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Regular Turán numbers and some Gan–Loh–Sudakov-type problems

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Abstract

Motivated by a Gan–Loh–Sudakov-type problem, we introduce the regular Turán numbers, a natural variation on the classical Turán numbers where we restrict ourselves to the class of regular graphs. Among other results, we prove a striking supersaturation version of Mantel's theorem in the case of a regular host graph of odd order. We also characterize the graphs for which the regular Turán numbers behave classically or otherwise.

KEYWORDS

cliques, Gan-Loh-Sudakov-type problem, order, regular graph, size, Turán number

1 | INTRODUCTION

Mantel's theorem [21], Turán's theorem [25] and the Erdős–Stone theorem [9] are foundational in extremal graph theory. The extremal graphs seem close to being regular. For example, the construction of C_4-free graphs with many edges using Sidon sets (see, e.g., [7]) is regular for even order and has difference 1 between the minimum and maximum degrees for odd order. Nevertheless, the restriction to regular graphs forces that the maximum number of edges, or, equivalently, maximum degree, when avoiding certain 3-chromatic graphs depends heavily on the parity of the order. To make this phenomenon more concrete, we use the following terminology.

Definition 1. The *regular Turán number* of a graph H is

$$\mathrm{ex}_r(n,H)=\max\{k: |V(G)|=n, G \text{ is } k\text{-regular and does not contain } H \text{ as a subgraph}\}.$$

For a family of graphs $\mathcal{H}$, $\mathrm{ex}_r(n,\mathcal{H})$ is defined similarly, so G must not contain any $H\in\mathcal{H}$.

The following result, focusing on odd cycles, is most illustrative.

Theorem 2. *For fixed $\ell \geq 2$ and $H = C_{2\ell-1}$, it holds for sufficiently large n that*

$$\mathrm{ex}_r(n,H)=\begin{cases}\frac{n}{2} & \text{if } n \text{ is even,}\\ 2\left\lfloor\frac{n}{2\ell+1}\right\rfloor & \text{if } n \text{ is odd.}\end{cases}$$

This is a generalization of a regular version of Mantel's theorem and we prove it in Section 2.

With an extremal result in hand, one may naturally pursue supersaturation results, as in, for example, [8,20,22,23]. The classical such result for Mantel's theorem states that the minimum number of triangles in a graph is linear in the number of additional edges. More precisely, if a graph has at least $\left\lfloor\frac{n^2}{4}\right\rfloor+\ell$ edges for some $0\leq\ell<\frac{n}{2}$, then it must contain at least $g_3(t_2(n)+\ell)=\ell\left\lfloor\frac{n}{2}\right\rfloor$ triangles. If a graph has at least γn^2 edges for $\gamma>\frac{1}{4}$, then the graph contains at least $\Omega_\gamma(n^3)$ triangles. In particular, if the average degree is $\frac{n}{2}+1$, there are at least $\Omega(n^2)$ triangles, while there are at least $\Omega(n^3)$ triangles when the average degree is at least $(0.5+\gamma)n$ for some constant $\gamma>0$.

On the other hand, supersaturation differs for regular Mantel's theorem, as the minimum number of triangles in a k-regular graph with odd order n is $\Theta(n^2)$ for every $\mathrm{ex}_r(n,C_3)<k<\mathrm{ex}_r(n,C_3)+\frac{n}{10}$. The following is shown in Section 3.

Theorem 3 (Supersaturated regular Mantel's theorem). *Let G be a k-regular graph on n vertices. If n is odd and $k>2\left\lfloor\frac{n}{5}\right\rfloor$, then G contains at least $\frac{1}{300}n^2$ triangles.*

Returning to regular Turán numbers, we show in Section 4 that for those H for which $\chi(H)\neq 3$ these numbers behave as in the classical Erdős–Stone theorem, that is, $\lim_{n\to\infty}\frac{1}{n}\mathrm{ex}_r(n,H)=1-\frac{1}{\chi(H)-1}$. There remains the question of which H with $\chi(H)=3$ also behaves classically, and so not like in Theorem 2. In Theorem 15, we consider two constructions—which roughly speaking are adaptations from complete bipartite graphs so that they have an odd number of vertices and are regular—and prove that they completely describe such H.

We describe our point of inspiration for the regular Turán numbers in Section 5, that is, some variations on a conjecture of Gan, Loh and Sudakov [10] (now confirmed, cf. [6]). In particular, we will consider a variation for which *both* order and size are prescribed, after a work by Kirsch and Radcliffe [17]. Although it does not behave as nicely as the original problem, the regular case (which one might consider as the most interesting of these variations) can be almost completely resolved with regular Turán numbers. A few other possible generalizations of the problem initiated in [10] are posed and briefly discussed in Section 6.

2 | REGULAR TURÁN NUMBERS OF ODD CYCLES

The relationship between the minimum degree of a graph and the existence of cycles or paths of certain lengths has been extensively studied. We begin by listing a few key results which are useful for determining the regular Turán numbers of odd cycles.

Theorem 4 (Andrásfai, Erdős and Sós [2]). *Let $\ell \geq 1$ and G be a nonbipartite graph with minimum degree $\delta > 2\frac{n}{2\ell+1}$, then G contains an odd cycle C_m with $m \leq 2\ell - 1$. In particular, if G is a nonbipartite graph with minimum degree $\delta > 2\frac{n}{5}$, then G contains a triangle.*

Theorem 5 (Voss and Zuluaga [26]). *Every 2-connected nonbipartite graph with minimum degree δ has an odd cycle of length at least* $\min\{2\delta - 1, n\}$.

Theorem 6 (Häggkvist [14]). *Let G be a graph with minimum degree $\delta > \frac{2n}{2\ell+1}$ and $n > \binom{\ell+1}{2}(2\ell + 1)(3\ell - 1)$. Then either G contains a $C_{2\ell-1}$ or it does not contain any odd cycle C_m for some $m > \frac{\ell}{2}$.*

Theorem 7 (Liu and Ma [19]). *Let G be a 2-connected bipartite graph, u, v two distinct vertices of G and d the minimum degree of the vertices in $G\backslash\{u, v\}$. Then there is a path between u and v of length at least* $2(d - 1)$.

2.1 | Regular Turán number of the triangle for all orders

Here we investigate the same problem as in [3], but from the standpoint of the order n instead of the regularity k.

