Fitting model parameters in conformal geometric algebra to Euclidean observation data

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Introduction to Geometric Algebra

Geometric algebra (also called Clifford algebra) is an extension to classical linear algebra, constructed over some vector space. Through the definition of an inner product it is equipped with a metric. The term geometric algebra emphasizes the idea that its elements are used to represent and transform geometric objects, which can be done in a coordinate-free manner. It has an invertible associative geometric product that can be decomposed into an inner and outer product.

In this chapter we provide an introduction to the algebraic structure of geometric algebra, first establishing the fundamental geometric product of the algebra in section 2.1 by describing its action on the elements of a vector space. In section 2.2 we introduce the concept of multivectors, algebraic entities that arise as the result of the geometric product between vectors and that are needed to close the algebra under the geometric product. We explain in section 2.3 how multivectors enable a compact representation of orthogonal transformations.

The algebraic entities of geometric algebra and their interaction readily allow geometric interpretation. The material presented in sections 2.1 through 2.3 is valid in all variants of geometric algebra. But in section 2.4 we introduce the conformal model of geometric algebra (CGA), a particular geometric algebra which can be used to represent a large number of geometric objects in Euclidean spaces and which unifies rigid body motions, Euclidean similarity transformations and conformal transformations as orthogonal linear transformations of a higher-dimensional space. Using techniques from section 2.3 these are then efficiently representable.
The work presented in sections 2.1 through 2.4 is entirely based on [HS84], [LD03] and [DFM07]. Sections 2.5 and 2.6 reflect the author’s perspective but do not contain original work. The concepts that are used can be found in the respective publications referenced there.

As opposed to the relatively young mathematical field of geometric algebra, classical linear algebra offers a host of tools — some of them analytically precise, others well established coping mechanisms — for solving equations exactly or approximately, analyzing transformations, dealing with inconsistencies etc. Being a multilinear algebra, geometric algebra yields to the application of some of these tools, while others have to be modified to acquire geometric algebra analogues. Section 2.5 provides a linear algebra perspective on geometric algebra.

On the other hand geometric algebra can be studied under the aspect of representing transformation groups. Some transformation groups of wide interest — such as the group of orthogonal linear transformations — are differentiable manifolds, which makes them amenable to differential geometry, a mathematical branch which comes with its own set of devices for analysis, calculation and description. In section 2.6 we briefly provide a view on geometric algebra from a differential geometry standpoint. This section also motivates the representation of versors by bivector exponentials.

Readers familiar with geometric algebra may skip sections 2.1 through 2.3. Section 2.4 introduces notation specific to the use of CGA in this thesis. In section 2.5 we consider the interface between geometric algebra and linear algebra in more detail. Therefore this section contributes to the central topic of this thesis. It may be especially interesting to readers with a strong background in linear algebra. Material presented in sections 2.5 and 2.6 prepares the reader for the analysis in chapter 4.

2.1. Inner, Outer and Geometric Product

Geometric algebra operates on a linear vector space just like “classical” linear algebra does. In addition to the known vector algebra, two concepts are introduced, namely that of the geometric product and that of multivectors. To understand these concepts it is useful to consider how geometric algebra works on vectors, e.g. in \( n \)-dimensional Euclidean space \( \mathbb{R}^n \).

There, the scalar- or dot-product between vectors \( \mathbf{a} \) and \( \mathbf{b} \), denoted \( \mathbf{a} \cdot \mathbf{b} \), is defined and specifies how to measure lengths and angles. This product is retained in geometric algebra, where it is called inner product. Applied to vectors, it behaves just like the well-known scalar product does in linear algebra.

In addition to the inner product, geometric algebra introduces an outer product or wedge product, which is identical to the outer product originally introduced by Grassmann. The outer product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \) results in a 2-blade \( \mathbf{a} \wedge \mathbf{b} \), which is an algebraic entity distinct from vectors and scalars. In its geometric interpretation the 2-blade \( \mathbf{a} \wedge \mathbf{b} \) gives the oriented area, which arises when vector \( \mathbf{a} \) is swept along vector \( \mathbf{b} \). Its magnitude is exactly that of the scalar content of the resulting area and it is oriented in the sense that the 2-blade changes sign, if the order of the vectors is inverted, i.e. \( \mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} \). Moreover, the 2-blade carries a sense of attitude, that is, if \( \mathbf{a} \) and \( \mathbf{b} \) are embedded in some higher-dimensional space,
\(a \wedge b\) captures the spatial pose of \(a\) and \(b\). Figure 2.1 shows a 2-blade and distributivity of the outer product over addition in three dimensions.

**Figure 2.1.** A 2-blade \(a \wedge b\) has the magnitude of the area of the parallelogram swept out by vector \(a\) along vector \(b\) and a direction. The outer product is distributive over addition: \((a + b) \wedge c = a \wedge c + b \wedge c\).

In the same manner the outer product is defined between three vectors \(a, b\) and \(c\), resulting in a 3-blade \(a \wedge b \wedge c\), which represents the oriented volume spanned by these. Even more general, the outer product between \(r\) vectors results in a general \(r\)-blade.

The outer product has the following properties. As said before, it is anti-commutative between vectors. It is also associative and distributive over addition:

\[
\begin{align*}
(a \wedge b) \wedge c &= a \wedge (b \wedge c), \\
(a \wedge (b + c)) &= a \wedge b + a \wedge c, \\
((a + b) \wedge c) &= a \wedge c + b \wedge c.
\end{align*}
\]

With respect to scalar multiplication, the following equation holds.

\[
a \wedge (\lambda b) = (\lambda a) \wedge b = \lambda (a \wedge b), \quad \lambda \in \mathbb{R}
\]

As a consequence of property (2.1), the outer product vanishes if two vectors are linearly dependent, especially

\[
\lambda(a \wedge a) = 0, \quad \text{for all vectors } a \text{ and all } \lambda \in \mathbb{R}.
\]

While the inner and outer product, respectively, may be familiar from different mathematical branches, the achievement of geometric algebra is to unite these two into one single operation, the geometric product introduced by Clifford. The geometric product between vectors \(a\) and \(b\) is denoted \(ab\). It can also be introduced independently of the inner and outer product. We choose this approach here and show a little later how the the geometric product can be decomposed into inner and outer product. The geometric product obeys the following axioms.

