The dynamics of imperfect information

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The validity problem for Dependence Logic (as well as for many of its variants examined in the previous chapters) is not decidable, as it follows at once by its equivalence to $\Sigma^1_1$ over sentences. One can develop axiomatic systems for fragments of these logics, as Kontinen and Väänänen did in [52] for the first order consequences of Dependence Logic formulas;\footnote{Another proof system for a fragment of Dependence Logic is the one developed by Ville Nurmi in [57]. However, it is not known if Nurmi’s system is complete for the corresponding fragment.} but it is not possible to generalize these results to full Dependence Logic under its usual semantics while preserving semidecidability.

However, Henkin developed in [35] a General Semantics for Second Order Logic, in which second order quantifiers range over an universe of discourse which is not necessarily the whole powerset; and furthermore, in the same paper, he developed a sound and complete axiom system for this logic.

In this chapter, we will first build a similar General Team Semantics in which not all teams belong to the universe of discourse; and afterwards, we will develop a proof system for Independence/Inclusion/Exclusion Logic $\mathcal{I}(\subseteq,|)$ which is sound and complete with respect to it. As we will see, the fact that our this formalism is contained in Existential Second Order Logic will be a big advantage for us, as it will allow us to focus exclusively on the least general models of our class.

\section{General Models}

\begin{definition}
Let $\Sigma$ be a first order signature. A general model with signature $\Sigma$ is a pair $(M, \mathcal{G})$, where $M$ is a first order model with signature $\Sigma$
\end{definition}
and \( \mathcal{G} \) is a set of teams over finite – but not necessarily identical, nor of the same size – domains, respecting the condition

- If \( n \in \mathbb{N} \) and \( \phi(x_1 \ldots x_n, \bar{m}, \bar{R}) \) is a first order formula, where \( \bar{m} \) is a tuple of constant parameters in \( \text{Dom}(M) \) and where \( \bar{R} \) is a tuple of “relation parameters” corresponding to teams in \( \mathcal{G} \), in the sense that each \( R_i \) is of the form
  \[ R_i = \text{Rel}(X_i) = \{ s(\bar{z}) : s \in X_i \} \]
  for some \( X_i \in \mathcal{G} \), then for
  \[ \| \phi(x_1 \ldots x_n, \bar{m}, \bar{R}) \|_M = \{ s : \text{Dom}(s) = \{ x_1 \ldots x_n \}, M \models s(\phi(x_1 \ldots x_n, \bar{m}, \bar{R})) \} \]
  it holds that \( \| \phi(x_1 \ldots x_n, \bar{m}, \bar{R}) \|_M \in \mathcal{G} \).

**Lemma 5.1.2.** Let \( \Sigma \) be a first order signature and let \((M, \mathcal{G})\) be a general model with signature \( \Sigma \). Then for all \( X \in \mathcal{G} \) and all variables \( y \), \( X[y/x] \in \mathcal{G} \).

**Proof.** Let \( \text{Dom}(X) = \bar{x} \), let \( R = \text{Rel}(X) \), and consider the formula \( \phi(\bar{x}, y) = \exists y R(\bar{x}). \) Then take any assignment \( s \) with domain \( \bar{x}y \): by construction, \( M \models s(\phi(\bar{x}, y) ) \Leftrightarrow \exists \bar{m} \) s.t. \( s[\bar{m}/y][\bar{x}] \in X \Leftrightarrow s \in X[M/y], \) as required.\(^2\)

We can easily adapt the standard Team Semantics to general models. We will report all the rules here, for ease of reference; but the only differences between this semantics and the previous one are in the cases \( \text{GMS-}\lor \) and \( \text{GMS-}\exists \).

In the case of the rule of the existential quantifier, a formulation somewhat different from the usual one will prove to be more convenient here:

**Definition 5.1.3.** Let \( X \) and \( X' \) be two teams on the same domain, and let \( x \in \text{Var} \) be a variable. Then we write \( X[x/X'] \) if and only if

1. \( \text{Dom}(X') = \text{Dom}(X) \cup \{ x \}; \)
2. \( \text{Rel}_{\text{Dom}(X)}(X') = \text{Rel}(X). \)

**Definition 5.1.4.** Let \((M, \mathcal{G})\) be a general model and let \( X \) be a team over it. Then

- **GMS-lit:** For all first order literals \( \alpha \), \((M, \mathcal{G}) \models_X \alpha \) if and only if \( s \in X, M \models s(\alpha) \) in the usual first order sense;

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\(^2\)Here by \( s[\bar{m}/y][\bar{x}] \) we intend the restriction of \( s(\bar{m}/y) \) to the domain \( \{ x_1 \ldots x_n \} \). If \( y \) is among \( x_1 \ldots x_n \), then this is the same of \( s(\bar{m}/y) \) itself; otherwise, it is simply \( s \).
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**GMS-inc, GMS-exc, GMS-ind:** For all inclusion atoms, exclusion atoms or independence atoms $\beta$, $(M, \mathcal{G}) \models X \beta$ if and only if $M \models X \beta$ in the usual Team Semantics sense;

**GMS-\lor:** For all $\psi_1$ and $\psi_2$, $(M, \mathcal{G}) \models X \psi_1 \lor \psi_2$ if and only if $X = Y \cup Z$ for some two teams $Y, Z \in \mathcal{G}$ such that $(M, \mathcal{G}) \models Y \psi_1$ and $(M, \mathcal{G}) \models Z \psi_2$;

**GMS-\land:** For all $\psi_1$ and $\psi_2$, $(M, \mathcal{G}) \models X \psi_1 \land \psi_2$ if and only if $M \models X \psi_1$ and $(M, \mathcal{G}) \models X \psi_2$;

**GMS-\exists:** For all $\psi$ and all $x \in \text{Var}$, $M \models X \exists x \psi$ if and only if there exists a $X' \in \mathcal{G}$ such that $X[x]X'$ and $(M, \mathcal{G}) \models X' \psi$;

**GMS-\forall:** For all $\psi$ and all $x \in \text{Var}$, $M \models X \forall x \psi$ if and only if $(M, \mathcal{G}) \models X[M/x] \psi$.

Let us verify that the same holds for General Model Semantics:

**Lemma 5.1.5.** Let $(M, \mathcal{G})$ be a general model, and let $X \in \mathcal{G}$ be such that $\text{Dom}(X) = \bar{x}\bar{y}$. Then $X_{|\bar{x}} = \{s : \text{Dom}(s) = \bar{x}, \exists \bar{m} \text{ st. } s[\bar{m}/\bar{y}] \in X\}$ is in $\mathcal{G}$.

Furthermore, let $Y \subseteq X_{|\bar{x}}$ be such that $Y \in \mathcal{G}$. Then the team

$$X(\bar{x} \in Y) = \{s \in X : s_{|\bar{x}} \in Y\}$$

is in $\mathcal{G}$.

**Proof.** By definition, $X_{|\bar{x}}$ is $\|\phi(\bar{x}, R)\|_M$, where $\phi$ is $\exists \bar{y}(R\bar{x}\bar{y})$ and $R = \text{Rel}(X)$. Therefore, $X_{|\bar{x}} \in \mathcal{G}$.

Similarly, $X(\bar{x} \in Y)$ is $\|\phi(\bar{x}\bar{y}, R_1, R_2)\|_M$, where $\phi$ is $R_1\bar{x}\bar{y} \land R_2\bar{x}$, $R_1$ is $\text{Rel}(X)$ and $R_2$ is $\text{Rel}(Y)$. \qed

**Theorem 5.1.6** (Locality). Let $(M, \mathcal{G})$ be a general model, let $X \in \mathcal{G}$ and let $\phi$ be a formula over the signature of $M$ with $\text{Free}(\phi) = \bar{z} \subseteq \text{Dom}(X)$. Then $(M, \mathcal{G}) \models X \phi$ if and only if $(M, \mathcal{G}) \models X_{|\bar{x}} \phi$.

**Proof.** The proof is by structural induction on $\phi$. We present only the passages corresponding to disjunction and existential quantification, as the others are trivial:

- Suppose that $(M, \mathcal{G}) \models X \phi_1 \lor \phi_2$. Then, by definition, there exist teams $Y$ and $Z$ in $\mathcal{G}$ such that $X = Y \cup Z$, $(M, \mathcal{G}) \models Y \phi_1$ and $M \models Z \phi_2$. By induction hypothesis, this means that $(M, \mathcal{G}) \models Y_{|\bar{x}} \phi_1$ and $(M, \mathcal{G}) \models Z_{|\bar{x}} \phi_2$. But $Y_{|\bar{x}} \cup Z_{|\bar{x}} = X_{|\bar{x}}$, and hence $(M, \mathcal{G}) \models X_{|\bar{x}} \phi_1 \lor \phi_2$. 

Conversely, suppose that \((M, G) \models X \psi_1 \lor \psi_2\). Then there exist teams \(Y', Z'\) in \(G\) such that \((M, G) \models Y' \psi_1\), \((M, G) \models Z' \psi_2\) and \(X|\vec{z} = X' \cup Y'\). Now let \(Y = X(\vec{z} \in Y')\) and \(Z = X(\vec{z} \in Z')\); by construction, \(Y \cup Z = X\), and furthermore \(Y' = Y|\vec{z}\) and \(Z' = Z|\vec{z}\), and, by the lemma, \(Y\) and \(Z\) are in \(G\). Thus, by induction hypothesis, \((M, G) \models Y \psi_1\) and \((M, G) \models Z \psi_2\), and finally \((M, G) \models X \psi_1 \lor \psi_2\), as required.

- Suppose that \((M, G) \models X \exists x \psi\). Then there exists a team \(Y \in G\) such that \(X[x]Y\) and \((M, G) \models Y \psi\). By induction hypothesis, this means that \((M, G) \models Y|\vec{x} \psi\) too; and since \(X[x]Y|\vec{x}\), this implies that \(M \models X|\vec{x} \exists x \psi\), as required.