Theorem 8 (Regular Mantel's theorem). *Let G be a k-regular, triangle-free graph on n vertices. When n is even, we have $k \leq \frac{n}{2}$. When n is odd, we have $k \leq 2\left\lfloor \frac{n}{5} \right\rfloor$. Moreover, these bounds are sharp. Put in another way,*

$$\mathrm{ex}_r(n, K_3) = \begin{cases} \dfrac{n}{2} & \textit{if } n \textit{ is even}, \\ 2\left\lfloor \dfrac{n}{5} \right\rfloor & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. When n is even, the result follows from the classical Mantel's theorem and complete bipartite graphs achieving equality. When n is odd, since G is regular, it cannot be bipartite.

By Theorem 4, we know $k \leq \frac{2}{5}n$. Due to the handshaking lemma, we know k has to be even and hence $k \leq 2\left\lfloor \frac{n}{5} \right\rfloor$ follows.

Now we show the sharpness of the result. Let $n = 5x + y$, with $0 \leq y \leq 4$ and $y < x$. Note that this will cover all cases where $n > 20$ is odd. Let S_1, S_2, S_3, S_4 and S_5 be stable sets of respective sizes $x + y, x, x - y, x$ and $x + y$. Add all edges between vertices of S_i and S_{i+1} for $1 \le i \le 4$ and let $G[S_1, S_5]$ be an x-regular bipartite graph (which can be obtained by removing y disjoint complete matchings of a complete bipartite graph $K_{x+y,x+y}$). Then the resulting graph G is a k-regular, triangle-free graph with $k = 2x = 2\left\lfloor \frac{n}{5} \right\rfloor$. Note that it is a classic blow-up of a 5-cycle when 5 divides n.

In the remaining cases we have $n \le 19$ (as we consider only odd n) and $n \ne 15$. Let the vertices be 1 up to n and connect i and j if $i - j \equiv \pm(2h + 1) \pmod{n}$ for some $0 \le h \le \left\lfloor \frac{n}{5} \right\rfloor - 1$. This results in a k-regular, triangle-free graph with $k = 2\left\lfloor \frac{n}{5} \right\rfloor$. For this note that $3\left(2\left\lfloor \frac{n}{5} \right\rfloor - 1\right) < n$ and no three numbers can have pairwise odd differences. □

2.2 | Asymptotic regular Turán numbers of odd cycles

We next show that the magnitude of $\mathrm{ex}_r(n, \mathcal{H})$ for different parity can differ by any factor. This is analogous to the main result in [27], but focusing on order instead of regularity.

Theorem 9. *For the family $\mathcal{H} = \{C_3, C_5 \ldots C_{2\ell-1}\}$, we have*

$$\mathrm{ex}_r(n, \mathcal{H}) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ 2\left\lfloor \frac{n}{2\ell + 1} \right\rfloor - o(1) & \text{if } n \text{ is odd}. \end{cases}$$

Proof. When n is odd, since G is regular, it cannot be bipartite. By Theorem 4, we know $k \le \frac{2}{2\ell + 1} n$. Due to the handshaking lemma, we know k has to be even and hence $k \le 2\left\lfloor \frac{n}{2\ell + 1} \right\rfloor$ follows.

Hence the main part is to show sharpness for large n, which will be obtained by taking a construction close to the blow-up of a $C_{2\ell+1}$. Let $M = 2\ell + 1$. Let $n = (2\ell + 1)x + y$ with $x > y$ and $0 \le y \le 2\ell$. Take $M = 2\ell + 1$ stable sets $S_1, S_2, \ldots, S_M$.

If ℓ is odd, equivalently $M \equiv 3 \pmod{4}$, we take them such that such that

$$|S_i| = \begin{cases} x & \text{if } i \text{ is odd}, \\ x + y & \text{if } i \equiv 2 \pmod{4}, \\ x - y & \text{if } i \equiv 0 \pmod{4}. \end{cases}$$

If ℓ is even, equivalently $M \equiv 1 \pmod{4}$, we take their sizes to be

$$|S_i| = \begin{cases} x & \text{if } i \text{ is even}, \\ x + y & \text{if } i \equiv 1 \pmod{4}, \\ x - y & \text{if } i \equiv 3 \pmod{4}. \end{cases}$$

For $1 \le i \le M$, connect every vertex in S_i with every vertex in S_{i+1}, where the indices are taken modulo M and remove y disjoint perfect matchings between S_1 and S_M. Now it is clear that the resulting graph is $2x$-regular, has odd girth M and order $Mx + y = n$. □

As we stated earlier in Theorem 2, it suffices for n large enough to exclude only the cycle $C_{2\ell-1}$ in Theorem 9.

To show this, we require the following.

Remark 10. For every unicyclic graph H having girth $2\ell - 1$, if a graph G with minimum degree larger than $|H|$ contains a $C_{2\ell-1}$, then it contains H as well, so $\mathrm{ex}_r(n, H) = \mathrm{ex}_r(n, C_{2\ell-1})$ for large n.

Theorem 11. *Let G be a graph of order $n > \binom{\ell+1}{2}(2\ell+1)(3\ell-1)$ with minimum degree $\delta > \frac{2n}{2\ell+1}$, such that deleting any collection of at most ℓ^3 edges of G does not result in a bipartite graph. Then G contains a $C_{2\ell-1}$.*

Proof. By Theorem 6, a counterexample G would not contain any odd cycle C_m for some $m > \frac{\ell}{2}$. First, iteratively, select cutvertices of the resulting graph and delete them. Note that one can have deleted at most $\ell - 1$ cutvertices at the end due to the minimum degree condition. Indeed, once having selected ℓ cutvertices there are at least $\ell + 1$ components, each contains a vertex of degree at least $\frac{2n}{2\ell+1} - \ell$, so this is a lower bound on its order. But now we would get that there the union of these components is at least $\frac{2(\ell+1)n}{2\ell+1} - \ell^2 > n$, contradiction.

For every component, look to the original part of the graph containing that component and the cutvertices adjacent to it. Every such part has at least $\frac{2n}{2\ell+1} + 1$ vertices, as it contains vertices (all vertices which were not a cutvertex) of degree at least $\frac{2n}{2\ell+1}$. In such a part, iteratively delete every (cut)vertex with minimum degree less than ℓ. Note that no noncutvertex can be deleted since $\delta > 2\ell$. The resulting part is 2-connected and has minimum degree at least ℓ. So if such a resulting part is nonbipartite, we can find an odd cycle of length larger than ℓ by Theorem 5, which gives the desired contradiction.

