UNIVERSITY OF AMSTERDAM

UvA-DARE (Digital Academic Repository)

Binary aggregation with integrity constraints

Grandi, U.

Publication date
2012

Link to publication

Citation for published version (APA):
Grandi, U. (2012). *Binary aggregation with integrity constraints*. [Thesis, fully internal, Universiteit van Amsterdam]. Institute for Logic, Language and Computation.

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: https://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, P.O. Box 19185, 1000 GD Amsterdam, The Netherlands. You will be contacted as soon as possible.

UvA-DARE is a service provided by the library of the University of Amsterdam (https://dare.uva.nl)

Download date:12 Nov 2026

Chapter 3

Paradoxes of Aggregation

Most work in Social Choice Theory started with the observation of paradoxical situations. From the Marquis de Condorcet (1785) and Jean-Charles de Borda (1781) to more recent American court cases (Kornhauser and Sager, 1986), a wide collection of paradoxes have been analysed and studied in the literature on Social Choice Theory (see, e.g., Nurmi, 1999). In this chapter we present some of the most well-known paradoxes that arise from the use of the majority rule in different contexts, and we show how they can be expressed in binary aggregation as instances of our Definition 2.1.9. Such a uniform representation of the most important paradoxes in Social Choice Theory enables us to make a crucial observation concerning the syntactic structure of paradoxical integrity constraints: they all feature a disjunction of literals of size at least 3. This observation will give rise to one of the main theorems of this dissertation (Theorem 4.4.8).

In Section 3.1, we introduce one of the most notable paradoxes in Social Choice Theory, the Condorcet paradox, and we show how settings of preference aggregation can be seen as instances of binary aggregation by devising a suitable integrity constraint. Section 3.2 repeats this construction for the framework of judgment aggregation and for the paradoxical example which gave rise to this area of research, namely the doctrinal paradox. In Section 3.3 we deal with the Ostrogorski paradox, in which a paradoxical feature of representative majoritarian systems is analysed. In Section 3.5 we introduce two further paradoxes generated by the use of the majority rule on multiple issues: the paradox of divided government and the paradox of multiple elections. In Section 3.6 we conclude.

3.1 The Condorcet Paradox and Preference Aggregation

During the Enlightment period in France, several active scholars dedicated themselves to the problem of collective choice, and in particular to the creation of

new procedures for the election of candidates. Although these are not the first documented studies of the problem of social choice (McLean and Urken, 1995), Marie Jean Antoine Nicolas de Caritat, the Marquis de Condorcet, was the first to point out a crucial problem of the most basic voting rule that was being used, the majority rule (Condorcet, 1785). The paradox he discovered, that now comes under his name, is explained in the following paragraphs:

> **Condorcet Paradox**. Three individuals need to decide on the ranking of three alternatives $\{\triangle, \bigcirc, \square\}$. Each individual expresses her own ranking in the form of a linear order, i.e., an irreflexive, transitive and complete binary relation over the set of alternatives. The collective outcome is then aggregated by pairwise majority: an alternative is preferred to a second one if and only if a majority of the individuals prefer the first alternative to the second. Consider the profile described in Table 3.1.

$$\begin{array}{c} \triangle <_1 \bigcirc <_1 \square \\ \square <_2 \triangle <_2 \bigcirc \\ \bigcirc <_3 \square <_3 \triangle \\ \hline \triangle < \bigcirc < \square < \triangle \end{array}$$

Table 3.1: The Condorcet paradox.

> When we compute the outcome of the pairwise majority rule on this profile, we notice that there is a majority of individuals preferring the circle to the triangle ($\triangle < \bigcirc$); that there is a majority of individuals preferring the square to the circle ($\bigcirc < \square$); and, finally, that there is a majority of individuals preferring the triangle to the square ($\square < \triangle$). The resulting outcome fails to be a linear order, giving rise to a circular collective preference between the alternatives.

Condorcet's paradox was rediscovered in the second half of the XXth century while a whole theory of *preference aggregation* was being developed, starting with the work of Black (1958) and Arrow's celebrated result (Arrow, 1963). In this section, we review the framework of preference aggregation, we show how this setting can be embedded into the framework of binary aggregation with integrity constraints, and we show how the Condorcet paradox can be seen as an instance of our general definition of paradox (Definition 2.1.9).