The geometric product is associative

\[
(ab)c = a(bc).
\]
The geometric product is *distributive over addition*

\[ a(b + c) = ab + ac, \quad (2.8) \]

\[ (a + b)c = ac + bc. \quad (2.9) \]

The square of any vector \( a \), i.e. the geometric product of \( a \) with itself, denoted \( a^2 \), is a scalar \[ a^2 = aa \in \mathbb{R}. \quad (2.10) \]

Note that axiom \((2.10)\) leads to the very useful fact that the geometric product between vectors is *invertible*. Just define the inverse of a vector \( a \) as \[ a^{-1} = \frac{a}{a^2}, \quad \text{for any vector } a \text{ with } a^2 \neq 0, \quad (2.11) \]
and we find that \[ aa^{-1} = a\frac{a}{a^2} = \frac{a^2}{a^2} = 1. \quad (2.12) \]

To derive one particular property of the geometric product, take the square of the sum of two vectors,

\[ (a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2. \quad (2.13) \]

By axiom \((2.10)\), \( a^2 \) and \( b^2 \) will be scalars and so must \( ab + ba \). We observe that this scalar term is bilinear and symmetric in \( a \) and \( b \), which allows us to identify it with the scalar product between vectors. A scalar factor of \( \frac{1}{2} \) is involved for reasons of consistency, to make sure that \( a^2 = a \cdot a \).

\[ a \cdot b = \frac{1}{2} (ab + ba). \quad (2.14) \]

The full geometric product contains this symmetric part, but also a remaining antisymmetric part, which we can identify with the outer product, because it fulfills all the required properties \((2.1) - (2.4)\);

\[ a \wedge b = \frac{1}{2} (ab - ba). \quad (2.15) \]

Combining equations \((2.14)\) and \((2.15)\) one sees that the geometric product between vectors can be written as a sum of the inner and outer product,

\[ ab = a \cdot b + a \wedge b. \quad (2.16) \]

Note that \( a \wedge b \) vanishes, if \( a \) and \( b \) are parallel and, if \( a \) and \( b \) are perpendicular, \( a \cdot b \) vanishes. In general, however, the geometric product between two vectors returns the sum of a scalar and a 2-blade. Note that the geometric product is associative, but not commutative.
2.2. Grades, Bases and Frames

The above considerations lead to the concept of a general multivector. By definition, the outer product of \( r \) vectors results in an \( r \)-blade

\[
B = b_1 \wedge \cdots \wedge b_r, \tag{2.17}
\]

Thus, by the outer product it is possible to create algebraic entities of different grade. Scalars are assigned grade zero and vectors grade 1; 2-blades are said to have grade 2, 3-blades have grade 3 and so on.

Now, a geometric algebra — as a mathematical structure — can be considered. It is a multilinear algebra, denoted \( \mathcal{G}(V) \), over a vector space \( V \) with the geometric product defined by the properties introduced above. In this thesis we will take the position that the vector space \( V \) does not inherently have to have a metric. However, the geometric algebra constructed over \( V \) defines an inner product, which induces a metric\(^2\). If the vector space \( V \) is assumed to have a metric, the geometric algebra constructed over it, inherits that metric. For example, in section 2.4 we will write \( \mathcal{G}(\mathbb{R}^{p,q}) \) to denote a geometric algebra constructed over \( \mathbb{R}^{p+q} \) with a \((p,q)\) signature.

The maximum grade that an \( r \)-blade in a geometric algebra can have depends on the dimension of the vector space the geometric algebra is based on (but not on its metric). In the geometric algebra over an \( n \)-dimensional vector space, the outer product of \( n + 1 \) vectors is always zero, because of property (2.6) of the outer product. So the maximum grade that a blade can assume is \( n \). The grade-\( n \) object created by taking the outer product of \( n \) vectors is unique up to a scalar factor for the geometric algebra. If it is normalized, it is denoted \( I \) and called the (unit) pseudoscalar. Thus, any grade-\( r \) object, \( A \), created this way generates its own geometric (sub-)algebra (borrowing the metric from the original space) sometimes denoted \( \mathcal{G}(A) = \mathcal{G}(\lambda I) = \mathcal{G}(I) \) and every geometric algebra determines a pseudoscalar \( I \).

As opposed to the outer product of vectors, the full geometric product returns a sum of objects of different grade, called a general multivector. Such a sum is assigned the grade of the object of the highest grade comprising it, e.g. the geometric product of two vectors returns a general multivector of grade 2. The blades of grade \( r \) introduced above are multivectors which can be written as the outer product of \( r \) vectors. Note that every multivector can be written as a sum of blades. We would like to stress that in the following when we invoke the term multivector without further qualification it expressly includes blades, vectors and scalars as special cases.

Many calculational and rewriting rules extend to blades of higher grades and — by linearity of the different products — to multivectors. We introduce a subset of these extensions insofar as we find them useful within the scope of this thesis. For a more comprehensive introduction we refer the reader to [HS84, LD03, DFM07].

For example, we note that (2.16) can be extended to the case where \( a \) is a vector and \( B \) is

\(^2\)To explicitly call attention to that fact one might denote the geometric algebra as \( \mathcal{G}(V, \cdot) \), but we drop the second argument as it is understood that a metric is defined on \( \mathcal{G}(V) \).
a multivector by

\[ aB = a \cdot B + a \wedge B. \]  

(2.18)

By convention the inner product of a vector with a scalar (i.e. a multivector of grade zero) is zero, while the outer product with a scalar returns a scalar multiple of the other factor.

If \( B \) is an \( r \)-blade, then the inner product of a vector \( a \) with it is given by

\[ a \cdot B = a \cdot (b_1 \wedge \cdots \wedge b_r) = \sum_{k=1}^{r} (-1)^{k-1} (a \cdot b_k) b_1 \wedge \cdots \wedge \hat{b}_k \wedge \cdots \wedge b_r, \] 

where the check over \( b_k \) indicates that this vector is missing in the outer product. The result is a blade of grade \( r - 1 \).