Note that in total, we have deleted at most $\ell(\ell - 1)$ edges in a single part and there are at most ℓ such parts. There are also no more than $\binom{\ell}{2}$ edges between cutvertices. Let G' be the graph obtained by deleting all the edges mentioned before. Since there are deleted at most $\ell^2(\ell - 1) + \binom{\ell}{2} \le \ell^3$ edges, G' is not a bipartite graph by the given assumption in the theorem. But since every part of G' is bipartite, there is an odd cycle which passes through multiple parts and hence cutvertices. Take the one containing the fewest number of cutvertices. This one will enter and leave every part exactly once. If not, the intersection of the odd cycle and a part contains at least 2 disjoint paths. Take the shortest path connecting two of these shortest paths. In the odd cycle, this connecting path divides the cycle in two, one of them being of odd length. But that path contains a smaller number of the cutvertices, from which the conclusion follows.

Take one such part H which has at least one edge in common with the smallest odd cycle and such that exactly two of its cutvertices u, v are on the odd cycle. By Theorem 7 we can find a path in H between u and v of length at least $2(\delta - \ell - 1) > \ell$. Since the length of every path between u and v will have the same parity, replacing the part of the odd cycle between u and v with this path gives the desired contradiction. □

Proof of Theorem 2. Note that if n is odd, one needs to delete at least $\frac{k}{2}$ edges from a k-regular graph on n vertices to obtain a bipartite graph. So if $n > \binom{\ell+1}{2}(2\ell+1)(3\ell-1)$ and $k > \frac{2n}{2\ell+1}$, the graph contains a $C_{2\ell-1}$ by Theorem 11. Again by the handshaking lemma, we know k is even and thus at most $2\left\lfloor \frac{n}{2\ell+1} \right\rfloor$. Sharpness for large n is due to the construction in the proof of Theorem 9. □

3 | SUPERSATURATION FOR REGULAR MANTEL'S THEOREM

Proof of Theorem 3. First note that $k \geq \frac{2n}{5} + \frac{2}{5}$ because k is even and $k/2 \geq \left\lfloor \frac{n}{5} \right\rfloor + 1 \geq \frac{n}{5} + \frac{1}{5}$.

We start with the following observation.

Claim 1. Every triangle of G intersects at least $\frac{1}{5}n - 2$ other triangles on an edge.

Proof. Let t be the number of triangles that intersect the triangle formed by the vertices u_1, u_2 and u_3 on an edge. Note that the number of triangles (including $u_1u_2u_3$) that contains the edge u_iu_j is equal to $|N(u_i) \cap N(u_j)|$, so $t = |N(u_1) \cap N(u_2)| + |N(u_2) \cap N(u_3)| + |N(u_3) \cap N(u_1)| - 3$. By the inclusion–exclusion principle,

$$n \geq |N(u_1) \cup N(u_2) \cup N(u_3)| \geq 3k - (t+3) \geq \frac{6}{5}n + \frac{6}{5} - t - 3.$$

It follows that $t > \frac{n}{5} - 2$. □

Let S be the set of vertices of G that are contained in at least $\frac{1}{10}n$ triangles. It follows from the definition that the total number of triangles is at least $\frac{1}{3} \cdot \frac{n}{10} \cdot |S|$, so the result holds if $|S| \geq \frac{n}{10}$. Thus, in the following, we assume that $|S| < \frac{n}{10}$.

Claim 2. Every triangle of G contains at least two vertices of S.

Proof. If otherwise, there is a triangle T with two vertices u_1 and u_2 that both intersect fewer than $\frac{1}{10}n$ triangles each, and then as every triangle intersecting T on an edge contains (at least) one of u_1 and u_2, it follows that T intersects fewer than $2 \cdot (\frac{1}{10}n - 1) = \frac{1}{5}n - 2$ other triangles on an edge, which contradicts Claim 1. □

Claim 3. $G \backslash S$ contains no C_5 as a subgraph.

Proof. Assume for a contradiction that C is a 5-cycle of $G \backslash S$. By Claim 2, we know that no triangle of G contains an edge of $G \backslash S$. It follows that $N_G(u) \cap N_G(v) = \varnothing$

whenever uv is an edge of $G\backslash S$. As a consequence, a vertex v of G is adjacent to at most $\alpha(C_5) = 2$ vertices of C.

It follows by double-counting that

$$|N(C)| \geq \frac{5k}{2} > n,$$

which yields a contradiction as $|N(C)| \leq n$. □

Claim 4. $G\backslash S$ is bipartite.

Proof. Assume $G\backslash S$ is not bipartite and let us prove the lower bound on $|S|$. Let $u_1 \ldots u_{2m+1}$ be an odd cycle of $G\backslash S$ with minimal length $2m+1$. We know from the triangle-freeness of $G\backslash S$ and Claim 3 that $2m+1 \geq 7$. Let us estimate the size of $N(\{u_1, u_2, u_{m+2}\})$ in G.

First note that u_1 and u_2 have no common neighbour in G because of Claim 2. Moreover, for $i \in \{1, 2\}$ the vertices u_{m+2} and u_i cannot have a common neighbour in $G\backslash S$ because it would yield cycles of lengths $m+2$ or $m+3$ in $G\backslash S$. As one of these lengths is odd, this would contradict the minimality of m. As a consequence, $(N(u_1) \cup N(u_2)) \cap N(u_{m+2})$ is a subset of S.

By the inclusion–exclusion principle, $3k - |S| \leq |N(\{u_1, u_2, u_{m+2}\})| \leq n$, so

$$|S| \geq (3k - n) > \frac{1}{5}n,$$

which contradicts our hypothesis. □

Let $A \cup B$ be a bipartition of $G\backslash S$.

Claim 5. There is a partition $S = S_1 \cup S_2$ such that there is no edge between S_2 and B, and no edge between S_1 and A.

Proof. If otherwise, then there are vertices $a \in A$, $b \in B$ and $s \in S$ such that as and bs are edges. As every triangle of G contains at least two vertices of S, the vertex a has no neighbour in $N(s) \cap B$. Similarly, the vertex b has no neighbour in $N(S) \cap A$. It follows that $N(s) \cap N(a)$, $N(s) \cap N(b)$ and $N(a) \cap N(b)$ are included in S. As a consequence,

$$3k \leq |N(a)| + |N(b)| + |N(s)| \leq n + 2|S|.$$

It follows that $|S| \geq \frac{1}{2}(3k - n) > \frac{n}{10}$, which contradicts our hypothesis on $|S|$. □

We may assume by symmetry that $|A \cup S_1| \leq |B \cup S_2|$. As $|A \cup S_1| + |B \cup S_2| = n$ and n is odd, it follows that $|A \cup S_1| \leq (n-1)/2$ and $|B \cup S_2| \geq (n+1)/2$.

