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Evolution of a Bose-condensed gas under variations of the confining potential

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We discuss the dynamic properties of a trapped Bose-condensed gas under variations of the confining field and find analytical scaling solutions for the evolving coherent state (condensate). We further discuss the characteristic features and the depletion of this coherent state. [S1050-2947(96)50209-X]

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The recent successful experiments on Bose-Einstein condensation (BEC) in trapped ultracold alkali atom gases [¹–³] open a unique possibility to investigate dynamic properties of a Bose-condensed phase. Of particular interest is the response of the system to time-dependent variations of the confining field. In an interacting Bose-condensed gas these properties are nontrivial. For example, if initially almost all trapped atoms are in the condensate (T = 0), then under adiabatically slow change of the trapping potential they remain in the condensate, which now corresponds to the ground state of the system in the instantaneous trapping field. At the other extreme, a fast change of the potential from the initial to final shape brings the system to an excited superpositional (ES) state, where the admixture of the (final) ground state can be small. Then, even assuming complete isolation of the system from the environment, there is the question of how the correlation properties change. Especially interesting are those responsible for the reduction of the probability of inelastic processes due to the presence of a Bose condensate [⁴].

Another question concerns trapped gases with negative scattering length. The trapping field stabilizes the condensate provided that the spacing between adjacent trap levels exceeds the interparticle interaction [⁵,⁶]. Will this hold in the ES state or will the system “collapse”? Of principal importance is the evolution of a completely isolated many-body system which proves to be in the ES state. Does the system undergo stochasticization and imitate relaxation behavior, at least for a large number of particles and large interaction between them compared to the level spacing in the potential well? This question is related to the well-known problem of the appearance of irreversibility in a quantum system with a large number of particles.

To answer most of the above questions we first consider the evolution of a Bose condensate in a parabolic trapping potential V(r) = mω²r²/2 (m is the atom mass) with frequency ω(t) varying from ω₀ to ω₁. We use ideas of the analysis of the quantum motion of a particle in a harmonic oscillator with time-dependent frequency (see [⁷]) and find the solution of the time-dependent nonlinear Schrödinger equation for the evolving coherent state. In certain cases our analytical results can be compared with numerical calculations, first performed in Ref. [⁵]. We further analyze the characteristic features of the evolving coherent state and discuss the problem of relaxation and the loss of coherence.

Let us consider a Bose gas with a fixed number of particles N in a symmetric harmonic potential with time-dependent frequency ω(t). We assume a pair interaction potential between atoms of the form U(r) = Uδ(r). In a three-dimensional (3D) gas U = 4πℏ²a/m, where a is the scattering length and m the atom mass. The Schrödinger equation for the Heisenberg field operator of atoms, Ĥ(r,t), reads

\[ iℏ \frac{\partial Ĥ}{\partial t} = -\frac{ℏ²}{2m} \Delta Ĥ + \frac{mω²(t)r²}{2} Ĥ + U0 Ĥ^† Ĥ. \]  

(1)

The field operator can be represented as a sum of the above-condensate part and the condensate wave function Ψ₀(Ψ), which is a c number: Ĥ = Ĥ + Ψ₀ (see [⁸]). The equation for Ĥ₀(r,t), obtained by averaging both sides in Eq. (1), in the mean-field approach has the form

\[ iℏ \frac{\partial Ĥ₀}{\partial t} = -\frac{ℏ²}{2m} \Delta Ĥ₀ + \frac{mω²(t)r²}{2} Ĥ₀ + U0 |Ψ₀|^2 + 2n' ]Ψ₀. \]  

(2)

where n'(r,t) = ⟨Ψ'Ψ'⟩. The mean-field equation of motion for Ĥ'(r,t) follows from Eqs. (1) and (2):

\[ iℏ \frac{\partial Ĥ'}{\partial t} = -\frac{ℏ²}{2m} \Delta Ĥ' + \frac{mω²(t)r²}{2} Ĥ' + 2U0 (|Ψ₀|^2 + n') Ĥ' + \frac{Ψ₀^2 Ψ²}{2}. \]  

(3)

In Eqs. (2) and (3), due to the condition n|a|³ ≪ 1 (n is the gas density), we omitted the terms containing anomalous averages ⟨Ψ'Ψ'⟩.