It is useful to introduce a grade selection operator, \( \langle \rangle_r \), which selects the components of grade \( r \) of a multivector. Multivectors with the property

\[ \langle A \rangle_r = A, \quad \text{for any } r \in \mathbb{Z} \]  

(2.20)

are called pure \( r \)-vectors. Not every pure \( r \)-vector is also an \( r \)-blade. For example, in four dimensions one can construct a bivector

\[ A = e_1 \wedge e_2 + e_3 \wedge e_4, \] 

(2.21)

which cannot be written as the outer product of any number of vectors, although \( \langle A \rangle_2 = A \). The scalar part \( \langle A \rangle_0 \) is sometimes abbreviated to \( \langle A \rangle \). By convention the grade selection operator results in zero when its index is negative.

To integrate entities of different grade into some closed form standard formulas (for example, in section 2.3) we need operators that change the sign of an entity depending on its grade. In particular, for blades of grade \( r = 0, 1, 2, \ldots \) we need the progressions of signs \( \{+1, -1, +1, -1, +1, -1, \ldots \} \) and \( \{+1, -1, -1, +1, +1, -1, -1, \ldots \} \). We introduce the involutions grade involution \( \hat{\cdot} \) and Clifford conjugation \( \bar{\cdot} \) on elements of \( G(I) \) by

\[ \hat{V} = \sum_r (-1)^r \langle V \rangle_r, \quad V \in G(I), \]  

(2.22)

\[ V = \sum_r (-1)^{\frac{r(r+1)}{2}} \langle V \rangle_r, \quad V \in G(I). \]  

(2.23)

The grade involution is an automorphism of \( G(I) \), while the Clifford conjugation is an anti-automorphism of \( G(I) \), i.e. \( \hat{V}W = \hat{W}V \) and \( VW = \bar{V} \bar{W} \) for \( V, W \in G(I) \). As a consequence of (2.22) and (2.23) we also get the progression of signs \( \{+1, +1, -1, -1, +1, +1, -1, -1, \ldots \} \) by

\[ \hat{V} = \sum_r (-1)^{\frac{r(r+3)}{2}} \langle V \rangle_r, \quad V \in G(I). \] 

(2.24)

Basis vectors of an \( n \)-dimensional Euclidean vector space are denoted \( \{e_k\} \), shorthand for
⟩ \bigcup_{k=1}^{n} \{e_k\}$. Such a set makes up a *frame* and makes it possible to write any vector in that space in terms of coordinates, denoted $\alpha^k$, i.e.

$$a = \alpha^1 e_1 + \cdots + \alpha^n e_n.$$  \hspace{1cm} (2.25)

An important notion is that of *duality*. For any $r$-vector $B$ the $|n-r|$-vector $(B \cdot I)$ is called the *dual of $B$ by the $n$-blade $I$*. We get the valuable relations

$$\begin{align*}
(a \cdot B)I &= a \wedge (BI) \quad \text{if } a \wedge I = 0, \\
(a \wedge B)I &= a \cdot (BI) \quad \text{if } a \wedge I = 0.
\end{align*}$$  \hspace{1cm} (2.26, 2.27)

Of course, our considerations in this section are not restricted to a basis of Euclidean $\mathbb{R}^n$. Any space can be described, and the definitions of inner, outer and geometric product hold.

### 2.3. Orthogonal Transformations in Geometric Algebra

Any invertible vector of a geometric algebra can be used to represent an orthogonal transformation using a sandwiching geometric product. Consider a vector space $V$, the vectors $v, w \in V$ and the invertible vector $u \in V$ and define the linear transformation $f_u : V \to V$ acting on vectors by

$$f_u(v) = -uvu^{-1}.$$  \hspace{1cm} (2.28)

This transformation can be interpreted geometrically as performing a reflection of the vector argument $v$ in a plane perpendicular to the vector parameter $u$. As a shorthand we shall call this a reflection *at* the vector $u$. See figure 2.2 for a visualization of this fact. To demonstrate

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{reflection.png}
\caption{The transformation $f_u(v) = -uvu^{-1}$ is linear in $v$. It can be interpreted as a reflection of the vector $v$ at the vector $u$.}
\end{figure}

it algebraically we use (2.18) and employ (2.19) to "distribute" the inner product over the outer product with appropriate sign changes,

$$u \cdot (v \wedge w) = (u \cdot v)w - (u \cdot w)v.$$  \hspace{1cm} (2.29)
We get

\[
\begin{align*}
    f_u(v) &= -uvu/u^2 \\
    &= (-u \cdot (vu) - u \wedge (vu))/u^2 \\
    &= (-u \cdot (v \cdot u + v \wedge u) - u \wedge (v \cdot u + v \wedge u))/u^2 \\
    &= (-u \cdot (v \wedge u) - u \wedge (v \cdot u))/u^2 \\
    &= (-u \cdot v)u + (u \cdot u)v - u(v \cdot u))/u^2 \\
    &= (u^2v - 2(u \cdot v)u)/u^2 \\
    &= v - 2(u \cdot v)u/u^2.
    \end{align*}
\]

(2.30)

By decomposing \( v \) into a component \( v_\perp \) perpendicular to \( u \) and a component \( v_\parallel \) parallel to \( u \) we have

\[
\begin{align*}
    f_u(v) &= f_u(v_\perp + v_\parallel) \\
    &= f_u(v_\perp) + f_u(v_\parallel) \\
    &= v_\perp - 2(u \cdot v_\perp)u/u^2 + v_\parallel - 2(u \cdot v_\parallel)u/u^2 \\
    &= v_\perp + v_\parallel - 2v_\parallel \\
    &= v_\perp - v_\parallel. 
\end{align*}
\]

(2.31)

Therefore, in a Euclidean geometric algebra, \( f_u(v) \) is a reflection of \( v \) in the hyperplane with normal vector \( u \).

