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QUANTUM SPECTRAL CURVE FOR THE GROMOV-WITTEN THEORY OF THE COMPLEX PROJECTIVE LINE

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Abstract. We construct the quantum curve for the Gromov-Witten theory of the complex projective line.

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1. Introduction

The purpose of this paper is to construct the *quantum curve* for the Gromov-Witten invariants of the complex projective line $\mathbb{P}^1$. Quantum curves are conceived in the physics literature, including [1, 4, 5, 12, 14, 16]. They quantize the *spectral curves* of the theory, and are conjectured to capture the information of many topological invariants, such as certain Gromov-Witten invariants, quantum knot invariants, and cohomology of instanton moduli spaces for 4-dimensional gauge theory. In this paper we show that the conjecture is indeed true for the Gromov-Witten theory of $\mathbb{P}^1$.

1.1. **Spectral curves and quantum curves.** When spectral curves appear in mathematics, they take various different forms, and even look as totally different objects. For example, they can be the mirror curve of a toric Calabi-Yau 3-fold, the SL_2-character variety of the fundamental group of a knot complement, or a Seiberg-Witten curve. In the context of the Gromov-Witten theory of $\mathbb{P}^1$, it is the *Landau-Ginzburg model*

$$x = z + \frac{1}{z}, \tag{1.1}$$

which is the homological mirror dual of $\mathbb{P}^1$ with respect to the standard Kähler structure. Our main theorem (Theorem 1.1 below) states that the quantization of (1.1), which we call a quantum curve, characterizes the exponential generating function of Gromov-Witten invariants of $\mathbb{P}^1$.

In a purely algebro-geometric setting, a quantum curve can be understood in the following way [7]. Let C be a non-singular complex projective algebraic curve, and η the tautological 1-form on the cotangent bundle T^*C. A spectral curve Σ is a complex 1-dimensional subvariety

$$\begin{array}{ccc} \iota : \Sigma & \longrightarrow & T^*C \\ & & \big\downarrow \pi \\ & & C \end{array} \tag{1.2}$$

in the cotangent bundle, which is automatically a Lagrangian subvariety with respect to the standard symplectic form $-d\eta$. A quantum curve is an $\hbar$-deformed D-module on the 1-parameter formal family $C[[\hbar]]$ of the curve C, whose *semi-classical limit* coincides with the spectral curve Σ. On an affine piece of the base curve C with coordinate x, we can choose a generator P of the D-module and consider a *Schrödinger-like* equation

$$P(x,\hbar)\Psi(x,\hbar) = 0. \tag{1.3}$$

The construction of the quantum curve in this setting is established in [7] for $SL(2,\mathbb{C})$ Hitchin fibrations.

The geometric situation we consider in this paper is slightly different. Instead of the cotangent bundle T^*C in (1.2), we have a surface X equipped with a $\mathbb{C}^*$-invariant holomorphic symplectic form and a spectral curve Σ is mapped into it. The base curve C is replaced by the quotient $X/\mathbb{C}^*$. For example, if a curve C admits a $\mathbb{C}^*$-action then the natural holomorphic symplectic form on $X = T^*C$ is $\mathbb{C}^*$-invariant. In local coordinates for T^*C, the $\mathbb{C}^*$-action is given by $c \cdot (w, z) = (cw, c^{-1}z)$ and the symplectic form is given by $dx \wedge (dz/z)$ where $x = wz$ is the quotient map $(w, z) \mapsto wz$ by the $\mathbb{C}^*$-action. Reflecting the $\mathbb{C}^*$-action, the quantum curve (1.3) becomes a differential equation of *infinite* order, or a *difference* equation.

We present in this paper the first rigorous example of a direct connection between Gromov-Witten theory and quantum curves. Our construction requires the fermionic Fock space representation of the Gromov-Witten invariants [18], and a subtle combinatorial analysis based on representation theory of symmetric groups.

1.2. **Main theorem.** Let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ denote the moduli space of stable maps of degree d from an n-pointed genus g curve to $\mathbb{P}^1$. This is an algebraic stack of dimension $2g-2+n+2d$. The dimension reflects the fact that a generic map from an algebraic curve to $\mathbb{P}^1$ has only simple ramifications, which we can see from the Riemann-Hurwitz formula. The descendant Gromov-Witten invariants of $\mathbb{P}^1$ are defined by

$$\left\langle \prod_{i=1}^{n} \tau_{b_i}(\alpha_i) \right\rangle_{g.n}^{d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{vir}} \prod_{i=1}^{n} \psi_i^{b_i} ev_i^*(\alpha_i), \tag{1.4}$$

where $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)]^{vir}$ is the virtual fundamental class of the moduli space,

$$ev_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \longrightarrow \mathbb{P}^1$$

is a natural morphism defined by evaluating a stable map at the i-th marked point of the source curve, $\alpha_i \in H^*(\mathbb{P}^1, \mathbb{Q})$ is a cohomology class of the target $\mathbb{P}^1$, and ψ_i is the tautological cotangent class in $H^2(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d), \mathbb{Q})$. We denote by 1 the generator of $H^0(\mathbb{P}^1, \mathbb{Q})$, and by $\omega \in H^2(\mathbb{P}^1, \mathbb{Q})$ the Poincaré dual to the point class. We assemble the Gromov-Witten invariants into particular generating functions as follows. For every (g, n) in the stable sector $2g - 2 + n > 0$, we define the *free energy* of type (g, n) by

$$F_{g,n}(x_1, \ldots, x_n) := \left\langle \prod_{i=1}^{n} \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x_i^{b+1}} \right) \right\rangle_{g,n}. \tag{1.5}$$

Here the degree d is determined by the dimension condition of the cohomology classes to be integrated over the virtual fundamental class. We note that (1.5) contains the class $\tau_0(1)$. For unstable geometries, we introduce two functions

$$S_0(x) := x - x\log x + \sum_{d=1}^{\infty} \left\langle -\frac{(2d-2)!\tau_{2d-2}(\omega)}{x^{2d-1}} \right\rangle_{0,1}^{d}, \tag{1.6}$$

$$S_1(x) := -\frac{1}{2}\log x + \frac{1}{2}\sum_{d=0}^{\infty} \left\langle \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x^{b+1}} \right)^2 \right\rangle_{0,2}^{d}. \tag{1.7}$$

The appearance of the extra terms, in particular the $\log x$ terms, will be explained in Section 3. We shall prove the following.

Theorem 1.1 (Main Theorem)**.** *The wave function*

$$\Psi(x,\hbar) := \exp\left(\frac{1}{\hbar}S_0(x) + S_1(x) + \sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}(x,\ldots,x) \right) \tag{1.8}$$

satisfies the quantum curve equation of an infinite order

$$\left[\exp\left(\hbar\frac{d}{dx}\right) + \exp\left(-\hbar\frac{d}{dx}\right) - x \right] \Psi(x,\hbar) = 0. \tag{1.9}$$

Moreover, the free energies $F_{g,n}(x_1,\ldots,x_n)$ as functions in n-variables, and hence all the Gromov-Witten invariants (1.4)*, can be recovered from the equation* (1.9) *alone, using the mechanism of the* ***topological recursion*** *of* [3, 13]*.*

Remark 1.2. Put

$$S_m(x) := \sum_{2g-2+n=m-1} \frac{1}{n!} F_{g,n}(x,\ldots,x). \tag{1.10}$$

Then our wave function is of the form

$$\Psi(x,\hbar) = \exp\left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x) \right), \tag{1.11}$$

which provides the WKB approximation of the quantum curve equation (1.9). Thus the significance of (1.5) is that the exponential generating function (1.8) of the descendant Gromov-Witten invariants of $\mathbb{P}^1$ gives the solution to the *exact* WKB analysis for the difference equation (1.9).

Remark 1.3. For the case of Hitchin fibrations [7], the Schrödinger-like equation (1.3) is a direct consequence of the generalized topological recursion. In our current context, the topological recursion does not play any role in establishing (1.9).

Remark 1.4. Although the shape of the operator in (1.9) has a similarity with the Lax operator of the Toda lattice equations that control the Gromov-Witten invariants of $\mathbb{P}^1$ [18], we are unable to find any direct relations between these two apparently different equations. We present a detailed comparison of these equations in Section 8.

1.3. **WKB approximation, topological recursion, and representation theory.** The WKB analysis provides a perturbative quantization method of a classical mechanical problem. We can recover the classical problem corresponding to (1.9) by taking its semi-classical limit, which is the singular perturbation limit

$$\lim_{\hbar\to 0}\left(e^{-\frac{1}{\hbar}S_0(x)}\left[\exp\left(\hbar\frac{d}{dx}\right)+\exp\left(-\hbar\frac{d}{dx}\right)-x\right]e^{\frac{1}{\hbar}S_0(x)}e^{\sum_{m=1}^{\infty}\hbar^{m-1}S_m(x)}\right) = \left(e^{S_0'(x)}+e^{-S_0'(x)}-x\right)e^{S_1(x)}=0. \tag{1.12}$$

In terms of new variables $y(x)=S_0'(x)$ and $z(x)=e^{y(x)}$, the semi-classical limit gives us an equation for the spectral curve

$$z\in\Sigma=\mathbb{C}^*\subset\mathbb{C}\times\mathbb{C}^*\xleftarrow{\exp}T^*\mathbb{C}=\mathbb{C}^2\ni(x,y)$$

by

$$\begin{cases} x=z+\frac{1}{z}\\ z=e^y\end{cases}. \tag{1.13}$$

This is the reason we consider (1.9) as the quantization of the Laudau-Ginzburg model (1.1).

It was conjectured in [17] that the stationary Gromov-Witten theory of $\mathbb{P}^1$ should satisfy the topological recursion of [3, 13] with respect to the spectral curve (1.13). We refer to [7, 8, 17] for a mathematical formulation of the topological recursion. The conjecture is solved in [10] as a corollary to its main theorem, which establishes the correspondence between the topological recursion and the Givental formalism.

The quantum curve equation (1.9) determines only the function Ψ, and by the $\hbar$-expansion, each coefficient $S_m(x)$. But then how do we possibly recover $F_{g,n}$ for each (g,n) as a function in n variables? Here comes the significance of the topological recursion of [3, 13], which was established in [10] for the case of the Gromov-Witten theory of $\mathbb{P}^1$. The scenario goes as follows. First we note that the semi-classical limit of (1.9) identifies the spectral curve (1.13). We then launch the topological recursion formalism of [3, 13] for this particular spectral curve, and obtain symmetric differential n-forms $\mathcal{W}_{g,n}(z_1,\dots,z_n)$ on Σ^n. In this paper we will present a canonical way to *integrate* these n-forms, which yields the free energy $F_{g,n}(x_1,\dots,x_n)$ for every (g,n) subject to $2g-2+n>0$. In this sense the single equation (1.9) knows the information of all Gromov-Witten invariants (1.4). This shows the power of quantum curves.

The key discovery of the present paper is that the quantum curve equation (1.9) is equivalent to a recursion equation

$$\frac{x}{\hbar}\left(e^{-\hbar\frac{d}{dx}}-1\right)X_d(x,\hbar)+\frac{1}{1+\frac{x}{\hbar}}e^{\hbar\frac{d}{dx}}X_{d-1}(x,\hbar)=0 \tag{1.14}$$

for a rational function

$$X_d(x,\hbar) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!}\right)^2 \prod_{i=1}^{\ell(\lambda)} \frac{x+(i-\lambda_i)\hbar}{x+i\hbar}. \tag{1.15}$$

Here λ is a partition of $d \geq 0$ with parts λ_i and $\dim \lambda$ denotes the dimension of the irreducible representation of the symmetric group S_d characterized by λ.

1.4. **Organization of the paper.** This paper is organized as follows. In Section 2 we start with a solution $\mathcal{W}_{g,n}$ to the topological recursion equation with respect to the spectral curve Σ of (1.13). It is a symmetric differential form of degree n on Σ^n. We then propose a unique mechanism to integrate $\mathcal{W}_{g,n}$ into a rational function. The goal of this section is to show that this primitive function is identical to (1.5). Then in Section 3, we re-write $\Psi(x,\hbar)$ in a different manner, only involving stationary Gromov-Witten invariants of $\mathbb{P}^1$. This formula allows us to express it in terms of a *semi-infinite wedge product* in Section 4. Using this formalism, we reduce the quantum curve equation (1.9) to a combinatorial equation (1.14) in Section 5. Equation (1.14) is then proved in Section 6 using representation theory of S_d, which in tern establishes (1.9). For completeness, we give an expression of (1.15) in terms of special values of the Laguerre polynomials in Section 7. Section 8 is devoted to the comparison of (1.9) and the Toda lattice equations of [18], in terms of the functions X_d of (1.15).

