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On the Hochschild homology of proper Lie groupoids

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Abstract. We study the Hochschild homology of the convolution algebra of a proper Lie groupoid by introducing a convolution sheaf over the space of orbits. We develop a localization result for the associated Hochschild homology sheaf, and we prove that the Hochschild homology sheaf at each stalk is quasi-isomorphic to the stalk at the origin of the Hochschild homology of the convolution algebra of its linearization, which is the transformation groupoid of a linear action of a compact isotropy group on a vector space. We then explain Brylinski's ansatz to compute the Hochschild homology of the transformation groupoid of a compact group action on a manifold. We verify Brylinski's conjecture for the case of smooth circle actions that the Hochschild homology is given by basic relative forms on the associated inertia space.

1. Introduction

Let M be a smooth manifold and $\mathcal{C}^\infty(M)$ the algebra of smooth functions on M. Connes' version [11] of the seminal Hochschild–Kostant–Rosenberg theorem [26] states that the Hochschild homology of $\mathcal{C}^\infty(M)$ is isomorphic to the graded vector space of differential forms on M. In this paper, we aim to establish tools for a general Hochschild–Kostant–Rosenberg-type theorem for proper Lie groupoids.

Recall that a Lie groupoid $\mathsf{G} \rightrightarrows M$ is proper if the map $\mathsf{G} \to M \times M$, $g \mapsto (s(g), t(g))$ is a proper map, where $s(g)$ and $t(g)$ are the source and target of $g \in \mathsf{G}$. When the source and target maps are both local diffeomorphisms, the groupoid $\mathsf{G} \rightrightarrows M$ is called étale. Through the efforts of many authors, notably [6, 9, 11–13, 20, 43, 49], the Hochschild and cyclic homology theory of étale Lie groupoids has been unveiled. The Hochschild and cyclic homology of a proper étale Lie groupoid was explicitly computed by Brylinksi and Nistor [9], and later extended and refined by Crainic [13] and Ponge [43] for general étale groupoids. Let us explain this result in the case of a finite group action on a smooth manifold using the transformation groupoid $\Gamma \ltimes M \rightrightarrows M$, where a finite group Γ acts on the manifold M.

The groupoid convolution algebra associated to the transformation groupoid $\Gamma \ltimes M \rightrightarrows M$ is the crossed product algebra $\mathcal{C}^\infty(M) \rtimes \Gamma$ which consists of $\mathcal{C}^\infty(M)$-valued functions on Γ equipped with the convolution product, e.g. for $f, g \in \mathcal{C}^\infty(M) \rtimes \Gamma$,

$$f * g(\gamma) = \sum_{\alpha\beta=\gamma} \beta^*\big(f(\alpha)\big) \cdot g(\beta).$$

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The algebra $\mathcal{C}^\infty(M) \rtimes \Gamma$ is naturally a Fréchet algebra. The Hochschild homology of the algebra $\mathcal{C}^\infty(M) \rtimes \Gamma$ as a bornological algebra is given by the following formula the proof of which is recalled in Corollary B.8:

$$HH_\bullet\big(\mathcal{C}^\infty(M) \rtimes \Gamma\big) \cong \Big(\bigoplus_{\gamma \in \Gamma} \Omega^\bullet(M^\gamma)\Big)^\Gamma,$$

where M^γ is the γ-fixed point submanifold and Γ acts on the disjoint union $\coprod_{\gamma \in \Gamma} M^\gamma$ by $\gamma'(\gamma, x) = (\gamma'\gamma(\gamma')^{-1}, \gamma' x)$. Recall that the so-called loop space $\Lambda_0(\Gamma, M)$ of the transformation groupoid $\Gamma \ltimes M \rightrightarrows M$ is defined as

$$\Lambda_0(\Gamma, M) := \coprod_{\gamma \in \Gamma} M^\gamma,$$

equipped with the same action of Γ as above. In other words, the Hochschild homology of $\mathcal{C}^\infty(M) \rtimes \Gamma$ is the space of differential forms on the quotient $\Lambda_0(\Gamma, M)/\Gamma$, which is called the associated inertia orbifold. We would like to remark that, just as the classical Hochschild–Kostant–Rosenberg theorem, the above identification can be realized as an isomorphism of sheaves over the quotient M/Γ. This makes Hochschild and cylic homology of $\mathcal{C}^\infty(M) \rtimes \Gamma$ the right object to work with in the study of orbifold index theory; see e.g. [38].

Our goal in this project is to extend the study of Hochschild homology of proper étale groupoids to general proper Lie groupoids, which are natural generalizations of transformation groupoids for proper Lie group actions. The key new challenge from the study of (proper) étale groupoids is that orbits of a general proper Lie groupoid have different dimensions. This turns the orbit space of a proper Lie groupoid into a stratified space with a significantly more complicated singularity structure than an orbifold.

Our main result is to introduce a sheaf $\mathcal{HH}_\bullet$ on the orbit space $X := M/\mathsf{G}$ of a proper Lie groupoid $\mathsf{G} \rightrightarrows M$, whose space of global sections computes the Hochschild homology of the convolution algebra of G. To achieve this, we start with introducing a sheaf $\mathcal{A}$ of convolution algebras on the orbit space X in Definition 2.1. Using the localization method from [4] we introduce the Hochschild homology sheaf $\mathcal{HH}_\bullet(\mathcal{A})$ for $\mathcal{A}$ as a sheaf of bornological algebras over X. Moreover, we prove the following sheafification theorem for the Hochschild homology of the convolution algebra $\mathcal{A}$ of the groupoid G.

Theorem 3.3. *Let $\mathcal{A}$ be the convolution sheaf of a proper Lie groupoid G. Then the natural map in Hochschild homology*

$$HH_\bullet\big(\mathcal{A}(X)\big) \to \mathcal{HH}_\bullet(\mathcal{A})(X) = \Gamma\big(X, \mathcal{HH}_\bullet(\mathcal{A})\big)$$

is an isomorphism.

To determine the homology sheaf $\mathcal{HH}_\bullet(\mathcal{A})$, we study its stalk at an orbit $\mathcal{O} \in X$. Using the linearization result of proper Lie groupoid developed by Weinstein and Zung (cf. [15,16,39,51,53]), we obtain a linear model of the stalk $\mathcal{HH}_{\bullet,\mathcal{O}}(\mathcal{A})$ in Proposition 4.5

as a linear compact group action on a vector space. This result leads us to focus on the Hochschild homology of the convolution algebra $\mathcal{C}^\infty(M) \rtimes G$ associated to a compact Lie group action on a smooth manifold M in the second part of this article.

The Hochschild homology of compact Lie group actions was studied by several authors, e.g. [3, 7, 8]. Brylinski [7, 8] proposed a geometric model of basic relative forms along the idea of the Grauert–Grothendieck forms to compute the Hochschild homology. However, a major part of the proof is missing in [7, 8]. We decided to turn this result into the main conjecture of this paper in Section 6.

Conjecture 6.7. *The Hochschild homology of the crossed product algebra $\mathcal{C}^\infty(M) \rtimes G$ associated to a compact Lie group action on a smooth manifold M is isomorphic to the space of basic relative forms on the loop space*

$$\Lambda_0(G \ltimes M) = \{(g, p) \in G \times M \mid gp = p\}.$$

Block and Getzler [3] introduced an interesting Cartan model for the cyclic homology of the crossed product algebra $\mathcal{C}^\infty(M) \rtimes G$. However, the Block–Getzler model is not a sheaf on the orbit space M/G, but a sheaf on the space of conjugacy classes of G. This makes it impossible to localize the sheaf to an orbit of the group action in the orbit space. It is worth pointing out that the truncation of the Block–Getzler Cartan model at E^1-page provides a complex to compute the Hochschild homology of $\mathcal{C}^\infty(M) \rtimes G$. However, the differential ι introduced in [3, Section 1] is nontrivial, and makes it challenging to explicitly identify the Hochschild homology of $\mathcal{C}^\infty(M) \rtimes G$ as the space of basic relative forms. We refer the reader to Remark 5.3 for a more detailed discussion about the Block–Getzler model.

In the last part of this paper, we prove Conjecture 6.7 in the case where the group G is $\mathbb{S}^1$; see Proposition 7.9. Our proof relies on a careful study of the stratification of the loop space $\Lambda_0(\mathbb{S}^1 \ltimes M) \subset \mathbb{S}^1 \times M$. The crucial property we use in our computation is that at its singular point, $\Lambda_0(\mathbb{S}^1 \ltimes M)$ locally looks like the union of the hyperplane $\{x_0 = 0\}$ and the line $\{x_1 = \cdots = x_n = 0\}$ in $\mathbb{R}^{n+1}$, which are transverse to each other. The loop space $\Lambda_0(G \ltimes M)$ for a general G-manifold M is much more complicated to describe. This has stopped us from extending our result for $\mathbb{S}^1$-actions to more general compact group actions. It is foreseeable that some combinatorial structures describing the stratifications of the loop spaces and real algebraic geometry tools characterizing basic relative forms on the loop spaces are needed to solve Conjecture 6.7 in full generality. We plan to come back to this problem in the near future.

As mentioned above, the Hochschild and cyclic homologies of the convolution algebra of a proper Lie groupoid are closely related to groupoid index theory; see e.g. [38, 40]. We expect that the study of the Hochschild homology and the generalized Connes–Hochschild–Kostant–Rosenberg theorem in this paper will lead to the correct definition of basic relative forms for proper Lie groupoids and to a geometric description of their (periodic) cyclic homology. We hope that eventually this will open up a path to establish a general index theorem for proper Lie groupoids.

2. The convolution sheaf of a proper Lie groupoid

Throughout this paper, $\mathsf{G} \rightrightarrows M$ denotes a Lie groupoid over a base manifold M. Elements of M are called points of the groupoid; those of G are its arrows. The symbols $s, t : \mathsf{G} \to M$ denote the source and target map, respectively, and $u : M \to \mathsf{G}$ the unit map. By definition of a Lie groupoid, s and t are assumed to be smooth submersions. This implies that the space of k-tuples of composable arrows

$$\mathsf{G}_k := \big\{(g_1, \dots, g_k) \in \mathsf{G}^k \mid s(g_i) = t(g_{i+1}) \text{ for } i = 1, \dots, k-1\big\}$$

is a regular submanifold of G^k, and multiplication of arrows

$$m : \mathsf{G}_2 \to \mathsf{G}, \quad (g_1, g_2) \mapsto g_1 g_2$$

is a smooth map.

If $g \in \mathsf{G}$ is an arrow with $s(g) = x$ and $t(g) = y$, we denote such an arrow sometimes by $g : y \leftarrow x$, and write $\mathsf{G}(y, x)$ for the space of arrows with source x and target y. The s-fiber over x, i.e., the manifold $s^{-1}(x)$, is denoted by $\mathsf{G}(-, x)$, the t-fiber over y by $\mathsf{G}(y, -)$. Note that for each object $x \in M$ multiplication of arrows induces on $\mathsf{G}(x, x)$ a group structure. This group is called the *isotropy group* of x and is denoted by G_x. The union of all isotropy groups

$$\Lambda_0 \mathsf{G} := \bigcup_{x \in M} \mathsf{G}_x = \big\{g \in \mathsf{G} \mid s(g) = t(g)\big\}$$

is called the *loop space* of G.

Given a Lie groupoid $\mathsf{G} \rightrightarrows M$, two points $x, y \in M$ are said to lie in the same orbit if there is an arrow $g : y \leftarrow x$. In the following, we will always write $\mathcal{O}_x$ for the orbit containing x, and M/G for the space of orbits of the groupoid G. We assume further that the orbit space always carries the quotient topology with respect to the canonical map $\pi : M \to M/\mathsf{G}$. Note that M/G need not be Hausdorff unless G is a proper Lie groupoid, which means that the map $(s, t) : \mathsf{G} \to M \times M$ is a proper map.

