivalent to approximately 12 years of daily data.

The first orthogonal matrix considered is the identity matrix. As it preserves independence, the components will be observed directly. In the second part, we simulate with an orthogonal matrix that induces dependence among the observed variables.

Independent multivariate GARCH

In this part, we test whether the models are able to detect the independent nature of the observed data. It is known that the O-GARCH model can not deal properly with virtually independent data. In contrast, GO-GARCH should be able to estimate a linear representation that induces weak dependence or even independence. The results are presented in Table 2.2.

As expected, O-GARCH was not able to detect the independence of the process. The estimated matrix is far from being diagonal, so that conditional dependence is 'brought into' the residuals. The substantial errors in the GARCH parameters estimates also indicate that O-GARCH did not extract the independent components, but some dependent variables instead. The GO-GARCH model, however, performs very well in this example. The estimated linkage correctly reflects the independent nature of the data. Also the GARCH parameters are estimated properly[9].

Dependent multivariate GARCH

The independent components are described by exactly the same process as in the first part. The key difference is that in this example they are not observed directly. The observed process will be an orthogonal representation of the components that exhibits strong dependence. In principle, the O-GARCH model could also perform well in this example, as the observed variables are no longer independent, while the associated matrix is orthog-

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[9]Note that some components might have been switched.

onal. However, note that some of the components have a similar scaling (unconditional variance). As a consequence, O-GARCH might still suffer from identification problems.

	O-GARCH				GO-GARCH			
$\widehat{Z^{-1}}$	0.39	−0.25	−0.64	−0.58	1.00	0.02	0.01	0.01
	−0.27	0.83	−0.06	−0.47	−0.01	1.00	0.01	0.01
	−0.88	−0.36	−0.32	−0.08	0.00	0.02	−1.00	0.00
	−0.06	−0.34	0.70	−0.66	0.00	0.00	0.04	−1.00
c	0.09	0.04	0.03	0.09	0.03	0.02	0.05	0.10
α	0.09	0.06	0.08	0.10	0.11	0.06	0.15	0.20
β	0.81	0.90	0.89	0.80	0.86	0.92	0.80	0.70

Table 2.2: *Estimates for the linkage and GARCH parameters*

The orthogonal matrix, denoted by Z, is constructed as a product of four rotation matrices, and is shown in Table 2.3. Table 2.4 summarizes the results.

map	Z
matrix	$\begin{pmatrix} \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2\sqrt{2}} \\ -\frac{\sqrt{3}}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{3}{4} \\ \frac{1}{4} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{4} \end{pmatrix}$

Table 2.3: *Orthogonal linkage*

	O-GARCH				GO-GARCH			
c	0.05	0.08	0.05	0.07	0.10	0.02	0.05	0.03
α	0.06	0.07	0.07	0.06	0.20	0.06	0.15	0.11
β	0.89	0.84	0.89	0.87	0.70	0.92	0.80	0.86

Table 2.4: *Estimates for the GARCH parameters*

In the case of O-GARCH, the estimates for the GARCH parameters are clearly different from the true parameters suggesting that the model was not able to identify the independent components. In the previous subsection we have seen that the trivial orthogonal matrix, namely identity, could also not be identified by O-GARCH. Thus even when the linkage is truly orthogonal, there is no guarantee that O-GARCH is able to identify it. The model additionally requires that all the components have a different scaling, which might often not be the case.

When we look at the estimates of the GO-GARCH model, we find that the GARCH parameters of the components are estimated with reasonable accuracy[10]. From this we conclude that the linkage estimated by GO-GARCH can not be far from the 'truth' as we build it.