2. The functions $F_{g,n}$ in terms of Gromov-Witten invariants

The significance of the idea of quantum curves is that the single equation (1.3) captures all information of the topological invariants of the theory. The key process from this single equation to the topological invariants is the *integral form* of the mechanism known as the topological recursion of [3, 13]. We refer to [7, 8, 17] for mathematical formulation of the topological recursion. This section is devoted to providing the unique mechanism to integrate the topological recursion, for the context of the Gromov-Witten theory of $\mathbb{P}^1$.

Let us begin with a solution $\mathcal{W}_{g,n}(z_1,\ldots,z_n)$ to the topological recursion of [3, 10, 13] associated with the spectral curve $\Sigma = \mathbb{C}^*$ defined by

$$\begin{cases} x(z) = z + \frac{1}{z} \\ y(z) = \log z \end{cases}. \tag{2.1}$$

This means that symmetric differential forms $\mathcal{W}_{g,n}(z_1,\ldots,z_n)$ of degree n on Σ^n for (g,n) in the stable range $2g-2+n>0$ are inductively defined by the following recursion formula:

$$\begin{aligned}(2.2)\quad &\mathcal{W}_{g,n}(z_1,\dots,z_n)\\ &= \frac{1}{2\pi i}\oint_{z=\pm1}\frac{\int_z^{1/z}\mathcal{W}_{0,2}(\,\cdot\,,z_1)}{\mathcal{W}_{0,1}(1/z)-\mathcal{W}_{0,1}(z)}\Bigg[\mathcal{W}_{g-1,n+1}(z,1/z,z_2,\dots,z_n)\\ &\qquad\qquad + \sum_{\substack{g_1+g_1=g\\ I\sqcup J=\{2,\dots,n\}}}^{\text{stable}}\mathcal{W}_{g_1,|I|+1}(z,z_I)\mathcal{W}_{g_2,|J|+1}(1/z,z_J)\Bigg],\end{aligned}$$

where the residue integral is taken with respect to the variable $z\in\Sigma$ on two small, positively oriented, closed loops around $z=1$ and $z=-1$, and for the index set $I\subset\{2,\dots,n\}$, we denote by $|I|$ its cardinality, and $z_I=(z_i)_{i\in I}$. For (g,n) in the unstable range, we define

$$\mathcal{W}_{0,1}(z) := y(z)dx(z), \tag{2.3}$$

$$\mathcal{W}_{0,2}(z_1,z_2) := \frac{dz_1dz_2}{(z_1-z_2)^2}-\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}. \tag{2.4}$$

The goal of this section is to derive the integral $F_{g,n}(z_1,\dots,z_n)$ of $\mathcal{W}_{g,n}(z_1,\dots,z_n)$ in a consistent and unique way that has the x-variable expansion (1.5).

Remark 2.1. The second term of the right-hand side of (2.4) does not play any role in the topological recursion (2.2). It is included here for the consistency of the primitive $F_{0,2}(z_1,z_2)$ to be discussed in Section 3.

Definition 2.2. For $2g-2+n>0$, we define the *primitive* $F_{g,n}(z_1,\dots,z_n)$ of the n-form $\mathcal{W}_{g,n}(z_1,\dots,z_n)$ to be a rational function on Σ^n that satisfies the following conditions:

$$d_1\cdots d_nF_{g,n}(z_1,\dots,z_n)=\mathcal{W}_{g,n}(z_1,\dots,z_n); \tag{2.5}$$

$$F_{g,n}(z_1,\dots,z_{i-1},1/z_i,z_{i+1},\dots,z_n)=-F_{g,n}(z_1,\dots,z_n),\quad i=1,\dots,n; \tag{2.6}$$

$$F_{g,n}(z_1,\dots,z_n)\big|_{z_1=\cdots=z_n=0}=0. \tag{2.7}$$

If it exists, then it is unique.

From now on, we need to relate functions or differential forms defined on the spectral curve $\Sigma=\mathbb{C}^*$ of (2.1) and on the base curve $\mathbb{C}$. We recall [8] that the inverse function of (1.1) for the branch near $z=0$ and $x=\infty$ is given by the generating function of the Catalan numbers

$$z=z(x)=\sum_{m=0}^{\infty}\frac{1}{m+1}\binom{2m}{m}\frac{1}{x^{2m+1}}. \tag{2.8}$$

By abuse of notation, for a function or a differential form $f(z)$ on Σ, we denote the pull-back via (2.8) simply by $f(x):=f(z(x))$.

It is established in [10, 17] that the solution $\mathcal{W}_{g,n}$ of the topological recursion has the following x-variable expansion in terms of the stationary Gromov-Witten invariants of $\mathbb{P}^1$:

$$\mathcal{W}_{g,n}(x_1,\dots,x_n) = \left\langle \prod_{i=1}^{n} \left(\sum_{b=0}^{\infty} (b+1)!\, \tau_b(\omega)\, \frac{dx_i}{x_i^{b+2}} \right) \right\rangle_{g,n} . \tag{2.9}$$

There is no systematic mechanism to integrate this expression to obtain (1.5). Instead, we establish the following theorem in this section.

Theorem 2.3. *For every (g,n) in the stable sector $2g-2+n>0$, there exists a primitive $F_{g,n}(z_1,\dots,z_n)$ in the sense of Definition 2.2, such that its x-variable expansion is given by*

$$F_{g,n}(x_1,\dots,x_n) = \left\langle \prod_{i=1}^{n} \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x_i^{b+1}} \right) \right\rangle_{g,n} . \tag{2.10}$$

Remark 2.4. We need a different treatment for the unstable primitives $F_{0,1}(z)$ and $F_{0,2}(z_1,z_2)$. They are calculated in Section 3.

The rest of this section is devoted to proving this theorem. We start with recalling some results of [10]. The most important one is the formula for $\mathcal{W}_{g,n}(z_1,\dots,z_n)$ in terms of the auxiliary functions $W_d^i(z)$ (defined below) with the *ancestor* Gromov-Witten invariants as its coefficients. We will then prove the existence of the anti-symmetric primitives of the functions W_d^i, and their x-expansions. This will then lead us to the proof of the above theorem, where we will also utilize the known relations between the ancestor and the descendant Gromov-Witten invariants.

2.1. **Some results from** [10]**.**

The ancestor Gromov-Witten invariants of $\mathbb{P}^1$ we need are

$$\left\langle \prod_{i=1}^{n} \bar\tau_{b_i}(\alpha_i) \right\rangle_{g,n}^{d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{vir}} \prod_{i=1}^{n} \bar\psi_i^{b_i} ev_i^*(\alpha_i), \tag{2.11}$$

where $\bar\psi_i$ denotes the pull back of the cotangent class on $\overline{\mathcal{M}}_{g,n}$ by the natural forgetful morphism

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d) \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Since we adopt a quantum field theoretic point of view in calculating Gromov-Witten invariants, we often call them *correlators* in this paper. The ancestor and descendant correlators do not agree. We will give a formula to determine one from the other in (2.16).

Let us define

$$W_0^1(z) := \frac{dz}{(1-z)^2}, \tag{2.12}$$

$$W_0^2(z) := \frac{idz}{(1+z)^2}, \tag{2.13}$$

$$W_k^i(z) := d\left(\left(-2\frac{d}{dx(z)}\right)^k \int W_0^i(z)\right), \qquad i = 1, 2; \quad k \geq 0. \tag{2.14}$$

Then for $g \geq 0$ and $n \geq 1$ with $2g-2+n > 0$, from Theorem 4.1 of [10] (as shown in the proof of Theorem 5.2 of [10]), we have

$$\mathcal{W}_{g,n}(z_1, \ldots, z_n) = \sum_{\vec{d}, \vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1}) \ldots \bar{\tau}_{d_n}(\tilde{e}_{i_n}) \rangle_g \frac{W_{d_1}^{i_1}(z_1)}{2^{d_1}\sqrt{2}} \cdots \frac{W_{d_n}^{i_n}(z_n)}{2^{d_n}\sqrt{2}}. \tag{2.15}$$

Here the sum over $\vec{d}$ and $\vec{i}$ are taken over all integer values $0 \leq d_k$ and $i_k = 1, 2$. Note that the coefficients of this expansion are the *ancestor* Gromov-Witten invariants. The cohomology basis for $H^1(\mathbb{P}^1, \mathbb{Q})$ is normalized as follows. First we denote by $e_1 = 1$ and $e_2 = \omega$. Using the normalization matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},$$

we define

$$\tilde{e}_i = \left(A^{-1}\right)_i^{\mu} e_\mu.$$

In this section we use the Einstein convention and take summation over repeated indices.

With the help of the Givental formula, Proposition 5.1 of [10] relates the ancestor and the descendant correlators for $\mathbb{P}^1$ by

$$\begin{aligned} &\sum_{\vec{d}, \vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1}) \ldots \bar{\tau}_{d_n}(\tilde{e}_{i_n}) \rangle_g v^{d_1, i_1} \ldots v^{d_n, i_n} \\ =&\sum_{\vec{d}, \vec{\mu}} \langle \tau_{d_1}(e_{\mu_1}) \ldots \tau_{d_n}(e_{\mu_n}) \rangle_g t^{d_1, \mu_1} \ldots t^{d_n, \mu_n}, \end{aligned} \tag{2.16}$$

where $v^{d,i}$ and $t^{d,\mu}$ are formal variables related by the following formula:

$$v^{d,i} = A_\mu^i \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})_\nu^\mu t^{m,\nu}. \tag{2.17}$$

Here $(\mathcal{S}_k)^\mu_\nu$ are the matrix elements of the Givental S-matrix and defined by

$$
\begin{aligned}
\mathcal{S}(\zeta^{-1}) = \sum_{k=0}^{\infty} \mathcal{S}_k \zeta^{-k} &= \mathrm{Id} + \zeta^{-1} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k}}{(k!)^2} \begin{pmatrix} 1 - 2k\left(\frac{1}{1} + \cdots + \frac{1}{k}\right) & 0 \\ 0 & 1 \end{pmatrix} \\
&+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k-1}}{(k!)^2} \begin{pmatrix} 0 & -2\left(\frac{1}{1} + \cdots + \frac{1}{k}\right) \\ \frac{1}{k+1} & 0 \end{pmatrix}.
\end{aligned}
\tag{2.18}
$$

In the proof of Theorem 5.2 of [10] it was shown that the x^{-1}-expansion of $W^i_d(z)$ near $z = 0$ was given by the following formula:

$$
W^i_d(z) = 2^d \sqrt{2}\, A^i_\mu \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^\mu_\nu \; \delta^\nu_2 \, (m+1)! \, \frac{dx}{x^{m+2}}, \tag{2.19}
$$

where δ^i_j is the Kronecker delta symbol. The above formula, together with formulas (2.15)-(2.18), implies (2.9).

The first step of integrating $\mathcal{W}_{g,n}$ is to identify a suitable primitive of the differential 1-forms $W^i_d(z)$.

Proposition 2.5. *For given $i = 1, 2$ and $d \geq 0$, there exists a uniquely defined rational function $\theta^i_d(z)$ on Σ such that*

$$
d\theta^i_d(z) = W^i_d(z), \tag{2.20}
$$

$$
\theta^i_d(1/z) = -\,\theta^i_d(z). \tag{2.21}
$$

Moreover, the x^{-1}-expansion of $\theta^i_d(z)$ near $z = 0$ is given by

$$
\theta^i_d(z(x)) = 2^d \sqrt{2}\, A^i_\mu \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^\mu_\nu \; \left(-\delta^\nu_1 \delta^m_0 \, \frac{1}{2} - \delta^\nu_2 \, m! \, \frac{1}{x^{m+1}} \right). \tag{2.22}
$$

2.2. **Proof of Proposition 2.5.** It is easy to see by direct computation that the rational functions

$$
\begin{aligned}
\theta^1_0 &:= \frac{1}{1-z} - \frac{1}{2} \\
\theta^2_0 &:= -\frac{i}{1+z} + \frac{i}{2}
\end{aligned}
\tag{2.23}
$$

are the unique solutions of (2.20) and (2.21) for $d = 0$.