Claim 6. The number e_2 of edges in the induced subgraph $G[S_2]$ is at least $k/2$.

Proof. Let e denote the number of edges from $A \cup S_1$ to $B \cup S_2$. It holds that $e \leq k \cdot |A \cup S_1| \leq k(n-1)/2$ and $2e_2 + e = k \cdot |B \cup S_2| \geq k(n+1)/2$. As a consequence,

$$2e_2 \geq k(n+1)/2 - k(n-1)/2 = k,$$

which proves the claim. □

We are now ready to conclude the proof.

First note that every edge uv of $G[S_2]$ is contained in at least $\frac{n}{5}$ triangles of G. Indeed, we know that $N(u)$ and $N(v)$ are subsets of $A \cup S$, so

$$|N(u) \cap N(v)| \geq |N(u)| + |N(v)| - |A| - |S| \geq 2k - \frac{n-1}{2} - \frac{n}{10} > \frac{n}{5}.$$

As there are at least $k/2$ such edges and a triangle contains at most three of them, we conclude that the number of triangles in G is at least

$$\frac{1}{3} \cdot \frac{k}{2} \cdot \frac{n}{5} > \frac{n^2}{75}.$$ □

We can show slightly more.

Theorem 12. *When n is odd and k is an even number with $2\left\lfloor \frac{n}{5} \right\rfloor < k \leq 2\left\lfloor \frac{n}{4} \right\rfloor$, every k-regular graph on n vertices has $\Omega(n^2)$ triangles. Moreover, this is sharp up to the multiplicative constant.*

Proof. The lower bound is proven in Theorem 3. So now we prove the sharpness of the result. Let $n = 2x + 1$ and $2\left\lfloor \frac{n}{5} \right\rfloor < k = x - y \leq 2\left\lfloor \frac{n}{4} \right\rfloor$. We construct a k-regular graph with $O(n^2)$ triangles. For this take a $K_{x,x}$, delete y disjoint perfect matchings and delete another disjoint matching of size $\frac{k}{2}$ (i.e., on k vertices). Now connect the endvertices of that last matching with an additional vertex v. Every triangle in the resulting graph contains the additional vertex v, from which the conclusion follows. For $k = 2\left\lfloor \frac{n}{5} \right\rfloor + 2$, this gives a construction with approximately $\frac{n^2}{50}$ triangles. □

4 | A REGULAR VERSION FOR ALL NONBIPARTITE GRAPHS

Theorem 13. *Let $r \geq 4$. For every $n \in \mathbb{N}$, there exists a k-regular $(r-1)$-partite graph with $k = \left(1 - \frac{1}{r-1} + o(1)\right)n$.*

Proof. Write $n = (r-1)x + y$ with x even and $0 \le y \le 2r-3$. As the statement is an asymptotic one, we only have to deal with n large and so we can assume $(r-2)x > y$. Construct a complete $(r-2)$-partite graph $K_{x,x,\dots,x}$ and remove a y-factor of it. Now connect all edges between a stable set of size $x+y$ and all vertices of this graph. The resulting graph is an $(r-2)x$-regular graph on n vertices. □

As a corollary to Theorem 13, the conclusions for the regular versions of Turán's theorem and the Erdős–Stone theorem are unchanged from their classical forms, if the chromatic number of the forbidden graph H satisfies $\chi(H) \neq 3$.

Theorem 14 (Regular Erdős–Stone theorem for $\chi H \neq 3$). *Let H be a graph with $\chi(H) \neq 3$. Then*

$$\mathrm{ex}_r(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)n.$$

We already saw in Theorem 2 that there are graphs H with $\chi(H) = 3$ for which the regular Erdős–Stone theorem differs from the classical statement. Next we characterize all such graphs H (with $\chi(H) = 3$).

We denote with $K^{=}_{2x,y}$ a complete bipartite graph $K_{2x,y}$ with a perfect matching in the part of size $2x$.

Theorem 15 (Regular Erdős–Stone theorem for $\chi H = 3$). *Let H be a graph with $\chi(H) = 3$.*

(i) *Suppose one of the following holds:*
- for every vertex v of H, the graph $H \backslash v$ is not bipartite; or
- H is not a subgraph of $K^{=}_{2|H|,|H|}$.

Then $\mathrm{ex}_r(n, H) = \frac{n}{2} + o(n)$.

(ii) *If neither of the above hold and n is odd, then* $\mathrm{ex}_r(n, H) \le 2\left\lfloor \frac{n}{5} \right\rfloor$.

Proof. We begin with the proof of (i). The upper bound is a consequence of the Erdős–Stone theorem. If n is even, the lower bound is given by the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$, so it is enough to give k-regular constructions of H-free graphs for odd n and $k = \frac{n}{2} + o(n)$.

We distinguish two cases depending on which condition holds.

- In the first case, namely, if $H \backslash v$ is not bipartite for every $v \in V(H)$, the k-regular construction in the proof of Theorem 12 for $k = 2\left\lfloor \frac{n}{4} \right\rfloor$ does the job. Indeed, one can remove one vertex from the resulting graph G such that it becomes bipartite, so all its subgraphs also have this property.
- In the second case, let $n = 2x + 1$ and take $K_{x+1,x}$ with $\left\lfloor \frac{x+1}{2} \right\rfloor$ disjoint edges added at the stable set of size $x+1$. If x is odd, this is a $(x+1)$ −regular graph $K^{=}_{x+1,x}$. If x is even, remove a maximum matching between the vertices of degree $x+1$ to get an

x-regular graph. In both cases, the obtained graph is a subgraph of $K_{2a,a}^{=}$ for some a and therefore does not contain H.

We proceed to the proof of (ii). Fix an odd number n and let $k > 2\left\lfloor \frac{n}{5} \right\rfloor$. Set $t = |H|$. Let G be a k-regular graph without H as a subgraph. It follows from this last hypothesis that every neighbourhood $N(u)$ in G contains no $H\backslash v$ as a subgraph, and therefore no $K_{t,t}$ because $H\backslash v$ is bipartite. By the Kövari–Sós–Turán theorem [18], it follows that $G[N(u)]$ contains at most $\frac{1}{2}(t-1)^{\frac{1}{t}}k^{2-\frac{1}{t}} + O(k)$ edges, which is smaller than $n^{2-\epsilon}$ if $\epsilon = \frac{1}{t}$ and n large enough. Equivalently, every vertex of G is contained in fewer than $n^{2-\epsilon}$ triangles.