2.4.2   Non-orthogonal invertible linkage

In this subsection, non-orthogonal invertible matrices are chosen to link the independent components with the observed process. This will be an important example, as we generalized the O-GARCH model to be able to expose linkages that are not orthogonal. It follows that lower bounds for the conditional correlations can be observed when the linkage matrices approach singularity.

component	c	α	β
1	0.05	0.15	0.80
2	0.05	0.25	0.70

Table 2.5: *The GARCH parameters*

map	Z_1	Z_2	Z_3	Z_4
matrix	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Table 2.6: *The invertible linkages*

map	Z_1	Z_2	Z_3	Z_4
V_i	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{5}{4} & 2 \\ 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$	$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$

Table 2.7: *The unconditional covariances*

map	Z_1	Z_2	Z_3	Z_4
W_i	$\begin{pmatrix} 1.41 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2.24 & -2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.47 & 0.75 \\ -0.94 & 0.75 \end{pmatrix}$	$\begin{pmatrix} -0.75 & 1.49 \\ 1.49 & -0.75 \end{pmatrix}$

Table 2.8: *The true linear representations*

In order to have a more controlled experiment, we confine ourselves to the 2-dimensional case. Similar to the first examples, we construct two independent components in order

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[10]Note that some components have been switched.

to build up a bivariate time series. The conditional variances of both components are specified by means of the same GARCH model, as in (2.18). Also the sample size is chosen to be identical, namely 3000 observations. Table 2.5 lists the values at which the GARCH parameters were set to simulate the independent components. It can easily be verified that both components have unit unconditional variance, so that their unconditional covariance matrix equals the identity matrix.

We will consider four different invertible linear maps for the linkage. The associated matrices, denoted by Z_1 till Z_4, are shown in Table 2.6. The unconditional covariance matrix of the observed process is simply given by: $V_i = Z_i Z_i^T$. The covariances V_1 till V_4 are listed in Table 2.7.

The observed data is commonly normalized to have unit variance by a diagonal matrix D, so that the covariance matrices of the normalized series $\tilde{V}_i = D_i Z_i Z_i^T D_i^T$ has 1's along the diagonal. In our example, the diagonal elements of D_1 till D_4 are easily seen to be $\left\{\frac{1}{\sqrt{2}}, 1\right\}$, $\left\{\frac{2}{\sqrt{5}}, \frac{1}{2}\right\}$, $\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{5}}\right\}$, and $\left\{\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\}$, respectively. It follows that the true matrix that links the normalized observed variables with its independent components, is given by $(D_i Z_i)^{-1}$. These matrices, denoted by W_i, are shown in Table 2.8.

Note that in all cases the orthonormal eigenvectors of the unconditional covariance matrix are given by $P = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, so that O-GARCH is expected to estimate scaled versions of P for the linkages.

O-GARCH

map	Z_1		Z_2		Z_3		Z_4	
$\hat{W}_i$	0.54	0.54	0.51	0.51	−0.61	−0.61	0.53	0.53
	1.31	−1.31	2.17	−2.17	0.86	−0.86	1.59	−1.59
$\hat{c}_i$	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.06
$\hat{\alpha}_i$	0.18	0.12	0.21	0.13	0.14	0.23	0.11	0.12
$\hat{\beta}_i$	0.76	0.83	0.73	0.82	0.82	0.72	0.84	0.81

Table 2.9: *The estimates for the linkages and the GARCH parameters*

GO-GARCH

map	Z_1		Z_2		Z_3		Z_4	
$\hat{W}_i$	0.01	0.99	0.02	0.98	−0.48	−0.73	1.49	−0.74
	1.41	−1.01	2.23	−2.00	0.94	−0.76	0.77	−1.51
$\hat{c}_i$	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
$\hat{\alpha}_i$	0.25	0.14	0.25	0.14	0.14	0.25	0.25	0.14
$\hat{\beta}_i$	0.71	0.81	0.71	0.81	0.81	0.71	0.71	0.81

Table 2.10: *The estimates for the linkages and the GARCH parameters*

The results for both the O-GARCH and the GO-GARCH model are presented in Table 2.9 and 2.10, respectively. This example illustrates that the GO-GARCH model is able to deal with decompositions that are not of the orthogonal type. The estimated linkages are in all cases very close[11] to the 'truth', the matrices from Table 2.8. The estimates for the GARCH parameters are also accurate.