Equation (2.14), together with condition (2.20), implies that if $\theta^i_d(z)$ exists, then it has to satisfy

$$
\theta^i_d(z) = \left(-2 \frac{d}{dx(z)} \right)^d \theta^i_0(z). \tag{2.24}
$$

Since x is symmetric under the coordinate change $z \longmapsto 1/z$, we see that the right-hand side of equation (2.24) satisfies (2.21). This means that $\theta^i_d(z)$ *defined* by (2.24) is, for given i and d, indeed the unique solution of (2.20) and (2.21).

We denote by $\tilde{\theta}^i_d$ the right-hand side of (2.22). We wish to prove that the x^{-1}-expansion of $\theta^i_d(z)$ near $z = 0$ is given by $\tilde{\theta}^i_d$. Let us introduce the following notation:

$$\eta^\mu_d := \frac{1}{2^d\sqrt{2}} \left(A^{-1}\right)^\mu_i \theta^i_d. \tag{2.25}$$

Then we have

$$\eta_0 = \left(\frac{1}{1-z^2} - \frac{1}{2}, \frac{z}{1-z^2}\right), \tag{2.26}$$

$$\eta^\mu_k(z) = \left(-\frac{d}{dx(z)}\right)^k \eta^\mu_0, \tag{2.27}$$

and condition (2.22) becomes equivalent to the condition that the x^{-1}-expansion of η^μ_d near $z = 0$ is equal to $\tilde{\eta}^\mu_d$, where

$$\tilde{\eta}^\mu_d := \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^\mu_\nu \left(-\delta^\nu_1 \delta^m_0 \frac{1}{2} - \delta^\nu_2\, m!\, \frac{1}{x^{m+1}}\right). \tag{2.28}$$

Let us prove formula (2.28) for $d = 0$. Note that $\mathcal{S}_0 = \mathrm{Id}$, so for the constant term of $\tilde{\eta}_0$ we have

$$\left[\frac{1}{x^0}\right] \tilde{\eta}^\mu_0 = -\delta^\mu_1 \frac{1}{2}. \tag{2.29}$$

It is easy to see from (2.26) that η^i_0 has the same constant term at $z = 0$.

For $k \geq 1$ we have

$$\begin{aligned}
\left[\frac{1}{x^{2k-1}}\right] \tilde{\eta}^1_0 &= -(2k-2)!\, (\mathcal{S}_{2k-2})^1_2 = 0, \\
\left[\frac{1}{x^{2k-1}}\right] \tilde{\eta}^2_0 &= -(2k-2)!\, (\mathcal{S}_{2k-2})^2_2 = -\frac{(2k-2)!}{((k-1)!)^2}, \\
\left[\frac{1}{x^{2k}}\right] \tilde{\eta}^1_0 &= -(2k-1)!\, (\mathcal{S}_{2k-1})^1_2 = -\frac{(2k-1)!}{k!\,(k-1)!}, \\
\left[\frac{1}{x^{2k}}\right] \tilde{\eta}^2_0 &= -(2k-1)!\, (\mathcal{S}_{2k-1})^2_2 = 0.
\end{aligned} \tag{2.30}$$

For the corresponding coefficients in the x^{-1}-expansion of η_0^μ near $z=0$ we have ($k \geq 1$):

$$\begin{aligned}
&\operatorname*{Res}_{z=0} x^{2k-2}(z)\,\eta_0^1\,dx(z) = -\operatorname*{Res}_{z=0} z^{-2k}\left(1+z^2\right)^{2k-2} dz = 0,\\
&\operatorname*{Res}_{z=0} x^{2k-2}(z)\,\eta_0^2\,dx(z) = -\operatorname*{Res}_{z=0} z^{-2k+1}\left(1+z^2\right)^{2k-2} dz = -\frac{(2k-2)!}{((k-1)!)^2},\\
&\operatorname*{Res}_{z=0} x^{2k-1}(z)\,\eta_0^1\,dx(z) = -\operatorname*{Res}_{z=0} z^{-2k-1}\left(1+z^2\right)^{2k-1} dz = -\frac{(2k-1)!}{k!\,(k-1)!},\\
&\operatorname*{Res}_{z=0} x^{2k-1}(z)\,\eta_0^2\,dx(z) = -\operatorname*{Res}_{z=0} z^{-2k}\left(1+z^2\right)^{2k-1} dz = 0.
\end{aligned} \tag{2.31}$$

We see that the coefficients in (2.30) precisely coincide with the ones in (2.31). This implies that the x^{-1}-expansion of η_0^μ is indeed given by $\tilde{\eta}_0^\mu$.

By virtue of (2.27), we see that the x^{-1}-expansion of η_k^μ near $z=0$ is given by the following formula (for $k \geq 1$):

$$\left(-\frac{d}{dx}\right)^k \eta_0^\mu = \sum_{m=0}^{\infty} (\mathcal{S}_m)^\mu_\nu \left(-\delta_2^\nu\,(m+k)!\,\frac{1}{x^{m+k}}\right)$$
$$= \sum_{m=d}^{\infty} (\mathcal{S}_{m-k})^\mu_\nu \left(-\delta_2^\nu\, m!\,\frac{1}{x^{m+1}}\right).$$

This coincides with the formula for $\tilde{\eta}_k^\mu$ for $k \geq 1$. Thus, we have proved that the x^{-1}-expansion of η_k^μ is given by $\tilde{\eta}_k^\mu$, which, in turn, implies that Equation (2.22) holds. This concludes the proof of the proposition.

2.3. **Proof of Theorem 2.3.** Recall Equation (2.15) for $\mathcal{W}_{g,n}$:

$$\mathcal{W}_{g,n}(z_1,\dots,z_n) = \sum_{\vec{d},\,\vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1})\dots\bar{\tau}_{d_n}(\tilde{e}_{i_n})\rangle_g \frac{W^{i_1}_{d_1}(z_1)}{2^{d_1}\sqrt{2}}\cdots\frac{W^{i_n}_{d_n}(z_n)}{2^{d_n}\sqrt{2}}.$$

Since we know how to integrate every $W^i_d(z)$, we simply define

$$F_{g,n}(z_1,\dots,z_n) := \sum_{\vec{d},\,\vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1})\dots\bar{\tau}_{d_n}(\tilde{e}_{i_n})\rangle_g \frac{\theta^{i_1}_{d_1}(z_1)}{2^{d_n}\sqrt{2}}\cdots\frac{\theta^{i_n}_{d_n}(z_n)}{2^{d_n}\sqrt{2}}. \tag{2.32}$$

Then from Proposition 2.5, we see that (2.5) and (2.6) are automatically satisfied. We also know from Proposition 2.5 that the x^{-1}-expansion of $F_{g,n}$ near $z_1=\dots=z_n=0$ is given by

$$F_{g,n}(x_1,\dots,x_n)$$
$$= \sum_{\vec{d},\,\vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1})\dots\bar{\tau}_{d_n}(\tilde{e}_{i_n})\rangle_g \prod_{k=1}^{n} A^{i_k}_{\mu_k} \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^{\mu_k}_{\nu_k} \left(-\delta_1^{\nu_k}\delta_0^m\,\frac{1}{2} - \delta_2^{\nu_k}\, m!\,\frac{1}{x_k^{m+1}}\right).$$

Using (2.16) and (2.17), we find

$$
\begin{aligned}
&F_{g,n}(x_1,\dots,x_n)\\
&= \sum_{\vec{d},\vec{i}} \langle \tau_{d_1}(e_{\mu_1})\dots\tau_{d_n}(e_{\mu_n})\rangle_g \prod_{i=1}^{n}\left(-\delta_1^{\mu_i}\delta_0^{d_i}\,\frac{1}{2} - \delta_2^{\mu_i}\, d_i!\,\frac{1}{x^{d_i+1}}\right)\\
&= \left\langle \prod_{i=1}^{n}\left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x_i^{b+1}}\right)\right\rangle_{g,n}.
\end{aligned}
\tag{2.33}
$$

The final condition (2.7) follows from the fact that $\langle(\tau_0(1))^n\rangle_{g,n} = 0$ for all g and n in the stable range. This concludes the proof of the theorem.

3. The shift of variable simplification

Let us now turn our attention toward proving (1.9) of Theorem 1.1. In this section, as the first step, we establish a formula for the wave function $\Psi(x,\hbar)$ of (1.8) involving only the stationary Gromov-Witten invariants.

Our starting point is

$$
\begin{aligned}
\log\Psi(x,\hbar) &= \frac{1}{\hbar}S_0(x) + S_1(x)\\
&\quad + \sum_{g,d=0}^{\infty}\sum_{\substack{n=1\\2g-2+n>0}}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\left\langle\left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x^{b+1}}\right)^n\right\rangle_{g,n}^{d}.
\end{aligned}
\tag{3.1}
$$

Using the string equation and some earlier results in [8], we shall give an expression for $\log\Psi(x,\hbar)$ purely in terms of the stationary sector. More precisely, we prove the following lemma.

Lemma 3.1. *The function* $\log\Psi(x,\hbar)$ *is a solution to the following difference equation:*

$$
\begin{aligned}
\exp\left(-\frac{\hbar}{2}\frac{d}{dx}\right)\log\Psi(x,\hbar) &= \frac{1}{\hbar}\left(x - x\log x\right)\\
&\quad + \sum_{g,d=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\left\langle\left(-\sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x^{b+1}}\right)^n\right\rangle_{g,n}^{d}.
\end{aligned}
\tag{3.2}
$$

3.1. **Expansion of S_0 and S_1.** The functions $S_0(x)$ and $S_1(x)$ of (1.6) and (1.7) are derived from the first steps of the WKB method, that is, they are just imposed by the quantum spectral curve equation. In this subsection, we represent them in terms of the unstable $(0,1)$- and $(0,2)$-Gromov-Witten invariants.

First let us calculate these functions from the WKB approximation (1.11). After taking the semi-classical limit (1.12), we can calculate $S_1'(x)$ as follows:

$$
\begin{aligned}
&e^{-\frac{1}{\hbar}S_0(x)-S_1(x)}\big(e^{\hbar\frac{d}{dx}}+e^{-\hbar\frac{d}{dx}}-x\big)e^{\frac{1}{\hbar}S_0(x)+S_1(x)}\\
&=e^{S_0'(x)+\hbar\left(\frac{1}{2}S_0''(x)+S_1'(x)\right)}e^{\hbar\frac{d}{dx}}+e^{-S_0'(x)+\hbar\left(\frac{1}{2}S_0''(x)-S_1'(x)\right)}e^{-\hbar\frac{d}{dx}}-x+O(\hbar^2)\\
&=e^{S_0'(x)}\left(1+\hbar\left(\frac{1}{2}S_0''(x)+S_1'(x)\right)\right)+e^{-S_0'(x)}\left(1+\hbar\left(\frac{1}{2}S_0''(x)-S_1'(x)\right)\right)\\
&\qquad -x+O(\hbar^2)\\
&=\hbar\left(\frac{S_0''(x)}{2}\left(e^{S_0'(x)}+e^{-S_0'(x)}\right)+S_1'(x)\left(e^{S_0'(x)}-e^{-S_0'(x)}\right)\right)+O(\hbar^2).
\end{aligned}
$$

The coefficient of $\hbar$ must vanish, hence we can solve for $S_1'(x)$. Since

$$
S_0''(x)=\frac{d}{dx}S_0'(x)=\frac{d}{dx}\log z=\frac{\frac{d}{dz}\log z}{x'(z)}=\frac{\frac{1}{z}}{1-\frac{1}{z^2}}=\frac{1}{z-\frac{1}{z}},
$$

we find

$$
S_1'(x)=-\frac{1}{2}\frac{1}{z-\frac{1}{z}}\frac{z+\frac{1}{z}}{z-\frac{1}{z}}=-\frac{1}{2}\frac{z(z^2+1)}{(z^2-1)^2}. \tag{3.3}
$$