We say that an edge of G is *thick* if it is contained in at least $\frac{n}{15}$ triangle. Let us show that G contains a set $A \subseteq E(G)$ of $\Omega(n^{\epsilon})$ disjoint thick edges.

We first proceed as in Claim 1 in the proof of Theorem 3 to show that every triangle of G contains a thick edge. Indeed, consider a triangle $u_1u_2u_3$ in G. By symmetry we may assume that $|N(u_1) \cap N(u_2)| \geq |N(u_2) \cap N(u_3)| \geq |N(u_1) \cap N(u_3)|$. By the inclusion–exclusion principle,

$$n \geq |N(u_1) \cup N(u_2) \cup N(u_3)| > 3k - 3|N(u_1) \cap N(u_2)|$$

and thus it follows that $|N(u_1) \cap N(u_2)| > \frac{n}{15}$, so u_1u_2 is thick.

Let T be a maximal set of vertex-disjoint triangles. Since T is maximal, every triangle of G intersects a vertex of a triangle of T. As each vertex is contained in at most $n^{2-\epsilon}$ triangles, it then follows from the hypothesis that G has at most $3|T| \cdot n^{2-\epsilon}$ triangles. Theorem 3 applied to G then yields

$$3|T| \cdot n^{2-\epsilon} \geq \frac{1}{300}n^2,$$

so $|T| \geq \frac{n^{\epsilon}}{900}$.

It then suffices to construct A by choosing a thick edge in each triangle of T.

To conclude the proof, consider the bipartite auxiliary graph F on $V(F) = A \cup V$ such that for every $e \in A$ and $v \in V$, the pair $\{e, v\}$ is an edge of F if and only if $e \cup \{v\}$ is a triangle. Note that in particular v is not an endpoint of e. As the edges of A are thick, every $e \in A$ has a degree at least $\frac{n}{15}$ in F, so $|E(F)| \geq \frac{n|A|}{15}$. By the asymmetric version of the Kővari–Sós–Turán theorem [18,29], this implies the existence of a $K_{t,t}$ as a subgraph of F provided that

$$(t-1)^{\frac{1}{t}}|A|n^{1-\frac{1}{t}} + (t-1)n < n\frac{|A|}{15},$$

which is true whenever n—and therefore A—is large enough. Since a copy of $K_{t,t}$ in F gives a copy of $K_{2t,t}^{=}$ in G, the graph G contains a copy of H, which yields a contradiction and concludes the proof. □

In Theorem 15(ii), there is possibly still room for improvement on the value of $\mathrm{ex}_r(n, H)$. On the basis of Theorem 2, it is natural to wonder if the value depends on the odd girth. More precisely, the following question would be worth investigating.

Question 16. Let H be a graph with $\chi(H) = 3$ such that there exists a vertex v for which $H \backslash v$ is bipartite and such that H is a subgraph of $K^{=}_{2|H|,|H|}$. Let the odd girth of the graph H be g. It is true that $\mathrm{ex}_r(n, H) = 2\left\lfloor \frac{n}{g+2} \right\rfloor + o(n)$?

5 | MAXIMIZING CLIQUE COUNT GIVEN ORDER AND SIZE

Given a graph G, define $k_t(G)$ to be the number of cliques K_t in G. Gan, Loh and Sudakov [10] proposed the problem of maximizing $k_t(G)$ in G given the order and the maximum degree of G. Motivated by the Gan–Loh–Sudakov problem, Kirsch and Radcliffe [17] proposed the problem of maximizing $k_t(G)$ in G given the size and the maximum degree of G. They also wondered about the problem of maximizing $k_t(G)$ in G given the order n and size m, as well as the maximum degree of G. (Up to the maximum degree condition, this question appeared, e.g., also in [11].) As will become apparent, the most interesting case here is that of regular graphs. This case is closely related to the Kahn–Zhao theorem [15,28], which is a natural predecessor to the Gan–Loh–Sudakov problem. When the order n is not much larger than the degree r of the regular graphs, by focusing on the complementary graph $\overline{G}$, some cases are related to a conjecture of Kahn [15]. This is the case when $2(n - r)|n$. For this note that $k_t(G) = i_t(\overline{G})$, where i_t is the number of independent sets of order t, as every clique in G is an independent set in $\overline{G}$ and vice versa.

Chase [6] solved the main problem in [10]. Chakraborti and Chen [5] recently solved (in a stronger form) the main conjecture in [17], and due to this the extremal graph for our problem with $n = a(r+1) + b$ (here $b \le r$) and $m \le a\binom{r+1}{2} + \binom{b}{2}$ is the union of K_{r+1}s and a colex graph. For the remaining cases, that is, in the critical regime (where one cannot have a copies of K_{r+1}), it is natural to pose the following analogous conjecture.

Conjecture 17. *Let $n = a(r+1) + b$ and $\frac{nr}{2} \ge m > a\binom{r+1}{2} + \binom{b}{2}$. Any graph maximizing k_t for a fixed t or $k = \sum_{t\ge 2} k_t$ among all graphs of order n, size m and maximum degree at most r can be represented as $(a-1)K_{r+1} + H$.*

There are some obstructions to a tidier conjecture. Examples 1 and 2 show that there might be several different kinds of extremal graph H, and for distinct t the extremal graphs might not correspond. This is in stark contrast to the cases of prescribed size or order alone.

Example 1. The graph G in Figure 1A satisfies $k_3(G) = 16$, $k_4(G) = 4$, $k_5(G) = 0$ and $k(G) = 20$. It is the unique graph maximizing $k_3(G)$ among all graphs with $(n, m, r) = (8, 18, 5)$. On the other hand, the graph G in Figure 1B satisfies $k_3(G) = 15$, $k_4(G) = 6$, $k_5(G) = 1$ and $k(G) = 22$. It is the unique graph maximizing $k(G)$ among all graphs with $(n, m, r) = (8, 18, 5)$ and maximizes k_4 and k_5 as well. For k_4 and k_5 there are, respectively, 2 and 3 extremal graphs.

As the $t = 3$ case was the main interest in [17], we can further focus on this case.