Conversely, suppose that \((M, G) \models X|\vec{x} \exists x \psi\). Then there exists a team \(Y'\), with domain \(\vec{x}\), such that \(M \models Y' \psi\) and \(X|\vec{x} \psi\). Now let \(Y = (X[M/x])(\vec{x} \in Y')\). By the lemma, \(Y \in G\); furthermore, \(Y|\vec{x} = Y'\), and hence by induction hypothesis \((M, G) \models Y \psi\). Finally, \(X[x]Y\): indeed, if \(s \in X\) then \(s[x][m/x] \in Y'\) for some \(m \in \text{Dom}(M)\), and hence \(s[m/x] \in Y\) for the same \(m\), and on the other hand, \(Y\) is contained in \(X[M/x]\), and hence if \(s[m/x] \in Y\) it follows that \(s \in X\).

Therefore \((M, G) \models X \exists x \psi\), as required.

As in the case of Second Order Logic, first-order models can be represented as a special kind of general model:

**Definition 5.1.7.** Let \((M, G)\) be a general model. Then it is said to be \textit{full} if and only if \(G\) contains all teams over \(M\).

The following result is then trivial.

**Proposition 5.1.8.** Let \((M, G)\) be a full model. Then for all suitable teams \(X\) and formulas \(\phi\), \((M, G) \models X \phi\) in General Team Semantics if and only if \(M \models X \phi\) in the usual Team Semantics.

**Proof.** Follows at once by comparing the rules of Team Semantics and General Team Semantics for the case that \(G\) contains all teams. 

How does the satisfaction relation in General Team Semantics change if we vary the set \(G\)? The following definition and result give us some information about this:
Definition 5.1.9. Let \((M, G)\) and \((M, G')\) be two general models. Then we say that \((M, G')\) is a refinement of \((M, G)\), and we write \((M, G) \subseteq (M, G')\), if and only if \(G \subseteq G'\).

Intuitively speaking, a refinement of a general model is another general model with more teams in it than the former. The following result shows that refinements preserve satisfaction relations:

Theorem 5.1.10. Let \((M, G)\) and \((M, G')\) be two general models with \((M, G) \subseteq (M, G')\), let \(X \in G\), and let \(\phi\) be a formula over the signature of \(M\) with \(\text{Free}(\phi) \subseteq \text{Dom}(X)\). Then

\[
(M, G) \models_X \phi \Rightarrow (M, G') \models_X \phi.
\]

Proof. The proof is an easy induction on \(\phi\).

1. If \(\phi\) is a first order literal or a non-first-order atom, the result is obvious, as the choice of the set of teams \(G\) (or \(G'\)) does not enter into the definition of its satisfaction condition.

2. If \((M, G) \models_X \psi_1 \lor \psi_2\) then there exist two teams \(Y, Z \in G\) such that \(X = Y \cup Z\), \((M, G) \models Y \psi_1\) and \((M, G) \models Z \psi_2\). But \(Y\) and \(Z\) are also in \(G'\), and by induction hypothesis we have that \((M, G') \models Y \psi_1\) and \((M, G') \models Z \psi_2\), and therefore \((M, G') \models X \psi_1 \lor \psi_2\).

3. If \((M, G) \models_X \psi_1 \land \psi_2\) then \((M, G) \models_X \psi_1\) and \((M, G) \models_X \psi_2\). Then, by induction hypothesis, \((M, G') \models_X \psi_1\) and \((M, G') \models_X \psi_2\), and finally \((M, G') \models_X \psi_1 \land \psi_2\).

4. If \((M, G) \models_X \exists x \psi\) then there exists a \(X' \in G\) such that \(X' \in G\) and \((M, G) \models X' \psi\). But then \(X'\) is also in \(G'\), and by induction hypothesis \((M, G') \models X' \psi\), and finally \((M, G') \models_X \exists x \psi\).

5. If \((M, G) \models_X \forall x \psi\) then \((M, G) \models X[M/x] \psi\). Then, by induction hypothesis, \((M, G') \models X[M/x] \psi\), and finally \((M, G') \models_X \forall x \psi\).

This result shows us that, as was to be expected from the equivalence between independence logic and existential second order logic, if we are interested in formulas which hold in all general models over a certain first-order model we only need to pay attention to the smallest (in the sense of the refinement relation) ones. But do such “least general models” exist? As the following result shows, this is indeed the case:
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Proposition 5.1.11. Let \( \{(M_i, \mathcal{G}_i) : i \in I\} \) be a family of general models with signature \( \Sigma \) and over the same first order model \( M \). Then \( (M, \bigcap_{i \in I} \mathcal{G}_i) \) is also a general model.

Proof. Let \( \phi(x_1 \ldots x_n, \bar{m}, \bar{R}) \) be a first order formula with parameters, where each \( R_i \) is of the form \( \text{Rel}(X) \) for some \( X \in \bigcap_i \mathcal{G}_i \). Then \( \|\phi(x_1 \ldots x_n, \bar{m}, \bar{R})\|_M \) is in \( \mathcal{G}_i \) for all \( i \in I \), and therefore it is in \( \bigcap_{i \in I} \mathcal{G}_i \), as required.

Therefore, it is indeed possible to talk about the least general model over a first order model.

Definition 5.1.12. Let \( M \) be a first order model. Then the least general model over \( M \) is the \( (M, \mathcal{L}) \), where

\[ \mathcal{L} = \bigcap \{ \mathcal{G} : (M, \mathcal{G}) \text{ is a general model.} \} \]

As an example of a least general model, let \( n \in \mathbb{N} \), and let \( M_n \) be a model with empty signature and domain \( \{1 \ldots n\} \). Then the least general model over \( M_n \) is actually the full general model \( (M_n, \mathcal{G}_n) \), where \( \mathcal{G}_n \) contains all teams over \( M_n \). Indeed, let \( \{v_1 \ldots v_k\} \) be a finite set of variables and let

\[ X = \{s_1 \ldots s_q\} = \begin{array}{c|ccc}
  v_1 & \ldots & v_k \\
  s_1 & a_{11} & \ldots & a_{1k} \\
  \vdots & \vdots & \ddots & \vdots \\
  s_q & a_{q1} & \ldots & a_{qk}
\end{array} \]

be any team over \( M_n \) with domain \( \{v_1 \ldots v_k\} \), where \( s_i(v_j) = a_{ij} \) for all \( i \in 1 \ldots q \) and all \( j \in 1 \ldots k \). Then clearly \( q \leq n^k \), and furthermore, for \( \phi(v_1 \ldots v_k) = \bigvee_{i=1}^q \bigwedge_{j=1}^k v_j = a_{qi} \) we have that

\[ \|\phi(v_1 \ldots v_k, a_{11} \ldots a_{qk})\|_M = \{s : \text{Dom}(s) = \{v_1 \ldots v_k\}, M \models s \phi\} = X \]

as required.

As this example shows, if \( M \) is finite then the least (and only) general model over it is the full one. Hence, if we are only interested in finite models, General Model Semantics is equivalent to the standard Team Semantics, and the same can be said about the Entailment Semantics which we will develop later in this chapter.

What is the purpose of least general models? The answer comes as a consequence of Theorem 5.1.10, and can be summarized by the following corollary:

Corollary 5.1.13. Let \( \Sigma \) be a first order signature, let \( M \) be a first order model over it and let \( (M, \mathcal{L}) \) be the least general model over it. Then, for all teams

\[ \cdots \]
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$X \in L$ and all formulas $\phi$ with signature $\Sigma$ and with free variables in $\text{Dom}(X)$,

$$(M, L) \models_X \phi \iff (M, G) \models_X \phi \text{ for all general models } (M, G) \text{ over } M.$$ 

Proof. Suppose that $(M, L) \models_X \phi$. Then take any general model $(M, G)$: by definition, we have that $(M, L) \subseteq (M, G)$, and hence by Theorem 5.1.10 we have that $(M, G) \models_X \phi$.

Conversely, suppose that $(M, G) \models_X \phi$ for all general models $(M, G)$; then in particular $(M, L) \models_X \phi$, as required.

We can also find a more practical characterization of this “least general model”.

**Proposition 5.1.14.** Let $M$ be a first order model. Then the least general model over it is $(M, L)$, where $L$ is the set of all $\|\phi(x, m)\|_M$, where $\phi$ ranges over all first order formulas and $m$ ranges over all tuples of variables of suitable length.

Proof. If $(M, G)$ is a general model then $L \subseteq G$ by definition; therefore, we only need to prove that $(M, L)$ is a general model.

Now, let $\phi(x, m, \overrightarrow{R})$ be a first order formula, and let each $R_i$ be $\text{Rel}(X_i)$ for some $X_i \in L$. So for each $R_i$, any assignment $s$ and any suitable tuple of terms $t, \overrightarrow{R}_i$ if and only if $M \models_s \psi_i(t, \overrightarrow{n}_i)$ for some first order formula $\psi_i$ with parameters $\overrightarrow{n}_i$. Now let $\phi'(x, m, \overrightarrow{n}, \overrightarrow{R})$ be the expression obtained by replacing, in $\phi$, each instance of $R_i(t)$ by $\psi_i(t, \overrightarrow{n}_i)$; by construction, we have that $M \models_s \phi(x, m, \overrightarrow{R})$ if and only if $M \models_s \phi'(x, m, \overrightarrow{n}, \overrightarrow{R})$, and therefore

$$\|\phi(x, m, \overrightarrow{R})\|_M = \|\phi'(x, m, \overrightarrow{n}, \overrightarrow{R})\|_M \in L$$

as required.

**Definition 5.1.15.** Let $\Sigma$ be a first order signature, let $V$ be a finite set of variables, and let $\phi$ be a formula of our language with free variables in $V$. Then $\phi$ is valid with respect to general models if and only if $(M, G) \models_X \phi$ for all general models $(M, \Sigma)$ with signature $\Sigma$ and for all teams $X \in G$ with $\text{Dom}(X) \supseteq \text{Free}(\phi)$. If this is the case, we write $\text{GMS} \models \phi$.