It is proved in [8, Equation (7.9) and Theorem 7.7] that

$$
\begin{aligned}
\sum_{d=0}^{\infty}\left\langle\left(-\sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x^{b+1}}\right)\right\rangle_{0,1}^{d}&=\sum_{d=1}^{\infty}\left\langle\left(-\frac{(2d-2)!\tau_{2d-2}(\omega)}{x^{2d-1}}\right)\right\rangle_{0,1}^{d}\\
&=-2z+\left(z+\frac{1}{z}\right)\log\left(1+z^2\right),
\end{aligned} \tag{3.4}
$$

and

$$
\sum_{d=0}^{\infty}\left\langle\prod_{i=1}^{2}\left(-\sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x_i^{b+1}}\right)\right\rangle_{0,2}^{d}=-\log\left(1-z_1z_2\right). \tag{3.5}
$$

One of the implications of the string equation is

$$
\langle\tau_0(1)\tau_{b+1}(\omega)\rangle_{0,2}^{d}=\langle\tau_b(\omega)\rangle_{0,1}^{d}.
$$

Using this form of the string equation and Equation (3.4), we calculate that

$$
\begin{aligned}
&\sum_{d=1}^{\infty}\left\langle\left(-\frac{1}{2}\tau_0(1)\right)\left(-\frac{(2d-1)!\tau_{2d-1}(\omega)}{x_i^{2d}}\right)\right\rangle_{0,2}^{d}\\
&=\frac{1}{2}\frac{d}{dx}\left(-2z+\left(z+\frac{1}{z}\right)\log\left(1+z^2\right)\right)\\
&=\frac{1}{2}\log x+\frac{1}{2}\log z.
\end{aligned} \tag{3.6}
$$

Note that the only condition we have for $S_0(x)$ is that $S_0'(x) = \log z$. Therefore, if we define

$$S_0(z) := F_{0,1}(z) = \int \mathcal{W}_{0,1}(z) = \int y(z)dx(z)$$

by formally applying (1.10) for $m = 0$, and impose the skew-symmetry condition (2.6) to the primitive $F_{0,1}(z)$, then from (3.4) we obtain

$$\begin{aligned} S_0(x) &= \frac{1}{z} - z + \left(z + \frac{1}{z}\right)\log z \\ &= (x - x\log x) + \sum_{d=1}^{\infty} \left\langle \left(-\frac{(2d-2)!\tau_{2d-2}(\omega)}{x_i^{2d-1}} \right) \right\rangle_{0,1}^{d} . \end{aligned} \tag{3.7}$$

The determination of $S_1(x)$ is trickier. Morally speaking, if we formally apply (1.10) for $m = 1$, then we obtain

$$S_1(x) = -\frac{1}{2} F_{0,2}(z, z) \tag{3.8}$$

for the primitive

$$\begin{aligned} F_{0,2}(z_1, z_2) &= \int^{z_1}\int^{z_2} \mathcal{W}_{0,2}(z_1, z_2) \\ &= \int^{z_1}\int^{z_2} \left(\frac{dz_1 dz_2}{(z_1 - z_2)^2} - \frac{dx_1 dx_2}{(x_1 - x_2)^2} \right) \\ &= -\log(1 - z_1 z_2) + f(z_1) + f(z_2) + c. \end{aligned} \tag{3.9}$$

Here we are imposing the condition that $F_{0,2}(z_1, z_2)$ is a symmetric function. The fact that $F_{0,2}$ is a primitive of $\mathcal{W}_{0,2}$ does not determine the function $f(z)$. Therefore, we are free to choose $f(z)$ so that the differential equation (3.3) holds. Obviously, we need to choose $f(z) = \frac{1}{2}\log z$. In this way, using (3.5) and (3.6) as well, we obtain

$$\begin{aligned} S_1(x) &= -\frac{1}{2}\log\left(1 - z^2\right) + \frac{1}{2}\log z \\ &= -\frac{1}{2}\log x + \frac{1}{2}\sum_{d=0}^{\infty} \left\langle \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x^{b+1}} \right)^2 \right\rangle_{0,2}^{d} . \end{aligned} \tag{3.10}$$

Remark 3.2. This adjustment of the choice of $S_1(x)$ also appears in the Hitchin fibration case of [7]. Still we have one degree of freedom for choosing a constant c of (3.9). It does not matter to the linear quantum curve equation (1.9), because the constant term c only affects on the overall constant factor of Ψ of (1.8).

3.2. **A new formula for** $\log\Psi$**.** We use Equations (3.7) and (3.10) to rewrite the formula (3.1) for $\log\Psi$ in the following way:

$$\log\Psi(x,\hbar)=\sum_{g,d=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}(-1)^n}{n!}\Theta^d_{g,n}, \tag{3.11}$$

where

$$\begin{aligned}\Theta^0_{0,1}:=-x+x\log x+\frac{\hbar}{2}\log x \\ +\sum_{k=2}^{\infty}\left\langle\tau_0(1)^k\tau_{k-2}(\omega)\right\rangle^0_{0,k+1}\frac{(-1)^k\hbar^k}{2^k k!}\frac{(k-2)!}{x^{k-1}}\end{aligned} \tag{3.12}$$

and

$$\Theta^d_{g,n}:=\sum_{k=0}^{\infty}\sum_{b_1,\dots,b_n=0}^{\infty}\left\langle\tau_0(1)^k\prod_{i=1}^n\tau_{b_i}(\omega)\right\rangle^d_{g,n+k}\frac{(-1)^k\hbar^k}{2^k k!}\frac{\prod_{i=1}^n b_i!}{x^{n+\sum_{i=1}^n b_i}}. \tag{3.13}$$

It is obvious that for dimensional reasons, $\Theta^0_{0,n}=0$ for any $n\geq 2$. Lemma 3.1 is then a direct corollary to the following statement.

Lemma 3.3. *The quantities defined in* (3.12) *and* (3.13) *are given by*

$$\Theta^0_{0,1}=-\left(x+\frac{\hbar}{2}\right)+\left(x+\frac{\hbar}{2}\right)\log\left(x+\frac{\hbar}{2}\right); \tag{3.14}$$

$$\Theta^d_{g,n}=\sum_{b_1,\dots,b_n}\left\langle\prod_{i=1}^n\tau_{b_i}(\omega)\right\rangle^d_{g,n}\frac{\prod_{i=1}^n b_i!}{\left(x+\frac{\hbar}{2}\right)^{n+\sum_{i=1}^n b_i}}, \tag{3.15}$$

where in the second equation the sum is taken over all $b_1,\dots,b_n\geq 0$ *such that* $\sum_{i=1}^n b_i=2g+2d-2$.

3.3. **Proof of Lemma 3.3.** Since the difference between the definitions (3.12)-(3.13) and the values (3.14)-(3.15) is simply the elimination of $\tau_0(1)$, we prove Lemma 3.3 by using the string equation for the Gromov-Witten invariants of $\mathbb{P}^1$:

$$\left\langle\tau_0(1)^k\prod_{i=1}^n\tau_{b_i}(\omega)\right\rangle^d_{g,n+k}=\sum_{\substack{j=1\\ b_j>0}}^n\left\langle\tau_0(1)^{k-1}\tau_{b_j-1}(\omega)\prod_{\substack{i=1\\ i\neq j}}^n\tau_{b_i}(\omega)\right\rangle^d_{g,n+k-1}, \tag{3.16}$$

where we assume $2g-2+n>1$ and $k>0$.

First, let us directly compute $\Theta^0_{0,1}$. Equation (3.16) implies that

$$\left\langle\tau_0(1)^k\tau_{k-2}(\omega)\right\rangle^0_{0,k+1}=\left\langle\tau_0(1)^{k-1}\tau_{k-3}(\omega)\right\rangle^0_{0,k}=\left\langle\tau_0(1)^2\tau_0(\omega)\right\rangle_{0,3}=1. \tag{3.17}$$

Therefore,

$$\sum_{k=2}^{\infty} \left\langle \tau_0(1)^k \tau_{k-2}(\omega) \right\rangle_{0,k+1}^{0} \frac{(-1)^k \hbar^k}{2^k k!} \frac{(k-2)!}{x^{k-1}}$$
$$= \sum_{k=2}^{\infty} \frac{(-1)^k \hbar^k}{2^k k!} \frac{(k-2)!}{x^{k-1}} = \left(x + \frac{\hbar}{2}\right) \log\left(\frac{x+\frac{\hbar}{2}}{x}\right) - \frac{\hbar}{2}.$$

This proves Equation (3.14).

The proof of Equation (3.15) goes as follows. Recall that $g + d > 0$ and $n > 0$. Equation (3.16) implies that any correlator $\left\langle \tau_0(1)^k \prod_{i=1}^n \tau_{b_i}(\omega) \right\rangle_{g,n+k}^{d}$ can be represented as a linear combination of the correlators $\langle \prod_{i=1}^n \tau_{b_i}(\omega) \rangle_{g,n}^{d}$ with $\sum_{i=1}^n b_i = 2g + 2d - 2$. Moreover, for any $k \geq 0$ and $c_1, \ldots, c_n \geq 0$ such that $\sum_{i=1}^n c_i = k$, the coefficient of a particular correlator $\langle \prod_{i=1}^n \tau_{b_i}(\omega) \rangle_{g,n}^{d}$ in $\left\langle \tau_0(1)^k \prod_{i=1}^n \tau_{b_i+c_i}(\omega) \right\rangle_{g,n+k}^{d}$ is equal to

$$\frac{k!}{c_1! \cdots c_n!}.$$

Therefore, the total coefficient of $\langle \prod_{i=1}^n \tau_{b_i}(\omega) \rangle_{g,n}^{d}$ in $\Theta_{g,n}^{d}$ is equal to

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_{\substack{c_1,\ldots,c_n=0 \\ c_1+\cdots c_n=k}}^{\infty} \frac{(-1)^k \hbar^k}{2^k k!} \frac{\prod_{i=1}^n (b_i+c_i)!}{x^{n+\sum_{i=1}^n (b_i+c_i)}} \frac{k!}{c_1! \cdots c_n!} \\ (3.18) \qquad &= \frac{\prod_{i=1}^n (b_i)!}{x^{n+\sum_{i=1}^n (b_i)}} \sum_{k=0}^{\infty} \left(\frac{-\hbar}{2x}\right)^k \sum_{\substack{c_1,\ldots,c_n \geq 0 \\ c_1+\cdots+c_n=k}} \prod_{i=1}^n \frac{(b_i+c_i)!}{b_i! c_i!}. \end{aligned}$$

On the other hand, expansion of the coefficient of $\langle \prod_{i=1}^n \tau_{b_i}(\omega) \rangle_{g,n}^{d}$ in Equation (3.15) is equal to

$$\begin{aligned} &\frac{\prod_{i=1}^n (b_i)!}{\left(x+\frac{\hbar}{2}\right)^{n+\sum_{i=1}^n (b_i)}} = \prod_{i=1}^n \frac{(b_i)!}{\left(x+\frac{\hbar}{2}\right)^{b_i+1}} \\ (3.19) \qquad &= \prod_{i=1}^n \frac{(b_i)!}{(x)^{b_i+1}} \sum_{c_i=0}^{\infty} \left(\frac{-\hbar}{2x}\right)^k \frac{(b_i+c_i)!}{b_i! c_i!}. \end{aligned}$$

Since (3.18) and (3.19) are identical, we have proved Equation (3.15). This completes the proof of Lemma 3.1.

4. Reduction to the semi-infinite wedge formalism

In this Section we represent the formula for $\Psi(x, \hbar)$ in terms of the semi-infinite wedge formalism. We use the formula of Okounkov-Pandharipande [18] that relates the stationary sector of the Gromov-Witten invariants of $\mathbb{P}^1$ to the expectation values of the so-called $\mathcal{E}$-operators. In order to include the extra combinatorial factors that we have in the expansion of $\log \Psi(x, \hbar)$, we consider the $\mathcal{E}$-operators with values in formal differential operators.

4.1. **Semi-infinite wedge formalism.** In this subsection we recall very briefly some basic facts about the semi-infinite wedge formalism. For more details we refer to [9, 18, 19].