Moreover, this transformation preserves the inner product. Algebraically this can be shown as follows.

\[
\begin{align*}
    f_u(v) \cdot f_u(w) &= (v_\perp - v_\parallel) \cdot (w_\perp - w_\parallel) \\
    &= v_\perp \cdot w_\perp - v_\parallel \cdot w_\perp - v_\perp \cdot w_\parallel + v_\parallel \cdot w_\parallel \\
    &= v_\perp \cdot w_\perp + v_\parallel \cdot w_\perp + v_\perp \cdot w_\parallel + v_\parallel \cdot w_\parallel \\
    &= (v_\perp + v_\parallel) \cdot (w_\perp + w_\parallel) \\
    &= v \cdot w, 
\end{align*}
\]

(2.32)

where \( v_\parallel \cdot w_\perp = v_\perp \cdot w_\parallel = 0 \). Thus, every linear transformation of the form (2.28) is an orthogonal transformation. Especially, every concatenation of linear transformations of the form (2.28) is an orthogonal transformation. Denoting the concatenation of functions by \( \circ \), we have

\[
    f_{u_1} \circ f_{u_2} \circ \cdots \circ f_{u_k}(v) = (-1)^k u_1 u_2 \cdots u_k v u_k^{-1} \cdots u_2^{-1} u_1^{-1}, \quad k \in \mathbb{N},
\]

(2.33)

in which we can substitute

\[
    u_1 u_2 \cdots u_k = S
\]

(2.34)

in order to obtain

\[
    f_S(v) = SvS^{-1},
\]

(2.35)
where the grade involution (2.22) enforces the proper sign changes $(-1)^k$ depending on the grade $k$ of $S$. The Cartan-Dieudonné theorem (e.g., [Mar87], chapter 13) states that every orthogonal transformation of an $n$-dimensional non-degenerate symmetric bilinear space over a field with characteristic not equal to 2 is a composition of (at most $n$) reflections. Therefore, every orthogonal transformation on such fields can be written in the form (2.34).

The geometric product $S$ of a number $k \in \mathbb{N}$ of invertible vectors is called a versor. If $k$ is even (respectively odd) $S$ is called an even (respectively odd) versor. Due to the properties of the grade involution (2.22) and the Clifford conjugation (2.23) a well as axiom (2.10), for any versor $S$,

$$S \hat{S} \in \mathbb{R} \setminus \{0\}. \quad (2.36)$$

We will call $S$ a unit versor if $S \hat{S} = 1$. For the sake of brevity we will call the sandwiching geometric product of the form (2.35) versor product. The versor product preserves the structure of the outer product, and a mapping $f_S : \mathcal{G}(V) \to \mathcal{G}(V)$ is induced. For arbitrary blades $X$ of grade $m$ we define

$$f_S(X) = (-1)^m SXS^{-1} = (-1)^mk SXS^{-1}. \quad (2.37)$$

For example, for the 2-blade $v \wedge w$ we then get,

$$f_S(v \wedge w) = f_S(v) \wedge f_S(w), \quad (2.38)$$

as we will show now using the properties of the geometric product introduced so far. We have

$$f_S(v \wedge w) = (-1)^{2k} u_k \ldots u_1(v \wedge w)u_1^{-1} \ldots u_k^{-1}$$

$$= u_k \ldots u_1(v \wedge w)u_1^{-1} \ldots u_k^{-1}$$

$$= u_k \ldots u_1(vw - v \cdot w)u_1^{-1} \ldots u_k^{-1}$$

$$= u_k \ldots u_1(vw)u_1^{-1} \ldots u_k^{-1} - u_k \ldots u_1(v \cdot w)u_1^{-1} \ldots u_k^{-1}$$

$$= u_k \ldots u_1(vw)u_1^{-1} \ldots u_k^{-1} - (v \cdot w)u_k \ldots u_1u_1^{-1} \ldots u_k^{-1}$$

$$= u_k \ldots u_1(vw)u_1^{-1} \ldots u_k^{-1} - (v \cdot w)$$

$$= u_k \ldots u_1u_1^{-1} \ldots u_k^{-1} - (v \cdot w)$$

$$= (-1)^k f_S(v) (-1)^k f_S(w) - f_S(v) \cdot f_S(w)$$

$$= (-1)^{2k} f_S(v) f_S(w) - f_S(v) \cdot f_S(w)$$

$$= f_S(v) \wedge f_S(w). \quad (2.39)$$

The versor product is therefore called an outermorphism in [HS84]. By (2.38) the versor product extends to blade arguments and by linearity to general multivector arguments, because any multivector can be written as a sum of blades.

Instead of constructing even versors by the geometric product of an even number of invertible vectors, they can be written as the exponential of a bivector. Every even unit versor can be
written as the exponential of a bivector and every bivector exponential results in an even unit versor. We will consider this notion in more detail in section 2.6. Here, we only point out that in this way it is possible to characterize a special orthogonal transformation by the minimal number of parameters. The number of degrees of freedom in a special orthogonal transformation is \( \binom{n}{2} \), the same as the number of degrees of freedom in a general bivector from the geometric algebra over an \( n \)-dimensional vector space.

Figure 2.3: Successive reflection in two hyperplanes \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) through the origin amounts to a rotation about the origin with the plane of rotation perpendicular to both hyperplanes. The versor \( S = \mathbf{n}_2 \mathbf{n}_1 \), which represents this rotation — and therefore is sometimes called a rotator — is a purely Euclidean multivector.

Since versors represent orthogonal linear transformations, they can be interpreted geometrically. In Euclidean space, for example, an even versor represents a rotation about the coordinate origin, i.e. a transformation that preserves lengths of vectors and angles between vectors (which are defined via the inner product); see figure 2.3 for an example. According to the geometrical transformation that they represent versors are often given specific names, such as “rotators” (for rotation versors), “translators” (for translation versors), “motors” (for rigid body motion versors) etc. For simplicity we will just call them versors, irrespective of the geometrical transformation they represent, emphasizing their algebraic properties rather than their geometric interpretation.