A priori we know that the O-GARCH model can not uncover the non-orthogonal linkages, as it restricts the matrix to be orthogonal. As a consequence, it extracts components that are not independent, which is reflected by the biased estimates for the GARCH parameters. Particularly in example 4, the O-GARCH estimates for the GARCH parameters show substantial error. The difference between the estimated correlations is therefore most notable in example 4, which can be seen in Figure 2.1.

In example 4, the GO-GARCH estimates for the correlations never fall below 0.8, say, whereas the correlations estimated by O-GARCH show much stronger declines and sometimes even drop till below 0.4. The reason for this effect is that the matrix from example 4 shows the strongest 'deviation' from an orthogonal matrix. The linkage from example 4 maps both independent components in almost the same direction which induces a strong correlation between the observed variables. Exactly the same feature is observed in the empirical example described in the next section. Indeed, it seems reasonable that observed variables that are strongly related exhibit high correlation at all times. As the linkage with the components that induces the high correlation is assumed constant over time, it will be surprising to observe periods in time where the variables suddenly appear almost uncorrelated. This feature is illustrated by a theoretical bivariate example, where the lower bound and upper bounds of the correlation are derived as a function of a characteristic parameter θ of the linkage Z_θ.

2.5   Empirical example

We include an example from real life, as an attempt to gain insight in the relation between observed economic and financial variables and the uncorrelated factors that are assumed to drive the market. Our example considers the Dow Jones Industrial Average (DJIA) versus the NASDAQ composite. The sample contains more than ten years of daily observations, starting at the first of January in 1990 and ending in October 2001. First, we estimate a

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[11]Neglect signs, as they do not yield a different representation. Also note that some components might have been switched.
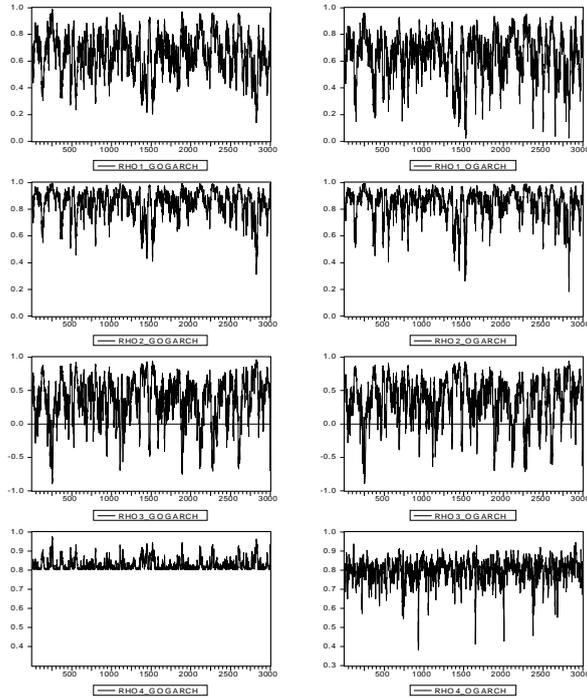
Figure 2.1: *The estimated time evolutions of the (conditional) correlations for all four examples. The figures on the left correspond with the GO-GARCH model, and the figures on the right with O-GARCH.*

first order[12] vector autoregressive (VAR) model to account for the linear structure present in the data. Subsequently, we use the residuals to estimate the conditional covariances from which the (conditional) correlations between the DJIA and the NASDAQ can be computed. Questions that arise naturally include: (i) Are non-orthogonal linkages common in real life examples? (ii) Will allowing for a more general linkage (all invertible matrices) typically induce a better description of the time-varying correlations between economic and financial variables?