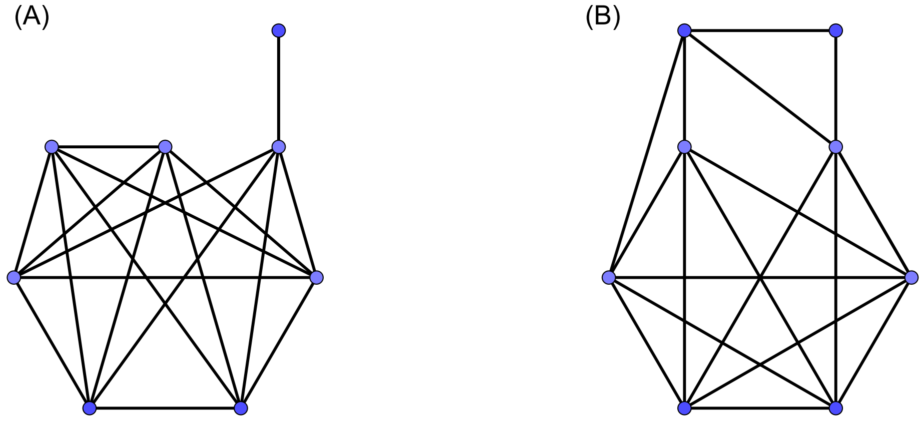

FIGURE 1 Graphs with $(n, m, r) = (8, 18, 5)$. (A) Extremal graph for k_3 and (B) extremal graph for k.

The following equality expresses $k_3(G)$ in terms of its order, the degrees and $k_3(\overline{G})$. It is basically proven in [13].

Claim 7. For any graph G of order n, we have

$$k_3(G) + k_3(\overline{G}) + \frac{1}{2}\sum_v \deg(v)(n - 1 - \deg(v)) = \binom{n}{3}.$$

Describing the extremal graphs in general seems to be hard as they are not unique and also $k_3(\overline{G})$ and the degree sequences can be different for different extremal graphs, as the next example shows.

Example 2. There are three graphs with the maximum number of triangles, 16, among all graphs of order 8, size 17 and maximum degree at most 5. These are presented in Figure 2. The number of triangles in their complement $\overline{G}$ is equal to $4, 1$ and 0, respectively, implying also that their degree sequences are different.

We also remark that in the critical regime, increasing m can imply both a decrease or increase in the number of triangles. This is also the case if one increases both m and n by 1.

Example 3. When $r = 4$, the maximum number of triangles among all graphs of order n and size m in the critical regime are given in Table 1.

We also give some positive results, for example, we can describe the extremal graphs for $n = r + 2$.

Proposition 18. *When* $n = r + 2$ *and* $\binom{r+1}{2} \le m \le \frac{(r+2)r}{2}$, *the extremal graph* G *attaining the maximum number of triangles among all graphs of order* n, *size* m *and maximum degree at most* r *is the one for which* $\overline{G}$ *is the union of a matching of size* $m - \binom{r+1}{2}$ *and a star of order* $(r + 1)^2 + 1 - 2m$.

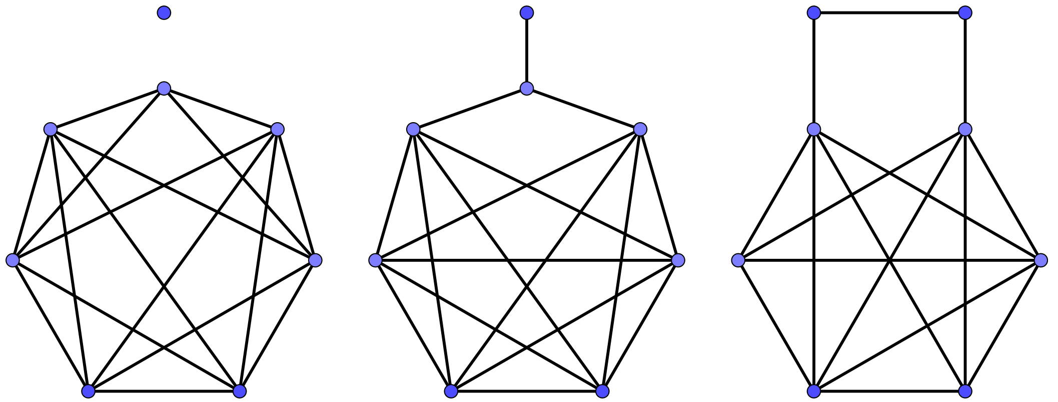

FIGURE 2 Graphs with $(n, m, r) = (8, 17, 5)$ maximizing k_3

TABLE 1 Maximum $k_3(G)$ given n and m when $r = 4$

$n\backslash m$	11	12	13	14	15	16
6	7	8				
7		8	7	7		
8				8	8	8

Proof. Note that the degrees $0 \le d_i \le r$ satisfy $\sum d_i = 2m$ and the function $f(x) = x(n-1-x)$ is a strictly concave function. By the inequality of Karamata [16], this implies that $\sum f(d_i) \ge (r+1)f(r) + f(2m - r(r+1))$. By Claim 7 this implies that $k_3(G) \le \binom{n}{3} - \frac{1}{2}(r+1)f(r) - \frac{1}{2}f(2m - r(r+1))$. Equality occurs if $k_3(\overline{G}) = 0$ and $\overline{G}$ has $r+1$ vertices of degree 1 and one vertex of degree $(r+1)^2 - 2m$, from which the characterization follows. □

For $n = r + 3$ one can get a similar characterization, up to a few exceptions, if $m \ge \frac{(r+2)r}{2}$. In that case, the complement $\overline{G}$ is in general a union of cycles, some of them having one vertex in common.

The case where $m = \frac{nr}{2}$, that is, when the graphs are regular, might be considered as the most interesting case because it is the extreme case which is most far apart from the edge case. Due to Conjecture 17 we focus on this case for $r + 2 \le n \le 2r + 1$.

Theorem 19. *Let $r + 2 \le n \le 2r + 1$ and $m = \frac{nr}{2}$. Every graph G maximizing the number of triangles among the graphs of order n and size m can be formed by taking the complement of an $(n - r - 1)$-regular graph on n vertices minimizing the number of triangles. In particular, if n is even or $n \le r + 1 + 2\left\lfloor \frac{r}{3} \right\rfloor$ is odd then the maximum number of triangles equals $k_3(G) = \binom{n}{3} - \frac{n}{2}r(n-1-r)$.*

Proof. By Claim 7 we know $k_3(G) = \binom{n}{3} - \frac{n}{2}r(n-1-r) - k_3(\overline{G})$. So the maximum is attained if the $(n-r-1)$-regular graph on n vertices $\overline{G}$ minimizes k_3. So Theorem 8 implies the exact result for n being even (take $\overline{G}$ bipartite) or n being odd and $n \le r + 1 + 2\left\lfloor \frac{r}{3} \right\rfloor$. Section 3 implies $k_3(G) = \binom{n}{3} - \frac{n}{2}r(n-1-r) - \Theta(n^2)$ in the remaining case. □

The exact result for the regular case would be known once proven Conjecture 17 and the following conjecture.