Sometimes, we need to specify to which groupoid a particular structure map belongs to. In such a situation we will write s_{G}, m_{G}, π_{G}, and so on.

In the following, we will define a sheaf of algebras $\mathcal{A}$ on M/G in such a way that the algebra $\mathcal{A}_c(M/\mathsf{G})$ of compactly supported global sections of $\mathcal{A}$ coincides with the smooth convolution algebra of the groupoid. To this end, we use a smooth left Haar system on G.

Recall that by a smooth left Haar system on G we understand a family of measures $(\lambda^x)_{x \in M}$ such that the following properties hold true.

(H1) For every $x \in \mathsf{G}_0$, λ^x is a positive measure on $\mathsf{G}(x, -)$ with $\operatorname{supp} \lambda^x = \mathsf{G}(x, -)$.

(H2) For every $g \in \mathsf{G}$, the family $(\lambda^x)_{x \in M}$ is invariant under left multiplication

$$L_g : \mathsf{G}\big(s(g), -\big) \to \mathsf{G}\big(t(g), -\big), \quad h \mapsto gh$$

or in other words

$$\int_{\mathsf{G}(s(g),-)} u(gh)\, d\lambda^{s(g)}(h) = \int_{\mathsf{G}(t(g),-)} u(h)\, d\lambda^{t(g)}(h) \quad \text{for all } u \in \mathcal{C}^\infty_c(\mathsf{G}).$$

(H3) The system is smooth in the sense that for every $u \in \mathcal{C}^\infty_c(\mathsf{G})$ the map

$$M \to \mathbb{C}, \quad x \mapsto \int_{\mathsf{G}(x,-)} u(h)\, d\lambda^x(h)$$

is smooth.

Let us fix a smooth left Haar system $(\lambda^x)_{x\in M}$ on G. Given an open set $U \subset M/\mathsf{G}$, we first put

$$U_0 := \pi^{-1}(U), \quad U_1 := s^{-1}(U_0) \subset \mathsf{G}_1, \quad \text{and} \quad U_{k+1} := \bigcap_{i=1}^{k} \sigma_i^{-1}(U_k) \subset \mathsf{G}_{k+1} \tag{2.1}$$

for all $k \in \mathbb{N}^*$, where $\sigma_i : \mathsf{G}_{k+1} \to \mathsf{G}_k$, $(g_1, \ldots, g_{k+1}) \mapsto (g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1})$. Then we define

$$\mathcal{A}(U) := \{ f \in \mathcal{C}^\infty(U_1) \mid \operatorname{supp} f \text{ is longitudinally compact} \}. \tag{2.2}$$

Hereby, a subset $K \subset \mathsf{G}$ is called *longitudinally compact* if for every compact subset $C \subset M/\mathsf{G}$ the intersection $K \cap s^{-1}\pi^{-1}(C)$ is compact. Obviously, every $\mathcal{A}(U)$ is a linear space, and the map which assigns to an open $U \subset M/\mathsf{G}$ the space $\mathcal{A}(U)$ forms a sheaf on M/G which in the following is denoted by $\mathcal{A}$ or by $\mathcal{A}_\mathsf{G}$ if we want to emphasize the underlying groupoid. The section space $\mathcal{A}(U)$ over $U \subset M/\mathsf{G}$ open becomes an associative algebra with the *convolution product*

$$f_1 * f_2(g) := \int_{\mathsf{G}(t(g),-)} f_1(h) f_2(h^{-1}g)\, d\lambda^{t(g)}(h), \quad f_1, f_2 \in \mathcal{A}(U),\ g \in \mathsf{G}. \tag{2.3}$$

The convolution product is compatible with the restriction maps, hence $\mathcal{A}$ becomes a sheaf of algebras on M/G.

Let us assume from now on that the groupoid G is proper. Recall from [39] that then the orbit space M/G carries the structure of a differentiable stratified space in a canonical way. The structure sheaf $\mathcal{C}^\infty_{M/\mathsf{G}}$ coincides with the sheaf of continuous functions $\varphi : U \to \mathbb{R}$ with $U \subset M/\mathsf{G}$ open such that $\varphi \circ \pi \in \mathcal{C}^\infty(U_1)$. Now observe that the action

$$\mathcal{C}^\infty_{M/\mathsf{G}}(U) \times \mathcal{A}(U) \to \mathcal{A}(U), \quad (\varphi, f) \mapsto \varphi f := \big(U_1 \ni g \mapsto \varphi\big(\pi s(g)\big) f(g) \in \mathbb{R}\big)$$

commutes with the convolution product, and turns $\mathcal{A}$ into a $\mathcal{C}^\infty_{M/\mathsf{G}}$-module sheaf.

Proposition and Definition 2.1. *Given a proper Lie groupoid $\mathsf{G} \rightrightarrows M$, the associated sheaf $\mathcal{A}$ is a fine sheaf of algebras over the orbit space M/G which in addition carries the structure of a $\mathcal{C}^\infty_{M/\mathsf{G}}$-module sheaf. The space $\mathcal{A}_c(M/\mathsf{G})$ of global sections of $\mathcal{A}$ with compact support coincides with the* smooth convolution algebra *of G. We call $\mathcal{A}$ the* convolution sheaf *of G.*

For later purposes, we equip the spaces $\mathcal{A}(U)$ with a locally convex topology and a convex vector bornology. To this end, observe first that for every longitudinally compact subset $K \subset U_1$ the space

$$\mathcal{A}(M/\mathsf{G}; K) := \{ f \in \mathcal{C}^\infty(\mathsf{G}) \mid \operatorname{supp} f \subset K \}$$

inherits from $\mathcal{C}^\infty(\mathsf{G})$ the structure of a Fréchet space. Moreover, since $\mathcal{C}^\infty(\mathsf{G})$ is nuclear, $\mathcal{A}(M/\mathsf{G}; K)$ has to be nuclear as well by [48, Proposition 50.1]. By separability of U there exists a (countable) exhaustion of U_1 by longitudinally compact sets, i.e., a family $(K_n)_{n\in\mathbb{N}}$ of a longitudinally compact subset of U_1 such that $K_n \subset K_{n+1}^\circ$ for all $n \in \mathbb{N}$, and such that $\bigcup_{n\in\mathbb{N}} K_n = U_1$. The space $\mathcal{A}(U)$ can then be identified with the inductive limit of the strict inductive system of nuclear Fréchet spaces $(\mathcal{A}(M/\mathsf{G}; K_n))_{n\in\mathbb{N}}$. It is straightforward to check that the resulting inductive limit topology on $\mathcal{A}(U)$ does not depend on the particular choice of the exhaustion $(K_n)_{n\in\mathbb{N}}$. Thus, $\mathcal{A}(U)$ becomes a nuclear LF-space, where nuclearity follows from [48, Proposition 50.1]. As an LF-space, $\mathcal{A}(U)$ carries a natural bornology given by the von Neumann bounded sets, i.e., by the sets $S \subset \mathcal{A}(U)$ which are absorbed by each neighborhood of 0. In other words, a subset $S \subset \mathcal{A}(U)$ is bounded if all $f \in S$ are supported in a fixed longitudinally compact subset $K \subset U_1$, and if the set of functions $D(S)$ is uniformly bounded for every compactly supported differential operator D on U_1.

Remark 2.2. (1) We refer to Appendix B and [27] for basic definitions and fundamentals on bornological vector spaces. Bornological tensor products and their completions are defined in Appendix B and [32].

(2) In this paper, we always assume the bornologies to be convex vector bornologies. We also often make use of the fact that for two nuclear LF-spaces V_1 and V_2 their completed bornological tensor product $V_1 \hat{\otimes} V_2$ naturally coincides (up to natural equivalence) with the completed inductive tensor product $V_1 \hat{\otimes}_\iota V_2$ endowed with the bornology of von Neumann bounded sets. Moreover, $V_1 \hat{\otimes}_\iota V_2$ is again a nuclear LF-space. We refer to [32, Section A.1.4] for a proof of these propositions. Note that for Fréchet spaces the projective and inductive topological tensor product coincide.

(3) For LF-spaces like the convolution algebras, we consider here that the projective and inductive topological tensor products do in general not coincide. The bornological point of view therefore is not only particularly convenient but even crucial when considering tensor products of such spaces, since the (completed) bornological tensor product is the natural distinguished tensor product which needs to be used when the projective topological tensor product fails to work and since it has all the necessary properties needed in cyclic homology theory; see [32] for details.

For our purposes, the following observation is fundamental.

Proposition 2.3. *Let $\mathsf{G} \rightrightarrows M$ and $\mathsf{H} \rightrightarrows N$ be proper Lie groupoids. Denote by M/G and N/H their respective orbit spaces. Then $M/\mathsf{G} \times N/\mathsf{H}$ is diffeomorphic as a differentiable stratified space to the orbit space of the product groupoid $\mathsf{G} \times \mathsf{H} \rightrightarrows M \times N$. Moreover,* *there is a natural isomorphism between the completed bornological tensor product of the convolution algebras over* G *and* H *and the convolution algebra of the product groupoid* $\mathsf{G} \times \mathsf{H}$. *More precisely, for every pair of open sets* $U \subset M/\mathsf{G}$ *and* $V \subset N/\mathsf{H}$, *there is a natural isomorphism*

$$\mathcal{A}_{\mathsf{G}}(U) \mathbin{\hat{\otimes}} \mathcal{A}_{\mathsf{H}}(V) \cong \mathcal{A}_{\mathsf{G}\times\mathsf{H}}(U \times V), \tag{2.4}$$

where $\hat{\otimes}$ *denotes the completed bornological tensor product.*

Proof. The first claim is a consequence of the fact that two elements $(x, y), (x', y') \in M \times N$ lie in the same $(\mathsf{G} \times \mathsf{H})$-orbit if and only if x and x' lie in the same G-orbit and y and y' lie in the same H-orbit. Let us prove the second claim. Let $(K_n)_{n\in\mathbb{N}}$ be an exhaustion of $U_1 := s_{\mathsf{G}}^{-1}\pi_{\mathsf{G}}^{-1}(U)$ by longitudinally compact subsets and $(L_m)_{m\in\mathbb{N}}$ an exhaustion of $V_1 := s_{\mathsf{H}}^{-1}\pi_{\mathsf{H}}^{-1}(V)$ by such sets. Since $\mathcal{A}_{\mathsf{G}}(U)$ coincides with the inductive limit $\operatorname{colim}_{n\in\mathbb{N}} \mathcal{A}_{\mathsf{G}}(M/\mathsf{G}; K_n)$ and $\mathcal{A}_{\mathsf{H}}(V)$ with $\operatorname{colim}_{m\in\mathbb{N}} \mathcal{A}_{\mathsf{H}}(N/\mathsf{H}; L_m)$, Corollary 2.30 in [32] entails that

$$\mathcal{A}_{\mathsf{G}}(U) \mathbin{\hat{\otimes}} \mathcal{A}_{\mathsf{H}}(V) \cong \operatorname*{colim}_{n\in\mathbb{N}} \mathcal{A}_{\mathsf{G}}(M/\mathsf{G}; K_n) \mathbin{\hat{\otimes}} \mathcal{A}_{\mathsf{H}}(N/\mathsf{H}; L_n). \tag{2.5}$$

Now observe that $\mathcal{A}_{\mathsf{G}}(M/\mathsf{G}; K_n) \mathbin{\hat{\otimes}} \mathcal{A}_{\mathsf{H}}(N/\mathsf{H}; L_m) \cong \mathcal{A}_{\mathsf{G}\times\mathsf{H}}(M/\mathsf{G} \times N/\mathsf{H}; K_n \times L_m)$ by [48, Proposition 51.6], and that $(K_n \times L_n)_{n\in\mathbb{N}}$ is an exhaustion of $U \times V$ by longitudinally compact subsets. Together with equation (2.5) this proves the claim. ■

3. Localization of the Hochschild chain complex

In this section, we apply the localization method in Hochschild homology theory, partially following [4], to the Hochschild chain complex of the convolution algebra.