**Definition 5.1.16.** Let $\Sigma$ be a first order signature, let $V$ be a finite set of variables, and let $\phi$ be a formula of our language over this signature with free variables in $V$. Then $\phi$ is valid with respect to least general models if and only if $(M, L) \models_X \phi$ for all least general models $(M, L)$ with signature $\Sigma$ and for all teams $X \in L$ with $\text{Dom}(X) \supseteq \text{Free}(\phi)$. If this is the case, we write $\text{LMS} \models \phi$. 

Lemma 5.1.17. Let $M$ be a first order model with signature $\Sigma$, and let $M'$ be another first order model with signature $\Sigma' \supseteq \Sigma$ such that the restriction of $M'$ to $\Sigma$ is precisely $M$. Then for all general models $G$ for $M'$, for all formulas $\phi$ with signature $\Sigma$ and for all $X \in G$,

$$(M,G) \models_X \phi \iff (M',G) \models_X \phi.$$ 

Proof. First of all, if $(M',G)$ is a general model then $(M,G)$ is also a general model. Then, the result is proved by observing that the truth conditions of our semantics depend only on the interpretations of the symbols in the signature of the formula (and on the choice of $G$, of course).

Lemma 5.1.18. Let $(M,G)$ be a general model with signature $\Sigma$, let $S \not\in \Sigma$ be a new relation symbol and let $X \in G$. Furthermore, let $M' = M[\text{Rel}(X)/S]$ be the extension of $M$ to the signature $\Sigma \cup \{S\}$ such that $S^{M'} = \text{Rel}(X)$. Then $(M',G)$ is a general model.

Proof. Let $\phi(\bar{x}, \bar{m}, \bar{R})$ be a first order formula with signature $\Sigma \cup \{S\}$ and parameters $\bar{m}$ and $\bar{R}$, where each $R_i$ is $\text{Rel}(X_i)$ for some $X_i \in G$. Then let $\phi'(\bar{x}, \bar{m}, \bar{R}, S)$ be the first order formula with signature $\Sigma$, where $S$ now stands for the relation $\text{Rel}(X)$. Now clearly

$$\|\phi(\bar{x}, \bar{m}, \bar{R})\|_{M'} = \|\phi'(\bar{x}, \bar{m}, \bar{R}, S)\|_M \in G,$$

as required.

Theorem 5.1.19. A formula $\phi$ is valid wrt general models if and only if it is valid wrt least general models.

Proof. The left to right direction is obvious. For the right to left direction, suppose that $\text{LMS} \models \phi$, let $(M,G)$ be a general model whose signature contains the signature of $\phi$, and let $X \in G$ be a team whose domain $\{x_1 \ldots x_n\}$ contains all free variables of $\phi$. Then consider the first order model $M' = M[\text{Rel}(X)/S]$, where $S$ is a new relation symbol, and take the least general model $(M', L)$ over it. We clearly have that $X \in L$, since

$$X = \{s : \text{Dom}(s) = \{x_1 \ldots x_n\}, M' \models_s Sx_1 \ldots x_n\}$$

and, therefore, $(M', L) \models_X \phi$ by hypothesis. Now, by Lemma 5.1.18, $(M', G)$ is a general model, and therefore by definition $L \subseteq G$, and hence by Theorem 5.1.10 $(M', G) \models_X \phi$ too. Finally, the relation symbol $S$ does not occur in $\phi$, and therefore by Lemma 5.1.17 $(M, G) \models_X \phi$, as required.
5.2 Entailment Semantics

Let \( M \) be a first order structure and let \((M, \mathcal{L})\) be the least general model over it. Then, as we saw, \( \mathcal{L} \) is the set of all teams corresponding to first order formulas with parameters. Therefore, in order to reason about satisfaction in a least general model, there is no need to carry around the teams themselves: rather, we can use the corresponding first order formulas. In this section, we will develop this idea, building up a new “Entailment Semantics” and proving its correspondence with General Model Semantics over least general models.

We will then construct a proof system and prove its soundness and completeness with respect to this semantics. Then, since – as we saw already – validity with respect to least general models is equivalent to validity with respect to general models, this proof system will also seen to be sound and complete with respect to General Model Semantics.

For the purposes of this chapter, Entailment Semantics acts as a bridge between General Model Semantics and our proof system: by allowing us to abstract away from higher-order objects such as teams, it will make it significantly easier for us to establish a connection between semantics and proof theory.

Furthermore, the semantics which we will build, with its more syntactic flavor, is of independent interest. The phenomena of dependence and independence whose study is among the principal reasons of being of dependence logic and independence logic are present in it, but the intrinsically higher-order nature of the usual Team Semantics is not. Entailment Semantics, in other words, can be seen as an attempt of examining the content of the notions of dependence and independence from a first-order perspective, rather than from the higher-order perspective implicit in the formulation of Team Semantics.

**Definition 5.2.1.** Let \( V_P = \{p_1, \ldots, p_n, \ldots\} \) and \( V_T = \{x, y, z, \ldots\} \) be fixed, disjoint, countably infinite sets of variables. We will call any \( p \in V_P \) a parameter variable, and we will call any \( x \in V_T \) a team variable. Furthermore, we will assume that any variable which occurs in any of our formulas is a team variable or a parameter variable.

**Definition 5.2.2.** Let \( \phi \) be any formula. Then \( \text{Free}_P(\phi) = \text{Free}(\phi) \cap V_P \) and \( \text{Free}_T(\phi) = \text{Free}(\phi) \cap V_T \).

Parameter variables clarify the interpretation of such expressions such as \( M \models_s \gamma(\vec{x}, m) \): this is simply a shorthand for \( M \models_{h, \vec{x}} \gamma(\vec{x}, \vec{p}) \), where \( h \) is
a parameter assignment with domain $\vec{p}$ and with $h(\vec{p}) = \vec{m}$. Team variables, instead, are going to be used in order to describe the variables in the domain of the team corresponding to a given first order expression: for any first order $\gamma(\vec{x}, \vec{p})$, where $\vec{x}$ are team variables and $\vec{p}$ are parameter variables, and for any $h$ with domain $\vec{p}$, we will therefore have $\|\gamma(\vec{x}, \vec{p})\|_{M,h} = \|\gamma(\vec{x}, h(\vec{p}))\|_M = \{ s : \text{Dom}(s) = \vec{x}, M \models_{h \cup s} \gamma \}$. For this reason, parameter variables will never occur in the domain of a team, and, hence, from this point on we will always assume that parameter variables never occur in independence logic formulas, but only in the first order team definitions.

After these preliminaries, we can now give our main definition for this section:

**Definition 5.2.3.** For all first order models $M$, all first order formulas $\gamma(\vec{x}, \vec{p})$ with $\text{Free}_T(\gamma) = \vec{x}$ and $\text{Free}_P(\gamma) = \vec{p}$, and all parameter assignments $h$ with domain $\vec{p}$

- **ES-lit:** For all first order literals $\alpha$, $M \models \gamma(h) \alpha$ if and only if for all assignments $s$ with domain $\text{Free}_T(\gamma) \cup \text{Free}_T(\alpha)$ such that $M \models h \cup s \gamma$ it holds that $M \models h \cup s \alpha$;

- **ES-inc, ES-exc, ES-ind:** For all inclusion, exclusion or independence atoms $\beta$, $M \models \gamma(h) \beta$ if and only if the team $\{ s : \text{Dom}(s) = \text{Free}_T(\gamma) \cup \text{Free}_T(\beta), M \models h \cup s \beta \}$ satisfies $\beta$ in the usual sense;

- **ES-$\lor$:** For all $\psi_1$ and $\psi_2$, $M \models \gamma(h) \psi_1 \lor \psi_2$ if and only if there exists a parameter assignment $h'$ extending $h$ and there exist first order formulas $\gamma_1$ and $\gamma_2$ such that
  
  - $\text{Free}_P(\gamma_1), \text{Free}_P(\gamma_2) \subseteq \text{Dom}(h')$;
  - $M \models_{\gamma(h')} \psi_1$;
  - $M \models_{\gamma(h')} \psi_2$;
  - $M \models_{h'} \forall \vec{v}(\gamma \leftrightarrow \gamma_1 \lor \gamma_2)$, where $\vec{v}$ is $\text{Free}_T(\gamma) \cup \text{Free}_T(\gamma_1) \cup \text{Free}_T(\gamma_2)$;

- **ES-$\land$:** For all $\psi_1$ and $\psi_2$, $M \models \gamma(h) \psi_1 \land \psi_2$ if and only if $M \models \gamma(h) \psi_1$ and $M \models \gamma(h) \psi_2$;

- **ES-$\exists$:** For all $x_n \in \text{Var}_T$ and all $\psi$, $M \models \gamma(h) \exists x_n \psi$ if and only if there exist a parameter assignment $h'$ extending $h$ and a first order formula $\gamma'$ with $\text{Free}_P(\gamma') \subseteq \text{Dom}(h')$ such that
  
  - $M \models \gamma(h') \psi$;

\[\text{That is, } \text{Dom}(h') \supseteq \text{Dom}(h), \text{ and } h'(\vec{p}) = h(\vec{p}).\]
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- $M \models h' \forall \vec{v}(\exists x_{n}\gamma' \leftrightarrow \exists x_{n}\gamma)$, where $\vec{v}$ is $\text{Free}_T(\gamma) \cup \text{Free}_T(\gamma')$;

**ES-$\forall$:** For all $x_n \in \text{Var}_T$ and all $\psi$, $M \models \gamma(h) \forall x_n \psi$ if and only if there exists a parameter assignment $h'$ extending $h$ and a first order formula $\gamma'$ with $\text{Free}_P(\gamma') \subseteq \text{Dom}(h')$ such that

- $M \models \gamma(h) \psi$;
- $M \models h' \forall \vec{v}(\exists x_{n}\gamma)$, where $\vec{v}$ is $\text{Free}_T(\gamma) \cup \text{Free}_T(\gamma')$.