3.1.1 Preference Aggregation

The framework of *preference aggregation* (see, e.g., Gaertner, 2006) considers a finite set of individuals $\mathcal{N}$ expressing preferences over a finite set of alternatives $\mathcal{X}$. A preference relation is represented by a binary relation over $\mathcal{X}$. Preference relations are traditionally assumed to be *weak orders*, i.e., reflexive, transitive and complete binary relations. In some cases, in order to simplify the framework, preferences are assumed to be *linear orders*, i.e., irreflexive, transitive and complete binary relations. In the first case, we write aRb for "alternative a is preferred to alternative b or it is equally preferred as b", while in the second case aPb stands for "alternative a is strictly preferred to b". In the sequel we shall assume that preferences are represented as linear orders. We refer to Chapter 5 for a more detailed presentation of other assumptions in preference aggregation.

Each individual submits a linear order P_i, forming a profile $\boldsymbol{P} = (P_1, \ldots, P_{|\mathcal{N}|})$. Let $\mathcal{L}(\mathcal{X})$ denote the set of all linear orders on $\mathcal{X}$. Aggregation procedures in this framework are called *social welfare functions* (SWFs):

Definition 3.1.1. Given a finite set of individuals $\mathcal{N}$ and a finite set of alternatives $\mathcal{X}$, a *social welfare function* is a function $F : \mathcal{L}(\mathcal{X})^{\mathcal{N}} \to \mathcal{L}(\mathcal{X})$.

Note that a SWF is defined for every logically possible profile of linear orders, a condition that traditionally goes under the name of universal domain, and that it always outputs a linear order. This last condition was given the name of "collective rationality" by Arrow (1963). As we have seen in Table 3.1, the Condorcet paradox proves that the pairwise majority rule is not a SWF because, in Arrow's words, it fails to be "collectively rational". In the following section we will formalise this observation by devising an integrity constraint that encodes the assumptions underlying Arrow's framework of preference aggregation.

3.1.2 Translation

Given a preference aggregation problem defined by a set of individuals $\mathcal{N}$ and a set of alternatives $\mathcal{X}$, let us now consider the following setting for binary aggregation. Define a set of issues $\mathcal{I}_{\mathcal{X}}$ as the set of all pairs (a, b) in $\mathcal{X}$. The domain $\mathcal{D}_{\mathcal{X}}$ of aggregation is therefore $\{0, 1\}^{|\mathcal{X}|^2}$. In this setting, a binary ballot B corresponds to a binary relation P over $\mathcal{X}$: $B_{(a,b)} = 1$ if and only if a is in relation to b (aPb). Given this representation, we can associate with every SWF for $\mathcal{X}$ and $\mathcal{N}$ an aggregation procedure that is defined on a subdomain of $\mathcal{D}_{\mathcal{X}}^{\mathcal{N}}$. We now characterise this domain as the set of models of a suitable integrity constraint.

Using the propositional language $\mathcal{L}_{PS}$ constructed over the set $\mathcal{I}_{\mathcal{X}}$, we can express properties of binary ballots in $\mathcal{D}_{\mathcal{X}}$. In this case the language consists of $|\mathcal{X}|^2$ propositional symbols, which we shall call p_{ab} for every issue (a, b). As we already anticipated in Example 2.1.4, the properties of linear orders can be

enforced on binary ballots using the following set of integrity constraints, which we shall call $\mathrm{IC}_<$:[1]

Irreflexivity: $\neg p_{aa}$ for all $a \in \mathcal{X}$

Completeness: $p_{ab} \vee p_{ba}$ for all $a \neq b \in \mathcal{X}$

Transitivity: $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$ for $a, b, c \in \mathcal{X}$ pairwise distinct

Note that the size of this set of integrity constraints is polynomial in the number of alternatives in $\mathcal{X}$. It is now straightforward to see that every SWF corresponds to an aggregation procedure that is collectively rational with respect to $\mathrm{IC}_<$ and *vice versa*.[2]

In case preferences are expressed using weak orders rather than linear orders, it is sufficient to modify the integrity constraint $\mathrm{IC}_<$ to obtain a similar correspondence between SWFs and aggregation procedures. Recall that a weak order is a reflexive, transitive and complete binary relation over $\mathcal{X}$. Let therefore $\mathrm{IC}_\leqslant$ be the following set of integrity constraints:

Reflexivity: p_{aa} for all $a \in \mathcal{X}$

Completeness: $p_{ab} \vee p_{ba}$ for all $a \neq b \in \mathcal{X}$

Transitivity: $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$ for $a, b, c \in \mathcal{X}$ pairwise distinct

3.1.3 The Condorcet Paradox in Binary Aggregation

The translation presented in the previous section enables us to express the Condorcet paradox in terms of Definition 2.1.9. Let $\mathcal{X} = \{\triangle, \bigcirc, \square\}$ and let $\mathcal{N}$ contain three individuals. Consider the profile $\boldsymbol{B}$ for $\mathcal{I}_\mathcal{X}$ described in Table 3.2, where we have omitted the values of the reflexive issues $(\triangle, \triangle)$ (always 0 by $\mathrm{IC}_<$), and specified the value of only one of $(\triangle, \bigcirc)$ and $(\bigcirc, \triangle)$ (the other can be obtained by taking the opposite of the value of the first), and accordingly for the other alternatives. Every individual ballot in Table 3.2 satisfies $\mathrm{IC}_<$, but the outcome obtained using the majority rule *Maj* (which corresponds to pairwise majority in preference aggregation) does not satisfy $\mathrm{IC}_<$: the formula $p_{\triangle\bigcirc} \wedge p_{\bigcirc\square} \rightarrow p_{\triangle\square}$ is falsified by the outcome. Therefore, $(\mathit{Maj}, \boldsymbol{B}, \mathrm{IC}_<)$ is a paradox by Definition 2.1.9.

The integrity constraint $\mathrm{IC}_<$ can be further simplified for the case of 3 alternatives $\{a, b, c\}$. The formulas encoding the transitivity of binary relations are equivalent to just two positive clauses: The first one, $p_{ba} \vee p_{cb} \vee p_{ac}$, rules out the cycle $a<b<c<a$, and the second one, $p_{ab} \vee p_{bc} \vee p_{ca}$, rules out the opposite cycle $c<b<a<c$. That is, these constraints correspond exactly to the two Condorcet cycles that can be created from three alternatives.

[1] We will use the notation IC both for a single integrity constraint and for a set of formulas—in the latter case considering as the actual constraint the conjunction of all the formulas in IC.

[2] A technicality: to every SWF correspond *many* binary aggregation procedures, depending on how we extend the procedure outside of $\mathrm{Mod}(\mathrm{IC}_<)^\mathcal{N}$.

	△○	○□	△□
Agent 1	1	1	1
Agent 2	1	0	0
Agent 3	0	1	0
Maj	1	1	0

Table 3.2: Condorcet paradox in binary aggregation.

3.2 The Discursive Dilemma and Judgment Aggregation

The discursive dilemma emerged from the formal study of court cases that was carried out in recent years in the literature on law and economics, generalising the observation of a paradoxical situation known as the "doctrinal paradox" (Kornhauser and Sager, 1986, 1993). Such a setting was first given mathematical treatment by List and Pettit (2002), giving rise to an entirely new research area in Social Choice Theory known as *judgment aggregation*. Earlier versions of this paradox can be found in work by Guilbaud (1952) and Vacca (1922). We now describe one of the most common versions of the discursive dilemma:

> **Discursive Dilemma**. A court composed of three judges has to decide on the liability of a defendant under the charge of breach of contract. According to the law, the individual is liable if there was a valid contract and her behaviour was such as to be considered a breach of the contract. The court takes three majority decisions on the following issues: there was a valid contract (α), the individual broke the contract (β), the defendant is liable ($\alpha \wedge \beta$). Consider a situation like the one described in Table 3.3.

	α	β	$\alpha \wedge \beta$
Judge 1	yes	yes	yes
Judge 2	no	yes	no
Judge 3	yes	no	no
Majority	yes	yes	no

Table 3.3: The discursive dilemma.

> All judges are expressing consistent judgments: they accept the third proposition if and only if the first two are accepted. However, when aggregating the judgments using the majority rule we obtain an inconsistent outcome: even if there is a majority of judges who believe that

> there was a valid contract, and even if there is a majority of judges who believe that the individual broke the contract, the individual is considered *not liable* by a majority of the individuals.

In this section we review the framework of judgment aggregation (List and Puppe, 2009), and we provide a characterisation of judgment aggregation procedures as collectively rational procedures with respect to a suitable set of integrity constraints. This in turn enables us to show that the discursive dilemma is also an instance of our general definition of paradox. For a more detailed introduction to the framework of judgment aggregation we refer to Chapter 6.