3.1. Sheaves of bornological algebras over a differentiable space

We start with a (reduced separated second countable) differentiable space $(X, \mathcal{C}^\infty)$ and assume that $\mathcal{A}$ is a sheaf of $\mathbb{R}$-algebras on X. We will denote by $A = \mathcal{A}(X)$ its space of global sections. We assume further that $\mathcal{A}$ is a $\mathcal{C}^\infty_X$-module sheaf and that every section space $\mathcal{A}(U)$ with $U \subset X$ open carries the structure of a nuclear LF-space such that each of the restriction maps $\mathcal{A}(U) \to \mathcal{A}(V)$ is continuous for every open subset V of U, and multiplication in $\mathcal{A}(U)$ is separately continuous. Finally, it is assumed that the action $\mathcal{C}^\infty(U) \times \mathcal{A}(U) \to \mathcal{A}(U)$ is continuous.

As a consequence of our assumptions, each of the spaces $\mathcal{A}(U)$ carries a natural bornology, namely the one consisting of all von Neumann bounded subsets, i.e., of all subsets $B \subset \mathcal{A}(U)$ which are absorbed by every neighborhood of the origin. Moreover, by [33, Lemma 1.30], separate continuity of multiplication in $\mathcal{A}(U)$ entails that the product map is a jointly bounded map, hence induces a bounded map $\mathcal{A}(U) \mathbin{\hat{\otimes}} \mathcal{A}(U) \to \mathcal{A}(U)$ on the completed bornological tensor product of $\mathcal{A}(U)$ with itself. We therefore call a sheaf of algebras $\mathcal{A}$ defined over the differentiable space $(X, \mathcal{C}^\infty_X)$ such that the above assumptions are fulfilled a *sheaf of bornological algebras over* X.

Definition 3.1. A sheaf of bornological algebras $\mathcal{A}$ over the differentiable space $(X, \mathcal{C}^\infty_X)$ is called

(i) a *sheaf of unital bornological algebras* or just *unital* if all section spaces $\mathcal{A}(U)$ are unital algebras and the restriction maps $\mathcal{A}(U) \to \mathcal{A}(V)$ are unital homomorphisms, and

(ii) a *sheaf of H-unital bornological algebras* or briefly *H-unital* if every section space $\mathcal{A}(U)$ is an H-unital algebra that is if the Bar complex of $\mathcal{A}(U)$ is acyclic.

Example 3.2. (1) The structure sheaf $\mathcal{C}^\infty_X$ of a differentiable space $(X, \mathcal{C}^\infty_X)$ is an example of a sheaf of unital bornological algebras over $(X, \mathcal{C}^\infty_X)$.

(2) Given a proper Lie groupoid G, the convolution sheaf $\mathcal{A}$ is a sheaf of H-unital bornological algebras over the orbit space $(X, \mathcal{C}^\infty_X)$ of the groupoid. This follows by construction of $\mathcal{A}$ and [14, Proposition 2], which entails H-unitality of each of the section spaces $\mathcal{A}(U)$.

3.2. The Hochschild homology sheaf

Assume that $\mathcal{A}$ is a sheaf of bornological algebras over the differentiable space $(X, \mathcal{C}^\infty_X)$. We will construct the Hochschild homology sheaf $\mathcal{HH}_\bullet(\mathcal{A})$ associated to $\mathcal{A}$ as a generalization of Hochschild homology for algebras; see [30] for the latter and Appendix B for basic definitions and notation used.

For each $k \in \mathbb{N}^*$, let $\mathcal{C}_k(\mathcal{A})$ denote the presheaf on X which assigns to an open $U \subset X$ the $(k+1)$-fold completed bornological tensor product $\mathcal{A}(U)^{\hat{\otimes}(k+1)}$. Note that in general, $\mathcal{C}_k(\mathcal{A})$ is not a sheaf. We denote by $\hat{\mathcal{C}}_k(\mathcal{A})$ the sheafification of $\mathcal{C}_k(\mathcal{A})$. Observe that for $V \subset U \subset X$ open the Hochschild boundary

$$b : \mathcal{C}_k(\mathcal{A})(U) \to \mathcal{C}_{k-1}(\mathcal{A})(U)$$

commutes with the restriction maps

$$r^U_V : \mathcal{C}_k(\mathcal{A})(U) \to \mathcal{C}_k(\mathcal{A})(V),$$

hence we obtain a complex of presheaves $(\mathcal{C}_\bullet(\mathcal{A}), b)$ and by the universal property of the sheafification a sheaf complex $(\hat{\mathcal{C}}_\bullet(\mathcal{A}), b)$. The Hochschild homology sheaf $\mathcal{HH}_\bullet(\mathcal{A})$ is now defined as the homology sheaf of $(\hat{\mathcal{C}}_\bullet(\mathcal{A}), b)$ that means

$$\mathcal{HH}_k(\mathcal{A}) := \ker\big(b : \hat{\mathcal{C}}_k(\mathcal{A}) \to \hat{\mathcal{C}}_{k-1}(\mathcal{A})\big) / \operatorname{im}\big(b : \hat{\mathcal{C}}_{k+1}(\mathcal{A}) \to \hat{\mathcal{C}}_k(\mathcal{A})\big).$$

By construction, the stalk $\mathcal{HH}_k(\mathcal{A})_{\mathcal{O}}$, $\mathcal{O} \in X$ coincides with the k-th Hochschild homology $HH_k(\mathcal{A}_{\mathcal{O}})$ of the stalk $\mathcal{A}_{\mathcal{O}}$. On the other hand, $HH_k(\mathcal{A}(X))$ need in general not coincide with the space $\mathcal{HH}_k(\mathcal{A})(X)$ of global sections of the k-th Hochschild homology sheaf. The main goal of this section is to prove the following result which is crucial for our study of the Hochschild homology of the convolution algebra of a proper Lie groupoid, but also might be interesting by its own. Its proof will cover the remainder of this section.

Theorem 3.3. *Let $\mathcal{A}$ be the convolution sheaf of a proper Lie groupoid* G. *Then the natural map in Hochschild homology*

$$HH_\bullet\big(\mathcal{A}(X)\big) \to \mathcal{HH}_\bullet(\mathcal{A})(X) = \Gamma\big(X, \mathcal{HH}_\bullet(\mathcal{A})\big)$$

is an isomorphism.

Before we can spell out the proof we need several auxiliary tools and results.

3.3. The localization homotopies

Throughout this paragraph, we assume that $\mathcal{A}(X)$ is an admissible sheaf of bornological algebras over the differentiable space $(X, \mathcal{C}^\infty_X)$.

To construct the localization morphisms, observe that the complex $C_\bullet(A)$ inherits from $A = \mathcal{A}(X)$ the structure of a $\mathcal{C}^\infty(X)$-module. More precisely, the corresponding action is given by

$$\begin{aligned} &\mathcal{C}^\infty(X) \times C_k(A) \to C_k(A), \\ &(\varphi, a_0 \otimes \cdots \otimes a_k) \mapsto (\varphi a_0) \otimes a_1 \otimes \cdots \otimes a_k. \end{aligned} \tag{3.1}$$

It is immediate from its definition that the $\mathcal{C}^\infty(X)$-action commutes with the operators b and b' and hence induces a chain map $\mathcal{C}^\infty(X) \times C_\bullet(A) \to C_\bullet(A)$. In a similar fashion, we define an action of $\mathcal{C}^\infty(X^{k+1}) \cong (\mathcal{C}^\infty(X))^{\hat{\otimes}(k+1)}$ on $C_k(A)$ by

$$(\varphi_0 \otimes \cdots \otimes \varphi_k, a_0 \otimes \cdots \otimes a_k) \mapsto (\varphi_0 a_0) \otimes \cdots \otimes (\varphi_k a_k). \tag{3.2}$$

This allows us to speak of the *support* of a chain $c \in C_k(A)$. It is defined as the complement of the largest open subset U in X^{k+1} such that $\varphi \cdot c = 0$ for all $\varphi \in \mathcal{C}^\infty(X)$ with $\operatorname{supp} \varphi \subset U$.

Next choose a metric $d : X \times X \to \mathbb{R}$ such that the function d^2 lies in $\mathcal{C}^\infty(X \times X)$. Such a metric exists by Corollary A.4. Then fix a smooth function $\varrho : \mathbb{R} \to [0, 1]$ which has support in $(-\infty, \frac{3}{4}]$ and satisfies $\varrho(r) = 1$ for $r \leq \frac{1}{2}$. For $\varepsilon > 0$, we denote by ϱ_ε the rescaled function $r \mapsto \varrho(\frac{s}{\varepsilon^2})$. Now define functions $\Psi_{k,i,\varepsilon} \in \mathcal{C}^\infty(X^{k+1})$ for $k \in \mathbb{N}$ and $i = 0, \ldots, k$ by

$$\Psi_{k,i,\varepsilon}(x_0, \ldots, x_k) = \prod_{j=0}^{i-1} \varrho_\varepsilon\big(d^2(x_j, x_{j+1})\big), \tag{3.3}$$

where $x_0, \ldots, x_k \in X$ and $x_{k+1} := x_0$. Moreover, put $\Psi_{k,\varepsilon} := \Psi_{k,k+1,\varepsilon}$. Using the $\mathcal{C}^\infty(X^{k+1})$-action on $C_k(A)$, we obtain for each $\varepsilon > 0$ a graded map of degree 0

$$\Psi_\varepsilon : C_\bullet(A) \to C_\bullet(A), \quad C_k(A) \ni c \mapsto \Psi_{k,\varepsilon} c.$$

We immediately check that Ψ_ε commutes with the face maps b_i and the cyclic operator t_k. Hence, Ψ_ε is a chain map. We even have more.