The reason why the above semantics is called “Entailment Semantics” is because its satisfaction relation describes a sort of entailment relation between a first order formula with parameters, which takes the role that teams have in the usual Team Semantics, and an independence logic formula. In particular, it is easy to see that according to our rule **ES-lit**, for all first order literals $\phi(\vec{x}, \vec{y})$, first order formulas with parameters $\gamma(\vec{x})$ and parameter assignments $h, M \models \gamma(h) \phi$ if and only if $M \models h \forall \vec{x} \vec{y}(\gamma(\vec{x}) \rightarrow \phi(\vec{x}, \vec{y}))$.

Furthermore, one can notice some analogies between Entailment Semantics and Database Theory: in particular, the role of $\gamma$ in an expression $M \models \gamma(h) \phi$ is to specify a relation in terms of a first order formula, much as in the Tuple Relational Calculus expression $\{\langle x_1 \ldots x_n \rangle : M \models \gamma(x_1 \ldots x_n, h(\vec{p}))\}$.

**Proposition 5.2.4.** Let $M$ be a first order model with signature $\Sigma$, let $\gamma(\vec{x}, \vec{p})$ be a first order formula with $\text{Free}_P(\gamma) = \vec{p}$, let $h$ be a parameter assignment with domain $\vec{p}$ and let $\phi$ be a formula over this signature and with free variables in $\vec{x}$. Furthermore, let $(M, L)$ be the least general model over $M$, and let $X = \|\gamma(\vec{x}, \vec{p})\|_{M, h} = \{s : \text{Dom}(s) = \vec{x}, M \models h(\bar{s}) \gamma(\vec{x}, \vec{m})\}$. Then, for all formulas $\phi$,

$$M \models \gamma(h) \phi \iff M \models \gamma(h') \phi.$$  

**Proof.** The proof is a straightforward induction over $\phi$. \hfill □

As the next result shows, Entailment Semantics is entirely equivalent to Least General Model Semantics:

**Theorem 5.2.5.** Let $\Sigma$ be a first order model, let $\gamma(\vec{x}, \vec{p})$ be a first order formula with $\text{Free}_P(\gamma) = \vec{p}$, let $h$ be a parameter assignment with domain $\vec{p}$ and let $\phi$ be a formula over this signature and with free variables in $\vec{x}$.

Furthermore, let $(M, L)$ be the least general model over $M$, and let $X = \|\gamma(\vec{x}, \vec{p})\|_{M, h} = \{s : \text{Dom}(s) = \vec{x}, M \models h(\bar{s}) \gamma(\vec{x}, \vec{m})\}$. Then

$$(M, L) \models X \phi \iff M \models \gamma(h) \phi.$$  

**Proof.** The proof is by structural induction on $\phi$, and presents no difficulties.
1. If $\phi$ is a first order literal, $(M, L) \models_X \phi$ if and only if, for all $s \in X$, it holds that $M \models_s \phi$. But $s \in X$ if and only if $M \models \gamma(\bar{x}, h(\bar{p}))$, and hence $(M, L) \models_X \phi$ if and only if $M \models \gamma$, as required.

2. If $\phi$ is an inclusion, exclusion or independence atom, the result is also obvious, and follows at once from a comparison of the rules $\text{GMS-inc}$ ($\text{GMS-exc}$, $\text{GMS-ind}$) and $\text{ES-inc}$ ($\text{ES-exc}$, $\text{ES-ind}$).

3. If $\phi$ is $\psi_1 \lor \psi_2$,

   \[(M, L) \models_X \psi_1 \lor \psi_2 \Leftrightarrow \exists Y, Z \in \mathcal{L} \text{ s.t. } X = Y \cup Z, (M, L) \models_Y \psi_1 \text{ and } (M, L) \models_Z \psi_2 \Leftrightarrow \]
   
   \[
   \exists h' = h[\bar{m}/\bar{q}] \text{ extending } h \text{ and } \exists \gamma_1 \gamma_2 \text{ s.t., for } Y = \|\gamma_1(\bar{x}, \bar{p})\|_{M, h'}, \]
   \[
   Z = \|\gamma_2(\bar{x}, \bar{p})\|_{M, h'} = X = \|\gamma(\bar{x}, \bar{p})\|_{M, h} = \|\gamma(\bar{x}, \bar{p})\|_{M, h'} = Y \cup Z,
   \]
   \[
   (M, L) \models_Y \psi_1 \text{ and } (M, L) \models_Z \psi_2 \Leftrightarrow \exists h' = h[\bar{m}/\bar{q}] \text{ extending } h \text{ and } \exists \gamma_1 \gamma_2 \text{ s.t. } M \models h' \forall \bar{v}(\gamma \leftrightarrow \gamma_1 \lor \gamma_2),
   \]
   \[
   M \models \gamma_1(h') \psi \text{ and } M \models \gamma_2(h') \theta \Leftrightarrow \exists h' = h[\bar{m}/\bar{q}] \text{ extending } h \text{ and } \exists \gamma_1 \gamma_2 \text{ s.t. } M \models h' \forall \bar{v}(\gamma \leftrightarrow \gamma_1 \lor \gamma_2),
   \]
   \[
   \Leftrightarrow M \models \gamma(h) \psi \lor \theta.
   \]

4. If $\phi$ is $\psi \land \theta$,

   \[(M, L) \models_X \psi \land \theta \Leftrightarrow (M, L) \models_X \psi \text{ and } (M, L) \models_X \theta \Leftrightarrow \]
   \[
   \Leftrightarrow M \models \gamma(h) \psi \text{ and } M \models \gamma(h) \theta \Leftrightarrow M \models \gamma(h) \psi \land \theta.
   \]

5. If $\phi$ is $\exists x_n \psi$,

   \[(M, L) \models_X \exists x_n \psi \Leftrightarrow \exists X' \in \mathcal{L} \text{ s.t. } X[x_n]X' \text{ and } (M, L) \models_{X'} \psi \Leftrightarrow \]
   \[
   \exists h' = h[\bar{m}/\bar{q}] \text{ extending } h \text{ and } \exists \gamma' \text{ s.t., for } X' = \|\gamma'(\bar{x}, \bar{p})\|_{M, h'},
   \]
   \[
   X[x_n]X' \text{ and } (M, L) \models_{X'} \psi \Leftrightarrow \exists h' = h[\bar{m}/\bar{q}] \text{ extending } h \text{ and } \exists \gamma' \text{ s.t. } M \models h' \forall \bar{v}(\exists x_n \gamma \leftrightarrow \exists x_n \gamma')
   \]
   \[
   \text{and } M \models \gamma'(h) \psi \Leftrightarrow \exists h' = h[\bar{m}/\bar{q}] \text{ extending } h \text{ and } \exists \gamma' \text{ s.t. } M \models h' \forall \bar{v}(\exists x_n \gamma \leftrightarrow \exists x_n \gamma') \text{ and } M \models \gamma(h) \exists x_n \psi; \]
   \[
   \Leftrightarrow M \models \gamma(h) \exists x_n \psi.
   \]
6. If $\phi$ is $\forall x_n \psi$,

\[
(M, \mathcal{L}) \models_X \forall x_n \psi \iff \exists X' \in \mathcal{L} \text{ s.t. } X' = X[M/x_n] \text{ and } (M, \mathcal{L}) \models_{X'} \psi \iff \\
\exists h' = h[\vec{m}/\vec{q}] \text{ extending } h \text{ and } \exists \gamma' \text{ s.t., for } X' = \|\gamma'(\vec{x}, \vec{p})||_{M,h'}, \\
X' = X[M/x_n] \text{ and } (M, \mathcal{L}) \models_{X'} \psi \iff \\
\exists h' = h[\vec{m}/\vec{q}] \text{ extending } h \text{ and } \exists \gamma' \text{ s.t. } \gamma'(\vec{x}, \vec{p}) \models_{M,h'} \forall x_n \gamma' \iff \\
\exists h' = h[\vec{m}/\vec{q}] \text{ extending } h \text{ and } \exists \gamma' \text{ s.t., for } X' = \|\gamma'(\vec{x}, \vec{p})||_{M,h'}, \\
X' = X[M/x_n] \text{ and } (M, \mathcal{L}) \models_{X'} \psi \iff \exists h'(h) \psi \iff \\
\exists M \models_{\gamma(h)} \forall x_n \psi.
\]

\[\Box\]

**Definition 5.2.6.** Let $\phi$ be a formula. Then $\phi$ is **valid** in Entailment Semantics if and only if $M \models_{\gamma(h)} \phi$ for all first order models $M$ with signature containing that of $\phi$, for all first order formulas $\gamma(\vec{x}, \vec{p})$ over the signature of $M$ and for all parameter assignments $h$ with domain $\vec{p}$. If this is the case, we write $ENS \models \phi$.

**Corollary 5.2.7.** For all formulas $\phi$, $ENS \models \phi$ if and only if $LMS \models \phi$ if and only if $GMS \models \phi$.

It will also be useful to have a slightly more general notion of validity in Entailment Semantics:

**Definition 5.2.8.** Let $\gamma(\vec{x}, \vec{p})$ be a first order formula and let $\phi$ be a formula. Then $\phi$ is **valid** with respect to $\gamma$ if and only if $M \models_{\gamma(h)} \phi$ for all first order models $M$ with signature containing those of $\gamma$ and $\phi$ and for all parameter assignments $h$ with domain $\vec{p}$. If this is the case, we write $\models_{\gamma} \phi$.

**Proposition 5.2.9.** Let $\phi$ be a formula with $\text{Free}_T(\phi) = \{x_1 \ldots x_k\}$, let $\vec{x} = x_1 \ldots x_k$, and let $R$ be a $k$-ary relation symbol not occurring in $\gamma$. Then $ENS \models \phi$ if and only if $M \models_{RF} \phi$.

**Proof.** Suppose that $ENS \models \phi$. Then in particular, for any model $M$ whose signature contains that of $\phi$ and $R$ we have that $M \models_{RF} \phi$, and hence $\models_{RF} \phi$.