3.2.1 Judgment Aggregation

Judgement aggregation (JA) considers problems in which a finite set of individuals $\mathcal{N}$ has to generate a collective judgment over a set of interconnected propositional formulas[3] (List and Puppe, 2009). Formally, given a finite propositional language $\mathcal{L}$, an *agenda* is a finite nonempty subset $\Phi \subseteq \mathcal{L}$ that does not contain any doubly-negated formulas and that is closed under complementation (i.e, $\alpha \in \Phi$ whenever $\neg\alpha \in \Phi$, and $\neg\alpha \in \Phi$ for every non-negated $\alpha \in \Phi$).

Each individual in $\mathcal{N}$ expresses a *judgment set* $J \subseteq \Phi$, as the set of those formulas in the agenda that she judges to be true. Every individual judgment set J is assumed to be *complete* (i.e., for each $\alpha \in \Phi$ either α or its complement are in J) and *consistent* (i.e., there exists an assignment that makes all formulas in J true). If we denote by $\mathcal{J}(\Phi)$ the set of all complete and consistent subsets of Φ, we can give the following definition:

Definition 3.2.1. Given a finite agenda Φ and a finite set of individuals $\mathcal{N}$, a *JA procedure* for Φ and $\mathcal{N}$ is a function $F : \mathcal{J}(\Phi)^{\mathcal{N}} \to 2^{\Phi}$.

Note that no additional requirement is imposed on the collective judgment set. A JA procedure is called *complete* if the judgment set it returns is complete on every profile. A JA procedure is called *consistent* if, for every profile, the outcome is a consistent judgment set.

3.2.2 Translation

Given a judgment aggregation framework defined by an agenda Φ and a set of individuals $\mathcal{N}$, let us now construct a setting for binary aggregation with integrity constraints that interprets it, generalising from our previous Example 2.1.5. Let the set of issues $\mathcal{I}_{\Phi}$ be equal to the set of formulas in Φ. The domain $\mathcal{D}_{\Phi}$ of aggregation is therefore $\{0,1\}^{|\Phi|}$. In this setting, a binary ballot B corresponds

[3] We shall not treat here the case of judgment aggregation in more general logics (Dietrich, 2007). We refer to Appendix A for a brief introduction of propositional logic.

to a judgment set: $B_\alpha = 1$ if and only if $\alpha \in J$. Given this representation, we can associate with every JA procedure for Φ and $\mathcal{N}$ a binary aggregation procedure on a subdomain of $\mathcal{D}_\Phi^\mathcal{N}$.

It is important to remark that is not exactly the standard way of interpreting JA in binary aggregation. The embedding that is given, for instance, by Dokow and Holzman (2009, 2010a), associates with every judgment set a binary ballot over a set of issues representing only the *positive* formulas in Φ, considering a rejection of the issue associated with a formula φ as an acceptance of its negation $\neg\varphi$. The same embedding is given by List and Puppe (2009, Section 2.3). In our translation we made the choice of introducing both an issue for φ and one for $\neg\varphi$, adding an additional integrity constraint to enforce the completeness of a judgment set. This allow us to easily generalise the framework to the case of incomplete ballots (see, e.g., Dietrich and List, 2008a), without having to resort to an additional symbol for abstention (see, e.g., Dokow and Holzman, 2010b)

As we did for the case of preference aggregation, we now define a set of integrity constraints for $\mathcal{D}_\Phi$ to enforce the properties of consistency and completeness of individual judgment sets. Recall that the propositional language is constructed in this case on $|\Phi|$ propositional symbols p_α, one for every $\alpha \in \Phi$. Call an inconsistent set of formulas each proper subset of which is consistent *minimally inconsistent set* (mi-set). Let IC_Φ be the following set of integrity constraints:

Completeness: $p_\alpha \vee p_{\neg\alpha}$ for all $\alpha \in \Phi$

Consistency: $\neg(\bigwedge_{\alpha\in S} p_\alpha)$ for every mi-set $S \subseteq \Phi$

While the interpretation of the first formula is straightforward, we provide some further explanation for the second one. If a judgment set J is inconsistent, then it contains a minimally inconsistent set, obtained by sequentially deleting one formula at the time from J until it becomes consistent. This implies that the constraint previously introduced is falsified by the binary ballot that represents J, as all issues associated with formulas in a mi-set are accepted. *Vice versa*, if all formulas in a mi-set are accepted by a given binary ballot, then clearly the judgment set associated with it is inconsistent.