Let us consider a vector space $V := \bigoplus_{c=-\infty}^{\infty} V_c$, where V_c is spanned by the basis vectors $\underline{a_1} \wedge \underline{a_2} \wedge \underline{a_3} \wedge \cdots$ such that $a_i \in \mathbb{Z} + 1/2$, $i = 1, 2, \ldots$, $a_1 > a_2 > a_3 \ldots$, and for all but a finite number of terms we have $a_i = 1/2 - i + c$. We denote by ψ_k the operator $\underline{k}\wedge\colon V_c \to V_{c+1}$, and by ψ_k^* the operator $\partial/\partial \underline{k}\colon V_c \to V_{c-1}$. Both are odd operators, and they satisfy the graded commutaion relation $[\psi_i, \psi_i^*] = 1$, with all other possible pairwise commutators equal to zero.

We denote by $:\psi_i\psi_j^*:$ the normally ordered product, that is, $:\psi_i\psi_j^* := \psi_i\psi_j^*$ for $j > 0$ and $:\psi_i\psi_j^* := -\psi_j^*\psi_i$ for $j < 0$. We introduce the operators $\mathcal{E}_n(z)$, $n \in \mathbb{Z}$ as

$$\mathcal{E}_n(z) := \sum_{k\in\mathbb{Z}+1/2} \exp\left(z\left(k - \frac{n}{2}\right)\right) :\psi_{k-r}\psi_k^*: + \frac{\delta_{n0}}{\zeta(z)}, \tag{4.1}$$

where $\zeta(z) = \exp(z/2) - \exp(-z/2)$. These operators satisfy the commutation relation $[\mathcal{E}_n(z), \mathcal{E}_m(w)] = \zeta(nw - mz)\mathcal{E}_{n+m}(z + w)$.

For any operator $\mathcal{A} = \mathcal{E}_{n_1}(z_1) \cdots \mathcal{E}_{n_m}(z_m)$ we denote by $\langle|\mathcal{A}|\rangle$ the coefficient of the vector $v_\emptyset := \underline{-1/2} \wedge \underline{-3/2} \wedge \underline{-5/2} \wedge \cdots$ in the basis expansion of $\mathcal{A}v_\emptyset$. If we want to compute a particular correlator $\langle|\mathcal{E}_{n_1}(z_1) \cdots \mathcal{E}_{n_m}(z_m)|\rangle$, first we use the commutation relation for the $\mathcal{E}$-operators, and then appeal to the simple fact that $\mathcal{E}_n(z)|\rangle = 0$ for $n > 0$, $\langle|\mathcal{E}_n(z) = 0$ for $n < 0$, and $\langle|\mathcal{E}_0(z_1) \cdots \mathcal{E}_0(z_n)|\rangle = 1/\left(\zeta(z_1) \cdots \zeta(z_n)\right)$. In this section we are mostly interested in correlators for the form

$$\langle|\mathcal{A}|\rangle = \left\langle \left| \mathcal{E}_1(0)^d \prod_{i=1}^n \mathcal{E}_0(z_i)\mathcal{E}_{-1}(0)^d \right| \right\rangle. \tag{4.2}$$

For the purpose of establishing the results in [18], Okounkov and Pandharipande considered the *disconnected* version of Gromov-Witten invariants and Hurwitz numbers. The disconnectedness here means we allow disconnected domain curves mapped to $\mathbb{P}^1$. For example, they establish in [18, Proposition 3.1, Equation 3.4] a formula for disconnected stationary Gromov-Witten invariants of $\mathbb{P}^1$, which reads

$$\sum_{b_1,\ldots,b_n\geq -2} \left\langle \prod_{i=1}^n \tau_{b_i}(\omega) \right\rangle^{\bullet\, d} \prod_{i=1}^n x_i^{b_i+1} = \frac{1}{(d!)^2} \left\langle \left| \mathcal{E}_1(0)^d \prod_{i=1}^n \mathcal{E}_0(x_i)\mathcal{E}_{-1}(0)^d \right| \right\rangle, \tag{4.3}$$

where $\langle\ \ \rangle^\bullet$ denotes the disconnected Gromov-Witten invariant. Counting the number of disconnected domain curves and connected ones are related simply by talking the logarithm. Thus we have

$$\sum_{g=0}^{\infty} \sum_{b_1,\ldots,b_n \geq -2} \left\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \right\rangle_{g,n}^{d} \prod_{i=1}^{n} x_i^{b_i+1} = \log \left(\sum_{b_1,\ldots,b_n \geq -2} \left\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \right\rangle^{\bullet\, d} \prod_{i=1}^{n} x_i^{b_i+1} \right).$$

This prompts us to introduce the *connected* correlator notation, corresponding to (4.3), as follows:

$$\sum_{g=0}^{\infty} \sum_{b_1,\ldots,b_n \geq -2} \left\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \right\rangle_{g,n}^{d} \prod_{i=1}^{n} x_i^{b_i+1} = \frac{1}{(d!)^2} \left\langle \left| \mathcal{E}_1(0)^d \prod_{i=1}^{n} \mathcal{E}_0(x_i) \mathcal{E}_{-1}(0)^d \right| \right\rangle^{\circ}. \tag{4.4}$$

The connected correlator is also known as the *cumulant* in probability theory, which is calculate via the inclusion-exclusion formula. In general, for an operator $\mathcal{A}$ of (4.2), we denote by $\langle|\mathcal{A}|\rangle^{\circ}$ the contribution coming from the single operator of the form $\mathcal{E}_0(\sum_{i=1}^{n} z_i)$ in the end, after applying the commutation relation successively. Of course in terms of generating functions, this simply means we take the logarithm of the expression. See [9, Definition 2.12, Definition 2.14] for more detail.

4.2. **A new formula for Ψ.** Noticing that $\exp\left(\frac{\hbar}{2}\frac{d}{dx}\right)$ is an automorphism, from (3.2) we find

$$\log \Psi(x,\hbar) = \exp\left(\frac{\hbar}{2}\frac{d}{dx}\right) T(x),$$

where

$$T(x) := \sum_{g,d=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \sum_{b_1,\ldots,b_n=0}^{\infty} \left\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \right\rangle_{g,n}^{d} \prod_{i=1}^{n} \left(-\frac{b_i!}{x^{b_i+1}} \right) + \frac{1}{\hbar} \langle \tau_{-2}(\omega) \rangle_{0,1}^{0} (x - x \log x).$$

Here we have used the convention of [18] that $\langle \tau_{-2}(\omega) \rangle_{0,1}^{0} = 1$ and $\tau_{-1}(\omega) = 0$. We are now ready to re-write the right-hand side in terms of expectation values of $\mathcal{E}$-operators. Corollary 4.2 of the following lemma is the main result of this section.

Lemma 4.1. *For any $d \geq 0$, $n \geq 1$, $(d,n) \neq (0,1)$, we have*

$$\sum_{g=0}^{\infty} \hbar^{2g-2+n} \sum_{b_1,\ldots,b_n=0}^{\infty} \left\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \right\rangle_{g,n}^{d} \prod_{i=1}^{n} \left(-\frac{b_i!}{x_i^{b_i+1}} \right) \tag{4.5}$$
$$= \frac{1}{(d!)^2 \hbar^{2d}} \left\langle \left| \mathcal{E}_1(0)^d \prod_{i=1}^{n} \mathcal{E}_0 \left(-\hbar \frac{\partial}{\partial x_i} \right) (\log x_i)\, \mathcal{E}_{-1}(0)^{-d} \right| \right\rangle^{\circ}.$$

For $d = 0$ and $n = 1$, we have

$$\frac{1}{\hbar} \langle \tau_{-2}(\omega) \rangle_{0,1}^{0} (x - x \log x) + \sum_{g=1}^{\infty} \hbar^{2g-1} \left\langle \prod_{i=1}^{n} \tau_{2g-2}(\omega) \right\rangle_{g,1}^{0} \left(-\frac{(2g-2)!}{x^{2g-1}} \right) \tag{4.6}$$
$$= \left\langle \left| \mathcal{E}_0 \left(-\hbar \frac{d}{dx} \right) (\log x) \right| \right\rangle^{\circ}.$$

Here we denote by $\langle \quad \rangle^{\circ}$ the connected expectation value. This means that after the successive application of the commutation relation, all differential operators appear in one correlator. Of course for $d = 0$, $n = 1$, we have $\langle \mathcal{E}_0 \rangle^{\circ} = \langle \mathcal{E}_0 \rangle$. The following corollary is a straightforward application of Lemma 4.1.

Corollary 4.2. *We have the following expression for* $\log \Psi$*:*

$$\log \Psi(x, \hbar) \tag{4.7}$$
$$= \sum_{d=0}^{\infty} \frac{1}{\hbar^{2d} (d!)^2} \left\langle \left| \mathcal{E}_1(0)^d \sum_{n=1}^{\infty} \frac{\left(\exp\left(\frac{1}{2} \hbar \frac{d}{dx} \right) \mathcal{E}_0 \left(-\hbar \frac{d}{dx} \right) (\log x) \right)^n}{n!} \mathcal{E}_{-1}(0)^d \right| \right\rangle^{\circ}.$$

4.3. **Proof of Lemma 4.1.** The starting point of the proof is (4.4). Note that only negative b_i contribution comes from $\langle \tau_{-2}(\omega) \rangle_{0,1}^{0} = 1$, which is the coefficient of x_i^{-1} in $\langle | \mathcal{E}_0 (x_i) | \rangle^{\circ}$.

Let $A(x) = \sum_{i=-1}^{\infty} a_i x^i$ be an arbitrary Laurent series. Observe that

$$A \left(-\hbar \frac{d}{dx} \right) (\log x) = a_{-1} \left(\frac{x - x \log x}{\hbar} \right) + a_0 \log x - \sum_{i=1}^{\infty} a_i \frac{(i-1)! \hbar^i}{x^i}. \tag{4.8}$$

We can apply this observation to the correlator

$$\frac{1}{(d!)^2} \left\langle \left| \mathcal{E}_1(0)^d \prod_{i=1}^{n} \mathcal{E}_0 (x_i)\, \mathcal{E}_{-1}(0)^{-d} \right| \right\rangle^{\circ} \tag{4.9}$$

and change $\mathcal{E}_0 (x_i)$ to

$$\mathcal{E}_0 \left(-\hbar \frac{\partial}{\partial x_i} \right) \log x_i.$$

If $(n,d) \neq (1,0)$, then we have a formal Laurent series in $x_1, \ldots, x_n$, where the degree of each variable in each term is less than or equal to -1. Together with the computation of the degree of $\hbar$, which is $\sum_{i=1}^{n}(b_i+1)-2d = 2g-2+n$, we establish Equation (4.5).

If $(n,d)=(1,0)$, then it is sufficient to observe that $\langle|\mathcal{E}_0(x)|\rangle^\circ = x^{-1}+O(x)$. Thus we have one additional term $(x - x\log x)/\hbar$ as in (4.8), which is exactly the first term in Equation (4.6).

This completes the proof of Lemma 4.1, and hence, Corollary 4.2.

5. Reduction to a combinatorial problem

The expression (4.7) of $\log\Psi$ in the form of the vacuum expectation value of the operator product allows us to convert the quantum curve equation (1.9) into a combinatorial formula.

Our starting point is the Ψ-function represented in the form

$$\Psi(x,\hbar) = 1 + \sum_{d=0}^{\infty} \frac{1}{\hbar^{2d}(d!)^2} \left\langle \left| \mathcal{E}_1(0)^d \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{A}(x)^n \mathcal{E}_{-1}(0)^d \right| \right\rangle^{\star}, \tag{5.1}$$

where

$$\begin{aligned} \mathcal{A}(x) &= \exp\left(\frac{\hbar}{2}\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x) \\ &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \exp\left(\left(-k+\frac{1}{2}\right)\hbar\frac{d}{dx}\right)(\log x) :\psi_k\psi_k^*: \\ &\quad + B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x - x\log x}{\hbar}\right). \end{aligned} \tag{5.2}$$

Here $B(t) := t/(e^t-1)$ in (5.2) is the generating series of the Bernoulli numbers, and the notation $\langle - \rangle^\star$ in (5.1) means that in the computation of this expectation value using the commutation relations, we never allow any $\mathcal{E}_1(0)$ and $\mathcal{E}_{-1}(0)$ to commute directly. We need this requirement since we exponentiate the series (4.7), which does not have terms without $\mathcal{E}_0$-operators. The goal of this section is to prove Corollary 5.2.