Lemma 3.4. *Let $\mathcal{A}$ be a sheaf of H-unital bornological algebras over the differentiable space $(X, \mathcal{C}^\infty)$, and put $A := \mathcal{A}(X)$. Let d be a metric on X such that d^2 is smooth and* *fix a smooth map $\varrho : \mathbb{R} \to [0,1]$ with support in $(-\infty, \frac{3}{4}]$ such that $\varrho|_{(\infty,\frac{1}{2}]} = 1$. Then, for each $\varepsilon > 0$, the chain map $\Psi_\varepsilon : C_\bullet(A) \to C_\bullet(A)$ is homotopic to the identity morphism on $C_\bullet(A)$.*

Proof. Let us first consider the case, where $\mathcal{A}$ is a sheaf of unital algebras. The Hochschild chain complex then is a simplicial module with face maps b_i and the degeneracy maps

$$s_{k,i} : C_k(A) \to C_{k+1}(A), \quad a_0 \otimes \cdots \otimes a_k \mapsto a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_k,$$

where $k \in \mathbb{N}$, $i = 0, \ldots, k$. Define $\mathcal{C}^\infty(X)$-module maps $\eta_{k,i,\varepsilon} : C_k(A) \to C_{k+1}(A)$ for $k \in \mathbb{N}$, $i = 1, \ldots, k+2$ and $\varepsilon > 0$ by

$$\eta_{k,i,\varepsilon}(c) := \begin{cases} \Psi_{k+1,i,\varepsilon} \cdot (s_{k,i-1}c) & \text{for } i \le k+1, \\ 0 & \text{for } i = k+2. \end{cases} \tag{3.4}$$

Moreover, put $C_{-1}(A) := \{0\}$ and let $\eta_{-1,1,\varepsilon} : C_{-1}(A) \to C_0(A)$ be the 0-map. For $k \ge 1$ and $i = 2, \ldots, k$, we then compute

$$\begin{aligned}(b\eta_{k,i,\varepsilon} + \eta_{k-1,i,\varepsilon}b)c = (-1)^{i-1}\Psi_{k,i-1,\varepsilon}c + \Psi_{k,i-1,\varepsilon}\sum_{j=0}^{i-2}(-1)^j s_{k-1,i-2}b_{k,j}c \\ + (-1)^i\Psi_{k,i,\varepsilon}c + \Psi_{k,i,\varepsilon}\sum_{j=0}^{i-1}(-1)^j s_{k-1,i-1}b_{k,j}c.\end{aligned}$$

For the case $i = 1$, we obtain

$$(b\eta_{k,1,\varepsilon} + \eta_{k-1,1,\varepsilon}b)c = c - \Psi_{k,1,\varepsilon}c + \Psi_{k,1,\varepsilon}s_{k-1,0}b_{k,0}c,$$

and for $i = k+1$

$$\begin{aligned}&(b\eta_{k,k+1,\varepsilon} + \eta_{k-1,k+1,\varepsilon}b)c \\ &\quad = \Psi_{k,k,\varepsilon}(-1)^k c + \Psi_{k,k,\varepsilon}\sum_{j=0}^{k-1}(-1)^j s_{k-1,k-1}b_{k,j}c + (-1)^{k+1}\Psi_{k,\varepsilon}c.\end{aligned}$$

Finally, we check for $k = 0$ and $i = 1$

$$(b\eta_{0,1,\varepsilon} + \eta_{-1,1,\varepsilon}b)c = b\eta_{0,1,\varepsilon}c = 0.$$

These formulas immediately entail that the maps

$$H_{k,\varepsilon} = \sum_{i=1}^{k+1}(-1)^{i+1}\eta_{k,i,\varepsilon} : C_k(A) \to C_{k+1}(A)$$

form a homotopy between the identity and the localization morphism Ψ_ε. More precisely,

$$(bH_{k,\varepsilon} + H_{k-1,\varepsilon}b)c = c - \Psi_\varepsilon c \quad \text{for all } k \in \mathbb{N} \text{ and } c \in C_k(A). \tag{3.5}$$

This finishes the proof of the claim in the unital case.

Now let us consider the general case, where $\mathcal{A}$ is assumed to be a sheaf of H-unital but not necessarily unital algebras. Consider the direct sum of sheaves $\mathcal{A} \oplus \mathcal{C}^\infty_X$, denote it by $\widetilde{\mathcal{A}}$, and put $\widetilde{A} := \widetilde{\mathcal{A}}(X)$. We turn $\widetilde{\mathcal{A}}$ into a sheaf of unital bornological algebras by defining the product of $(f_1, h_1), (f_2, h_2) \in \widetilde{\mathcal{A}}(U)$ as

$$(f_1, h_1) \cdot (f_2, h_2) := (h_1 f_1 + h_2 f_1 + f_1 f_2, h_1 h_2). \tag{3.6}$$

We obtain a split short exact sequence in the category of bornological algebras

$$0 \longrightarrow A \longrightarrow \widetilde{A} \underset{i}{\overset{q}{\rightleftarrows}} \mathcal{C}^\infty(X) \longrightarrow 0.$$

This gives rise to a diagram of chain complexes and chain maps

$$\begin{array}{ccccccc} 0 \longrightarrow & \ker_\bullet q_* & \hookrightarrow & C_\bullet(\widetilde{A}) & \underset{i_*}{\overset{q_*}{\rightleftarrows}} & C_\bullet\big(\mathcal{C}^\infty(X)\big) & \longrightarrow 0 \\ & \iota \uparrow \downarrow \kappa & & & & & \\ & C_\bullet(A), & & & & & \end{array} \tag{3.7}$$

where the row is split exact, and ι denotes the canonical embedding. Since A is H-unital, ι is a quasi-isomorphism. Because the chain complexes $\ker_\bullet q_*$ and $C_\bullet(A)$ are bounded from below, there exists a chain map κ which is left inverse to ι. Note that the components κ_k need not be bounded maps between bornological spaces. By construction, Ψ_ε acts on each of the chain complexes within the diagram, and all chain maps (besides possibly κ) commute with this action. By the first part of the proof we have an algebraic homotopy $H : C_\bullet(\widetilde{A}) \to C_{\bullet+1}(\widetilde{A})$ such that

$$\mathrm{id} - \Psi_\varepsilon = bH + Hb.$$

Define $F : C_\bullet(A) \to C_{\bullet+1}(A)$ by $F := \kappa(\mathrm{id} - i_* q_*) H \iota$. Note that F is well defined indeed, since $q_*(\mathrm{id} - i_* q_*) = 0$. Now compute for $c \in C_k(A)$

$$(bF + Fb)c = \kappa(\mathrm{id} - i_* q_*)(bH + Hb)\iota c = \kappa(\mathrm{id} - i_* q_*)(\iota c - \Psi_\varepsilon \iota c) = c - \Psi_\varepsilon c.$$

Hence F is a homotopy between the identity and Ψ_ε and the claim is proved. ∎

Lemma 3.5. *Let $\mathcal{A}$ be a sheaf of H-unital bornological algebras over the differentiable space $(X, \mathcal{C}^\infty)$, put $A := \mathcal{A}(X)$, and let the metric d and the cut-off function ϱ as in the preceding lemma. Assume that $(\varphi_l)_{l \in \mathbb{N}}$ is a smooth locally finite partition of unity and that $(\varepsilon_l)_{l \in \mathbb{N}}$ is a sequence of positive real numbers. Then*

$$\Psi : C_\bullet(A) \to C_\bullet(A), \quad C_k(A) \ni c \mapsto \sum_{l \in \mathbb{N}} \varphi_l \Psi_{\varepsilon_l} c \tag{3.8}$$

is a chain map and there exists a homotopy between the identity on $C_\bullet(A)$ and Ψ.

Proof. Recall that the action of $\mathcal{C}^\infty(X)$ commutes with the Hochschild boundary and that each Ψ_{ε_l} is a chain map. Since $(\varphi_l)_{l\in\mathbb{N}}$ is a locally finite smooth partition of unity, Ψ then has to be a chain map by construction.

Now assume that $\mathcal{A}$ is a sheaf of unital algebras. Let $H_{\bullet,\varepsilon_l} : C_\bullet(A) \to C_{\bullet+1}$ be the homotopy from the preceding lemma which fulfills equation (3.5) with $\varepsilon = \varepsilon_l$. For all $k \in \mathbb{N}$, let H_k be the map

$$H_k : C_k(A) \to C_{k+1}(A), \quad c \mapsto \sum_{l\in\mathbb{N}} H_{k,\varepsilon_l}\varphi_l c.$$

Then

$$(bH_k + H_{k-1}b)c = \sum_{l i\in\mathbb{N}} (\varphi_l c - \Psi_{\varepsilon_l}\varphi_l c) = c - \Psi c \quad \text{for all } k \in \mathbb{N} \text{ and } c \in C_k(A). \tag{3.9}$$

Hence H is a homotopy between the identity and Ψ which proves the claim in the unital case.

In the non-unital case, define the unitalizations $\widetilde{\mathcal{A}}$ and $\widetilde{A}$ as before and let q_*, i_*, ι, κ denote the chain maps as in diagram (3.7). Let $H : C_\bullet(\widetilde{A}) \to C_{\bullet+1}(\widetilde{A})$ be the algebraic homotopy constructed for the unital case. In particular, this means that

$$\mathrm{id} - \Psi = bH + Hb.$$

Defining $F : C_\bullet(A) \to C_{\bullet+1}(A)$ by $F := \kappa(\mathrm{id} - i_* q_*)H\iota$ then gives a homotopy between the identity on $C_\bullet(A)$ and Ψ. ∎

Lemma 3.6. *Let $\mathcal{A}$ be a sheaf of H-unital bornological algebras over the differentiable space $(X, \mathcal{C}^\infty)$, put $A := \mathcal{A}(X)$, and let $c \in C_k(A)$ be a Hochschild cycle. If the support of c does not meet the diagonal, then c is a Hochschild boundary.*

Proof. Assume that the support of the Hochschild cycle c does not meet the diagonal and let $U = X^{k+1} \setminus \operatorname{supp} c$. Then U is an open neighborhood of the diagonal. By Corollary A.4, there exists a complete metric $d : X \times X \to \mathbb{R}$ such that $d^2 \in \mathcal{C}^\infty(X \times X)$. Choose a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of X which means that each K_n is compact,

$$K_n \subset K_{n+1}^\circ \quad \text{for all } n \in \mathbb{N} \text{ and } \bigcup_{n\in\mathbb{N}} K_n = X.$$

For each $n \in \mathbb{N}$, there then exists a $\varepsilon_n > 0$ such that all $(x_0, \dots, x_k) \in K_n^{k+1}$ are in U whenever $d(x_j, x_{j+1}) < \varepsilon_n$ for $j = 0, \dots, k$ and $x_{k+1} := x_0$. Choose a locally finite smooth partition of unity $(\varphi_l)_{l\in\mathbb{N}}$ subordinate to the open covering $(K_n^\circ)_{n\in\mathbb{N}}$ and let $\Psi : C_\bullet(A) \to C_\bullet(A)$ be the associated chain map defined by (3.8). According to Lemma 3.5, there then exists a chain homotopy H between the identity on $C_\bullet(A)$ and Ψ. Since the support of c does not meet U, we obtain

$$c = c - \Psi_\varepsilon c = bH(c),$$

so c is a Hochschild boundary indeed. ∎

Proposition 3.7. *Consider a proper Lie groupoid with orbit space X and convolution sheaf $\mathcal{A}$. Let $A = \mathcal{A}(X)$ and let $\widehat{\mathcal{C}}_\bullet(\mathcal{A})$ be the sheaf complex of Hochschild chains. Denote for each $\mathcal{O} \in X$ and each chain $c \in C_\bullet(\mathcal{A}(U))$ defined on a neighborhood $U \subset X$ of $\mathcal{O}$ by $[c]_\mathcal{O}$ the germ of c at $\mathcal{O}$ that is the image of c in the stalk $\widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}) = \operatorname{colim}_{V\in\mathcal{N}(\mathcal{O})} C_\bullet(\mathcal{A}(V))$, where $\mathcal{N}(\mathcal{O})$ denotes the filter basis of open neighborhoods of $\mathcal{O}$. Then the chain map*

$$\eta : C_\bullet(A) \to \Gamma\big(X, \widehat{\mathcal{C}}_\bullet(\mathcal{A})\big), \quad c \mapsto \big([c]_\mathcal{O}\big)_{\mathcal{O}\in X}$$

is a quasi-isomorphism.