Note that the size of IC_Φ might be exponential in the size of the agenda. This is in agreement with considerations of computational complexity (see, e.g., Papadimitriou, 1994): Since checking the consistency of a judgment set is NP-hard, while model checking on binary ballots is polynomial, the translation from JA to binary aggregation must contain a superpolynomial step (unless P=NP). A more detailed discussion of the computational complexity of these two frameworks can be found in Section 7.5.

In conclusion, the same kind of correspondence we have shown for SWFs holds between complete and consistent JA procedures and binary aggregation procedures that are collectively rational with respect to IC_Φ.

3.2.3 The Discursive Dilemma in Binary Aggregation

The same procedure that we have used to show that the Condorcet paradox is an instance of our general definition of paradox applies here for the case of the discursive dilemma. Let Φ be the agenda $\{\alpha, \beta, \alpha \wedge \beta\}$, in which we have omitted negated formulas, as for any $J \in \mathcal{J}(\Phi)$ their acceptance can be inferred from the acceptance of their positive counterparts. Consider the profile $\boldsymbol{B}$ for $\mathcal{I}_\Phi$ described in Table 3.4.

	α	β	$\alpha \wedge \beta$
Judge 1	1	1	1
Judge 2	0	1	0
Judge 3	1	0	0
Maj	1	1	0

Table 3.4: The discursive dilemma in binary aggregation.

Every individual ballot satisfies IC_Φ, while the outcome obtained by using the majority rule contradicts one of the constraints of consistency, namely $\neg(p_\alpha \wedge p_\beta \wedge p_{\neg(\alpha \wedge \beta)})$. Hence, $(Maj, \boldsymbol{B}, \mathrm{IC}_\Phi)$ constitutes a paradox by Definition 2.1.9.

3.3 The Ostrogorski Paradox

Another paradox listed by Nurmi (1999) as one of the main paradoxes of the majority rule on multiple issues is the Ostrogorski paradox. Ostrogorski (1902) published a treaty in support of procedures inspired by direct democracy, pointing out several fallacies that a representative system based on party structures can encounter. Rae and Daudt (1976) later focused on one such situation, presenting it as a paradox or a dilemma between two equivalently desirable procedures (the direct and the representative one), giving it the name of "Ostrogorski paradox". This paradox, in its simplest form, occurs when a majority of individuals are supporting a party that does not represent the view of a majority of individuals on a majority of issues.

> **Ostrogorski Paradox**. Consider the following situation: there is a two party contest between the Mountain Party (MP) and the Plain Party (PP); three individuals (or, equivalently, three equally big groups in an electorate) will vote for one of the two parties if their view agrees with that party on a majority of the three following issues: economic policy (E), social policy (S), and foreign affairs policy (F). Consider the situation described in Table 3.5.

	E	S	F	Party supported
Voter 1	MP	PP	PP	PP
Voter 2	PP	PP	MP	PP
Voter 3	MP	PP	MP	MP
Maj	MP	PP	MP	PP

Table 3.5: The Ostrogorski paradox.

> The result of the two party contest, assuming that the party that has the support of a majority of the voters wins, declares the Plain Party the winner. However, we notice that a majority of individuals support the Mountain Party both on the economic policy E *and* on the foreign policy F. Thus, the elected party, the PP, is in disagreement with a majority of the individuals on a majority of the issues.

Bezembinder and van Acker (1985) generalised this paradox, defining two different rules for compound majority decisions. The first, the representative outcome, outputs as a winner the party that receives support by a majority of the individuals. The second, the direct outcome, outputs the party that receives support on a majority of issues by a majority of the individuals. An instance of the Ostrogorski paradox occurs whenever the outcome of these two procedures differ.

Stronger versions of the paradox can be devised, in which the losing party represents the view of a majority on *all* the issues involved (see, e.g., Rae and Daudt, 1976; see also our Table 3.7). Further studies of the "Ostrogorski phenomenon" have been carried out by Deb and Kelsey (1987) as well as by Eckert and Klamler (2009). The relation between the Ostrogorski paradox and the Condorcet paradox has been investigated in several papers (Kelly, 1989; Rae and Daudt, 1976), while a comparison with the discursive dilemma was carried out by Pigozzi (2005).