We estimate both the O-GARCH and GO-GARCH model, and compare their results. The estimates are summarized in Table 2.11.

To address the question whether non-orthogonal linkages can be found in financial data,

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[12]Higher order specifications do not significantly contribute to a better linear fit.

we first verify whether the estimated unrestricted matrix is approximately orthogonal. Let $\widehat{Z_{go}^{-1}}$ denote the unrestricted representation, then $(\widehat{Z_{go}^{-1}}(\widehat{Z_{go}^{-1}})^T = \begin{pmatrix} 1.49 & -1.09 \\ -1.09 & 1.95 \end{pmatrix}$ should be close to the identity matrix, which it is not. This confirms our conjecture that the orthogonality assumption of O-GARCH is probably too restrictive, in that it might exclude many of the linkages 'observed' in financial markets. In order to quantify the impact of the restrictions imposed by O-GARCH, we compute the Likelihood Ratio statistic (LR) to test the O-GARCH specification against GO-GARCH for several lengths of the time series. The results are listed in Table 2.12. For all lengths of the time-series considered, the hypothesis of an orthogonal linkage is rejected at a 1% level.

model	O-GARCH		GO-GARCH	
$\widehat{Z^{-1}}$	-0.55	-0.55	-1.18	0.32
	1.19	-1.19	0.58	-1.27
c	0.009	0.003	0.009	0.003
α	0.070	0.039	0.054	0.079
β	0.922	0.957	0.939	0.915

Table 2.11: *The estimates for the linkages and GARCH parameters*

Length	250	500	1000	3082
LR	23.7	13.8	42.6	731.4

The critical value of χ_1^2 at a 1% level is 6.63.

Table 2.12: *Likelihood-Ratio Test of O-GARCH against GO-GARCH*

Even though the GO-GARCH model provides a better fit when compared with O-GARCH, it might be that our more general model is still seriously misspecified. For this reason, we include a simple test for misspecification. Since we are interested in heteroskedasticity in particular, we estimate a VAR model on the squares and the product of the two standardized residuals to verify whether the conditional covariance has been modelled correctly. The results, which are shown in Table 2.13, suggest that GO-GARCH is not seriously misspecified. The remaining structure found in the standardized GO-GARCH residuals is fairly weak. In contrast, the residuals from the O-GARCH model still seem to exhibit significant persistence in volatility.

To examine to what extent the restrictions on the linkage affect the (conditional) correlations, we compare the implied correlations of both models. The time evolution of the correlations is shown in Figure 2.2, which reveals several interesting features. Perhaps

model	O-GARCH			GO-GARCH		
variable	$\varepsilon_{1,t}^2$	$\varepsilon_{2,t}^2$	$\varepsilon_{1,t}\varepsilon_{2,t}$	$\varepsilon_{1,t}^2$	$\varepsilon_{2,t}^2$	$\varepsilon_{1,t}\varepsilon_{2,t}$
c	4.11	4.29	4.22	3.55	4.39	3.89
$\varepsilon_{1,t-1}^2$	1.40**	0.08	-0.27**	1.09*	0.52	0.70
$\varepsilon_{2,t-1}^2$	1.06**	0.36	-0.09	0.92	0.90	0.81
$\varepsilon_{1,t}\varepsilon_{2,t}$	-2.44**	-0.44	0.36	-1.99*	-1.46	-1.52
$adj.R^2$	0.14	0.00	0.01	0.00	0.00	0.00

*,** significant at the 5% and 1% level.