Frequency variations change the time and distance scales in Eqs. (2) and (3). Let us formally introduce new operators taking this into account \[ ρ = r/(b(t)); \]

\[ Ĥ₀(r,t) = [b(t)]^{-d²} x₀(ρ, τ(t))exp[iΦ(r,t)], \]

\[ Ĥ'(r,t) = [b(t)]^{-d²} x'(ρ, τ(t))exp[iΦ(r,t)]. \]  

(4)

where d is the dimension of the system and the phase

\[ Φ(r,t) = (m²/2ℏ)b(t)b(t). \]  

(5)

The equations of motion for the operators x₀ and x' in new coordinate (ρ) and time (τ) variables take the form
where $\tilde{\eta}'(p, \tau) = \langle \tilde{\eta}' (p, \tau) \rangle$. Note that the choice of
phase in the form (5) leads to the cancellation of terms proportional to $\nabla_{\tilde{p}} \tilde{\eta}'$ and $\nabla_{\tilde{p}} \tilde{\chi}'$. It is important that the phase be the
same for both $\Psi_0$ and $\Psi'$. Let us first consider $d = 2$. We choose $\tau(t)$ and $b(t)$ such that they are governed by the equations
\[ \tau(t) = \int^t dt' / b^2(t'), \] (8)
\[ \tilde{b}(t) + \omega^2(t)b(t) = \omega_0^2 / b^3(t), \] (9)
where $\omega_0 = \omega(-\infty)$ is the initial frequency. Then Eqs. (6) and (7) are reduced to a universal form
\[ i \hbar \frac{\partial \chi_0}{\partial \tau} = -\frac{\hbar^2}{2m} \Delta \rho \chi_0 + \frac{m \omega_0^2 \rho^2}{2} \chi_0 + \tilde{U} |\chi_0|^2 |\tilde{\eta}'| \chi_0, \] (10)
\[ i \hbar \frac{\partial \tilde{\chi}'}{\partial \tau} = -\frac{\hbar^2}{2m} \Delta \rho \tilde{\chi}' + \frac{m \omega_0^2 \rho^2}{2} \tilde{\chi}' + 2 \tilde{U} |\chi_0|^2 |\tilde{\eta}'| \tilde{\chi}' + \tilde{U} |\chi_0|^2 \tilde{\chi}''', \] (11)
Equations (10) and (11) are universal in the sense that in the variables $\rho, \tau$ the problem is reduced to an interacting Bose
gas in a harmonic well with constant frequency. Once we find the solution in the initial potential well, we know the
answer at any $t$. One should only solve the simple equation (9) with initial conditions $b(-\infty) = 1, \tilde{b}(-\infty) = 0$. For ex-
ample, the expression for $\Psi_0(r,t)$ reads
\[ \Psi_0(r,t) = \frac{1}{b(t)} \Psi_0 \left( \frac{r}{b(t)} \right) \exp \left( i \frac{mr^2 \tilde{b}(t)}{2 \hbar} - i \mu \tau(t) \right), \] (12)
where $\mu$ is the initial chemical potential and $\Psi_0(r)$ is the stationary solution for the condensate wave function at
$t \to -\infty$. Using the Bogolyubov transformation, generalized for an inhomogeneous case (see, e.g., [9]), on the basis of
Eqs. (10) and (11) we can describe the evolution of the spectrum and wave functions of elementary excitations.

Neglecting the excitations, Eq. (12) describes the 2D (radial) evolution of $\Psi_0$ in long samples (axial frequency is
much smaller than the radial one) under variations of the radial frequency; as to a first approximation one may omit
the dependence of $\Psi_0$ on the axial coordinate.