Of special interest as a versor is the mid-plane between two linearly independent vectors of equal length. Let \( \mathbf{v} \) and \( \mathbf{w} \) be such vectors, i.e. \( \mathbf{v}^2 = \mathbf{w}^2 \). Then the mid-plane between the two is the vector \( \mathbf{u} = \mathbf{w} - \mathbf{v} \). What makes the mid-plane useful is the fact that reflection in it
aligns two vectors of equal length, i.e. \( w = uv\hat{u}^{-1} \), as we can see algebraically by

\[
uv\hat{u}^{-1} = -uvu/u^2 \\
= -(w - v)(w - v)/(w - v)^2 \\
= (-wvw + vvw + wvv - vv)v/(w - v)^2 \\
= (v^2w + v^2w - v^2v - vvvw)/(w - v)^2 \\
= (w^2w + v^2w - w^2v - vvww)/(w - v)^2 \\
= w(w^2 + v^2 - vv - vw)/(w - v)^2 \\
= w. \tag{2.40}
\]

2.4. The Conformal Model of Geometric Algebra (CGA)

The conformal model of geometric algebra (CGA) is the smallest algebra we know that represents rigid body motions minimally as the exponential of bivectors. It is the geometric algebra \( G(\mathbb{R}^{n+1,1}) \) constructed over the Minkowski space \( \mathbb{R}^{n+1,1} \), that is, a real vector space with \( n + 1 \) basis vectors that square to a positive real number and one basis vector that squares to a negative real number. We will use this model to represent entities — such as points, circles and directions — residing in Euclidean vector space \( \mathbb{R}^n \), which is a subspace of the Minkowski vector space. For brevity’s sake we will simply call vectors from this Euclidean space-to-be-represented \textit{Euclidean vectors} and denote them by lower case bold italic font. A general vector from CGA we will call a \textit{conformal vector} and denote it by lower case italic font. Note that conformal vectors can, and in general do, contain Euclidean parts. Multivectors will be denoted in uppercase italic font.

In order to work with CGA one can introduce a vector basis \( \{e_+, e_1, \ldots, e_n, e_-\} \) with \( e_i^2 = 1 \), for \( i \in \{1, \ldots, n\} \cup \{+\} \) and \( e_0^2 = -1 \), as well as \( e_i \cdot e_j = 0 \), for \( i \neq j \). This basis is called the \textit{orthonormal basis}, but it is not the only basis used in practice. The mixed signature of the Minkowski space allows for the construction of \textit{null} or \textit{isotropic} vectors, which square to zero. Equation (2.11) implies that null vectors are not invertible.

For CGA it is common to introduce the \textit{null basis} \( \{n_0, e_1, \ldots, e_n, n_{\infty}\} \) with orthonormal Euclidean vectors \( e_i, i = 1, \ldots, n \) and \( e_i \cdot e_j = 0 \), for \( i = 1, \ldots, n, \ j = 0, \infty \), while \( n_0 \) and \( n_{\infty} \) are defined in terms of the orthonormal basis as

\[
n_0 = \frac{1}{2}(e_- + e_+), \tag{2.41}
\]

\[
n_{\infty} = e_- - e_+, \tag{2.42}
\]

which implies that

\[
n_0^2 = n_{\infty}^2 = 0 \tag{2.43}
\]

and

\[
n_0 \cdot n_{\infty} = -1. \tag{2.44}
\]
The conformal model of geometric algebra can be used to represent a variety of geometric objects in Euclidean space [DFM07, HJ03]. The lowest-dimensional geometric object in Euclidean space, a Euclidean point, is represented in CGA by a (non-invertible) null vector of the form

$$ p = \alpha \left( n_0 + p + \frac{1}{2} p^2 n_\infty \right), \quad (2.45) $$

where \( p \) denotes the Euclidean position of the point being represented and \( \alpha \in \mathbb{R} \) is called the point’s weight, which can be viewed as a homogeneous degree of freedom. As a result, \( n_0 \) represents the point at the origin while \( n_\infty \) represents the point at infinity. When we talk about a conformal vector representing a point, we will call it a conformal point \( p \) in order to set it apart from its position in Euclidean space \( p \). Note that that position can be retrieved from a conformal point by

$$ p = \frac{p}{-n_\infty \cdot p}, \quad (2.46) $$

where \(-n_\infty \cdot p\) retrieves the point’s weight \( \alpha \).

The inner product between two conformal points, \( p \) and \( q \) with unit weights results in

$$ p \cdot q = \left( -\frac{1}{2} q^2 + p \cdot q - \frac{1}{2} p^2 \right) $$

$$ = -\frac{1}{2} (p - q)^2, \quad (2.47) $$

which is proportional to the squared Euclidean distance between the points. This useful fact is one of the main reasons for assuming a Minkowski metric and for representing points as (2.45) in CGA. It follows that for conformal points \( p \) and \( q \), \( p \cdot q = 0 \) if and only if \( p = q \) up to a scalar factor.

A general conformal vector does not necessarily obey (2.45), in particular the relationship between the Euclidean part and the \( n_\infty \)-coordinate. But we can bring an arbitrary conformal vector into the form

$$ s = \alpha \left( n_0 + s + \frac{1}{2} (s^2 - \rho^2) n_\infty \right). \quad (2.48) $$

This represents a hypersphere [DFM07, HJ03] with its center at Euclidean location \( s \) and squared radius \( \rho^2 \) in the following sense. For any conformal point \( p \),

$$ p \cdot s = \alpha_p \left( n_0 + p + \frac{1}{2} p^2 n_\infty \right) \cdot \alpha_s \left( n_0 + s + \frac{1}{2} (s^2 - \rho^2) n_\infty \right) $$

$$ = -\alpha_p \alpha_s \frac{1}{2} (s^2 - \rho^2) - \alpha_p \alpha_s \frac{1}{2} p^2 + \alpha_p \alpha_s p \cdot s $$

$$ = -\alpha_p \alpha_s \frac{1}{2} \left( (s - p)^2 - \rho^2 \right), \quad (2.49) $$

which for nonzero weights is zero if and only if \( (s - p)^2 = \rho^2 \), i.e. the conformal point \( p \) lies on the hypersphere \( s \). Note that (2.48) allows the interpretation of conformal points as hyperspheres with zero radius. That is, substituting \( \rho = 0 \) into (2.48) results in (2.45). It is possible to construct a conformal vector of the form (2.48) with an \( n_\infty \)-component greater
than $\frac{1}{2}s^2$, which could be interpreted as a hypersphere with imaginary radius, i.e. $\rho^2 < 0$. Even though of seemingly little real geometric value, imaginary spheres have to be included in the conformal model of geometric algebra for closure, as they appear as the result of certain calculations.