Conjecture 20. *Let G be a k-regular graph on n vertices, with* $n = 2p + 1$ *being odd and* $2\left\lfloor \frac{n}{5} \right\rfloor < k = p - q \le 2\left\lfloor \frac{n}{4} \right\rfloor$ *being even. Then* $k_3(G) \ge \frac{k}{2}\left(\frac{k}{2} - q - 1\right)$.

Equality for this conjecture holds when G is a graph formed by a $K_{p,p}$ and an additional vertex v connected to $\frac{k}{2}$ vertices of both stable sets of $K_{p,p}$ and deleting edges between the k neighbours of v on the one hand and between the $2p - k$ nonneighbours of v on the other hand, such that the final graph is k-regular.

We prove one case of this conjecture.

Proposition 21. *Let G be a k-regular graph on* $n = 2k + 1$ *vertices, with k being even. Then* $k_3(G) \ge \frac{k}{2}\left(\frac{k}{2} - 1\right)$. *Equality holds if and only if G is a* $K_{k,k}$ *minus a matching of size* $\frac{k}{2}$, *where all endvertices of the matching are connected with an additional vertex.*

Proof. Assume there is such a graph G with $k_3(G) < \frac{k}{2}\left(\frac{k}{2} - 1\right)$. Take a vertex v of G for which the number of triangles containing $v, k_3(v)$, is minimal. In particular, $x = k_3(v) < \frac{3}{2k+1}\frac{k}{2}\left(\frac{k}{2} - 1\right) < \frac{k}{2} - 1$. Since $G = (V, E)$ is k-regular, $|N(v)| = k$ and $|N_2(v)| = k$, where $N_2(v) = V \backslash N[v]$. There are x edges in $G[N(v)]$ and $x + \frac{k}{2}$ edges in $G[N_2(v)]$. Note that the two endvertices of any of these $x + \frac{k}{2}$ edges in $G[N_2(v)]$ have at least $2k - (x + \frac{k}{2} + 1) - k = \frac{k}{2} - x - 1$ common neighbours in $N(v)$. Similarly, every edge among the x edges in $G[N(v)]$ is contained in at least $2k - (x+1) - (k+1) = k - x - 2$ triangles. This implies that G contains at least $x(k - x - 2) + (x + \frac{k}{2})(\frac{k}{2} - x - 1) = \frac{k}{2}\left(\frac{k}{2} - 1\right) + x(k - 2x - 3)$ triangles. Since $2x \le k - 4$, the result follows. We attain equality if $k_3(v) = 0$ and $G[N_2(v)]$ is a star and the vertices in $N(v)$ are connected to all vertices in $N_2(v)$ except one in such a way that they have total degree k, so the extremal graph is of the desired form. □

6 | SOME OTHER GAN–LOH–SUDAKOV-TYPE PROBLEMS

The general Turán-type study of Alon and Shikhelman [1] asks to determine the quantity $\mathrm{ex}(n, T, H)$, the maximum number of copies of T in an H-free graph on n vertices. The Gan–Loh–Sudakov problem can be formulated thus as the special case of determining

$ex(n, K_t, K_{1,r+1})$. If n is a multiple of $r+1$, it is trivial that the union of disjoint K_{r+1} is extremal since for every vertex v the construction attains the maximum number of copies of K_t containing v. By looking to the neighbourhood of any vertex, the following cases are immediate as well.

Proposition 22. *The quantity* $ex(n, K_{1,s}, K_{1,r+1})$ *is maximized by any* r*-regular graph on* n *vertices.*

For every tree T *with maximum degree at most* r *and diameter* d, *the quantity* $ex(n, T, K_{1,r+1})$ *is maximized by any* r*-regular graph of girth at least* $d+1$.

Note that if nr is odd, an extremal graph will have exactly one vertex with degree $r-1$.

When n is not a multiple of $r+1$, Chase's theorem [6] (formerly the conjecture of Gan, Loh and Sudakov [10]) implies that the extremal graph is the union of the maximum number of copies of K_{r+1}, being the unique graph maximizing $\frac{ex(n, K_t, K_{1,r+1})}{n}$ and a residue graph (which is a complete graph as well). The maximum of this normalized quantity can be found easily for complete bipartite graphs as well by looking locally to the neighbourhood of any vertex.

Proposition 23. *For every* $r \geq a, b \geq 2$, $\frac{ex(n, K_{a,b}, K_{1,r+1})}{n}$ *is maximized by the graph* $K_{r,r}$. *Furthermore this is the unique connected extremal graph for the quantity.*

In particular, we know the extremal graphs for the quantity $\frac{ex(n, H, K_{1,r+1})}{n}$ when $H \in \{C_3, C_4\}$. So one can wonder about cycles in general.

Question 24. For every even cycle C_m, for sufficiently large r, is $\frac{ex(n, C_m, K_{1,r+1})}{n}$ maximized by the graph $K_{r,r}$?

For every odd cycle C_m, for sufficiently large r, is $\frac{ex(n, C_m, K_{1,r+1})}{n}$ maximized by the graph K_{r+1}?

If the latter question is positive for the cycle C_5, the following proposition would imply that the analogue of Chase's theorem would not hold, as for example, $K_{r+1} + K_1$ is not necessarily maximizing the number of C_5s for $n = r+2$ as the following analysis shows.

Proposition 25. *Let* $r \geq 6$. *Then*

$$ex(r+2, C_5, K_{1,r+1}) = \begin{cases} 12\binom{r+1}{5} & \text{for } r \text{ being odd,} \\ \dfrac{r(r^2-4)(r^2-5r+9)}{10} & \text{for } r \text{ being even.} \end{cases}$$

The extremal graphs are, respectively, K_{r+1} *and* $K_{r+2} \backslash M$ *for a matching* M.