If the nonlinear interaction terms in Eqs. (6) and (7) can be omitted the above universal scaling takes place for any $d$. This requires at least a small ratio of interparticle interaction to the level spacing in the initial potential,
$\eta = n_0 \tilde{U} / \hbar \omega_0 \ll 1$, where $n_0 = |\Psi_0(0, -\infty)|^2$ (see below).

In the 3D case for arbitrary $\eta$ there is no universality. On the other hand, in the limit $\eta \gg 1$, neglecting the excitations, Eq. (6) can be again reduced to a universal form. In this case the kinetic energy term in Eq. (6) is comparatively small and can be omitted (see below). Then, introducing, instead of (8) and (9), the equations
\[ \tau(t) = \int^t dt' / b^3(t'), \] (13)
\[ \tilde{b}(t) + \omega^2(t)b(t) = \omega_0^2 / b^3(t), \] (14)
Eq. (6) is transformed to
\[ i \hbar \frac{\partial \chi_0}{\partial \tau} = -\frac{m \omega_0^2 \rho^2}{2} \chi_0 + \tilde{U} |\chi_0|^2 \chi_0. \] (15)
For $t \to -\infty$ Eq. (15) has a solution $\chi_0(t) = \tilde{\chi}_0(r) \exp(-i \mu t)$, where $\tilde{\chi}_0$ is given by the well known expression [10,11]
\[ \tilde{\chi}_0(r) = \frac{1}{U^{1/2}} \left( \frac{m \omega_0^2}{2 \mu r^2} \right)^{1/2}, \quad r \leq \left( \frac{2 \mu}{m \omega_0^2} \right)^{1/2} \] (16)
and zero otherwise. As follows from Eq. (15), for any $t$ the condensate wave function has the form
\[ \Psi_0(r,t) = \frac{1}{b(t)} \tilde{\Psi}_0 \left( \frac{r}{b(t)} \right) \exp \left( i \frac{mr^2 \tilde{b}(t)}{2 \hbar} - i \mu \tau(t) \right), \] (17)
where $b(t)$ is governed by Eq. (14) with $b(-\infty) = 1, \tilde{b}(-\infty) = 0$.

Equations (12) and (17) conserve the norm $\int |\Psi_0(r,t)|^2 d^d r = N_0$, where $N_0$ is the initial number of particles in
the condensate. Universal solutions of Eqs. (10) and (11) conserve the norm $\int [|\Psi_0(r,t)|^2 + n'(r,t)] d^d r = N$.

We should emphasize that Eqs. (12) and (17) describe a coherent evolution of $\Psi_0$. Generally speaking, it is very dif-
ficult from the condensate wave function corresponding to the ground state of the system in the potential well with an
instantaneous value of $\omega$ or the final value $\omega_1$, even if $\omega(t)$ returns to the initial frequency $\omega_0$. The
time dependence of $\Psi_0$ and normal excitations for $d = 2$ (or at any $d$ for $\eta \ll 1$, neglecting the nonlinear interaction
terms) is determined by the solution of Eq. (9). The latter can be found in a general case (see, e.g., [7]). Note that
the classical equation for a harmonic oscillator with time-dependent frequency
\[ \tilde{\xi}(t) + \omega^2(t) \tilde{\xi}(t) = 0, \] (18)
where $\tilde{\xi}(t) = \langle \tilde{\xi}(t) \rangle$.
leaves to Eq. (9), if one sets \( \xi(t) \equiv b(t) \exp(\pm i\omega_0 t) \). With \( \xi(t) = \exp(i\omega_0 t) \) for \( t \to -\infty \) and \( t \) interpreted as a 'spatial coordinate,' Eq. (18) is equivalent to the one-dimensional Schrödinger equation for the reflection of a particle with 'energy' \( \omega_0^2 \) from the 'potential' \( \omega_0^2 - \omega^2(t) \). At times where \( \omega \) is already constant (\( \omega_1 \)) we have

\[
b^2(t) = |\xi(t)|^2 = \frac{\omega_0}{\omega_1} \left[ 1 + \frac{2R^{1/2}}{1 - R} \cos(2\omega_1 t + \delta) \right],
\]

where \( R \) is the reflection coefficient and \( \delta \) the phase.