Conversely, suppose that $\models_{RF} \phi$, let $M$ be a first order model$^4$, and let $X \in \mathcal{L}$ be any team with domain $\{x_1 \ldots x_k\}$. Let us then consider the model $M'$ obtained by adding to $M$ the $k$-ary symbol $R$ with $R^{M'} = \text{Rel}(X)$. By hypothesis, $M' \models_{RF} \phi$, and furthermore since $R^{M'}$ is in $\mathcal{L}$ already the least general model over $M'$ is $(M', \mathcal{L})$ for the same $\mathcal{L}$.

$^4$Without loss of generality, we can assume that the signature of $M$ does not contain the symbol $R$. 

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Now \((M', L) \models X \varphi\), and therefore, as \(R\) occurs nowhere in \(\varphi\), \((M, L) \models X \varphi\) too. This holds for all \(X\) with domains \(\{x_1 \ldots x_k\}\); therefore by the Locality Theorem (Theorem 5.1.6), the same holds for all domains containing \(\text{Free}_T(\varphi)\), and hence \(\text{LMS} \models \varphi\). This implies that \(\text{ENS} \models \varphi\), as required.

In the next section, we will develop a sound and complete proof system for this notion of validity with respect to a team definition.

### 5.3 The Proof System

In this section, we will develop a proof system for \(I(\subseteq, |)\) and prove its soundness and completeness.

**Definition 5.3.1.** Let \(\Gamma\) be a finite first order theory with only parameter variables among its free ones, let \(\gamma(\vec{x}, \vec{p})\) be a first order formula and let \(\varphi\) be a formula with free variables in \(\text{Var}_T\). Then the expression

\[\Gamma \models \gamma(\vec{h}) \vdash \varphi\]

is a sequent.

The intended semantics of a sequent is the following one:

**Definition 5.3.2.** Let \(\Gamma \models \gamma \vdash \varphi\) be a sequent. Then \(\Gamma \models \gamma \vdash \varphi\) is valid if and only if for all models \(M\) and all parameter assignments \(h\) with domain \(\text{Free}_P(\Gamma) \cup \text{Free}_P(\gamma)\) such that \(M \models \gamma(\vec{h})\) it holds that \(M \models \varphi(\vec{h})\).

For example, the sequent \(\emptyset \models y = f(x) \vdash (x, y)\) is valid, as any team in which \(y = f(x)\) satisfies the condition corresponding to \(= (x, y)\) (or, equivalently, to the independence atom \(y \perp x\)); and similarly, the sequent \(\exists q \forall u(Rpu \rightarrow (u = q \lor u = r)) \models Rpx \vdash (x) \lor = (x)\) is valid, because if \(|\{m \in \text{Dom}(M) : M \models h Rpm\}| \leq 2\) then the team \((Rpx)(h) = \{s : M \models h Rpx\}\) assigns no more than two different values for \(x\) and hence satisfies \(= (x) \lor = (x)\).

However, \(\emptyset \models Rpx \vdash (x) \lor = (x)\) is not valid: indeed, let \(\text{Dom}(M) = \{1, 2, 3\}\), let \(R^M\) be \(\text{Dom}(M) \times \text{Dom}(M)\), and let \(h\) be such that \(h(p) = 1\). Then \((Rpx)(h)\) is exactly \(\text{Dom}(M) = \{1, 2, 3\}\), which does not satisfy \(= (x) \lor = (x)\).

The following result is then clear:

**Proposition 5.3.3.** For all \(\gamma\) and \(\varphi\), \(\models \gamma \varphi\) if and only if \(\emptyset \models \gamma \vdash \varphi\) is valid.
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Now, all we need to do is develop some syntactic rules for finding valid sequents.
We can do this as follows:

**Definition 5.3.4.** The axioms of our proof system are

**PS-lit:** If \( \phi \) is a first order literal with no free parameter variables (that is, \( \text{Free}_P(\phi) = \emptyset \)) then
\[
\forall \vec{v}(\gamma \rightarrow \phi) \mid \gamma \vdash \phi
\]
for all first order formulas \( \gamma \), where \( \vec{v} = \text{Free}_T(\gamma) \cup \text{Free}_T(\phi) \);

**PS-inc:** If \( \vec{t}_1 \) and \( \vec{t}_2 \) are tuples of terms of the same length with no parameter variables then
\[
\forall \vec{v}_1\forall \vec{v}_2((\gamma(\vec{v}_1) \land \gamma(\vec{v}_2)) \rightarrow \vec{v}_1(\vec{v}_1) = \vec{v}_2(\vec{v}_2)) \mid \gamma \vdash \vec{t}_1 \subseteq \vec{t}_2
\]
for all \( \gamma \), where \( \vec{v}_1 \) and \( \vec{v}_2 \) are tuples of variables of the same lengths of \( \vec{v} = \text{Free}_T(\gamma) \cup \text{Free}_T(\vec{t}_1 \vec{t}_2), \vec{t}_i(\vec{v}_i) \) is the tuple obtained by substituting \( \vec{v}_i \) for \( \vec{v} \) in \( \vec{t}_i \), and the same holds for \( \gamma(\vec{v}_i) \);

**PS-exc:** If \( \vec{t}_1 \) and \( \vec{t}_2 \) are tuples of terms of the same length with no parameter variables then
\[
\forall \vec{v}_1\forall \vec{v}_2((\gamma(\vec{v}_1) \land \gamma(\vec{v}_2)) \rightarrow \vec{v}_1(\vec{v}_1) \neq \vec{v}_2(\vec{v}_2)) \mid \gamma \vdash \vec{t}_1 \mid \vec{t}_2;
\]

**PS-ind:** If \( \vec{t}_1, \vec{t}_2 \) and \( \vec{t}_3 \) are tuples of terms with no parameter variables then
\[
\forall \vec{v}_1\forall \vec{v}_2((\gamma(\vec{v}_1) \land \gamma(\vec{v}_2) \land \vec{t}_1(\vec{v}_1) = \vec{t}_2(\vec{v}_2)) \rightarrow \exists \vec{v}_3((\gamma(\vec{v}_3) \land \vec{t}_1(\vec{v}_3) = \vec{t}_2(\vec{v}_3)) \land \vec{t}_1(\vec{v}_3) = \vec{t}_2(\vec{v}_3))) \mid \gamma \vdash \vec{t}_2 \bot \vec{t}_1 \vec{t}_3.
\]

The rules of our proof system are

**PS-\lor:** If \( \Gamma_1 \mid \gamma \vdash \phi_1 \) and \( \Gamma_2 \mid \gamma \vdash \phi_2 \) then, for all \( \gamma \), we have
\[
\Gamma_1, \Gamma_2, \forall \vec{v}(\gamma \leftrightarrow (\gamma_1 \lor \gamma_2)) \mid \gamma \vdash \phi_1 \lor \phi_2
\]
where \( \vec{v} = \text{Free}_T(\gamma) \cup \text{Free}_T(\gamma_1) \cup \text{Free}_T(\gamma_2) \);

**PS-\land:** If \( \Gamma_1 \mid \gamma \vdash \phi_1 \) and \( \Gamma_2 \mid \gamma \vdash \phi_2 \) then \( \Gamma_1, \Gamma_2 \mid \gamma \vdash \phi_1 \land \phi_2 \);

**PS-\exists:** If \( \Gamma \mid \gamma' \vdash \phi \) and \( x \) is a team variable then, for all \( \gamma \),
\[
\Gamma, \forall \vec{v}(\exists x\gamma' \leftrightarrow \exists x\gamma) \mid \gamma \vdash \exists x\phi
\]
where \( \vec{v} = \text{Free}_T(\gamma) \cup \text{Free}_T(\gamma') \);

**PS-\forall:** If \( \Gamma | \gamma' \vdash \phi \) and \( x \) is a team variable then, for all \( \gamma \),

\[
\Gamma, \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma) | \gamma \vdash \forall x \phi
\]

where, as in the previous case, \( \vec{v} = \text{Free}_T(\gamma) \cup \text{Free}_T(\gamma') \);

**PS-ent:** If \( \Gamma | \gamma \vdash \phi \) and \( \bigwedge \Gamma' \models \bigwedge \Gamma \) holds in First Order Logic then \( \Gamma' | \gamma \vdash \phi \);

**PS-depar:** If \( \Gamma | \gamma \vdash \phi \) and \( p \) is a parameter variable which does not occur free in \( \gamma \) then \( \exists p \bigwedge \Gamma | \gamma \vdash \phi \);

**PS-split:** If \( \Gamma_1 | \gamma \vdash \phi \) and \( \Gamma_2 | \gamma \vdash \phi \) then \( (\bigwedge \Gamma_1) \lor (\bigwedge \Gamma_2) | \gamma \vdash \phi \).

**Definition 5.3.5.** Let \( \Gamma | \gamma \vdash \phi \) be a sequent. A *proof* of this sequent is a finite list of sequents

\[
(\Gamma_1 | \gamma_1 \vdash \phi_1), \ldots, (\Gamma_n | \gamma_n \vdash \phi_n) = (\Gamma | \gamma \vdash \phi)
\]

such that, for all \( i = 1 \ldots n \), \( \Gamma_i | \gamma_i \vdash \phi_i \) is either an instance of **PS-lit**, **PS-inc**, **PS-exc**, **PS-ind** or it follows from \( \{ \Gamma_j | \gamma_j \vdash \phi_j : j < i \} \) through one application of the rules of our proof system.

Given a proof \( P = S_1 \ldots S_n \), where each \( S_i \) is a sequent, we define its length \( |P| \) as \( n - 1 \), that is, as the number of sequents in the proof minus one.

Before examining soundness and completeness for this proof system, it will be useful to derive a general rule for first order formulas.

**Proposition 5.3.6. PS-FO:** If \( \phi \) is a first order formula with no free parameter variables, \( \forall \vec{v}(\gamma \rightarrow \phi) | \gamma \vdash \phi \) is provable for all \( \gamma \), where \( \vec{v} = \text{Free}_T(\gamma) \cup \text{Free}_T(\phi) \);

*Proof.* The proof is by structural induction on \( \phi \).