By the same token as in (2.49), a hyperplane $\pi$ is represented by its unit normal vector $n$ and its scalar offset $\delta \in \mathbb{R}$ from the origin as

$$\pi = \alpha (n + \delta n_\infty). \tag{2.50}$$

We evaluate the inner product with an arbitrary conformal point $p$ as

$$p \cdot \pi = \alpha_p \left( n_0 + p + \frac{1}{2}p^2 n_\infty \right) \cdot \alpha_\pi (n + \delta n_\infty)$$

$$= -\alpha_p \alpha_\pi \delta + \alpha_p \alpha_\pi p \cdot n$$

$$= \alpha_p \alpha_\pi (p \cdot n - \delta), \tag{2.51}$$

which for nonzero weights is zero if and only if $p \cdot n = \delta$, i.e. $p$ lies on the hyperplane $\pi$. It can be shown that by a limiting process (2.48) allows the interpretation of hyperplanes as hyperspheres centered at infinity with infinite radius.

For obvious reasons, this way of representing a geometric object as the locus of all conformal points which have a zero inner product with the representing vector is called the *inner product null space* (IPNS) representation by Perwass [Per08]. The inner product can also be used to define local angles of intersection between two represented objects [HJ03]. For example, two hyperspheres of unit weight, $s_1$ and $s_2$, intersect locally under an angle of

$$\theta = \arccos(s_1 \cdot s_2). \tag{2.52}$$

So far we have focused on simple conformal vectors, but a number of geometric objects can be represented employing the outer product between conformal vectors. We will introduce some of them in as far as they are useful in the context of this thesis. For a more comprehensive account see, for example, [DFM07].

A (weighted) direction can be represented by

$$D = \alpha (d \wedge n_\infty). \tag{2.53}$$

This null 2-blade has the property that only the point at infinity is incident with it (i.e. $n_\infty \cdot D = 0$, as well as the property that $D^2 = D \cdot D = 0$, but for no other point $p$ is $p \cdot D = 0$). A *tangent vector* can be identified with a Euclidean direction vector anchored at a given Euclidean point. It is represented by a *conformal tangent*, $t$, a null 2-blade that retains information about a Euclidean direction, $t$, as well as a location, $p$, represented by a conformal point $p$. Formally,

$$t = -p \cdot (p \wedge t \wedge n_\infty). \tag{2.54}$$
In chapter 3 we will make use of a localized frame, i.e. a set of linearly independent conformal tangents anchored at a common point.

Another type of object that can be represented directly in CGA are circles, which are curves uniquely determined by three conformal points \( p_1, p_2, p_3 \). The equation

\[ C = p_1 \wedge p_2 \wedge p_3 \]  

(2.55)

represents a circle in the sense that any conformal point \( p \) lying on it yields a zero outer product, i.e.

\[ p \wedge C = 0, \]  

(2.56)

if and only if \( p \) lies on \( C \). Note that this representation uses the outer product rather than the inner product to determine incidence and therefore is called the outer product null space (OPNS) representation of a circle. By using the duality relationship (2.27) we could equivalently use the IPNS representation of the circle, but we will adhere to the definition (2.55) instead, because it is geometrically more intuitive to construct a circle representation as the outer product of three points lying on it. Moreover, a conformal tangent \( t \) to a conformal circle \( C \) in a conformal point \( p \) on \( C \) can be easily determined by

\[ t = p \cdot C. \]  

(2.57)

Note that a conformal line is a special case of a conformal circle containing the point at infinity. We can write

\[ L = p_1 \wedge p_2 \wedge n_\infty = p_1 \wedge t \wedge n_\infty, \]  

(2.58)

where \( t = p_2 - p_1 \).

We would like to emphasize that these constructions hold in any dimension, but unlike hyperspheres (who are named for their co-dimension) circles are always one-dimensional entities in a two-dimensional “carrier” space. There are no “hypercircles.” See figure 2.4 for a visualization in 3D. In chapter 3 we will use intersecting circles to determine a localized frame.

Since the versor product introduced in section 2.3 represents the reflection at a vector and invertible vectors in CGA represent hyperspheres, a versor in CGA represents a series of inversions — i.e. generalized reflections — in hyperspheres. See figure 2.5 for an example of an inversion.

The concept of the mid-plane, introduced at the end of section 2.3, holds in CGA. Let \( p \) and \( p' \) be two distinct conformal points. Since conformal points are represented by null vectors, both of them square to the same number, namely zero. Then \( s = p' - p \) represents a Euclidean hypersphere, inversion in which makes the points coincide. If moreover \( p \) and \( p' \) have equal weight, \( s = p' - p \) represents the Euclidean mid-hyperplane between them. More generally, \( p \) and \( p' \) can be taken to represent Euclidean hyperspheres. If and only if they square to the same number, is \( s = p' - p \) the versor that aligns the two. Note that for simplicity we will
2.4 The Conformal Model of Geometric Algebra (CGA)

Figure 2.4.: A conformal circle $C$ represents a circle in Euclidean space directly by the outer product of three conformal points $p_1$, $p_2$, and $p_3$. The conformal tangent $t$ to the oriented circle in a conformal point $p$ on $C$ is given by $t = p \cdot C$. Here the circle is shown with the 2-dimensional subspace of $\mathbb{R}^3$ in which it lies. The conformal point at the origin, $n_0$, is shown for reference.