Proof. Consider a section $s \in \Gamma(X, \widehat{\mathcal{C}}_k(\mathcal{A}))$. Then there exists a (countable) open covering $(U_i)_{i\in I}$ of the orbit space X and a family $(c_i)_{i\in I}$ of k-chains $c_i \in C_k(\mathcal{A}(U_i))$ such that $[c_i]_\mathcal{O} = s(\mathcal{O})$ for all $i \in I$ and $\mathcal{O} \in U_i$. After possibly passing to a finer (still countable) and locally finite covering, we can assume that there exists a partition of unity $(\varphi_i)_{i\in I}$ by functions $\varphi_i \in \mathcal{C}^\infty(X)$ such that $\operatorname{supp}\varphi_i \Subset U_i$ for all $i \in I$. If s is a cycle, then we can achieve after possible passing to an even finer locally finite covering that each c_i is a Hochschild cycle as well. Choose a metric $d : X \times X \to \mathbb{R}$ such that $d^2 \in \mathcal{C}^\infty(X \times X)$. For each i, there then exists $\varepsilon_i > 0$ such that the space of all $\mathcal{O} \in X$ with $d(\mathcal{O}, \operatorname{supp}\varphi_i) \le (k+1)\varepsilon_i$ is a compact subset of U_i. The chain $\Psi_{\varepsilon_i}(\varphi_i c_i)$ then has compact support in U_i^{k+1}. Extend it by 0 to a smooth function on X^{k+1} and denote the thus obtained k-chain also by $\Psi_{\varepsilon_i}(\varphi_i c_i)$. Now put

$$c := \sum_{i\in I} \Psi_{\varepsilon_i}(\varphi_i c_i). \tag{3.10}$$

Then $c \in C_k(A)$ is well defined since the sum in the definition of c is locally finite. For every $\mathcal{O} \in X$ now choose an open neighborhood $W_\mathcal{O}$ meeting only finitely many of the elements of the covering $(U_i)_{i\in I}$. Denote by $I_\mathcal{O}$ the set of indices $i \in I$ such that $U_i \cap W_\mathcal{O} \neq \emptyset$. Then each $I_\mathcal{O}$ is finite. Next let $H_i : C_\bullet(\mathcal{A}(U_i)) \to C_{\bullet+1}(\mathcal{A}(U_i))$ be the homotopy operator constructed in the proof of Lemma 3.4 such that

$$bH_i + H_i b = \mathrm{id} - \Psi_{\varepsilon_i}.$$

Let $e_i = H_i(\varphi_i c_i)$ for $i \in I_\mathcal{O}$ and put $e_\mathcal{O} = \sum_{i\in I_\mathcal{O}} e_i|_{W_\mathcal{O}^{k+2}}$. Then $e_\mathcal{O} \in C_{k+1}(\mathcal{A}(W_\mathcal{O}))$. Now compute for $\mathcal{Q} \in W_\mathcal{O}$

$$\begin{aligned} s(\mathcal{Q}) - [c]_\mathcal{Q} &= \sum_{i\in I_\mathcal{O}} [\varphi_i c_i](\mathcal{Q}) - \big[\Psi_{\varepsilon_i}(\varphi_i c_i)\big]_\mathcal{Q} = \sum_{i\in I_\mathcal{O}} [be_i]_\mathcal{Q} + \big[H_i(\varphi_i bc_i)\big]_\mathcal{Q} \\ &= [be_\mathcal{O}]_\mathcal{Q} + \sum_{i\in I_\mathcal{O}} \big[H_i(\varphi_i bc_i)\big]_\mathcal{Q}. \end{aligned}$$

Hence we obtain, whenever s is a cycle,

$$s(\mathcal{Q}) - [c]_\mathcal{Q} = [be_\mathcal{O}]_\mathcal{Q} \quad \text{for all } \mathcal{O} \in X,\ \mathcal{Q} \in W_\mathcal{O}.$$

This means that s and $\eta(c)$ define the same homology class. So the induced morphism between homologies $H_\bullet\eta : HH_\bullet(A) \to H_\bullet(\Gamma(X, \widehat{\mathcal{C}}_\bullet(\mathcal{A})))$ is surjective. It remains to show that $H_\bullet\eta$ is injective. To this end assume that $e \in C_k(A)$ is a cycle such that $H_\bullet\eta(e) = 0$. Then $\eta(e) = bs$ for some $s \in \Gamma(X, \hat{\mathcal{C}}_{k+1}(\mathcal{A}))$. As before, associate to s a sufficiently fine locally finite open cover $(U_i)_{i\in I}$ together with a subordinate smooth partition of unity $(\varphi_i)_{i\in I}$ and $c_i \in C_{k+1}(\mathcal{A}(U_i))$ such that $[c_i]_{\mathcal{O}} = s(\mathcal{O})$ for all $\mathcal{O} \in U_i$. Let $W_{\mathcal{O}}$ and $I_{\mathcal{O}}$ also be as above. Define $c \in C_{k+1}(A)$ by equation (3.10). Now compute for $\mathcal{Q} \in W_{\mathcal{O}}$

$$\begin{aligned}[bc - e]_{\mathcal{Q}} &= \sum_{i\in I_{\mathcal{O}}} \big[b\Psi_{\varepsilon_i}(\varphi_i c_i)\big]_{\mathcal{Q}} - [\varphi_i e]_{\mathcal{Q}} = \sum_{i\in I_{\mathcal{O}}} \big[\Psi_{\varepsilon_i}(\varphi_i b c_i)\big]_{\mathcal{Q}} - [\varphi_i e]_{\mathcal{Q}} \\ &= \sum_{i\in I_{\mathcal{O}}} [\varphi_i b c_i]_{\mathcal{Q}} - [\varphi_i e]_{\mathcal{Q}} = \sum_{i\in I_{\mathcal{O}}} (\varphi_i bs)(\mathcal{Q}) - (\varphi_i bs)(\mathcal{Q}) = 0.\end{aligned}$$

Therefore, $bc - e \in C_k(A)$ is a k-cycle such that its support does not meet the diagonal. By Lemma 3.6, $bc - e$ is a boundary which means that the homology of e is trivial. Hence $H_\bullet\eta$ is an isomorphism. ■

Now we have all the tools to verify our main localization result.

Proof of Theorem 3.3. First note that we can regard every chain complex of sheaves $\mathcal{D}_\bullet$ as a cochain complex of sheaves under the duality $\mathcal{D}^n := \mathcal{D}_{-n}$ for all integers n. We therefore have the hypercohomology $\mathbb{H}_n(X, \mathcal{D}_\bullet) := \mathbb{H}^{-n}(X, \mathcal{D}^\bullet)$ (see [50, Appendix]), where the case of cochain complexes of sheaves not necessarily bounded below as we have it here is considered. Observe that $(\hat{\mathcal{C}}_\bullet(\mathcal{A}), b)$ and $(\mathcal{HH}_\bullet(\mathcal{A}), 0)$ are quasi-isomorphic sheaf complexes, hence their hypercohomologies coincide. Recall that for a cochain complex of fine sheaves $\mathcal{D}^\bullet$

$$\mathbb{H}^n(X, \mathcal{D}^\bullet) = H^n\big(\Gamma(X, \mathcal{D}^\bullet)\big).$$

Since both $\hat{\mathcal{C}}_\bullet(\mathcal{A})$ and $\mathcal{HH}_\bullet(\mathcal{A})$ are complexes of fine sheaves, these observations together with Proposition 3.7 now entail for natural n that

$$\begin{aligned} HH_n\big(\mathcal{A}(X)\big) &= H_n\big(\Gamma\big(X, \hat{\mathcal{C}}_\bullet(\mathcal{A})\big)\big) = \mathbb{H}_n\big(X, \hat{\mathcal{C}}_\bullet(\mathcal{A})\big) \\ &= \mathbb{H}_n\big(X, \mathcal{HH}_\bullet(\mathcal{A})\big) = H_n\big(\Gamma(X, \mathcal{HH}_\bullet)\big) \\ &= \Gamma\big(X, \mathcal{HH}_n(\mathcal{A})\big).\end{aligned}$$

This is the claim. ■

4. Computation at a stalk

Recall that $\mathsf{G} \rightrightarrows M$ is a proper Lie groupoid, X is its orbit space, and $\mathcal{A}_G$ is the convolution sheaf of G (Definition 2.1). Given an orbit $\mathcal{O} \in X$ of G, we introduce in this section a linear model of the groupoid around the stalk and use it in Proposition 4.5 to construct a quasi-isomorphism between the stalk complex $\hat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_{\mathsf{G}})$ and the corresponding of the linear model. We divide the construction into two steps.

4.1. Reduction to the linear model

Let us recall the linearization result for the groupoid G around an orbit $\mathcal{O}$. Let $N\mathcal{O} \to \mathcal{O}$ be the normal bundle of the closed submanifold $\mathcal{O}$ in M, and let $\mathsf{G}_{|\mathcal{O}} \rightrightarrows \mathcal{O}$ be the restriction of the groupoid G to $\mathcal{O}$. $\mathsf{G}_{|\mathcal{O}}$ acts on $N\mathcal{O}$ canonically. And we use $\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O} \rightrightarrows N\mathcal{O}$ to denote the associated transformation groupoid. As in Definition 2.1, let $\mathcal{A}_{N\mathcal{O}}$ be the sheaf of convolution algebras on $X_{N\mathcal{O}} = N\mathcal{O}/\mathsf{G}_{|\mathcal{O}}$, with the orbit space associated to the groupoid $\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O}$. Accordingly, we can consider the presheaf of chain complexes $\mathcal{C}_\bullet(\mathcal{A}_{N\mathcal{O}})$ and the associated sheaf complex $\widehat{\mathcal{C}}_\bullet(\mathcal{A}_{N\mathcal{O}})$ as in Proposition 3.7. In what follows, we will explain how to identify the stalk $\widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_\mathsf{G})$ with the linearized model $\widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_{N\mathcal{O}})$, which is the stalk of the sheaf $\widehat{\mathcal{C}}_\bullet(\mathcal{A}_{N\mathcal{O}})$ at the zero section of $N\mathcal{O}$.

The main tool to identify the above two stalks are the linearization results for proper Lie groupoids of Weinstein [51] and Zung [53] (see also [15, 16, 39]). The particular approach we take below is from [39]. Fix a transversely invariant Riemannian metric g on M. Given a function $\delta : \mathcal{O} \to \mathbb{R}_{>0}$, let $T^\delta_{\mathcal{O},N\mathcal{O}}$ be the δ-neighborhood of the zero section in $N\mathcal{O}$. According to [39, Theorem 4.1], there exists a continuous map $\delta : \mathcal{O} \to \mathbb{R}_{>0}$ such that the exponential map

$$\exp_{|T^\delta_{\mathcal{O},N\mathcal{O}}} : T^\delta_{\mathcal{O},N\mathcal{O}} \to T^\delta_{\mathcal{O}} := \exp(T^\delta_{\mathcal{O},N\mathcal{O}}) \subset M$$

is a diffeomorphism. Furthermore, the exponential map $\exp_{|T^\delta_{\mathcal{O},N\mathcal{O}}}$ lifts to an isomorphism Θ of the following groupoids

$$\Theta : (\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O})_{|T^\delta_{\mathcal{O},N\mathcal{O}}} \to \mathsf{G}_{|T^\delta_{\mathcal{O}}}. \tag{4.1}$$

Lemma 4.1. *For each orbit $\mathcal{O} \subset M$, the pullback map Θ^* defines a quasi-isomorphism $\Theta_{\bullet,\mathcal{O}}$ from the stalk complex $\widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_\mathsf{G})$ to the stalk complex $\widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_{N\mathcal{O}})$.*

Proof. We explain how $\Theta_{\bullet,\mathcal{O}}$ is defined on $\widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_\mathsf{G})$. Let $[f_0 \otimes \cdots \otimes f_k] \in \widehat{\mathcal{C}}_{k,\mathcal{O}}(\mathcal{A}_\mathsf{G})$ be a germ of a k-chain at $\mathcal{O} \in X$. Let U be a neighborhood of $\mathcal{O}$ in X such that $f_0 \otimes \cdots \otimes f_k$ is a section of $\mathcal{C}_k(\mathcal{A}(U))$ which is mapped to $[f_0 \otimes \cdots \otimes f_k]$ in the stalk complex $\widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_\mathsf{G})$ under the canonical map η from Proposition 3.7. By (2.2), the support of each of the maps $f_0, \ldots, f_k$ is longitudinally compact. In particular,

$$\operatorname{supp}(f_i) \cap s^{-1}(\mathcal{O}) \quad (i = 0, \ldots, k)$$

is compact. Therefore,

$$s\big(\operatorname{supp}(f_i) \cap s^{-1}(\mathcal{O})\big) = t\big(\operatorname{supp}(f_i) \cap s^{-1}(\mathcal{O})\big)$$

and the union $K_{f_0,\ldots,f_k} := \bigcup_{i=0}^k s(\operatorname{supp}(f_i) \cap s^{-1}(\mathcal{O}))$ is also compact in $\mathcal{O}$.