Table 2.13: *Misspecification test: VAR model on the squares and the product of the two residuals*
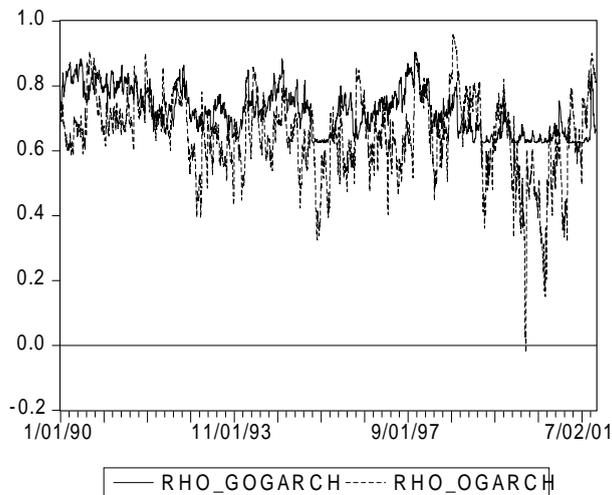

Figure 2.2: *The estimated correlations between the DJIA and the NASDAQ*

most striking is that the correlations estimated by GO-GARCH are much less volatile. Furthermore, the GO-GARCH correlations never seem to fall below 0.6, say, whereas the more volatile correlations estimated by O-GARCH show several substantial drops. In the beginning of 2000, the correlation according to O-GARCH is even just below zero where the GO-GARCH correlations show no decline at all. Since the matrix estimated by GO-GARCH is explicitly not orthogonal, we have reason to believe that O-GARCH often underestimates the correlations. Indeed, it seems plausible that the DJIA and the NAS-DAQ, which are strongly related, exhibit high correlation at all times. Assuming that the linkage does not change over time, it will be surprising to observe periods where the DJIA

and the NASDAQ appear to be almost unrelated. The differences observed for the year 2000, however, are kind of extreme. This might suggest that the linkage is not constant over time, or stronger, that it was subject to a structural change. Indeed, at the beginning of 2000 we experienced a technological boom which could explain our findings. To test the assumption of a constant linkage will be left for further research.

2.6   Concluding remarks

A new type of multivariate GARCH model is proposed that can best be seen as a generalization of the O-GARCH model. It supports the assumption that the observed variables are driven by some unobserved uncorrelated components, linked by a linear map. In order to identify these components, we only need invertibility of the associated matrix. Under the null of O-GARCH, however, the matrix is assumed orthogonal which only covers a very small subset of all possible invertible matrices. Moreover, even when the matrix is truly orthogonal, the estimator proposed by O-GARCH is not always able to identify it. The GO-GARCH model considers every invertible matrix as a possible linkage, which will be parameterized in such a way that it is not expected to complicate estimation while excluding any identification problems.

   The model is tested on both artificial and financial data. The simulation results show that the model correctly estimates both orthogonal and non-orthogonal invertible linkages. The results are not affected by the scaling of the uncorrelated factors or a possible weak dependence among the observed variables. The latter is known to be responsible for the identification problems of O-GARCH, which is confirmed by our experimental results. The nature of the linkage, for example whether it is orthogonal or not, is strongly related with the implied correlations between the observed variables. This relation is made explicit by a theoretical example, and illustrated by some of the simulation results. The effect of the linkage on the correlations is also observed in the empirical example, the Dow Jones Industrial Average versus the NASDAQ. A likelihood ratio test rejects the hypothesis that the associated matrix is orthogonal. In addition a simple test for misspecification suggests that the GO-GARCH model provides a better description of the process, and the conditional correlations in particular. We argue that by restricting the matrix to be orthogonal, O-GARCH will often underestimate the correlations. The differences are kind of extreme during the year 2000, which coincides with the technology boom that initiated early that year. It could be that the linkage is not constant over time and that

it experienced a structural change during the technological bust in 2000. A test for such a structural break, and perhaps even extending the model to allow for a time-varying linkage, is left for further research. Probably the most important question that remains, is to what extent is GO-GARCH able to improve the modelling of very large covariance matrices. Indeed, it would be very interesting to compare the model with other recently developed multivariate GARCH models, such as the Dynamic Conditional Correlation model of Engle (2002). The misspecification tests recently proposed by Tse (2002) can be used to measure performance.