Figure 2.5.: An inversion in a hypersphere is a generalized reflection. The point at infinity is mapped to the hypersphere’s center. Here we see a point $p$ and a line $L$ mapped to $p'$ and $L'$, respectively, by an inversion in a circle $s$, i.e. a 2-dimensional hypersphere

continue calling this versor the mid-plane, even though in general it does not represent a plane, nor does it necessarily lie in the (Euclidean) middle of $p$ and $p'$.

By Liouville’s theorem every conformal transformation of Euclidean space is made up of inversions — i.e. reflections in suitably chosen spheres or planes. In conjunction with the Cartan-Dieudonné theorem this means that the orthogonal transformations of $\mathbb{R}^{n+1,1}$ are the conformal transformations of the Euclidean space $\mathbb{R}^n$ which it represents. A first in-depth algebraical treatment of the versor representation of the conformal group using Clifford algebra can be found in [Ang80]. We will return to conformal transformations in more detail in chapter 3.
2.5. Relation to Linear Algebra

Due to the linearity of the geometric product — and its more specialized companions, the inner and outer product — a geometric algebra over an \( n \)-dimensional vector space can be expressed as a linear algebra over a \( 2^n \)-dimensional vector space with the basis consisting of a number of basis blades of all possible grades. For example the geometric algebra over the three-dimensional Euclidean space, \( \mathcal{G}(\mathbb{R}^3) \), can be expressed as a vector space with the basis \( \{1, e_1, e_2, e_3, e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2, e_1 \wedge e_2 \wedge e_3\} \). Any multivector from \( \mathcal{G}(\mathbb{R}^3) \) can be written as a linear combination of these basis elements. Linear algebra would then study the linear mappings — e.g. represented by \( 8 \times 8 \)-matrices acting on the basis components — between those vector elements. The inner, outer and geometric product are particular instances of such mappings.

An obvious advantage of this approach is that linear algebra is a well established framework. Techniques for solving equations in it are readily available, accepted criteria for assessing the quality of a solution exist and have been well studied, methods for analysis such as the singular value decomposition and a number of other matrix decompositions are commonplace. When it comes to implementation on a computer, one finds highly optimized hardware, specialized sub-processors and a vast number of software libraries and tools for facilitating calculations in linear algebra. For an efficient implementation of geometric algebra based on linear algebra, see for example [Fon07]. Software for automatic code generation using this geometric algebra implementation (Gaigen 2.5 [Fon]), for optimizing geometric algebra based algorithms (Gaalop [PH]) and for geometric algebra based scientific calculation and visualization (CLUCalc [Per]) is available online and uses linear algebra implementations.

A reason why geometric algebra interfaces with classical linear algebra is that they are both based on similar principles. Just like the geometric product, matrix multiplication is associative but not commutative. For two multivectors \( A \) and \( B \) their geometric product \( AB \) is a transformation linear in \( A \) as well as in \( B \). Since in classical linear algebra linear transformations are represented by matrices, the geometric product \( AB \) is either the matrix that represents multiplication on the left with \( A \) acting on \( B \) or the matrix that represents multiplication on the right with \( B \) acting on \( A \), depending on which viewpoint we take.

In order to faithfully represent the geometric product between general multivectors we have to resort to the language of tensors. For any given geometric algebra Perwass [Per08] introduces the geometric product tensor \( \Gamma_{ij}^k \). Given a \( 2^n \)-dimensional blade basis, multivectors can be represented by a collection of their coefficients, similar to the representation of vectors in classical linear algebra. With the scalar components \( \alpha^i \) of \( A \) and \( \beta^j \) of \( B \) the components of the multivector \( AB \) that is the geometric product between \( A \) and \( B \) are given by \( \gamma^k = \Gamma_{ij}^k \alpha^i \beta^j \), where summation over repeated indices is implied. The derived inner and outer product and the grade selection operator can be represented similarly by specific tensors.

While this enables us to apply linear algebra techniques, casting geometric algebra problems into linear algebra introduces a number of disadvantages. Besides the high dimensionality (\( 2^n \)) of the formulation, the involved matrices have to be designed specifically for the problem at
hand using the geometric product tensor. Moreover, when using classical linear algebra on the representation of geometric algebra concepts, one enforces a (often tacitly assumed Euclidean) distance metric on the vector representation of multivectors. As a result entities from the geometric algebra are blended in ways that may not be admissible in the problem’s context or not geometrically meaningful — e.g. what is the “proper” distance metric between a vector and, say, a 2-blade, or between a conformal point and a versor?

Additionally, it is not easy to switch back and forth between the different formulations. Consider a versor $S$ acting on a vector $v$ by the versor product $Sv\hat{S}^{-1}$. The linear algebra formulation of this transformation would be the orthogonal matrix $M$ with $M_{jk} = \Gamma_{ij}^k \hat{\Gamma}_{lm}^j \sigma^i \sigma'^m$, where $\sigma$ and $\sigma'$ denote the components of $S$ and $\hat{S}^{-1}$, respectively. In geometric algebra the vector argument $v$ is positioned awkwardly in the transformation equation and it is not easy to isolate it or solve the equation for the versor components. In classical linear algebra the term $Mv$ is easier to handle, different techniques to invert matrix $M$ can be attempted. On the other hand, recovering the versor’s components from such a matrix is a very difficult task — as is tracking the changes of the matrix entries with changes of the versor or assigning a meaningful distance measure to both representations.

Finally, some established linear algebra techniques allow us to find a solution to some system of equations, which is even optimal by some criterion, say, in a least squares sense. But the solution may not be admissible according to the problem statement, for example a recovered matrix may not be orthogonal or some internal constraints on the matrix’ entries may be violated. Such constraints have to be enforced explicitly, while often the original geometric algebra formulation implicitly precluded such solutions.