1. If \( \phi \) is a first order literal, this follows at once from rule **PS-lit**.

2. If \( \phi = \psi_1 \lor \psi_2 \), by induction hypothesis we have that

\[
\forall \vec{v}(\gamma \land \psi_1 \rightarrow \psi_1) | \gamma \land \psi_1 \vdash \psi_1 \text{ and } \forall \vec{v}(\gamma \land \psi_2 \rightarrow \psi_2) | \gamma \land \psi_2 \vdash \psi_2 \text{ are provable. But then we can prove } \forall \vec{v}(\gamma \rightarrow \phi_1 \lor \phi_2) | \gamma \vdash \phi \text{ as follows:}
\]

\[(a) \ \forall \vec{v}(\gamma \land \psi_1 \rightarrow \psi_1) | \gamma \land \psi_1 \vdash \psi_1 \text{ (Derived before)} \]
\[(b) \ \forall \vec{v}(\gamma \land \psi_2 \rightarrow \psi_2) | \gamma \land \psi_2 \vdash \psi_2 \text{ (Derived before)} \]
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(c) \( \gamma \land \psi_1 \vdash \psi_1 \) (PS-ent, from (a), because \( \models \forall \bar{v}((\gamma \land \psi_1) \rightarrow \psi_1) \) in First Order Logic)

(d) \( \gamma \land \psi_2 \vdash \psi_2 \) (PS-ent, from (b), because \( \models \forall \bar{v}((\gamma \land \psi_2) \rightarrow \psi_2) \) in First Order Logic)

(e) \( \forall \bar{v}(\gamma \leftrightarrow (\gamma \land \psi_1) \lor (\gamma \land \psi_2)) \land \gamma \vdash \psi_1 \lor \psi_2 \) (PS-V, from (c) and (d))

(f) \( \forall \bar{v}(\gamma \rightarrow (\psi_1 \lor \psi_2)) \land \gamma \vdash \psi_1 \lor \psi_2 \) (PS-ent: from (e), because \( \forall \bar{v}(\gamma \rightarrow (\psi_1 \lor \psi_2)) \) entails \( \forall \bar{v}(\gamma \leftrightarrow (\gamma \land \psi_1) \lor (\gamma \land \psi_2)) \) in First Order Logic).

3. If \( \phi \) is \( \psi_1 \land \psi_2 \), by induction hypothesis we have that \( \forall \bar{v}(\gamma \rightarrow \psi_1) \land \gamma \vdash \psi_1 \) and \( \forall \bar{v}(\gamma \rightarrow \psi_2) \land \gamma \vdash \psi_2 \) are provable. But then

(a) \( \forall \bar{v}(\gamma \rightarrow \psi_1) \land \gamma \vdash \psi_1 \) (derived before)

(b) \( \forall \bar{v}(\gamma \rightarrow \psi_2) \land \gamma \vdash \psi_2 \) (derived before)

(c) \( \forall \bar{v}(\gamma \rightarrow \psi_1), \forall \bar{v}(\gamma \rightarrow \psi_2) \land \gamma \vdash \psi_1 \land \psi_2 \) (PS-\( \land \), (a), (b))

(d) \( \forall \bar{v}(\gamma \rightarrow \psi_1 \land \psi_2) \land \gamma \vdash \psi_1 \land \psi_2 \) (PS-ent, (c))

as required.

4. If \( \phi \) is \( \exists x \psi \), by induction hypothesis we have that
\( \forall \bar{v} \forall x(((\exists x \gamma) \land \psi) \rightarrow \psi) \land (\exists x \gamma) \lor \psi \vdash \psi \) is provable. But then

(a) \( \forall \bar{v} \forall x(((\exists x \gamma) \land \psi) \rightarrow \psi) \land (\exists x \gamma) \land \psi \vdash \psi \) (derived before)

(b) \( \forall \bar{v} \forall x(((\exists x \gamma) \land \psi) \rightarrow \psi) \land (\exists x \gamma) \lor \psi \vdash \psi \) (PS-ent, from (a))

(c) \( \forall \bar{v} \forall x((\exists x \gamma) \land \psi) \rightarrow \exists x \gamma) \land \gamma \vdash \exists x \psi \) (PS-\( \exists \), from (b))

(d) \( \forall \bar{v}((\exists x \gamma) \land (\exists x \psi)) \rightarrow \exists x \gamma) \land \gamma \vdash \exists x \psi \) (PS-ent, from (c))

(e) \( \forall \bar{v}(\gamma \rightarrow \exists x \psi) \land \gamma \vdash \psi \) (PS-ent, from (d))

as required, where the last passage uses the fact that \( \forall \bar{v}(\gamma \rightarrow \exists x \psi) \models \forall \bar{v}(((\exists x \gamma) \land (\exists x \psi)) \rightarrow \exists x \gamma) \) in First Order Logic.

5. If \( \phi \) is \( \forall x \psi \), by induction hypothesis we have that
\( \forall \bar{v} \forall x((\exists x \gamma) \rightarrow \psi) \land \exists x \gamma \lor \psi \vdash \psi \) is provable. But then

(a) \( \forall \bar{v} \forall x((\exists x \gamma) \rightarrow \psi) \land \exists x \gamma \lor \psi \vdash \psi \) (derived before)

(b) \( \forall \bar{v} \forall x((\exists x \gamma) \rightarrow \psi), \forall \bar{v}(\exists x \gamma \leftrightarrow \exists x \gamma) \land \gamma \vdash \forall x \psi \) (PS-\( \forall \), from (a))

(c) \( \forall \bar{v} \forall x((\exists x \gamma) \rightarrow \psi) \land \gamma \vdash \forall x \psi \) (PS-ent, from (c))

(d) \( \forall \bar{v}(\gamma \rightarrow \forall x \psi) \land \gamma \vdash \forall x \psi \) (PS-ent, from (d))
where the last two passages hold because $\forall \bar{v}(\exists x \gamma \leftrightarrow \exists x \gamma)$ is valid and because $\forall \bar{v}(\gamma \rightarrow \forall x \psi)$ entails $\forall \bar{v}\forall x((\exists x \gamma) \rightarrow \psi)$ in first order logic, where $\bar{v} = \text{Free}_T(\gamma) \cup \text{Free}_T(\psi)$ (and, therefore, if $x$ is free in $\gamma$ then $x$ is in $\bar{v}$).

\[ \square \]

**Theorem 5.3.7** (Soundness). Suppose that $\Gamma \models \gamma \vdash \phi$ is provable. Then it is valid.

**Proof.** If $S$ is a provable sequent then there exists a proof $S_1 \ldots S_nS$ for it. Then we go by induction of the length $n$ of this proof:

**Base case:** Suppose that the proof has length 0. Then $S$ is an instance of $\text{PS-lit}$, of $\text{PS-inc}$, of $\text{PS-exc}$ or of $\text{PS-ind}$. Assume first that it is the former, that is, that $S = \forall \bar{v}(\gamma \rightarrow \phi) \mid \gamma \vdash \phi$

for some first order $\gamma$ and some first order literal $\phi$, where $\bar{v} = \text{Free}_T(\gamma) \cup \text{Free}_T(\phi)$ and $\phi$ has no parameter variables. Now suppose that $M \models h(\forall \bar{v}(\gamma \rightarrow \phi))$; then, by definition, if $s$ is an assignment over team variables such that $M \models h \cup s \gamma$ then $M \models s \phi$. Therefore, by $\text{ES-lit}$, $M \models \gamma_{(s)} \phi$ in Entailment Semantics, as required.

The other cases are treated in an entirely similar manner.

**Induction case:** Let $S_1S_2 \ldots S_nS$ be our proof. For each $i \leq n$ we have that $S_1 \ldots S_i$ is a valid proof for $S_i$, and hence by induction hypothesis that $S_i$ is valid. Now let us consider which rule $r$ was been used to derive $S$ from $S_1 \ldots S_n$:

1. If $r$ was $\text{PS-lit}$ or $\text{PS-ind}$ then $(S)$ is a proof for $S$ already, and hence by our base case $S$ is valid;
2. If $r$ was $\text{PS-\lor}$ then $S$ is $\Gamma_1, \Gamma_2, \forall \bar{v}(\gamma \leftrightarrow (\gamma_1 \lor \gamma_2)) \mid \gamma \vdash \phi_1 \lor \phi_2$, and there exist two $i, j \leq n$ such that $S_i = (\Gamma_1 \mid \gamma \vdash \phi_1)$ and $S_j = (\Gamma_2 \mid \gamma_2 \vdash \phi_2)$. By induction hypothesis, these sequents are valid.

Now suppose that $M \models \forall \bar{v}(\gamma \leftrightarrow (\gamma_1 \lor \gamma_2))$. Then, since $M \models \forall \bar{v}(\gamma \leftrightarrow (\gamma_1 \lor \gamma_2))$, we have that $M \models h \forall \bar{v}(\gamma \leftrightarrow (\gamma_1 \lor \gamma_2))$, and, analogously, since $M \models h \forall \bar{v}(\gamma \leftrightarrow (\gamma_1 \lor \gamma_2))$, and therefore by rule $\text{ES-\lor}$ we have that $M \models \forall \bar{v}(\gamma \leftrightarrow (\gamma_1 \lor \gamma_2))$, as required.
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3. If \( r \) was \( \text{PS-} \land \) then \( S_n \) is of the form \( \Gamma_1, \Gamma_2 \mid \gamma \vdash \phi_1 \land \phi_2 \) and, by induction hypothesis, \( \Gamma_1 \mid \gamma \vdash \phi_1 \) and \( \Gamma_2 \mid \gamma \vdash \phi_2 \) are valid. Now suppose that \( M \models_h \Gamma_1, \Gamma_2 \); then \( M \models_{\gamma(h)} \phi_1 \) and \( M \models_{\gamma(h)} \phi_2 \), and therefore \( M \models_{\gamma(h)} \phi_1 \land \phi_2 \) by \( \text{ES-} \land \).