Let K be a precompact open subset of $\mathcal{O}$ containing $K_{f_0,\ldots,f_k}$ as a proper subset. Observe that the closure of K is compact in $\mathcal{O}$. Hence, there is a positive constant ε such that the ε-neighborhood T^ε_K of K is contained inside the δ-neighborhood $T^\delta_{\mathcal{O}}$, the range of the linearization map Θ in (4.1). Applying the homotopy map Ψ_ε defined in Lemma 3.4 to

$f_0 \otimes \cdots \otimes f_k$, we may assume without loss of generality that the support of $f_0, \ldots, f_n$ is contained inside T_K^ε, and therefore inside the δ-neighborhood $T_{\mathcal{O}}^\delta$. Accordingly, the pull-back function $\Theta^*(f_0 \otimes \cdots \otimes f_k)$ is well defined and supported in

$$(\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O})_{|\Theta^{-1}(T_K^\varepsilon)} \times \cdots \times (\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O})_{|\Theta^{-1}(T_K^\varepsilon)}.$$

Let $U_{\mathcal{O}}^\varepsilon$ be the ε-neighborhood of $\mathcal{O}$ in $N\mathcal{O}/\mathsf{G}_{|\mathcal{O}}$. By the definition of Θ, it is not difficult to check that $\Theta^*(f_i)$ is supported inside $(\mathsf{G}|_{\mathcal{O}} \rtimes N\mathcal{O})|_{\Theta^{-1}(T_K^\varepsilon)}$ for $i = 0, \ldots, k$ and therefore $\Theta^*(f_0 \otimes \cdots \otimes f_k)$ is a well-defined k-chain in $\mathcal{C}_k(\mathcal{A}_{N\mathcal{O}}(U_{\mathcal{O}}^\varepsilon))$. Define $\Theta_{\bullet,\mathcal{O}}([f_0 \otimes \cdots \otimes f_k]) \in \widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_{N\mathcal{O}})$ to be the germ of $\Theta^*(f_0 \otimes \cdots \otimes f_k)$ at the point $\mathcal{O}$ in the orbit space $X_{N\mathcal{O}} = N\mathcal{O}/\mathsf{G}_{|\mathcal{O}}$. It is worth pointing out that the construction of $\Theta_{\bullet,\mathcal{O}}([f_0 \otimes \cdots \otimes f_k])$ is independent of the choices of the subset K and the constant ε. Analogously, using the inverse map Θ^{-1}, we can construct the inverse morphism $(\Theta^{-1})_{\bullet,\mathcal{O}}$ from $\widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_{N\mathcal{O}})$ to $\widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_{\mathsf{G}})$, and therefore prove that $\Theta_{\bullet,\mathcal{O}}$ is a quasi-isomorphism. We leave the details to the diligent reader. ■

4.2. Computation of the linear model

We compute in this subsection the cohomology of $C_\bullet(\mathcal{A}_{N\mathcal{O}})$. Our method is inspired by the work of Crainic and Moerdijk [14].

To start with, recall that we prove in [39, Corollary 3.11] that for a proper Lie groupoid $\mathsf{G} \rightrightarrows M$, given $x \in M$, there is a neighborhood U of x in M diffeomorphic to $O \times V_x$, where O is an open ball in the orbit $\mathcal{O}$ through x centered at x, and V_x is a G_x – the isotropy group of G at x – invariant open ball in $N_x\mathcal{O}$ centered at the origin. Under this diffeomorphism, $\mathsf{G}_{|U}$ is isomorphic to the product of the pair groupoid $O \times O \rightrightarrows O$ and the transformation groupoid $\mathsf{G}_x \ltimes V_x \rightrightarrows V_x$. Applying this result to the transformation groupoid $\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O} \rightrightarrows N\mathcal{O}$, we conclude that given any $x \in \mathcal{O}$, there is an open ball O of x in $\mathcal{O}$ such that the restricted normal bundle $U_x := N\mathcal{O}_{|O}$ is diffeomorphic to $N_x\mathcal{O} \times O$ and $(\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O})_{|U_x}$ is isomorphic to the product of the pair groupoid $O \times O$ and the transformation groupoid $\mathsf{G}_x \ltimes N_x\mathcal{O}$.

Following the above local description of $\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O}$, we choose an open covering $(O_x)_{x\in\mathcal{O}}$ of the orbit $\mathcal{O}$, and therefore also a covering $(U_x)_{x\in\mathcal{O}}$, $U_x := O_x \times N_x\mathcal{O}$, of $N\mathcal{O}$. Furthermore, we choose a locally finite countable subcovering $\mathfrak{U} := (O_i)_{i\in I}$ of $\mathcal{O}$ and the associated covering $(U_i)_{i\in I}$ of $N\mathcal{O}$. Choose $\varphi_i \in \mathcal{C}_c^\infty(\mathcal{O})$ such that $(\varphi_i^2)_{i\in I}$ is a partition of unity subordinate to the open covering $(O_i)_{i\in I}$ of $\mathcal{O}$. Lift $\varphi_i \in \mathcal{C}_c^\infty(\mathcal{O})$ to $\widetilde{\varphi}_i \in \mathcal{C}^\infty(N\mathcal{O})$ that is let it be constant along the fiber direction. As φ_i is compactly supported, $\widetilde{\varphi}_i$ is longitudinally compactly supported and therefore belongs to $\mathcal{A}_{N\mathcal{O}}$. Now consider the groupoid $\mathsf{H}_{\mathfrak{U}}$ over the disjoint union $\bigsqcup U_i$, such that arrows from U_i to U_j are arrows in $\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O}$ starting from U_i and ending in U_j. Observe that $\mathsf{H}_{\mathfrak{U}}$ inherits the Lie groupoid structure from the Lie groupoid $\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O}$ and thus becomes Morita equivalent to $\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O}$. As a consequence of this, the orbit spaces of the groupoids $\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O}$ and $\mathsf{H}_{\mathfrak{U}}$ are naturally homeomorphic, actually even diffeomorphic in the sense of differentiable spaces. We therefore identify them.

The following lemma is essentially due to Crainic and Moerdijk [14].

Lemma 4.2. *The map* $\Lambda : A(\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O}) := \Gamma(\mathcal{A}_{\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O}}) \to A(\mathsf{H}_{\mathfrak{U}}) := \Gamma(\mathcal{A}_{\mathsf{H}_{\mathfrak{U}}})$ *defined by*

$$\Lambda(f) := (\widetilde{\varphi}_i f \widetilde{\varphi}_j)_{i,j}$$

is an algebra homomorphism which induces a quasi-isomorphism $\Lambda_\bullet$ *of complexes of Hochschild chains from* $C_\bullet(A(\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O}))$ *to* $C_\bullet(A(\mathsf{H}_{\mathfrak{U}}))$*. In addition,* Λ *induces a quasi-isomorphism of sheaf complexes*

$$\Lambda_\bullet : \widehat{\mathcal{C}}_\bullet(\mathcal{A}_{\mathsf{G}_{|\mathcal{O}}\ltimes}) \to \widehat{\mathcal{C}}_\bullet(\mathcal{A}_{\mathsf{H}_{\mathfrak{U}}})$$

over their joint orbit space $N\mathcal{O}/\mathsf{G}_{|\mathcal{O}} \cong (\mathsf{H}_{\mathfrak{U}})_0/\mathsf{H}_{\mathfrak{U}}$.

Proof. The proof of the claim is a straightforward generalization of the one of [14, Lemma 5]. The slight difference here is that we work with the algebras $A(\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O})$ and $A(\mathsf{H}_{\mathfrak{U}})$ instead of the algebra of compactly supported functions. We skip the proof here to avoid repetition. ∎

Next, the groupoid $\mathsf{H}_{\mathfrak{U}}$ can be described more explicitly as follows. Firstly, index the open sets in the covering $(U_i)_{i\in I}$ by natural numbers meaning that either $I = \{1, \dots, N\}$ or that I coincides with the set of positive integers. Secondly, given i, write $x \in U_i$ as $(x_{\mathrm{v}}, x_{\mathrm{o}})$ where $x_{\mathrm{v}} \in N_{x_i}\mathcal{O}$ and $x_{\mathrm{o}} \in O_i$. Choose a diffeomorphism $\psi_i : O_i \to \mathbb{R}^k$, where $k = \dim(\mathcal{O})$. Thirdly, for any $i \in I \setminus \{1\}$ choose an arrow $g_i \in \mathsf{G}$ from x_1 to x_i. The arrow g_i induces an isomorphism between $N_{x_1}\mathcal{O}$ and $N_{x_i}\mathcal{O}$, and conjugation by g_i defines an isomorphism from G_{x_i} to G_{x_1}. Accordingly, g_i induces a groupoid isomorphism between $\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O}$ and $\mathsf{G}_{x_i} \ltimes N_{x_i}\mathcal{O}$.