### 2.6. Relation to Differential Geometry

Focusing on the even unit versors of a given geometric algebra $\mathcal{G}(V)$, we can treat them as a representation of the special orthogonal group of the underlying vector space $V$, that is, the transformation group which preserves the inner product. This transformation group is a Lie group [DHSvA93] which, in turn, can be viewed as a differentiable manifold, forming a bridge to the field of differential geometry. We would like to spend a few words elucidating this relationship.

There, *differential 1-forms* are defined, linear functions mapping vectors from $V$ to real numbers. It is well known that differential 1-forms form a vector space, the so called *dual space* $V^*$ to the vector space $V$ upon which they act.

If $\varphi \in V^*$ is a differential 1-form and $v \in V$ a vector, one finds many different notations for the action of $\varphi$ on $v$. Since $\varphi$ is a linear function, a common notation is $\varphi(v)$. Physics texts often use the bracket notation $(\varphi, v)$ or $(\varphi|v)$, sometimes denoting $\varphi$ by $\langle \varphi \rangle$ and $v$ by $|v\rangle$ in order to keep track of the space of which they are elements. Confusingly, $\langle \ldots \rangle$ is also the notation used for the inner product of a space with a bilinear form in some texts.

Differential geometry also defines the exterior product (or “wedge product”) between differential forms. The wedge product between two 1-forms, $\varphi$ and $\psi$, results in a differential 2-form
\( \varphi \wedge \psi \), which linearly maps two vectors to a scalar. That is, if \( \varphi, \psi \in V^* \) and \( v, w \in V \), then
\[
(\varphi \wedge \psi)(v, w) = \varphi(v)\psi(w) - \varphi(w)\psi(v) \in \mathbb{R}.
\]
This can be extended to arbitrary differential \( r \)-forms.

The properties of the wedge product between differential forms match the behavior of the outer product introduced between vectors in section 2.1. Together with the properties of the inner product, this leads to the fact that, in purely Euclidean as well as in conformal geometric algebras, the vector basis of \( V \) can be identified with the dual basis of \( V^* \), the action of the differential 1-forms being given by the inner product between vectors [HS84].

When [HS84] introduces the notion of reciprocal vectors, this identification of \( V \) and \( V^* \) happens implicitly. The reciprocal vector \( e^j \in V \) to the vector \( e_j \in V \) is the unique vector for which \( e^j \cdot e_i = \delta^j_i \), the Kronecker delta, for all vectors \( e_i \) in the basis of \( V \). Thus, the vector \( e^j \) takes the place of the basic differential 1-form \( \varphi^j \in V^* \) for which \( \langle \varphi^j, e_i \rangle = \delta^j_i \). In that respect the outer product used in geometric algebra is not (just) an abuse of notation, the similarity with the outer product between differential forms not coincidental. It is the (implicit) identification of the basis of a vector space with the basis of its dual that reconciles the two frameworks.

That being said, we turn our attention to tangent spaces to the manifold \( M \) which is the orthogonal group of \( V \). It should be noted here that the orthogonal group is usually not connected, i.e. consist of more than one connected component. Considering the component \( M^0 \) which contains the identity, the tangent space at the identity \( T_1 M^0 \) is also called the Lie algebra of \( M^0 \) and can be represented as a pure bivector algebra [DHSvA93], i.e. the subalgebra of elements of the geometric algebra which can be written as a sum of 2-blades. Every element of the Lie algebra of \( M^0 \) can be mapped to an even unit versor (and vice versa) by the exponential map (respectively, the logarithmic map). In terms of geometric algebra, the exponential map of a pure bivector \( B \) — as well as of any multivector — can be calculated by the power series [HS84]
\[
\exp(B) = \sum_{k=0}^{\infty} \frac{B^k}{k!},
\]
where \( B^k \) means the \( k \)-fold geometric product of \( B \) with itself.

The union over all tangent spaces, \( TM = \bigcup_S T_S M \), is called the tangent bundle of the manifold \( M \) and can be identified with the full geometric algebra \( \mathcal{G}(V) \) over \( V \).

In differential geometry, a metric is the assignment of an inner product to the tangent space \( T_p M \) at every point \( p \in M \) of the manifold \( M \) — as opposed to a distance metric, which is a measure between two points and fulfills the metric axioms, such as commutativity and the triangle inequality. A metric is often used to define a notion of lengths and angles of intersection.

The properties of geometric algebra and, in particular, CGA that are beneficial for the representation of geometric objects and transformations appear from a differential geometry standpoint as follows. For a geometric algebra the inner product introduced in section 2.1 works on arbitrary elements and thus is a global metric on the whole tangent bundle. In section 2.4 we explained how it can be used to define incidence relationships between different geometric
objects. The fact that the action of the orthogonal group preserves the inner and outer product means that the versor product acts on geometric objects in a structure preserving manner. More specifically, hyperspheres are mapped to hyperspheres and incidence relationships as well as local angles of intersection are preserved.

2.7. Summary

We have explained that by suitably choosing an embedding function the conformal model of geometric algebra is capable of representing points in Euclidean space by null vectors of a higher-dimensional Minkowski space. The properties of the inner product with which the underlying Minkowski space is endowed ensure that the inner product between unit weight conformal points is proportional to the squared Euclidean distance between the represented Euclidean points. Moreover, the inner product can be used to define the representation of a number of geometric objects, such as hyperspheres and hyperplanes in Euclidean space. The inner product is also used to define a notion of incidence and angles of intersection between these objects.

We have demonstrated how versors are used to represent orthogonal transformations, which preserve the inner product. Moreover, they also preserve the structure of the outer product by acting as an outermorphism. The outer product is used to define more complicated geometric objects such as tangents and circles, which are therefore also preserved as respective classes by the versor product.

The Lie group structure of the group of even geometric algebra versors and the fact that its Lie algebra is representable as an algebra of bivectors allows a minimal parametrization of the special orthogonal transformations as bivector exponentials. All of these facts make geometric algebra suitable for representing geometric objects and transformations in Euclidean spaces. Geometric algebra extends classical linear algebra in a natural and powerful way. This makes it a desirable interface between geometric intuition used to describe and reason about geometric entities at one end and the mathematical methods used to utilize the algebraic description of those entities on the other end.