For slowly changing frequency (on a time scale \( \tau_0 \gg \omega_1^{-1} \)) the coefficient \( R \) is exponentially small, and

\[
b = \sqrt{\omega_0 / \omega_1}.
\]

In this case the initial condensate is adiabatically transformed to the ground state of the system in the final trapping field, without oscillations. If the condition \( \omega_0 \tau_0 \gg 1 \) is not valid, for at least one of the frequencies, the scaling parameter \( b(t) \) will oscillate with a constant amplitude given by Eq. (19). There is no damping of the oscillations of the condensate density, unless relaxation is included (see below).

The instantaneous size of the evolving condensate is related to the initial size by \( r_0(t) = r_0 b(t) \left[ r_0 = r_0(-\infty) \right] \). In the case of abrupt change of the frequency we have

\[
R = (\omega_0 - \omega_1)^2 / (\omega_0 + \omega_1)^2, \quad \delta = 0,
\]

and the function \( b(t) \) oscillates from 1 to \( \omega_0 / \omega_1 \). For \( \omega_0 \gg \omega_1 \) there is a large expansion of the condensate and then compression to the initial shape. At times \( t \ll \omega_1^{-1} \) the expansion is practically free, and Eqs. (19) and (20) yield

\[
b(t) = (1 + \omega_0^2 t^2)^{1/2}, \quad \tau(t) = (1/\omega_0) \arctan(\omega_0 t).
\]

If \( \omega_1 = 0 \) the compression does not occur. We have an expanding condensate described by Eq. (22) at any \( t \).

At times \( t \gg \omega_1^{-1} \) the characteristic velocity of free expansion \( v_0 = r_0(t) = \omega_0 r_0 \). As follows from Eq. (16), for \( \eta \gg 1 \) the initial size of the condensate \( r_0 = (2 \mu / m \omega_0^2)^{1/2} \), and we have \( v_0 = \sqrt{2\mu/m} \). Since in this case \( \mu = n_0 \bar{U} \), the velocity is determined by the interaction between particles. For \( \eta \ll 1 \) the initial size \( r_0 \approx l_0 \), where \( l_0 = (\hbar/m \omega_0)^{1/2} \) is the amplitude of zero-point oscillations in the initial potential, and \( v_0 = \sqrt{\hbar n_0 / m} \) is much smaller than for \( \eta \gg 1 \).

The same picture holds for the free evolution of a condensate in the opposite limit, where initially \( \eta \gg 1 \), although we should use the scaling transformation following from Eqs. (13) and (14). The latter has no analog in the quantum theory of scattering and should be solved directly. We again obtain a periodic function \( b(t) \), and oscillations of the condensate density will be determined by Eq. (17) which was derived assuming a large ratio of interparticle interaction to the kinetic-energy term (IK ratio) in Eq. (6). For \( \eta \gg 1 \) the IK ratio varies as \( \eta^1 b(t) \) and definitely remains large if the frequency increases \( [b(t) \ll 1] \). As follows from Eq. (14), fast decrease of the frequency to \( \omega_1 = \omega_0 \) leads to large oscil-
retains the effect of partial suppression of inelastic processes, characteristic of the static condensate. Since any loss of coherence will lead to an increasing inelastic rate, there is an interesting possibility to study the depletion of the evolving condensate through the measurement of the rates of intrinsic or light-induced inelastic collisional processes.