Lemma 4.3. *The groupoid* $\mathsf{H}_{\mathfrak{U}}$ *is isomorphic to the product groupoid*

$$\mathsf{H}_{\mathfrak{U},I} := (\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O}) \times (I \times I) \times (\mathbb{R}^k \times \mathbb{R}^k).$$

Proof. We define groupoid morphisms

$$\Phi : \mathsf{H}_{\mathfrak{U}} \to \mathsf{H}_{\mathfrak{U},I} \quad \text{and} \quad \Psi : \mathsf{H}_{\mathfrak{U},I} \to \mathsf{H}_{\mathfrak{U}}.$$

Given an arrow $h \in \mathsf{H}_{\mathfrak{U}}$ with source in U_i and target in U_j, we consider

$$\big(s(h)_{\mathrm{o}}, x_i\big) \in O_i \times O_i \quad \text{and} \quad \big(t(h)_{\mathrm{o}}, x_j\big) \in O_j \times O_j.$$

Define $h_{x_i} \in (\mathsf{G}_{x_i} \ltimes N_{x_i}\mathcal{O}) \times (O_i \times O_i)$ (and $h_{x_j} \in (\mathsf{G}_{x_j} \ltimes N_{x_j}\mathcal{O}) \times (O_i \times O_i)$) by $h_{x_i} = ((\mathrm{id}, 0), (s(h)_{\mathrm{o}}, x_i))$ (and $h_{x_j} = ((\mathrm{id}, 0), (t(h)_{\mathrm{o}}, x_j))$). The arrow $g_j^{-1} h_{x_j}^{-1} h h_{x_i} g_i$ belongs to $\mathsf{H}_{\mathfrak{U}|U_1}$ and its component in $O_1 \times O_1$ is (x_1, x_1). The arrow $\Phi(h)$ now is defined to be

$$\Phi(h) := \big(g_j^{-1} h_{x_j}^{-1} h h_{x_i} g_i, (i, j), \psi\big(s(h_{ij}), t(h_{ij})\big)\big) \in \mathsf{H}_{\mathfrak{U},I}.$$

Similarly, given $(k, (i, j), (y_i, y_j)) \in \mathsf{H}_{\mathfrak{U},I}$, define

$$h_{y_i} := \big((\mathrm{id}, 0), \big(\psi_i^{-1}(y_i), x_i\big)\big) \in \mathsf{G}_{|U_i}, \quad h_{y_j} := \big((\mathrm{id}, 0), \big(\psi_j^{-1}(y_j), x_j\big)\big) \in \mathsf{G}_{|U_j},$$

and $h_1 := (k, (x_1, x_1)) \in \mathsf{G}_{|U_1}$. Notice that $g_j h_1 g_i^{-1}$ is an arrow in $\mathsf{H}_{\mathfrak{U}}$ starting from x_i and ending at x_j. We can now define $\Psi(k, (i, j), (y_i, y_j))$ to be

$$\Psi\big(k, (i, j), (y_i, y_j)\big) := h_{y_j} g_j h_1 g_i^{-1} h_{y_i}^{-1} \in \mathsf{H}_{\mathfrak{U}}.$$

It is straightforward to check that Φ and Ψ are groupoid morphisms and inverse to each other. ∎

Let $A(\mathsf{H}_{\mathfrak{U},I})$ be the space of global sections of the convolution sheaf $\mathcal{A}_{\mathsf{H}_{\mathfrak{U},I}}$. With the maps Φ and Ψ introduced in Lemma 4.3, we have the following induced isomorphisms of chain complexes:

$$\Phi_\bullet : C_\bullet\big(A(\mathsf{H}_{\mathfrak{U},I})\big) \to C_\bullet\big(A(\mathsf{H}_{\mathfrak{U}})\big), \quad \Psi_\bullet : C_\bullet\big(A(\mathsf{H}_{\mathfrak{U}})\big) \to C_\bullet\big(A(\mathsf{H}_{\mathfrak{U},I})\big).$$

Since they are induced by an isomorphism of groupoids, we also obtain isomorphisms of sheaf complexes that are inverses of each other:

$$\Phi_\bullet : \hat{\mathcal{C}}_\bullet(\mathcal{A}_{\mathsf{H}_{\mathfrak{U},I}}) \to \hat{\mathcal{C}}_\bullet(\mathcal{A}_{\mathsf{H}_{\mathfrak{U}}}), \quad \Psi_\bullet : \hat{\mathcal{C}}_\bullet(\mathcal{A}_{\mathsf{H}_{\mathfrak{U}}}) \to \hat{\mathcal{C}}_\bullet(\mathcal{A}_{\mathsf{H}_{\mathfrak{U},I}}).$$

Observe that both groupoids $I \times I$ and $\mathbb{R}^k \times \mathbb{R}^k$ have only one orbit. Therefore, longitudinally compactly supported functions on them are the same as compactly supported functions. Observe that $\mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O})$ is the algebra of longitudinally compactly supported smooth functions on $\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O}$. By Lemma 4.3, the groupoid algebra $A(\mathsf{H}_{\mathfrak{U}})$ is isomorphic to $A(\mathsf{H}_{\mathfrak{U},I})$. The latter can be identified with

$$\mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O}) \mathbin{\hat{\otimes}} \mathbb{R}^{I\times I} \mathbin{\hat{\otimes}} \mathcal{C}_c^\infty(\mathbb{R}^k \times \mathbb{R}^k),$$

where $\mathbb{R}^{I\times I}$ is the space of finitely supported functions on $I \times I$. Note that $I \times I$ and $\mathbb{R}^k \times \mathbb{R}^k$ both carry the structure of a pair groupoid, so the corresponding products on $\mathbb{R}^{I\times I}$ and $\mathcal{C}_c^\infty(\mathbb{R}^k \times \mathbb{R}^k)$ are given in both cases by convolution which we denote as usual by $*$. Let τ_I be the trace on $\mathbb{R}^{I\times I}$ defined by

$$\tau_I(d) := \sum_i d_{ii}, \quad d = (d_{ij})_{i,j\in I} \in \mathbb{R}^{I\times I}$$

and let $\tau_{\mathbb{R}^k}$ be the trace on $\mathcal{C}_c^\infty(\mathbb{R}^k \times \mathbb{R}^k)$ given by

$$\tau_{\mathbb{R}^k}(\alpha) := \int_{\mathbb{R}^k} \alpha(x, x)\, dx, \quad \alpha \in \mathcal{C}_c^\infty(\mathbb{R}^k \times \mathbb{R}^k),$$

where dx is the Lebesgue measure on $\mathbb{R}^k$. Define a map

$$\tau_m : C_m\big(\mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O}) \mathbin{\hat{\otimes}} \mathbb{R}^{I\times I} \mathbin{\hat{\otimes}} \mathcal{C}_c^\infty(\mathbb{R}^k \times \mathbb{R}^k)\big) \to C_m\big(\mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O})\big)$$

as follows:

$$\begin{aligned}\tau_m\big((f_0 \otimes \cdots \otimes f_m) \otimes (d_0 \otimes \cdots \otimes d_m) \otimes (\alpha_0 \otimes \cdots \otimes \alpha_m)\big)&\\ := \tau_I(d_0 * \cdots * d_m)\tau_{\mathbb{R}^k}(\alpha_0 * \cdots * \alpha_m) f_0 \otimes \cdots \otimes f_m&,\end{aligned}$$

$$f_0, \ldots, f_m \in \mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O}), \quad d_0, \ldots, d_m \in \mathbb{R}^{I\times I}, \quad \alpha_0, \ldots, \alpha_m \in \mathcal{C}_c^\infty(\mathbb{R}^k \times \mathbb{R}^k).$$

It is easy to check using the tracial property of τ_I and $\tau_{\mathbb{R}^k}$ that $\tau_\bullet$ is a chain map. Moreover, observe that the whole argument works not only for the global section algebra $\mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O})$ but for any of the section algebras $\mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes V)$ with $V \subset N_x\mathcal{O}$ an open G_{x_1}-invariant subspace. So eventually we obtain a morphism of sheaf complexes

$$\tau_\bullet : \widehat{\mathcal{C}}_\bullet(\mathcal{A}_{\mathcal{C}^\infty(\mathsf{H}_{\mathfrak{U},I})}) \to \widehat{\mathcal{C}}_\bullet(\mathcal{A}_{\mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O})})$$

over the orbit space $N_{x_1}\mathcal{O}/\mathsf{G}_{x_1}$.

Lemma 4.4. *The above chain map $\tau_\bullet$ is a quasi-isomorphism. More generally,*

$$\tau_\bullet : \widehat{\mathcal{C}}_\bullet(\mathcal{A}_{\mathcal{C}^\infty(\mathsf{H}_{\mathfrak{U},I})}) \to \widehat{\mathcal{C}}_\bullet(\mathcal{A}_{\mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O})})$$

is an isomorphism of complexes of sheaves.

Proof. Choose a function $\beta \in \mathcal{C}^\infty_c(\mathbb{R}^k)$ such that

$$\int_{\mathbb{R}^k} \beta^2(x)\, dx = 1.$$

Let $\alpha \in \mathcal{C}^\infty_c(\mathbb{R}^k \times \mathbb{R}^k)$ be the function $\beta \otimes \beta$. Define an algebra morphism

$$j_\alpha : \mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O}) \to \mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O}) \mathbin{\hat{\otimes}} \mathbb{R}^{I\times I} \mathbin{\hat{\otimes}} \mathcal{C}^\infty_c(\mathbb{R}^k \times \mathbb{R}^k)$$

by

$$j_\alpha(f) = f \otimes \delta_{(1,1)} \otimes \alpha,$$

where $\delta_{(1,1)}$ is the function on $I \times I$ that is 1 on $(1,1)$ and 0 otherwise. $j_{\alpha,\bullet}$ is the induced map on the cochain complex. It is easy to check that $\tau_\bullet \circ j_{\alpha,\bullet} = \mathrm{id}$. Applying $j_{\alpha,\bullet} \circ \tau_\bullet$ to

$$(f_0 \otimes \cdots \otimes f_m) \otimes (d_0 \otimes \cdots \otimes d_m) \otimes (\alpha_0 \otimes \cdots \otimes \alpha_m)$$

gives

$$\tau_I(d_0 * \cdots * g_m)\tau_{\mathbb{R}^k}(\alpha_0 * \cdots * \alpha_m)(f_0 \otimes \cdots \otimes f_m) \otimes (\delta_{1,1} \otimes \cdots \otimes \delta_{1,1}) \otimes (\alpha \otimes \cdots \otimes \alpha).$$

Following the proof of Lemma 3.4, we consider the unital algebra $\widetilde{\mathcal{C}}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O})$ which is the direct sum of $\mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O})$ with $\mathcal{C}^\infty(N_{x_1}\mathcal{O})^{\mathsf{G}_{x_1}}$ and product structure given by equation (3.6). We then have the following split exact sequence in the category of bornological algebras

$$0 \to \mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O}) \to \widetilde{\mathcal{C}}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O}) \to \mathcal{C}^\infty(N_{x_1})^{\mathsf{G}_{x_1}} \to 0. \tag{4.2}$$

It is not hard to see that the chain maps $\tau_\bullet$ and $j_{\alpha,\bullet}$ extend to the corresponding versions of the algebras $\widetilde{\mathcal{C}}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O})$ and $\mathcal{C}^\infty(N_{x_1})^{\mathsf{G}_{x_1}}$. As both algebras are unital, the homotopy maps constructed in the proof of [14, Lemma 6] can be applied to conclude that $j_{\alpha,\bullet}\tau_\bullet$ is a quasi-isomorphism for $\widetilde{\mathcal{C}}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O})$ and $\mathcal{C}^\infty(N_{x_1})^{\mathsf{G}_{x_1}}$. As the algebra $\mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1})$ is H-unital, we consider the long exact sequence associated to the short exact sequence (4.2). As $j_{\alpha,\bullet}$ and $\tau_\bullet$ are quasi-isomorphisms on $\widetilde{\mathcal{C}}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O})$ and $\mathcal{C}^\infty(N_{x_1})^{\mathsf{G}_{x_1}}$, we conclude by the five lemma that $\tau_\bullet$ and $j_{\alpha,\bullet}$ are also quasi-isomorphisms for $\mathcal{C}^\infty(\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O})$. The argument generalizes immediately to the sheaf case. ∎

Combining Lemma 4.1–Lemma 4.4, we thus obtain the following local model for the stalk complex $\widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_{\mathsf{G}})$.