Lemma 5.1. *We have*

$$\exp\left(\frac{1}{\hbar^2}\right)\Psi(x,\hbar) = \exp\left(B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x - x\log x}{\hbar}\right)\right)X, \tag{5.3}$$

where $X := \sum_{d=0}^{\infty} X_d/\hbar^{2g}$, *and* X_d *is given by*

(5.4)

$$\begin{aligned} X_d &= \frac{1}{(d!)^2}\left\langle \left| \mathcal{E}_1(0)^d \exp\left(\sum_{k\in\mathbb{Z}+\frac{1}{2}} \log\left(x - \left(k-\frac{1}{2}\right)\hbar\right) :\psi_k\psi_k^*:\right)\mathcal{E}_{-1}(0)^d \right| \right\rangle \\ &= \sum_{\lambda\vdash d}\left(\frac{\dim\lambda}{d!}\right)^2 \prod_{i=1}^{\infty}\frac{x+(i-\lambda_i)\hbar}{x+i\hbar}. \end{aligned}$$

Corollary 5.2. *The quantum spectral curve equation*

$$\left[\exp\left(\hbar\frac{d}{dx}\right)+\exp\left(-\hbar\frac{d}{dx}\right)-x\right]\Psi(x,\hbar)=0$$

is equivalent to the following equation for the function X:

$$\left[\frac{1}{x+\hbar}\exp\left(\hbar\frac{d}{dx}\right)+x\exp\left(-\hbar\frac{d}{dx}\right)-x\right]X=0. \tag{5.5}$$

Proof of Lemma 5.1. Corollary 4.2 implies that

$$\sum_{d=0}^{\infty}\frac{\left\langle\left|\mathcal{E}_1(0)^d\exp\left(\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\right)\mathcal{E}_{-1}(0)^d\right|\right\rangle^\circ}{\hbar^{2d}(d!)^2} \tag{5.6}$$
$$=\log\Psi(x,\hbar)+\frac{1}{\hbar^2}+1.$$

Indeed, we add terms with $n=0$, and it is easy to see that

$$\left\langle\left|\mathcal{E}_1(0)^d\mathcal{E}_{-1}(0)^d\right|\right\rangle^\circ=0,\qquad d\geq 2,$$

and $\langle|\mathcal{E}_1(0)\mathcal{E}_{-1}(0)|\rangle^\circ=\langle|\mathrm{Id}|\rangle^\circ=1$. Therefore,

$$\exp\left(\frac{1}{\hbar^2}\right)\Psi(x,\hbar)= \tag{5.7}$$
$$\sum_{d=0}^{\infty}\frac{\left\langle\left|\mathcal{E}_1(0)^d\exp\left(\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\right)\mathcal{E}_{-1}(0)^d\right|\right\rangle}{\hbar^{2d}(d!)^2}.$$

From the definition of the operator $\mathcal{E}_0$, we have

$$\begin{aligned}
&\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\\
&=\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\left(\sum_{k\in\mathbb{Z}+1/2}\log\left(x-k\hbar\right):\psi_k\psi_k^*:\right)\\
&\quad+\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\frac{-\hbar\frac{d}{dx}}{\exp\left(-\frac{1}{2}\hbar\frac{d}{dx}\right)-\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)}\left(\frac{x-x\log x}{\hbar}\right)\\
&=\sum_{k\in\mathbb{Z}+1/2}\log\left(x-\left(k-\frac{1}{2}\right)\hbar\right):\psi_k\psi_k^*:+B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x-x\log x}{\hbar}\right).
\end{aligned} \tag{5.8}$$

Now define

$$A_1=\sum_{k\in\mathbb{Z}+1/2}\log\left(x-\left(k-\frac{1}{2}\right)\hbar\right):\psi_k\psi_k^*: \tag{5.9}$$

$$A_2=B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x-x\log x}{\hbar}\right). \tag{5.10}$$

Since A_1 and A_2 commute, we have $\exp(A_1+A_2) = \exp(A_2)\exp(A_1)$. Furthermore, since A_2 is a scalar operator, we have

$$\sum_{d=0}^{\infty} \frac{\left\langle \left| \mathcal{E}_1(0)^d \exp(A_2)\exp(A_1)\mathcal{E}_{-1}(0)^d \right| \right\rangle}{\hbar^{2d}(d!)^2}$$

$$= \exp(A_2) \sum_{d=0}^{\infty} \frac{\left\langle \left| \mathcal{E}_1(0)^d \exp(A_1)\mathcal{E}_{-1}(0)^d \right| \right\rangle}{\hbar^{2d}(d!)^2}.$$

This is exactly the right-hand side of Equation (5.3). □

Proof of Corollary 5.2. We just have to show that

$$\exp(-A_2)\exp\left(\hbar\frac{d}{dx}\right)\exp(A_2) = \frac{1}{x+\hbar}\exp\left(\hbar\frac{d}{dx}\right);$$
$$\exp(-A_2)\exp\left(-\hbar\frac{d}{dx}\right)\exp(A_2) = x\exp\left(-\hbar\frac{d}{dx}\right);$$
$$\exp(-A_2)x\exp(A_2) = x.$$

The last equality is tautological, and the first two are obtained by a straightforward computation. □

For completeness, let us also explain Equation (5.4). It is based on several standard facts about the semi-infinite wedge formalism. For any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots)$ we associate a basis vector $v_\lambda \in V_0$ given by

$$\underline{\left(\lambda_1 - \frac{1}{2}\right)} \wedge \underline{\left(\lambda_2 - \frac{3}{2}\right)} \wedge \underline{\left(\lambda_3 - \frac{5}{2}\right)} \wedge \cdots. \tag{5.11}$$

Then, we have $\mathcal{E}_{-1}(0)^d v_\emptyset = \sum_{\lambda \vdash d} \dim\lambda \cdot v_\lambda$, $\left\langle \left| \mathcal{E}_1(0)^d v_\lambda \right. = \dim\lambda$, and the fact that for any constants a_n, $n \in \mathbb{Z}+1/2$, v_λ is an eigenvector of the operator $\sum_{n\in\mathbb{Z}+1/2} a_n : \psi_n\psi_n^* :$ with the eigenvalue $\sum_{i=1}^{\infty}\left(a_{\lambda_i - i+1/2} - a_{-i+1/2}\right)$. Therefore, v_λ is an eigenvector of the operator

$$A_1 = \exp\left(\sum_{k\in\mathbb{Z}+\frac{1}{2}} \log\left(x - \left(k-\frac{1}{2}\right)\hbar\right) : \psi_k\psi_k^* :\right) \tag{5.12}$$

with the eigenvalue

$$\exp\left(\sum_{i=1}^{\infty} \log\left(x+(i-\lambda_i)\hbar\right) - \log\left(x+i\hbar\right)\right) = \prod_{i=1}^{\infty}\frac{x+(i-\lambda_i)\hbar}{x+i\hbar}, \tag{5.13}$$

and the total weight of the vector v_λ in $\left\langle \left| \mathcal{E}_1(0)^d A_1 \mathcal{E}_{-1}(0)^d \right| \right\rangle$ is $(\dim\lambda)^2$. This implies Equation (5.4).

6. Key combinatorial argument

We have shown that the quantum curve equation (1.9) is equivalent to a combinatorial equation (5.5), which is indeed a first-order recursion equation for X_d of (5.4) with respect to the index d. In this section we prove (5.5).

Let $\lambda \vdash d$ be a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0)$ of $d \geq 1$. We can always append it with $d - \ell(\lambda)$ zeros $\lambda_{\ell(\lambda)+1} := 0, \ldots, \lambda_d := 0$ at the end so that we would have a partition of d of length d with non-negative parts. Throughout this section we use this convention that a partition of d has length d.

Consider the following sum over all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d)$ of $d \geq 1$:

$$X_d := \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \prod_{i=1}^{d} \frac{x + (i - \lambda_i)\hbar}{x + i\hbar}. \tag{6.1}$$

Here $H_\lambda := \prod_{ij} h_{ij}$, where h_{ij} is the hook length at the vertex (ij) of the corresponding Young diagram, so that $d!/\prod h_{ij}$ is the dimension of the irreducible representation corresponding to λ. Or equivalently, it is the number of the standard Young tableaux of this shape. We use the convention that $X_0 := 1$.

In this Section we prove the following key combinatorial lemma.

Lemma 6.1. *The series $X := \sum_{d=0}^{\infty} X_d/\hbar^{2g}$ satisfies the following equation:*

$$\left[\frac{1}{x+\hbar} \exp\left(\hbar \frac{d}{dx}\right) + x \exp\left(-\hbar \frac{d}{dx}\right) - x\right] X = 0. \tag{6.2}$$

Proof. In fact, (6.2) is a direct consequence of the following more refined statement.

Lemma 6.2. *For any $d \geq 1$ we have*

$$\frac{1}{x/\hbar + 1} \exp\left(\hbar \frac{d}{dx}\right) X_{d-1} + \left[\frac{x}{\hbar} \exp\left(-\hbar \frac{d}{dx}\right) - \frac{x}{\hbar}\right] X_d = 0. \tag{6.3}$$

Indeed, since $\left[x \exp\left(-h\frac{d}{dx}\right) - x\right] X_0 = 0$, the sum of Equation (6.3) for all $d \geq 1$ with coefficients $1/\hbar^{2d-1}$ yields Lemma 6.1. $\square$

To prove Lemma 6.2, we need to recall some standard facts on the hook length formula as well as a recent result of Han [15].

6.1. Hook lengths and shifted parts of partition.

We use the following result from [15]. For a partition $\lambda \vdash d$, $d \geq 1$, we define the so-called g-function:

$$g_\lambda(y) := \prod_{i=1}^{d} (y + \lambda_i - i). \tag{6.4}$$

For any $\lambda \vdash d$, $d \geq 1$, we denote by $\lambda \setminus 1$ the set of all partitions of $d-1$ that can be obtained from λ (or rather the corresponding Young diagram) by removing one corner of λ.

Lemma 6.3 (Han [15]). *For every partition λ we have*

$$\frac{1}{H_\lambda}\left(g_\lambda(y+1)-g_\lambda(y)\right)=\sum_{\mu\in\lambda\backslash 1}\frac{1}{H_\mu}g_\mu(y). \tag{6.5}$$

Here y is a formal variable.

We need the following corollary of this lemma.