In an isolated system the relaxation of the evolving coherent state will be accompanied by the appearance of an effective temperature. Assuming zero initial temperature and $\omega_1 \ll \omega_0$, a complete depletion of the condensate would lead to the effective temperature $T_{\text{eff}} \sim \mu$ for $\eta \gg 1$ and $T_{\text{eff}} \sim h/\omega_0$ for $\eta \ll 1$. In the 3D case the BEC transition temperature in the final potential $T_c \sim h/\omega_0 N_0^{1/3}[12]$. For $\eta \ll 1$ the condition of complete depletion of the condensate, $T_c/T_{\text{eff}} = (\omega_1/\omega_0) N_0^{1/3} \ll 1$, is rather strong although there is an upper bound for the number of particles, $N_0 \ll l_0/a$. For $\eta \gg 1$, with $\mu = \hbar \omega_0$ and $\eta \sim (N\alpha l_0)^{2/5}$ [6], we obtain a weaker condition which can easily be fulfilled:

$$T_c/T_{\text{eff}} = (\omega_1/\omega_0) (l_0/a)^{2/5} N_0^{-1/5} \ll 1.$$ 

The question of relaxation and the loss of coherence in a quantum system with a large number of particles has several nontrivial aspects, especially with regard to dynamic evolution of a completely isolated system. In the latter case we can discuss the imitation of stochasticization and the relaxation picture, although there are a number of reasons for the real relaxation. Thus far our analysis has assumed the mean-field approach. The relaxation can only appear beyond this approach: One should at least consider the exact Hamiltonian and include in the analysis the interaction terms proportional to $U^2$.

Fast decrease of the frequency to $\omega_1 \ll \omega_0$ brings the system to a high ES state of the final potential. In the case of a large interaction ($\eta \gg 1$) the characteristic energy of expansion is $\sim \mu$. For maximum expansion ($b \approx \omega_0/\omega_1$) the evolution is almost quasistationary and the size $r_0(t)$ is close to the size of stationary states with energy $\sim \mu$ in the final potential. The corresponding one-particle density of states $g(\mu) \sim \mu^2/2(h\omega_1)^3$ ($d=3$) will be very large, ensuring that the spacing between adjacent levels $\delta e \sim \mu(h\omega_1/\mu)^3$ is much smaller than the interparticle interaction at maximum expansion $\bar{U}_{\text{min}} \sim \mu(\omega_1/\omega_0)^3$. The many-particle density of states grows exponentially, with the exponent depending on the number of particles $N$. Under these conditions even a small external influence can lead to "mixing" of states and insert irreversibility.

The characteristic relaxation time in this case will be determined by the collisional time $\tau_c \sim [n\sigma v(\mu)]^{-1}$, where $\sigma = 8\pi a^2$ is the elastic cross section, $v(\mu)$ the particle velocity at energy $\mu$, and $n \sim n_{\text{min}}$. Strictly speaking, $\tau_c$ represents the minimum relaxation time and it would be interesting to find possibilities to observe a larger time of relaxation. One may assume that even in the absence of any external influence $\tau_c$ will be responsible for the formation of the state which imitates relaxation. The imitation can be promoted by the deviation of the external field from harmonicity or spherical symmetry.

The characteristic time of dynamic evolution is determined by $\omega_1^{-1}$, and we obtain a dimensionless parameter characterizing the relaxation:

$$\tau_c \sim (l_0/a) (l_0/N_{\text{min}})^{3/5} (\omega_0/\omega_1)^2.$$ 

This estimate shows the possibility of both fast and slow relaxation. For $\tau_c \omega_1 \gg 1$ one can expect to observe oscillations of the condensate.

For a small interaction ($\eta = N_0\alpha l_0 \ll 1$) the characteristic energy of expansion $\sim \hbar \omega_0$, and we have $(\tau_c \omega_1) \sim (l_0/a) \times (l_0/N_{\text{min}}) (\omega_0/\omega_1)^2 \gg 1$. In this limit the relaxation, and hence the loss of coherence, is always slower.

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