4. If \( r \) was \( \text{PS-} \exists \) then \( S_n \) is of the form \( \Gamma, \forall \vec{v}(\exists x \gamma' \leftrightarrow \exists x \gamma) \mid \gamma \vdash \exists x \phi \), where \( \Gamma \mid \gamma' \vdash \phi \) is valid by induction hypothesis. Now suppose that \( M \models_h \Gamma, \forall \vec{v}(\exists x \gamma \leftrightarrow \exists x \gamma') \); then \( M \models_{\gamma'(h)} \phi \) and \( M \models_h \forall \vec{v}(\exists x \gamma \leftrightarrow \exists x \gamma') \), and therefore \( M \models_{\gamma(h)} \exists x \phi \) by rule \( \text{ES-} \exists \).

5. If \( r \) was \( \text{PS-} \forall \) then \( S_n \) is of the form \( \Gamma, \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma) \mid \gamma \vdash \forall x \phi \), where \( \Gamma \mid \gamma' \vdash \phi \) is valid by induction hypothesis. Now, suppose that \( M \models_h \Gamma, \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma) \). Then \( M \models_{\gamma'(h)} \phi \), and furthermore \( M \models_h \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma) \). Therefore, by rule \( \text{ES-} \forall \), \( M \models_{\gamma(h)} \forall x \phi \), as required.

6. If \( r \) was \( \text{PS-sent} \) then \( S_n \) is of the form \( \Gamma' \mid \gamma \vdash \phi \), where \( \Gamma \mid \gamma \vdash \phi \) is valid by induction hypothesis and where \( \bigwedge \Gamma \models \bigwedge \Gamma' \) holds in first order logic. Now suppose that \( M \models_h \Gamma' \); then \( M \models_h \Gamma \), and hence \( M \models_{\gamma(h)} \phi \), as required.

7. If \( r \) was \( \text{PS-depar} \) then \( S_n \) is of the form \( \exists p \bigwedge \Gamma \mid \gamma \vdash \phi \), where \( \Gamma \mid \gamma \vdash \phi \) holds by induction hypothesis and where the parameter variable \( p \) does not occur free in \( \gamma \). Now suppose that \( M \models_h \exists p \bigwedge \Gamma \); then there exists an element \( m \in \text{Dom}(M) \) such that, for \( h' = h[m/p] \), \( M \models_{h'} \Gamma \). Then \( M \models_{\gamma'(h')} \phi \); but as \( p \) does not occur free in \( \gamma \) we then have, by Proposition 5.2.4, that \( M \models_{\gamma(h')} \phi \) as required.

8. If \( r \) was \( \text{PS-split} \) then \( S_n \) is of the form \( (\bigwedge \Gamma_1) \lor (\bigwedge \Gamma_2) \mid \gamma \vdash \phi \), where \( \Gamma_1 \mid \gamma \vdash \phi \) and \( \Gamma_2 \mid \gamma \vdash \phi \) by induction hypothesis. Now suppose that \( M \models_h (\bigwedge \Gamma_1) \lor (\bigwedge \Gamma_2) \). Then \( M \models_h \Gamma_1 \) or \( M \models_h \Gamma_2 \); and in either case, \( M \models_{\gamma(h)} \phi \), as required.

\( \square \)

In order to prove completeness, we first need a lemma:

**Lemma 5.3.8.** Suppose that \( M \models_{\gamma(h)} \phi \). Then there exists a finite \( \Gamma \) such that \( \Gamma \mid \gamma \vdash \phi \) is provable and such that \( M \models_h \Gamma \).

**Proof.** The proof is by structural induction on \( \phi \).

1. If \( \phi \) is a first order literal or an inclusion/exclusion/independence atom, this follows immediately.
2. If \( \phi \) is \( \psi_1 \lor \psi_2 \) and \( M \models_{\gamma(h)} \phi \) then, by definition, there exists an assignment \( h' \) extending \( h \) and two first order formulas \( \gamma_1, \gamma_2 \) such that \( M \models_{\gamma_1(h') \psi_1}, M \models_{\gamma_2(h') \psi_2} \) and \( M \models_{h'} \forall \gamma (\gamma_1 \lor \gamma_2) \). Let \( \vec{p} \) be the tuple of parameters in \( \text{Dom}(h') \backslash \text{Dom}(h) \); now, by induction hypothesis we have that there exist \( \Gamma_1 \) and \( \Gamma_2 \) such that \( \Gamma_1 \mid \gamma_1 \vdash \psi_1 \) and \( \Gamma_2 \mid \gamma_2 \vdash \psi_2 \) are provable, and such that furthermore \( M \models_{h'} \Gamma_1 \) and \( M \models_{h'} \Gamma_2 \).

But then the following is a correct proof:

\[
\begin{align*}
(a) & \quad \Gamma_1 \mid \gamma_1 \vdash \psi_1 \quad \text{(Derived before)} \\
(b) & \quad \Gamma_2 \mid \gamma_2 \vdash \psi_2 \quad \text{(Derived before)} \\
(c) & \quad \Gamma_1, \Gamma_2, \forall \vec{v}(\gamma \leftrightarrow \gamma_1 \lor \gamma_2) \mid \gamma \vdash \phi \quad \text{(PS-\lor, (a), (b))} \\
(d) & \quad \exists \vec{p} \land \Gamma_1 \land \bigwedge \Gamma_2 \land \forall \vec{v}(\gamma \leftrightarrow \gamma_1 \lor \gamma_2)) \mid \gamma \vdash \phi \quad \text{(PS-depar, (c))}^5
\end{align*}
\]

Finally, \( M \models_{h'} \exists \vec{p} \land \Gamma_1 \land \bigwedge \Gamma_2 \land \forall \vec{v}(\gamma \leftrightarrow \gamma_1 \lor \gamma_2) \), as required, because there exists a tuple of elements \( \vec{m} \) such that \( h[\vec{m} / \vec{p}] = h' \).

3. If \( \phi \) is \( \psi_1 \land \psi_2 \) and \( M \models_{\gamma(h)} \phi \), then \( M \models_{\gamma(h)} \psi_1 \) and \( M \models_{\gamma(h)} \psi_2 \). Then, by induction hypothesis, there exist \( \Gamma_1 \) and \( \Gamma_2 \) such that \( \Gamma_1 \mid \gamma \vdash \psi_1 \) and \( \Gamma_2 \mid \gamma \vdash \psi_2 \) are provable and such that \( M \models_{h'} \Gamma_1 \Gamma_2 \). Then by rule \( \text{PS-\land} \), \( \Gamma_1 \Gamma_2 \mid \gamma \vdash \psi_1 \land \psi_2 \), as required.

4. If \( \phi \) is \( \exists x \psi \) and \( M \models_{\gamma(h)} \phi \), then there exists a tuple \( \vec{p} \) of parameter variables not in the domain of \( h \), a tuple \( \vec{m} \) of elements of the model and a formula \( \gamma' \) such that, for \( h' = h[\vec{m} / \vec{p}] \), \( M \models_{\gamma'(h')} \psi \) and \( M \models_{h'} \forall \vec{v}(\exists x \gamma' \leftrightarrow \exists x \gamma) \). By induction hypothesis, we then have a \( \Gamma' \) such that \( \Gamma' \mid \gamma' \vdash \psi \) and \( M \models_{h'} \Gamma' \).

Then the following is a valid proof:

\[
\begin{align*}
(a) & \quad \Gamma' \mid \gamma' \vdash \psi \quad \text{(Derived before)} \\
(b) & \quad \Gamma', \forall \vec{v}(\exists x \gamma' \leftrightarrow \exists x \gamma) \mid \gamma \vdash \exists x \psi \quad \text{(PS-\exists)} \\
(c) & \quad \exists \vec{p} \land \Gamma' \land \bigwedge \vec{v}(\exists x \gamma' \leftrightarrow \exists x \gamma)) \mid \gamma \vdash \exists x \psi \quad \text{(PS-depar)}
\end{align*}
\]

Furthermore, \( M \models_{h'} \exists \vec{p} \land \Gamma' \land \bigwedge \vec{v}(\exists x \gamma' \leftrightarrow \exists x \gamma) \), as required.

5. If \( \phi \) is \( \forall x \psi \) and \( M \models_{\gamma(h)} \phi \), then there exists a tuple \( \vec{p} \) of parameter variables not in the domain of \( h \), a tuple \( \vec{m} \) of elements of the model and a formula \( \gamma' \) such that \( M \models_{\gamma'(h')} \psi \) and \( M \models_{h'} \forall \vec{v}(\gamma' \leftrightarrow \exists x \gamma) \), where

\[\text{To be entirely formal, this passage consists of } |\vec{p}| \text{ distinct applications of } \text{PS-depar}, \text{ all of which are correct because none of the parameters in } \vec{p} \text{ appear in } \gamma.\]
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\[ h' = h[\vec{m}/\vec{p}] \]  

By induction hypothesis, we can then find a \( \Gamma' \) such that \( \Gamma' \vdash \gamma' \psi \) is provable and \( M \models_{h'} \Gamma' \).

Then the following is a valid proof:

(a) \( \Gamma' \vdash \gamma' \psi \) (Derived before)

(b) \( \Gamma', \forall \vec{v}(\gamma' \iff \exists x \gamma) \vdash \forall \vec{x} \psi \) (PS-\( \forall \))

(c) \( \exists \vec{v}(\bigwedge \Gamma' \land \forall \vec{v}(\gamma' \iff \exists x \gamma)) \vdash \gamma \land \forall \vec{x} \psi \) (PS-depar)

And, once again, the assignment \( h \) satisfies the antecedent of the last sequent, as required.

\[ \square \]

The completeness of our proof system follows from the above lemma and from the compactness and the Löwenheim-Skolem theorem for First Order Logic:

**Theorem 5.3.9 (Completeness).** Suppose that \( \Gamma \vdash \phi \) is valid, where \( \Gamma \) is finite. Then it is provable.