3.3.1 The Ostrogorski Paradox in Binary Aggregation

In this section, we provide a binary aggregation setting that represents the Ostrogorski paradox as a failure of collective rationality with respect to a suitable integrity constraint.

Let $\{E, S, F\}$ be the set of issues at stake, and let the set of issues $\mathcal{I}_O = \{E, S, F, A\}$ consist of the same issues plus an extra issue A to encode the support for the first party (MP).[4] A binary ballot over these issues represents the

[4]We hereby propose a model that can be used for instances of the Ostrogorski paradox concerning at most two parties. In case the number of parties is bigger than two, the framework can be extended adding one extra issue for every party.

individual view on the three issues E, S and F: if, for instance, $b_E = 1$, then the individual supports the first party MP on the first issue E. Moreover, it also represents the overall support for party MP (in case issue A is accepted) or PP (in case A is rejected). In the Ostrogorski paradox, an individual votes for a party if and only if she agrees with that party on a majority of the issues. This rule can be represented as a rationality assumption by means of the following integrity constraint IC_O:

$$p_A \leftrightarrow [(p_E \wedge p_S) \vee (p_E \wedge p_F) \vee (p_S \wedge p_F)]$$

An instance of the Ostrogorski paradox can therefore be represented by the profile $\boldsymbol{B}$ described in Table 3.6.

	E	S	F	A
Voter 1	1	0	0	0
Voter 2	0	0	1	0
Voter 3	1	0	1	1
Maj	1	0	1	0

Table 3.6: The Ostrogorski paradox in binary aggregation.

Each individual in Table 3.6 accepts issue A if and only if she accepts a majority of the other issues. However, the outcome of the majority rule is a rejection of issue A, even if a majority of the issues gets accepted by the same rule. Therefore, the triple $(Maj, \boldsymbol{B}, \mathrm{IC}_O)$ constitutes a paradox by Definition 2.1.9.

Using this formalism we can easily devise a stronger version of the Ostrogorski paradox, in which the winning party disagrees with a majority of the individuals on *all* issues. Such a profile is described in Table 3.7.

	E	S	F	A
Voter 1	1	0	0	0
Voter 2	0	1	0	0
Voter 3	0	0	1	0
Voter 4	1	1	1	1
Voter 5	1	1	1	1
Maj	1	1	1	0

Table 3.7: Strong version of the Ostrogorski paradox.

3.4 The Common Structure of Paradoxical Integrity Constraints

Let us make an important remark concerning the syntactic structure of the integrity constraints that formalise the three paradoxes we have presented so far. The first formula, encoding the transitivity of a preference relation in the Condorcet paradox, is the implication $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$. This formula is equivalent to $\neg p_{ab} \vee \neg p_{bc} \vee p_{ac}$, which is a clause of size 3, i.e., it is a disjunction of three different literals. The second formula, presented in Section 3.2 to represent the discursive dilemma, is also equivalent to a clause of size 3, namely $\neg p_\alpha \vee \neg p_\beta \vee \neg p_{\neg(\alpha \wedge \beta)}$. The last formula, which formalises the majoritarian constraint underlying the Ostrogorski paradox, is equivalent to the following conjunction of clauses of size 3:

$$(p_A \vee \neg p_E \vee \neg p_F) \wedge (p_A \vee \neg p_E \vee \neg p_S) \wedge (p_A \vee \neg p_S \vee \neg p_F) \wedge \\ \wedge (\neg p_A \vee p_E \vee p_F) \wedge (\neg p_A \vee p_E \vee p_S) \wedge (\neg p_A \vee p_S \vee p_F)$$

The observation that the integrity constraints formalising the most classical paradoxes in aggregation theory all feature a clause of size at least 3 is not a coincidence. In Section 4.4.2 we will formalise this observation with a theorem that characterises the class of integrity constraints that are lifted by the majority rule as those and only those that can be expressed as a conjunction of clauses of maximal size 2 (see Theorem 4.4.8).[5]

3.5 Further Paradoxes on Multiple Issues

In this section we describe two further paradoxes that can be analysed using our framework of binary aggregation with integrity constraints: the paradox of divided government and the paradox of multiple elections. Both situations concern a paradoxical outcome obtained by using the majority rule on an aggregation problem defined on multiple issues. The first paradox can be seen as an instance of a more general behaviour described by the second paradox.