Corollary 6.4. *For an integer $d\geq 1$ we have*

$$\sum_{\lambda\vdash d+1}\frac{1}{H_\lambda^2}\left(g_\lambda(y+1)-g_\lambda(y)\right)=\sum_{\mu\vdash d}\frac{1}{H_\mu^2}g_\mu(y). \tag{6.6}$$

Proof. We recall that for any $\mu\vdash d$, $d\geq 1$, we have:

$$\sum_{\substack{\lambda\vdash d+1\\ \lambda\backslash 1\ni\mu}}\frac{1}{H_\lambda}=\frac{1}{H_\mu}. \tag{6.7}$$

Therefore,

$$\begin{aligned}\sum_{\mu\vdash d}\frac{1}{H_\mu^2}g_\mu(y) &= \sum_{\mu\vdash d}\frac{1}{H_\mu}\sum_{\substack{\lambda\vdash d+1\\ \lambda\backslash 1\ni\mu}}\frac{1}{H_\lambda}g_\mu(y)\\ &= \sum_{\lambda\vdash d+1}\frac{1}{H_\lambda}\sum_{\substack{\mu\vdash d\\ \mu\in\lambda\backslash 1}}\frac{1}{H_\lambda}g_\mu(y)\\ &= \sum_{\lambda\vdash d+1}\frac{1}{H_\lambda^2}\left(g_\lambda(y+1)-g_\lambda(y)\right).\end{aligned}$$

$\square$

6.2. **Reformulation of Lemma 6.2 in terms of g-functions.** We make the following substitution: $y:=-x/\hbar$. Then we see that

$$X_d=\sum_{\lambda\vdash d}\frac{1}{H_\lambda^2}\frac{g_\lambda(y)}{\prod_{i=1}^d(y-i)}.$$

Moreover,

$$\begin{aligned}&\frac{1}{x/\hbar+1}\exp\left(\hbar\frac{d}{dx}\right)X_{d-1}+\left[\frac{x}{\hbar}\exp\left(-\hbar\frac{d}{dx}\right)-\frac{x}{\hbar}\right]X_d\\ &=\frac{-1}{y-1}\exp\left(-\frac{d}{dy}\right)X_{d-1}+\left[-y\exp\left(\frac{d}{dy}\right)+y\right]X_d.\end{aligned} \tag{6.8}$$

Observe that

$$\begin{aligned}
\frac{-1}{y-1}\exp\left(-\frac{d}{dy}\right)X_{d-1} &= -\sum_{\lambda\vdash d-1}\frac{1}{H_\lambda^2}\frac{g_\lambda(y-1)}{\prod_{i=1}^d(y-i)}; \\
-y\exp\left(\frac{d}{dy}\right)X_d &= (d-y)\sum_{\lambda\vdash d}\frac{1}{H_\lambda^2}\frac{g_\lambda(y+1)}{\prod_{i=1}^d(y-i)}; \\
yX_d &= y\sum_{\lambda\vdash d}\frac{1}{H_\lambda^2}\frac{g_\lambda(y)}{\prod_{i=1}^d(y-i)}.
\end{aligned} \tag{6.9}$$

Using Corollary 6.4 we can rewrite the right hand side of Equation (6.9) as

$$\frac{-1}{y-1}\exp\left(-\frac{d}{dy}\right)X_{d-1} = \sum_{\lambda\vdash d}\frac{1}{H_\lambda^2}\frac{g_\lambda(y-1)-g_\lambda(y)}{\prod_{i=1}^d(y-i)}. \tag{6.10}$$

Therefore, the right hand side of Equation (6.8) is equal to

$$\frac{Y_d(y)}{\prod_{i=1}^d(y-i)}, \tag{6.11}$$

where

$$Y_d(y) := \sum_{\lambda\vdash d}\frac{(d-y)g_\lambda(y+1)+(y-1)g_\lambda(y)+g_\lambda(y-1)}{H_\lambda^2}. \tag{6.12}$$

Note that $Y_d(y)$ is a polynomial in y of degree $\leq d+1$, and Lemma 6.2 is equivalent to the following statement:

Lemma 6.5. *For any $d\geq 1$ we have $Y_d(y)\equiv 0$.*

6.3. **Proof of Lemma 6.5.** In this subsection we prove Lemma 6.5 and, therefore, Lemmas 6.2 and 6.1.

First of all, it is easy to check that for any $d\geq 1$ the polynomial $Y_d(y)$ has at least one root. Namely,

$$Y_d(d) = \sum_{\lambda\vdash d}\frac{(d-1)g_\lambda(d)+g_\lambda(d-1)}{H_\lambda^2} = 0. \tag{6.13}$$

Indeed, $g_\lambda(d)$ is not equal to zero only for $\lambda=(1,1,\dots,1)$. In this case $g_\lambda(d)=d!$, $H_\lambda=d!$, and $(d-1)g_\lambda(d)/H_\lambda^2=(d-1)/d!$. Notice that $g_\lambda(d-1)$ does not vanish only for $\lambda=(2,1,1,\dots,1,0)$. In this case $g_\lambda(d-1)=-d\cdot(d-2)!$, $H_\lambda=d\cdot(d-2)!$, and $g_\lambda(d-1)/H_\lambda^2=-(d-1)/d!$. Thus we see that always $Y_d(d)=0$, establishing (6.13).

Now we proceed by induction. It is easy to check that $Y_1(y) \equiv 0$. Assume that we know that $Y_d(y) \equiv 0$. Corollory 6.4 then implies that

$$\begin{aligned}
Y_d(y) = &\sum_{\lambda \vdash d} \frac{(d-y)g_\lambda(y+1) + (y-1)g_\lambda(y) + g_\lambda(y-1)}{H_\lambda^2} \\
&= \sum_{\lambda \vdash d+1} \frac{(d-y)g_\lambda(y+2) + (2y-d-1)g_\lambda(y+1) + (2-y)g_\lambda(y) - g_\lambda(y-1)}{H_\lambda^2} \\
&= \sum_{\lambda \vdash d+1} \frac{((d+1)-(y+1))g_\lambda(y+2) + ((y+1)-1)g_\lambda(y+1) + g_\lambda(y)}{H_\lambda^2} \\
&\quad - \sum_{\lambda \vdash d+1} \frac{((d+1)-y)g_\lambda(y+1) + (y-1)g_\lambda(y) + g_\lambda(y-1)}{H_\lambda^2} \\
&= Y_{d+1}(y+1) - Y_{d+1}(y).
\end{aligned}$$

By assumption, we have $Y_d(y) \equiv 0$. Therefore, $Y_{d+1}(y+1) = Y_{d+1}(y)$ for any y. HenceY_{d+1} is constant. Since we have shown that $Y_{d+1}(d+1) = 0$, we conclude that $Y_{d+1} \equiv 0$.

This completes the proof of Lemmas 6.5, 6.2, and 6.1. Thus we have established the main theorem of this paper.

7. Laguerre polynomials

In this section we prove a combinatorial expression for the functions

$$X_d := \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \prod_{i=1}^{d} \frac{x + (i - \lambda_i)\hbar}{x + i\hbar}$$

in terms of the *Laguerre polynomials* $L_n^{(\alpha)}(z)$.

The Laguerre polynomial is a solution of the differential equation

$$z\frac{d^2}{dz^2}L_n^{(\alpha)}(z) + (\alpha + 1 - z)\frac{d}{dz}L_n^{(\alpha)}(z) + nL_n^{(\alpha)}(z) = 0,$$

and has a closed expression

$$L_n^{(\alpha)}(z) = \sum_{i=0}^{n} (-1)^i \binom{n+a}{n-i} \frac{z^i}{i!}.$$

Here is a list of some properties of the Laguerre polynomials:

$$\frac{L_n^\alpha(z)}{\binom{n+\alpha}{n}} = 1 - \sum_{j=1}^{n} \frac{z^j}{a+j} \frac{L_{n-j}^{(j)}(z)}{(j-1)!}; \tag{7.1}$$

$$nL_n^\alpha(z) = (n+\alpha)L_{n-1}^\alpha(z) - zL_{n-1}^{\alpha+1}(z); \tag{7.2}$$

$$L_n^\alpha(z) = L_n^{\alpha+1}(z) - L_{n-1}^{\alpha+1}(z); \tag{7.3}$$

$$\binom{n+\alpha}{n} = \sum_{i=0}^{n} \frac{z^i}{i!} L_{n-i}^{(\alpha+i)}(z). \tag{7.4}$$

Proposition 7.1. *For any $d \geq 0$ we have:*

$$X_d = \frac{1}{d!}\left(1 - \sum_{m=1}^{d} \frac{1}{(m-1)!} L_{d-m}^{(m)}(1) \frac{\hbar}{x+m\hbar}\right).$$

Remark 7.2. This equation can be rewritten as

$$X_d = \frac{1}{d!} \frac{L_d^{x/\hbar}(1)}{L_d^{x/\hbar}(0)}.$$

Indeed, we just apply the identity (7.1) for $\alpha = x/\hbar$ and $z = 1$ and further observe that $\binom{n+x/\hbar}{n} = L_n^{(x/\hbar)}(0)$.

Proof of Proposition 7.1. It is obvious that both X_d and

$$\tilde{X}_d := \frac{1}{d!}\left(1 - \sum_{m=1}^{d} \frac{1}{(m-1)!} L_{d-m}^{(m)}(1) \frac{\hbar}{x+m\hbar}\right) \tag{7.5}$$

are rational functions with simple poles at $x/\hbar = -1, -2, \ldots, -d$. We have defined $X_0 := 1$, and it is easy to see that $Z_0 = 1$. Then we know (see Lemma 6.2) that all X_d are unambiguously determined by the equation

$$\frac{1}{\frac{x}{\hbar}+1} \exp\left(\hbar \frac{d}{dx}\right) X_d + \left[\frac{x}{\hbar} \exp\left(-\hbar \frac{d}{dx}\right) - \frac{x}{\hbar}\right] X_{d+1} = 0 \tag{7.6}$$

for all $d \geq 0$. In order to prove the proposition we check that $\{\tilde{X}_d\}_{d \geq 0}$ also satisfy this equation.

Indeed, observe that

$$\begin{aligned}
&\frac{1}{\frac{x}{\hbar}+1} \exp\left(\hbar \frac{d}{dx}\right) \tilde{X}_d \\
&= \frac{1}{d!}\left(\frac{1}{\frac{x}{\hbar}+1} - \sum_{m=1}^{d} \frac{L_{d-m}^{(m)}(1)}{(m-1)!} \frac{1}{(\frac{x}{\hbar}+1)(\frac{x}{\hbar}+m+1)}\right) \\
&= \frac{1}{\frac{x}{\hbar}+1} \frac{1}{d!}\left(1 - \sum_{m=1}^{d} \frac{L_{d-m}^{(m)}(1)}{m!}\right) + \frac{1}{d!} \sum_{m=2}^{d+1} \frac{L_{d+1-m}^{(m-1)}(1)}{(m-1)!} \frac{1}{(\frac{x}{\hbar}+m)};
\end{aligned} \tag{7.7}$$

$$
\begin{aligned}
(7.8)\qquad & \frac{x}{\hbar}\exp\left(-\hbar\frac{d}{dx}\right)\tilde{X}_{d+1} \\
&= \frac{1}{(d+1)!}\left(\frac{x}{\hbar}-\sum_{m=1}^{d+1}\frac{L^{(m)}_{d+1-m}(1)}{(m-1)!}\frac{\frac{x}{\hbar}}{\frac{x}{\hbar}+m-1}\right) \\
&= \frac{\frac{x}{\hbar}}{(d+1)!}-\frac{1}{(d+1)!}\sum_{m=1}^{d+1}\frac{L^{(m)}_{d+1-m}(1)}{(m-1)!}+\frac{1}{(d+1)!}\sum_{m=1}^{d}\frac{L^{(m+1)}_{d-m}(1)}{(m-1)!}\frac{1}{\frac{x}{\hbar}+m};
\end{aligned}
$$

and

$$
\begin{aligned}
(7.9)\qquad -\frac{x}{\hbar}\tilde{X}_{d+1} &= \frac{1}{(d+1)!}\left(-\frac{x}{\hbar}+\sum_{m=1}^{d+1}\frac{L^{(m)}_{d+1-m}(1)}{(m-1)!}\frac{\frac{x}{\hbar}}{\frac{x}{\hbar}+m}\right) \\
&= \frac{-\frac{x}{\hbar}}{(d+1)!}+\frac{1}{(d+1)!}\sum_{m=1}^{d+1}\frac{L^{(m)}_{d+1-m}(1)}{(m-1)!}-\frac{1}{(d+1)!}\sum_{m=1}^{d+1}\frac{L^{(m)}_{d+1-m}(1)}{(m-1)!}\frac{m}{\frac{x}{\hbar}+m}.
\end{aligned}
$$

It is obvious that in the sum of the expressions (7.7), (7.8), and (7.9) the coefficient of $x/\hbar$ and the constant term vanish. So, we have to prove that the coefficient of each $1/(x/\hbar+m)$, $m=1,\dots,d+1$, vanishes.

The coefficient of $1/(x/\hbar+d+1)$ is equal to

$$
\frac{1}{d!}\frac{L^{(d)}_0(1)}{d!}-\frac{1}{(d+1)!}\frac{(d+1)L^{(d+1)}_0(1)}{d!},
$$

which is equal to zero since $L^{(d)}_0=L^{(d+1)}_0=1$.

The coefficient of $1/(x/\hbar+m)$, $2\le m\le d$, is equal to

$$
\frac{1}{d!}\frac{L^{(m-1)}_{d+1-m}(1)}{(m-1)!}+\frac{1}{(d+1)!}\frac{L^{(m+1)}_{d-m}(1)}{(m-1)!}-\frac{1}{(d+1)!}\frac{mL^{(m)}_{d+1-m}(1)}{(m-1)!}. \tag{7.10}
$$

First we use Equation (7.2) for $z=1$, $\alpha=m$, and $n=d+m-1$:

$$
(d+1-m)L^{(m)}_{d+1-m}(1)=(d+1)L^{(m)}_{d-m}(1)-L^{(m+1)}_{d-m}(1). \tag{7.11}
$$

We see then that expression (7.10) is equal to

$$
\frac{1}{d!(m-1)!}\left(L^{(m-1)}_{d+1-m}(1)-L^{(m)}_{d+1-m}(1)+L^{(m)}_{d-m}(1)\right).
$$

And this is equal to zero due to Equation (7.3) for $n=d+1-m$, $\alpha=m-1$, and $z=1$.