**Proof.** Since \( \Gamma \vdash \phi \) is valid, for any first order model \( M \) over the signature of \( \Gamma, \gamma \) and \( \phi \) and for all \( h \) such that \( M \models_{h \Gamma} \) we have that \( M \models_{\gamma(h)} \phi \), and hence by the lemma that \( M \models_{h \Gamma} \Gamma_{M,h} \) for some finite \( \Gamma_{M,h} \) such that \( \Gamma_{M,h} \vdash \phi \) is provable.

Then consider the first order, countable\(^6\) theory

\[ T = \{ \bigwedge \Gamma \} \cup \{ \neg \bigwedge \Gamma_{M,h} : M \text{ is a countable model, } \]

\[ h \text{ is an assignment s.t. } M \models_{h \Gamma} \}. \]

This theory is unsatisfiable. Indeed, suppose that \( M_0 \) is a model that satisfies \( \bigwedge \Gamma \) under the assignment \( h_0 \); then, by the Löwenheim-Skolem theorem, there exists a countable elementary submodel \( (M'_0, h'_0) \) of \( (M_0, h_0) \).

Now, \( M'_0 \models_{h'_0} \Gamma \) and \( M'_0 \) is countable, and hence by definition \( M'_0 \models_{h'_0} \Gamma_{M'_0,h'_0} \).

But then \( M_0 \models_{h_0} \Gamma_{M'_0,h'_0} \) too, and therefore \( M_0 \) is not a model of \( T \).

By the compactness theorem, this implies that there exists a finite subset \( T_0 = \{ \neg \bigwedge \Gamma_{M_1,h_1}, \ldots, \neg \bigwedge \Gamma_{M_n,h_n} \} \) of \( T \) such that \( \{ \bigwedge \Gamma \} \cup T_0 \) is unsatisfiable,

\(^6\)The fact that it is countable follows at once from the fact that it is a first order theory over a countable vocabulary.
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that is, such that

$$\Gamma \models (\bigwedge \Gamma_{M_1,h_1}) \lor \ldots \lor (\bigwedge \Gamma_{M_n,h_n}).$$

Now, for each $i$, $\Gamma_{M_i,h_i} \models \gamma \vdash \phi$ can be proved. Therefore, by rule $\textbf{PS-split}$, we have that $(\bigwedge \Gamma_{M_1,s_1}) \lor \ldots \lor (\bigwedge \Gamma_{M_n,s_n}) \models \gamma \vdash \phi$ is also provable; and finally, by rule $\textbf{PS-ent}$ we can prove that $\Gamma \models \gamma \vdash \phi$, as required. 

Thus, we succeeded in designing a proof system which is sound and complete with respect to our semantics; and as we saw, with respect to finite models our semantics is identical to the standard one, and furthermore even with respect to infinite models it is a natural generalization of Team Semantics.

Using essentially the same method, it is also possible to prove a “compactness” result for our semantics:

**Theorem 5.3.10.** Suppose that $\Gamma \models \gamma \vdash \phi$ is valid. Then there exists a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \gamma \vdash \phi$ is provable (and valid).

**Proof.** Let $\kappa = \max(|\Gamma|, \aleph_0)$, and consider the theory

$$T = \Gamma \cup \{ \neg \bigwedge \Gamma_{M,h} : |M| \leq \kappa, M \models \Gamma \}$$

where, as in the previous proof, $\Gamma_{M,h}$ is a finite theory such that $M \models_h \Gamma_{M,h}$ and such that $\Gamma_{M,h} \models \gamma \vdash \phi$ is provable in our system.

Then $T$ is unsatisfiable: indeed, if $T$ had a model then it would have a model $(M, h)$ of cardinality at most $\kappa$, and since that model would satisfy $\Gamma$ it would satisfy $\Gamma_{M,h}$ too, which contradicts our hypothesis.

Hence, by the compactness theorem, there exists a finite set $\{ \bigwedge \Gamma_{M_1,h_1}, \ldots, \bigwedge \Gamma_{M_n,h_n} \}$ and a finite $\Gamma_0 \subseteq \Gamma$ such that

$$\Gamma_0 \models \bigwedge \Gamma_{M_1,h_1} \lor \ldots \lor \bigwedge \Gamma_{M_n,h_n}.$$ 

But by rule $\textbf{PS-split}$, we have that $\bigwedge \Gamma_{M_1,h_1} \lor \ldots \lor \bigwedge \Gamma_{M_n,h_n} \models \gamma \vdash \phi$ is provable, and hence by rule $\textbf{PS-ent}$ $\Gamma_0 \models \gamma \vdash \phi$ is also provable, as required. 

5.4 Adding More Teams

The proof system that we developed in the previous section is, as we saw, sound and complete with respect to its intended semantics. However, this semantics
is perhaps rather weak: all that we know is that the teams which correspond to parametrized first order formulas belong in our general models.

Rather than adding more and more axioms to our proof system in order to guarantee the existence of more teams, in this section we will attempt to separate our assumptions about team existence from our main proof system. This will allow us to make our formalism modular: depending on our needs, we may want to assume the existence of more or of less teams in our general model.

The natural language for describing assertions about the existence of relations is of course, existential second order logic. The following definitions show how it can be used for our purposes:

**Definition 5.4.1.** A relation existence theory $\Theta$ is a set of existential second order sentences of the form $\exists \vec{R} \phi(\vec{R})$, where $\phi$ is first order.

**Definition 5.4.2.** Let $(M, \mathcal{G})$ be a general model, and let $\Theta$ be a relation existence theory. Then $(M, \mathcal{G})$ is $\Theta$-closed if and only if for all $\exists \vec{R} \phi(\vec{R})$ in $\Theta$ there exists a tuple of teams $\vec{X} \in \mathcal{G}$ such that $M \models \phi[\vec{Rel}(\vec{X})/\vec{R}]$.

**Definition 5.4.3.** Let $\Gamma \vdash \gamma \vdash \phi$ be a sequent and let $\Theta$ be a relation existence theory. Then $\Gamma \vdash \gamma \vdash \phi$ is valid if and only if for all $\Theta$-closed models $(M, \mathcal{G})$ and all parameter assignments $h$ with domain $\text{Free}(\Gamma) \cup \text{Free}(\gamma)$ such that $M \models h \Gamma$ it holds that

$$(M, \mathcal{G}) \models \|\gamma\|_h \phi.$$
such that $M \models_h \Gamma$. By definition, there exists a tuple of teams $\vec{X} \in G$ such that $M \models \bigwedge \Gamma_{1}[\vec{X}]$. Now let $M'$ be $M[\vec{R}\vec{1}(\vec{X})/\vec{S}]$: since $\vec{X}$ is in $G$, it is not difficult to see that $(M', G)$ is a general model. Furthermore, it is $\Theta$-closed, $M' \models \Gamma_{1}$, and $M' \models_h \Gamma$. Hence, $(M', G) \models \phi$; but since the relation symbols $\vec{S}$ do not occur in $\gamma$ or in $\phi$, this implies that $(M, G) \models \phi$.

In order to prove completeness, we first need a definition and a simple lemma.

**Definition 5.4.5.** Let $\Theta$ be a relation existence theory. Then $\Theta^{FO}$ is the theory $\{\theta_{i}[\vec{S}_{i}/\vec{R}] : \exists \vec{R}\theta_{i}(\vec{R}) \in \Theta\}$, where the tuples of symbols $\vec{S}_{i}$ are all disjoint and otherwise unused.

**Lemma 5.4.6.** Let $\Theta$ be a relation existence theory and let $M$ be a model such that $M \models \Theta^{FO}$. Then the least general model $(M, L)$ over it is $\Theta$-closed.

**Proof.** Consider any $\exists \vec{R}\theta(\vec{R}) \in \Theta$. Then $M \models \theta(\vec{S})$, for some tuple of relation symbols $\vec{S}$ in the signature of $M$. Then, the teams $\vec{X}$ associated to the corresponding relations are in $L$, and for these teams we have that $M \models \theta[\vec{R}\vec{1}(\vec{X})/\vec{R}]$, as required.

**Theorem 5.4.7** (Completeness). Suppose that $\Gamma \models \gamma \vdash \phi$ is $\Theta$-valid. Then it is provable in our proof system plus $PS-\Theta$.

**Proof.** Let $M$ be any first order model satisfying $\Theta^{FO}$, where we assume that the relation symbols used in the construction of $\Theta^{FO}$ do not occur in $\Gamma$, in $\gamma$ or in $\phi$. Then, by the lemma, $(M, L)$ is $\Theta$-closed, and this implies that, for all assignments $h$ such that $M \models_h \Gamma$, $M \models \phi$.

Therefore, $\Theta^{FO}, \Gamma \models \gamma \vdash \phi$ is valid; and hence, for some finite $\Delta \subseteq \Theta^{FO}$ it holds that $\Delta, \Gamma \models \gamma \vdash \phi$ is provable. Now we can get rid of $\Delta$ through repeated applications of rule $PS-\Theta$ and, therefore, prove that $\Gamma \models \gamma \vdash \phi$, as required.

**5.5 Conclusions**

We began this chapter by defining a general semantics for a logic of imperfect information. Then we proved that – owing to the relationships between it and existential second order logic – in order to study validity with respect to this semantics it suffices to examine least general models. We then showed that, because of the correspondence between teams in least general models and first order formulas with parameters, we could limit ourselves to study entailments between first-order team-defining formulas and independence logic formulas. Finally, we developed a sound and complete proof system for this semantics,
and we showed that this system can easily be strengthened by assuming the existence of more teams.

As we said, the correspondence between our logic and existential second order logic is of essential importance for the construction which we described: extending our approach to such logics as team logic or intuitionistic dependence logic promises to be nontrivial, although certainly not impossible. The relationship between our approach and the one developed by Kontinen and Väänänen in [52] is also certainly worth investigating.

Furthermore, Entailment Semantics – the key ingredient of our construction, and our “bridge” between General Model Semantics and the proof system – is, as we wrote, of independent interest for a more syntactic approach to the study of dependence and independence, and more in general to the study of this interesting family of logics.