3.5.1 The Paradox of Divided Government

The paradox of divided government is a failure of collective rationality that was pointed out for the first time by Brams et al. (1993). Here we follow the presentation of Nurmi (1997).

[5]This observation is strongly related to a result proven by Nehring and Puppe (2007) in the framework of judgment aggregation, which characterises the set of paradoxical agendas for the majority rule as those agendas containing a minimal inconsistent subset of size at least 3. See also our previous work (Grandi, 2012).

The paradox of divided government. Suppose that 13 voters (equivalently, groups of voters) can choose for Democratic (D) or Republican (R) candidate for the following three offices: House of Representatives (H), Senate (S) and the governor (G). It is a common assumption that in case the House of Representatives gets a Republican candidate, then at least one of the remaining offices should go to Republicans as well. Consider now the profile in Table 3.8.

	H	S	G
Voters 1-3	D	D	D
Voter 4	D	D	R
Voter 5	D	R	D
Voter 6	D	R	R
Voters 7-9	R	D	R
Voters 10-12	R	R	D
Voter 13	R	R	R
Maj	R	D	D

Table 3.8: The paradox of divided government.

As shown in Table 3.8, it is exactly the combination that had to be avoided (i.e., RDD) that is elected, even if no individual voted for it.

This paradox can be easily seen as a failure of collective rationality: it is sufficient to replace the letters D and R with 0 and 1, and to formulate the integrity constraint as $\neg(p_H \wedge \neg p_S \wedge \neg p_G)$. The binary ballot $(1, 0, 0)$ is therefore ruled out as irrational, encoding the combination (R,D,D) that needs to be avoided.

This type of paradox can be observed in cases like the elections of a committee, such as in our Example 2.1.7. Even if it is recognised by every individual that a certain committee structure is unfeasible (i.e., it will not work well together), this may be the outcome of aggregation if the majority rule is being used.

In view of our discussion in Section 2.1.5, we may consider the constraint underlying the paradox of divided government as a feasibility constraint, rather than a constraint of rationality. Under such an interpretation this situation would cease to be paradoxical, while still showing the failure of the majority rule to output a feasible outcome.

3.5.2 The Paradox of Multiple Elections

Whilst the Ostrogorski paradox was devised to stage an attack against representative systems of collective choice based on parties, the paradox of multiple

elections (MEP) is based on the observation that when voting directly on multiple issues, a combination that was not supported nor liked by any of the voters can be the winner of the election (Brams et al., 1998; Lacy and Niou, 2000). While the original model takes into account the full preferences of individuals over combinations of issues, if we focus on only those ballots that are submitted by the individuals, then an instance of the MEP can be represented as a paradox of collective rationality. Let us consider a simple example described in Table 3.9.

> **Multiple election paradox**. Suppose three voters need to take a decision over three binary issues A, B and C. Their ballots are described in Table 3.9.

	A	B	C
Voter 1	1	0	1
Voter 2	0	1	1
Voter 3	1	1	0
Maj	1	1	1

Table 3.9: The multiple election paradox (MEP).

> The outcome of the majority rule in Table 3.9 is the acceptance of all three issues, even if this combination was not voted for by any of the individuals.

While there seems to be no integrity constraint directly causing this paradox, we may represent the profile in Table 3.9 as a situation in which the three individual ballots are bound by a budget constraint $\neg(p_A \wedge p_B \wedge p_C)$ (like in our Example 2.1.10). Even if all individuals are giving acceptance to two issues each, the result of the aggregation is the unfeasible acceptance of all three issues.

As can be deduced from our previous discussion, every instance of the MEP gives rise to several instances of a binary aggregation paradox for Definition 2.1.9. To see this, it is sufficient to find an integrity constraint that is satisfied by all individuals and not by the outcome of the aggregation.[6] On the other hand, every instance of Definition 2.1.9 in binary aggregation represents an instance of the MEP, as the irrational outcome cannot have been voted for by any of the individuals. In Section 7.2 we define an interesting aggregation procedure, called the average voter rule, which avoids both the MEP and any other failure of collective rationality.

[6]Such a formula always exists. Consider for instance the disjunction of the formulas specifying each of the individual ballots. This integrity constraint forces the result of the aggregation to be equal to one of the individual ballots on the given profile, thus generating a binary aggregation paradox from a MEP.