The coefficient of $1/(x/\hbar+1)$ is equal to

$$
\frac{1}{d!}\left(1-\sum_{m=1}^{d}\frac{L^{(m)}_{d-m}(1)}{m!}\right)+\frac{1}{(d+1)!}L^{(2)}_{d-1}(1)-\frac{1}{(d+1)!}L^{(1)}_{d}(1).
$$

Using that $L^{(2)}_{d-1}(1) = -dL^{(1)}_d(1)+(d+1)L^{(1)}_{d-1}(1)$ (which is Equation (7.2) for $z = 1$, $n = d$, and $\alpha = 1$) and $-L^{(1)}_d(1) + L^{(1)}_{d-1}(1) = -L^{(0)}_d(1)$ (which is Equation (7.3) for $z = 1$, $d = n$, and $\alpha = 0$), we see that this coefficient is equal to

$$\frac{1}{d!}\left(1-\sum_{m=0}^{d}\frac{L^{(m)}_{d-m}(1)}{m!}\right).$$

This is equal to zero due to Equation (7.4) for $z = 1$, $\alpha = 0$, and $n = d$.

Thus we see that the sum of expressions (7.7), (7.8), and (7.9) is equal to zero. So, the functions $\tilde{X}_d$ satisfy Equation (7.6), and, therefore, $\tilde{X}_d = X_d$ for all $d \geq 0$. □

8. Toda lattice equation

In this section we recall, for completeness, the Toda lattice equation for the partition function of the Gromov-Witten invariants of $\mathbb{P}^1$. We show that the Toda lattice equation implies a quadratic relation for the functions X_d, $d \geq 0$, considered in the previous sections. It is an open question whether it is possible to relate the Toda lattice equation to the quantum spectral curve equation.

It is convenient to include a degree variable q in the free energy of the Gromov-Witten invariants of $\mathbb{P}^1$. Define

$$\mathcal{F}_g := \sum_d q^d \left\langle \exp\left\{\sum_{i\geq 0}^{\infty}\tau_i(\omega)t_i + \tau_0(1)t\right\}\right\rangle^d_g \tag{8.1}$$

(where we have switched off $\tau_k(1)$ for $k > 0$). Then $\mathcal{F} = \sum_{g=0}^{\infty}\mathcal{F}_g$ satisfies the Toda lattice equation:

$$\exp(\mathcal{F}(t+1) + \mathcal{F}(t-1) - 2\mathcal{F}(t)) = \frac{1}{q}\frac{\partial^2}{\partial t_0^2}\mathcal{F}(t), \tag{8.2}$$

which was conjectured by Eguchi-Yang [11] and proven by Dubrovin-Zhang [6] and Okounkov-Pandharipande [18, Equation (4.11)].

We specialise

$$\Phi(x,\hbar,q) := F\left(q = \frac{q}{\hbar^2},\ t = -\frac{1}{2},\ t_i = -i!\left(\frac{\hbar}{x}\right)^{i+1}\right) \tag{8.3}$$

and consider the function $\exp\{\Phi(x,\hbar,q) - \Phi(x,\hbar,0)\}$.

Lemma 8.1. *We have:*

$$\exp\{\Phi(x,\hbar,q) - \Phi(x,\hbar,0)\} = \sum_{d=0}^{\infty}\left(\frac{q}{\hbar^2}\right)^d X_d. \tag{8.4}$$

Proof. Indeed, tracing back the arguments of Sections 3 and 4 and taking q into account this time, it is easy to see that

$$\Phi(x,\hbar,q) = \sum_{d=0}^{\infty} \left(\frac{q}{\hbar^2}\right)^d \sum_{g=0}^{\infty} \left\langle \exp\left(-\frac{1}{2}\tau_0(1) - \sum_{i=0}^{\infty} \tau_i(\omega) i! \left(\frac{\hbar}{x}\right)^{i+1}\right)\right\rangle_g^d . \tag{8.5}$$

The proof of Lemma 5.1 implies that

$$\begin{aligned} \Phi(x,\hbar,0) &= \sum_{g=0}^{\infty} \left\langle \exp\left(-\frac{1}{2}\tau_0(1) - \sum_{i=0}^{\infty} \tau_i(\omega) i! \left(\frac{\hbar}{x}\right)^{i+1}\right)\right\rangle_g^0 \\ &= B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x - x\log x}{\hbar}\right) \end{aligned} \tag{8.6}$$

These two observations and Equation (5.3) imply Equation (8.4). ☐

By abuse of notation, we denote the function $\exp\{\Phi(x,\hbar,q) - \Phi(x,\hbar,0)\}$ also by X (so-called degree-weighted X). The quantum spectral curve equation for this degree-weighted X reads

$$\left[\frac{q}{x+\hbar}\exp\left(\hbar\frac{d}{dx}\right) + x\exp\left(-\hbar\frac{d}{dx}\right) - x\right] X = 0. \tag{8.7}$$

The Toda lattice equation combined with the string and the divisor equations implies the following equation for X:

Proposition 8.2. *We have:*

$$\frac{X(x+\hbar)X(x-\hbar)}{X(x)^2} = \frac{x+\hbar}{x}\frac{\partial}{\partial q}\left(q\frac{\partial}{\partial q}\log X(x)\right). \tag{8.8}$$

Proof. We recall the divisor equation

$$\frac{\partial \mathcal{F}}{\partial t_0} = \frac{1}{2}t^2 + q\frac{\partial \mathcal{F}}{\partial q}. \tag{8.9}$$

Consider Equations (8.2). The result of Section 3 implies that the shifts in $\tau_0(1)$-coefficient t can be replaced by shifts of variable x with factor $\hbar$. Using this and Equation (8.9) we obtain equation for $\Phi(x,\hbar,q)$:

$$\exp\left(\Phi(x+\hbar,\hbar,q) + \Phi(x-\hbar,\hbar,q) - 2\Phi(x,\hbar,q)\right) = \frac{\partial}{\partial q}\left(q\frac{\partial}{\partial q}\Phi(x,\hbar,q)\right). \tag{8.10}$$

From the definition of degree-weighted X it follows that

$$\frac{\partial}{\partial q}\left(q\frac{\partial}{\partial q}\log X(x)\right) = \frac{\partial}{\partial q}\left(q\frac{\partial}{\partial q}\Phi_q(x)\right). \tag{8.11}$$

Therefore, in order to prove Equation (8.8), it is enough to show that

$$\frac{\exp\left(\Phi(x+\hbar,\hbar,0)\right)\exp\left(\Phi(x-\hbar,\hbar,0)\right)}{\exp\left(2\Phi(x,\hbar,0)\right)} = \frac{x}{x+\hbar}. \tag{8.12}$$

Indeed, using Equation (8.6) we can represent the left hand side of Equation (8.12) in the following form:

$$\begin{aligned}\exp\left(\left[\left(e^t+e^{-t}-2\right)B(t)\right]\Big|_{t=-\hbar\frac{d}{dx}} \left(\frac{x-x\log x}{\hbar}\right)\right) \\ = \exp\left(\left[\left(e^{t/2}-e^{-t/2}\right)^2 \frac{t}{e^t-1}\right]\Bigg|_{t=-\hbar\frac{d}{dx}} \left(\frac{x-x\log x}{\hbar}\right)\right) \\ = \exp\left(\log(x)-\log(x+\hbar)\right).\end{aligned} \tag{8.13}$$

□

The homogeneous in q part of the Toda lattice equation (8.8) for the series expansion given by Equation (8.4) can be rewritten as

$$\frac{x}{x+\hbar}\sum_{a+b=d} X_a(x+\hbar)X_b(x-\hbar) = \sum_{a+b=d+1} X_a(x)X_b(x)\frac{(a-b)^2}{2} \tag{8.14}$$

for any $d \geq 0$. We can use Equation (6.3) in order to rewrite this equation as

$$\begin{aligned}\sum_{a+b=d+1}\left(X_a(x)X_b(x-\hbar)-X_a(x-\hbar)X_b(x-\hbar)\right) \\ = \frac{\hbar^2}{x^2}\sum_{a+b=d+1} X_a(x)X_b(x)\frac{(a-b)^2}{2},\end{aligned} \tag{8.15}$$

for any $d \geq 0$. It would be interesting to see whether Equation (8.14) or Equation (8.15) can be related to Equation (6.3).

References

[1] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Mariño, and C. Vafa, *Topological Strings and Integrable Hierarchies*, Comm. Math. Phys. **261**, 451–516 (2006).

[2] G. Carlet, B. Dubrovin, and Y. Zhang, *The extended Toda hierarchy*, Mosc. Math. J. **4**, no. 2, 313–332 (2004).

[3] L. Chekhov, B. Eynard, and N. Orantin, *Free energy topological expansion for the 2-matrix model*, JHEP 0612:053 (2006).

[4] R. Dijkgraaf, L. Hollands, and P. Sułkowski, *Quantum curves and $\mathcal{D}$-modules*, Journal of High Energy Physics **0810.4157**, 1–58 (2009).

[5] R. Dijkgraaf, L. Hollands P. Sułkowski, and C. Vafa, *Supersymmetric gauge theories, intersecting branes and free Fermions*, Journal of High Energy Physics **0802.106**, (2008).

[6] B. Dubrovin and Y. Zhang, *Virasoro symmetries of the extended Toda hierarchy*, Comm. Math. Phys. **250**, no. 1, 161–193 (2004).

[7] O. Dumitrescu and M. Mulase, *Quantum curves for Hitchin fibrations and the Eynard-Orantin theory*, arXiv:1310.6022 math.AG (2013).

[8] O. Dumitrescu, M. Mulase, B. Safnuk, and A. Sorkin, *The spectral curve of the Eynard-Orantin recursion via the Laplace transform*, in "Algebraic and Geometric Aspects of Integrable Systems and Random Matrices," Dzhamay, Maruno and Pierce, Eds. Contemporary Mathematics **593**, 263–315 (2013).

[9] P Dunin-Barkowski, M Kazarian, N Orantin, S Shadrin, and L Spitz, *Polynomiality of Hurwitz numbers, Bouchard-Mariño conjecture, and a new proof of the ELSV formula*, arXiv preprint arXiv:1307.4729 (2013).
[10] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz, *Identification of the Givental formula with the spectral curve topological recursion procedure*, arXiv:1211.4021 (2012).
[11] T. Eguchi and S.-K. Yang, *The topological $\mathbb{P}^1$ model and the large-N matrix integral*, Modern Phys. Lett. A **9**, 2893–2902 (1994).
[12] B. Eynard and M. Mariño, *A holomorphic and background independent partition function for matrix models and topological strings*, J. Geom. Phys. **61**, 1181–1202 (2011).
[13] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Communications in Number Theory and Physics **1**, 347–452 (2007).
[14] S. Gukov and P. Sułkowski, *A-polynomial, B-model, and quantization*, Journal of High Energy Physics, 2012:70 (2012).
[15] Guo-Niu Han, Hook lengths and shifted parts of partitions, Ramanujan J. **23**, no. 1-3, 127–135 (2010).
[16] L. Hollands, *Topological strings and quantum curves*, Ph.D. Thesis, University of Amsterdam arXiv:0911.3413 [hep-th] (2009).
[17] P. Norbury and N. Scott, Gromov-Witten invariants of $\mathbf{P}^1$ and Eynard-Orantin invariants, arXiv:1106.1337 (2011).
[18] A. Okounkov and R. Pandharipande, Gromov-Witten theory, Hurwitz theory, and completed cycles. Ann. of Math. (2) **163**, no. 2, 517–560 (2006).
[19] S. Shadrin, L. Spitz, and D. Zvonkine, On double Hurwitz numbers with completed cycles. J. Lond. Math. Soc. (2) **86**, no. 2, 407–432 (2012).

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