Proof. We start with some observations to get some structure of the extremal graphs. Note that a graph G of order $n = r+2$ has a maximum degree at most r if and only if the complement $\overline{G}$ has a minimum degree 1. If H is a subgraph of G, then the number of C_5s in G is at least the number of C_5s in H. So if $\overline{G}$ has an edge for which both of its endvertices have degree at least 2, we can delete that edge without decreasing the number of C_5s in G. Repeating this, we end with $\overline{G}$ being the disjoint union of stars. Let $\overline{G} = \sum_{i=1}^{k} S_{a_i+1}$. Here $a_i \geq 1$ for every i. Note that $A = \sum_i a_i = n - k$ and $k \leq \frac{n}{2}$. Using the principle of inclusion–exclusion, we find that the number of C_5s in G equals

$$12\binom{n}{5} - 6A\binom{n-2}{3} + 2\sum_i \binom{a_i}{2}\binom{n-3}{2} + 2\sum_{i\neq j} a_i a_j (n-4) - 2\sum_{i \neq j}\binom{a_i}{2} a_j. \tag{1}$$

Claim 8. Let $n \geq 9$. For fixed k, Equation (1) attains its maximum over all $a_i \geq 1$ if and only if all but at most one a_i are equal to 1.

Proof. Note that this is obviously true for $k = 1$. Also we note that $A = n - k$ is fixed. Now assume $k \geq 2$ and $a_i, a_j > 1$. The part of Equation (1) which depends on a_i and a_j for fixed sum $a_i + a_j$, equals

$$\begin{aligned}
&((n-3)(n-4) - 2A)\left(\binom{a_i}{2} + \binom{a_j}{2}\right) + 4a_i a_j (n-4) + 2a_i\binom{a_i}{2} + 2a_j\binom{a_j}{2} \\
&= ((n-3)(n-4) - 2A - 4(n-5))\left(\binom{a_i}{2} + \binom{a_j}{2}\right) \\
&\quad + 4(n-5)\binom{a_i + a_j}{2} + 4a_i a_j + 2a_i\binom{a_i}{2} + 2a_j\binom{a_j}{2}.
\end{aligned}$$

We have

$$\begin{aligned}
(n-3)(n-4) - 2A - 4(n-5) &\geq (n-3)(n-4) - 2(n-2) - 4(n-5) \\
&= n^2 - 13n + 36
\end{aligned}$$

which is nonnegative for $n \geq 9$. Also $\binom{a_i + a_j - 1}{2} + \binom{1}{2} > \binom{a_i}{2} + \binom{a_j}{2}$ when $a_i, a_j > 1$ since $x(x-1)$ is a strictly convex function. Furthermore, let $f(x, y) = 4xy + 2x\binom{x}{2} + 2y\binom{y}{2}$. Then $f(a_i + a_j - 1, 1) - f(a_i, a_j) = 3(a_i - 1)(a_j - 1)(a_i + a_j - 1) > 0$. So substituting (a_i, a_j) by $(a_i + a_j - 1, 1)$ implies an increase of Equation (1) from which the result follows. □

Now we can focus on $a_1 = a_2 = \cdots = a_{k-1} = 1$ and $a_k = n - 2k + 1$. In this case Equation (1) reduces to

$$\begin{aligned}
g(n,k) = 12\binom{n}{5} - 6(n-k)\binom{n-2}{3} + 2\binom{n-2k+1}{2}\binom{n-3}{2} \\
+ 4\left(\binom{k-1}{2} + (k-1)(n-2k+1)\right)(n-4) - 2(k-1)\binom{n-2k+1}{2}.
\end{aligned}$$

Note that $\frac{d^2 g(n,k)}{dk^2} = 4n^2 - 24k - 32n + 108 \geq 4n^2 - 44n + 108 = 4(n-2)(n-9) + 36$ is positive for $n \geq 9$. This implies that $g(n, k)$ is strictly convex and hence takes its maximum at $k = 1$ or $k = \left\lfloor \frac{n}{2} \right\rfloor$. Since $g\left(n, \frac{n}{2}\right) > g(n, 1) > g\left(n, \frac{n-1}{2}\right)$ for $n \geq 9$, we conclude. For every edge in both K_{n-1} and $K_n \backslash M$ there is a C_5 containing that edge, from which we conclude that the extremal graphs are unique in these cases. For $n \leq 8$, the extremal graphs can easily be computed with computer software such as Sage. □

7 | CONCLUSION

Our work was motivated by a Gan–Loh–Sudakov-type problem where we are given both the number of edges and vertices, in addition to the maximum degree, after [17]. By focusing on the regular case and looking to the complement of the extremal graphs, this led us to the notion of regular Turán numbers. This has resulted in a number of interesting regular versions of classical Turán-type results.

The Gan–Loh–Sudakov conjecture was solved by Chase [6] in 2019, very shortly after the appearance of our manuscript on arXiv. Also the size variation proposed in [17] was solved by Chakraborti and Chen [5]. The variations which we considered here are still open. We note that the main ingredients used in [5,6,10] are not enough to tackle the problem with both order and size. In particular, our observations in Section 5 show that the extremal graphs are not that easily described so as to start computations in an inductive way.

Some related questions have arisen, which we suspect should provoke further investigations, particularly with respect to the regular Turán numbers. In particular, it would be interesting to resolve Question 16, as this would more precisely characterize the regular Turán numbers for graphs of chromatic number 3. It would also be natural to investigate bipartite graphs.

We also highlight Conjecture 20, which would imply both the exact saturation result of the regular Mantel's theorem and the exact form in the regular case of the Gan–Loh–Sudakov-type question given both the order and size. A last natural problem is Question 24, being morally the right Gan–Loh–Sudakov question for cycles instead of cliques.

Notes added: During the preparation of this manuscript, we learned of the concurrent and independent works by Gerbner, Patkós, Tuza and Vizer [12] and by Caro and Tuza [4]. With a different application in mind, the regular Turán number was introduced in [12] in an alternative formulation as the maximum number of edges in a regular H-free graph: $\text{rex}(n, H) = \frac{2}{n}\text{ex}_r(n, H)$. Caro and Tuza [4] have also determined the regular Turán numbers of complete graphs.

We point out that Theorem 2 proves Conjecture 1 of [4] for large n. For small n, the conjecture is false. For example, C_m does not contain a C_g when $g < m < 2g + 4$ nor does the disjoint union of b cliques K_m do for $m < g$, leading to another counterexample when $bm < \frac{(m-1)}{2}(g + 2)$. We also note that Theorem 15 provides progress towards Problem 1 in [4]. This problem has been reduced to a more concrete form in Question 16.

A number of relevant works have appeared following the public posting of our manuscript. Besides the works of Chase [6] and Chakraborti and Chen [5] already mentioned, we point out that Tait and Timmons [24] have already made the first effort at investigating the regular Turán numbers for bipartite graphs.

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