The multiple election paradox gives rise to a different problem than that of consistency, to which this dissertation is dedicated, as it is not directly linked to an integrity constraint established in advance. The problem formalised by the MEP is rather the *compatibility* of the outcome of aggregation with the individual ballots. Individuals in such a situation may be forced to adhere to a collective choice which, despite it being rational, they do not perceive as representing their views (Grandi and Pigozzi, 2012).

In their paper, Brams et al. (1998) provide many versions of the multiple election paradox, varying the number of issues and the presence of ties. Lacy and Niou (2000) enrich the model by assuming that individuals have a preference order over combinations of issues and submit just their top candidate for the election. They present situations in which, e.g., the winning combination is a Condorcet loser (i.e., it loses in pairwise comparison with all other combinations). Some answers to the problem raised by the MEP have already been proposed in the literature on Artificial Intelligence. For instance, a sequence of papers have studied the problem of devising sequential elections to avoid the MEP in case the preferences of the individuals over combinations of multiple issues are expressed in a suitable preference representation language (Lang, 2007; Lang and Xia, 2009; Xia et al., 2011; Conitzer and Xia, 2012).

3.6 Conclusions

The first lesson that can be drawn from this chapter dedicated to paradoxes of aggregation is that the majority rule is not a good aggregation procedure to be employed when dealing with collective choices over multiple issues. This fact stands out as a counterpart to May's Theorem (1952), which proves that the majority rule is the only aggregation rule *for a single binary issue* that satisfies a set of highly desirable conditions. The sequence of paradoxes we have analysed in this chapter shows that this is not the case when multiple issues are involved. While this fact may not add anything substantially new to the existing literature, the wide variety of paradoxical situations encountered in this chapter stresses even further the negative features of the majority rule for multi-issue domains.

A second conclusion is that most paradoxes of Social Choice Theory share a common structure, and that this structure is formalised by our Definition 2.1.9, which stands out as a truly general definition of paradox in aggregation theory. Moreover, by analysing the integrity constraints that underlie some of the most classical paradoxes, we were able to identify a common syntactic feature of paradoxical constraints (cf. Section 3.4). This observation is the starting point of the following chapter, in which we build a systematic theory of collective rationality depending on the syntactic properties of integrity constraints.

The paradoxical situations presented in this chapter constitute a fragment of the problems that can be encountered in the formalisation of collective choice

problems. First, all the paradoxes we presented feature the majority rule as the procedure used for aggregation. Paradoxical situations can be encountered in the study of many other aggregation procedures, e.g., in the case of the Borda paradox (McLean and Urken, 1995). Second, all paradoxes concern problems of aggregation in which the input given by the individuals is of the same form as the output. Paradoxical situations concerning voting procedures (Nurmi, 1999), which take as input a set of preferences and output a set of winning candidates, are therefore not included in our analysis.

We close this section with an observation regarding the interpretation of some of the paradoxes presented in this chapter. We have already remarked how some of these examples have been employed in the literature to show weaknesses and advantages of either the direct approach to democratic choice (represented by issue-by-issue aggregation) or the representative one. The last two paradoxes especially (the paradox of divided government and the MEP) seem to suggest that direct decisions over multiple issues should be avoided, at least when issues are not completely independent from one another. In our view, elections over multi-issue domains cannot be escaped: not only do they represent a model for the aggregation of more complex objects like preferences and judgments, as seen in Section 3.1 and 3.2, but they also stand out as one of the biggest challenges to the design of more complex automated systems for collective decision making.

A crucial problem in the modelling of real-world situations of collective choice is that of identifying the set of issues that best represent a given domain of aggregation, and devising an integrity constraint that models correctly the correlations between those issues. This problem obviously represents a serious obstacle to a mechanism designer, and is moreover open to manipulation. However, we believe that structuring collective decision problems with more detailed models *before* the aggregation takes place, e.g., by discovering a shared order of preferential dependencies between issues (Lang and Xia, 2009; Airiau et al., 2011), facilitates the definition of collective choice procedures on complex domains without having to elicit the full preferences of individuals. Such models can be employed in the design and the implementation of automated decision systems, in which a safe aggregation, i.e., one that avoids paradoxical situations, is of the utmost necessity. One of the main aims of this dissertation is exactly to provide tools allowing to stage direct elections on correlated issues in a safe way.