Proposition 4.5. *For every orbit $\mathcal{O} \in X$, the composition $L_{\bullet,\mathcal{O}} := \tau_{\bullet,0} \circ \Psi_{\bullet,0} \circ \Lambda_{\bullet,0} \circ \Theta_{\bullet,\mathcal{O}}$, where $\tau_{\bullet,0}$, $\Psi_{\bullet,0}$, and $\Lambda_{\bullet,0}$ denote the respective sheaf morphisms localized at the zero sections, is a quasi-isomorphism,*

$$L_{\bullet,\mathcal{O}} : \widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_{\mathsf{G}}) \xrightarrow{\Theta_{\bullet,\mathcal{O}}} \widehat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}_{\mathsf{G}_{|\mathcal{O}} \ltimes N\mathcal{O}}) \xrightarrow{\Lambda_{\bullet,0}} \widehat{\mathcal{C}}_{\bullet,0}(\mathcal{A}_{\mathsf{H}\mathfrak{U}}) \xrightarrow{\Psi_{\bullet,0}} \widehat{\mathcal{C}}_{\bullet,0}(\mathcal{A}_{\mathsf{H}\mathfrak{U},I}) \xrightarrow{\tau_{\bullet,0}} \widehat{\mathcal{C}}_{\bullet,0}(\mathcal{A}_{\mathsf{G}_{x_1} \ltimes N_{x_1}\mathcal{O}}).$$

5. Basic relative forms

Let M be a smooth manifold equipped with a left action of a compact Lie group G which we write as $(g,x) \mapsto gx$, for $g \in G, x \in M$. Associated to this action is the Lie groupoid $G \ltimes M \rightrightarrows M$ with source map given by the projection $(g,x) \mapsto x$ and target given by the action $(g,x) \mapsto gx$. The *loop space* $\Lambda_0(G \ltimes M) \subset G \times M$ coincides in this case with the disjoint union of all fixed point sets $M^g \subset M$ for $g \in G$:

$$\Lambda_0(G \ltimes M) := \big\{(g,p) \in G \times M \mid gp = p\big\} = \bigcup_{g \in G} \{g\} \times M^g.$$

For fixed $g \in G$, the connected components of the fixed point subset $M^g \subset M$ are closed submanifolds which can wildly vary as g runs through G. Therefore, the loop space $\Lambda_0(G \ltimes M)$ is a singular subset of $G \times M$. Actually, $\Lambda_0(G \ltimes M)$ carries even the structure of a stratified space as shown in [18, 19]. If one lets G act on $G \times M$ by

$$h \cdot (g,p) := (hgh^{-1}, hp), \quad h \in G,\ (g,p) \in G \times M,$$

this action preserves $\Lambda_0(G \ltimes M) \subset G \times M$ and sends M^g to $M^{hgh^{-1}}$. In [7,8], Brylinski introduces the notion of *basic relative forms* of which we will give a sheafified version in the following. Intuitively, a basic relative k-form is a smooth family $(\omega_g)_{g \in G} \in \prod_{g \in G} \Omega^k(M^g)$ of differential forms on fixed point subspaces which are

(i) *horizontal* that is $i_{\xi_{M^g}} \omega_g = 0$ for all $g \in G$ and $\xi \in \mathrm{Lie}(G_g)$, and

(ii) *G-invariant* which means that $h^* \omega_g = \omega_{h^{-1}gh}$ for all $g, h \in G$.

Here, $G_g := Z_G(g)$ denotes the centralizer of $g \in G$, which acts on M^g. Because of the singular nature of Λ_0, one needs to make sense of what is exactly meant by a *smooth* family of differential forms. There are two solutions for this illustrated in the following.

(A) Sheaf theory. In the sense of Grauert–Grothendieck and following Brylinski [8], we define the sheaf of relative forms on $\Lambda_0(G \ltimes M)$ as the quotient sheaf

$$\Omega^k_{\mathrm{rel},\Lambda_0} := \iota^{-1}\big(\Omega^k_{G \ltimes M \to G} / (\mathcal{J}\Omega^k_{G \ltimes M \to G} + d_{\mathrm{rel}}\mathcal{J} \wedge \Omega^{k-1}_{G \ltimes M \to G})\big).$$

Here, $\Omega^k_{G\ltimes M\to G}$ denotes the sheaf of k-forms on $G\times M$ relative to the projection $\mathrm{pr}_1 : G\times M\to G$ and ι the canonical injection

$$\Lambda_0(G\ltimes M)\hookrightarrow G\ltimes M.$$

A form $\omega\in\Omega^k_{G\ltimes M\to G}(\widetilde{U})$ for $\widetilde{U}\subset G\ltimes M$ open is given by a smooth global section of the vector bundle $s^*\wedge^k T^*M$ that is by an element $\omega\in\Gamma^\infty(\widetilde{U},s^*\wedge^k T^*M)$. The de Rham differential on M defines a differential

$$d_{\mathrm{rel}}:\Omega^k_{G\ltimes M\to G}\to\Omega^{k+1}_{G\ltimes M\to G}.$$

Finally, $\mathcal{J}$ denotes the vanishing ideal of smooth functions on $G\times M$ that restrict to zero on $\Lambda_0(G\ltimes M)\subset G\times M$. Note that

$$\mathcal{J}\Omega^\bullet_{G\ltimes M\to G}+d_{\mathrm{rel}}\mathcal{J}\wedge\Omega^\bullet_{G\ltimes M\to G}$$

is a differential graded ideal in the sheaf complex $(\Omega^k_{G\ltimes M\to G},d_{\mathrm{rel}})$, so $\Omega^\bullet_{\mathrm{rel},\Lambda_0}$ becomes a sheaf of differential graded algebras on the loop space. For open $U\subset\Lambda_0(G\ltimes M)$, an element of $\Omega^k_{\mathrm{rel},\Lambda_0}(U)$ can now be understood as an equivalence class $[\omega]_{\Lambda_0}$ of forms $\omega\in\Omega^k_{G\ltimes M\to G}(\widetilde{U})$ defined on some open $\widetilde{U}\subset G\ltimes M$ such that $U=\widetilde{U}\cap\Lambda_0(G\ltimes M)$. This explains the definition of the sheaf complex of relative forms on the singular space $\Lambda_0(G\ltimes M)$; confer also to [41]. Next observe that the map which associates to each $p\in M$ the conormal space $N^*_p:=(T_pM/T_p\mathcal{O}_p)^*$ is a generalized subdistribution of the cotangent bundle T^*M in the sense of Stefan–Suessmann; cf. [28,46,47]. In the language of [17], N^* is a cosmooth generalized distribution. The restriction of N^* to each orbit, and even to each stratum of M of a fixed isotropy type, is a vector bundle; cf. [39]. Henceforth, the pullback distribution $s^*\wedge^k N^*$ is naturally a cosmooth generalized subdistribution of $\wedge^k T^*G\ltimes M$. We define the space $\Omega^k_{\mathrm{hrel},\Lambda_0\mathsf{G}}(U)$ of *horizontal relative k-forms on the loop space* (over U) as the subspace

$$\Omega^k_{\mathrm{hrel},\Lambda_0\mathsf{G}}(U):=\left\{[\omega]_{\Lambda_0}\in\Omega^k_{\mathrm{rel},\Lambda_0\mathsf{G}}(U)\mid\omega_{(g,p)}\in\wedge^k N^*_p\text{ for all }(g,p)\in U\right\}.$$

This implements the above condition (i). Observe that the action of G on TN leaves the orbits invariant, hence induces also an action on the conormal distribution N^* in a canonical way [39, Section 3]. Call a section $[\omega]_{\Lambda_0}\in\Omega^k_{\mathrm{hrel},\Lambda_0}(U)$ *invariant* if

$$\omega_{hgh^{-1},hp}(hv_1,\dots,hv_k)=\omega_{(g,p)}(v_1,\dots,v_k)\tag{5.1}$$

for all $(g,p)\in U\subset\Lambda_0\mathsf{G}$, $h\in G$ such that $(hgh^{-1},hp)\in U$ and $v_1,\dots,v_k\in N_p$. Note that the invariance of $[\omega]_{\Lambda_0}$ does not depend on the particular choice of the representative ω such that $\omega_p\in\wedge^k N^*_p$. Condition (ii) is covered by defining the space $\Omega^k_{\mathrm{brel},\Lambda_0}(U)$ of *basic relative k-forms on the loop space* (over U) now as the space of all invariant horizontal relative k-forms $[\omega]_{\Lambda_0}\in\Omega^k_{\mathrm{hrel},\Lambda_0\mathsf{G}}(U)$. Obviously, one thus obtains sheaves $\Omega^k_{\mathrm{hrel},\Lambda_0}$ and $\Omega^k_{\mathrm{brel},\Lambda_0}$ on the loop space $\Lambda_0(G\ltimes M)$. We will call the push forward $\pi_*s_*\Omega^k_{\mathrm{brel},\Lambda_0}$ by the source map s and canonical projection $\pi:M\to X=M/G$ sheaf of basic relative functions as well and denote it also by the symbol $\Omega^k_{\mathrm{brel},\Lambda_0}$. This will not lead to any confusion. The interpretation of basic relative forms as smooth families of forms on the fixed point manifolds is still missing, but will become visible in the following approach.

(B) Differential geometry. From a more differential geometric perspective, we consider the family of vector bundles $F \to \Lambda_0$ defined by $F_{(g,p)} := T_p^* M^g$ for $(g, p) \in \Lambda_0(G \ltimes M)$. Of course, this does not define a (topological) vector bundle over the inertia space $\Lambda_0(G \ltimes M)$ because in general the rank jumps discontinuously but it is again a cosmooth generalized distribution. Using the canonical projection $s^*T^*M|_{\Lambda_0} \to F$, we say that a local section $\omega \in \Gamma(U, \wedge^k F)$ over $U \subset \Lambda_0$ is *smooth* if for each $(g, p) \in U$ there exist open neighborhoods $O \subset G$ of g and $V \subset M$ of p together with a *locally representing* smooth k-form $\omega_{O,V} \in \Gamma^\infty(O \times V, \wedge^k s^*T^*M)$ such that $(O \times V) \cap \Lambda_0 \subset U$ and $\omega_{(h,q)} = [\omega_{O,V}]_{(h,q)}$ for all $(h,q) \in (O \times V) \cap \Lambda_0(G \ltimes M)$. Hence a smooth section ω can be identified with the smooth family $(\omega_g)_{g \in \mathrm{pr}_G(U)}$ of forms $\omega_g \in \Omega^k(s(U \cap (\{g\} \times M^g)))$ which are uniquely defined by the condition that $\omega_{g|V^g} = \iota_{V^g}^* \omega_{O,V}$ for all $g \in O$ and all pairs (O, V) with locally representing forms $\omega_{O,V}$ as before. The $\iota_{V^g} : V^g \hookrightarrow V$ hereby are the canonical embeddings of the fixed point manifolds V^g. We denote the space of all smooth sections of $\wedge^k F$ over U by $\Gamma^\infty(U, \wedge^k F)$ or $\Gamma^\infty_{\wedge^k F}(U)$. Obviously, $\Gamma^\infty_{\wedge^k F}$ becomes a sheaf on Λ_0.

Proposition 5.1. *The canonical sheaf morphism*

$$\theta^k : \iota^{-1}\Gamma^\infty_{\wedge^k s^*T^*M} \to \Gamma^\infty_{\wedge^k F}$$

factors through a unique epimorphism of sheaves $\Theta^k : \Omega^\bullet_{\mathrm{rel},\Lambda_0} \to \Gamma^\infty_{\wedge^k F}$ *making the following diagram commutative:*

$$\begin{array}{ccc} \iota^{-1}\Gamma^\infty_{\wedge^k s^*T^*M} & \xrightarrow{\theta^k} & \Gamma^\infty_{\wedge^k F} \\ \downarrow & \nearrow_{\Theta^k} & \\ \Omega^\bullet_{\mathrm{rel},\Lambda_0} & & \end{array}$$

Proof. The claim follows by showing that for open $\widetilde{U} \subset G \times M$ and $U := \widetilde{U} \cap \Lambda_0(G \ltimes M)$ the canonical map $\theta^k_{\widetilde{U}} : \Gamma^\infty(\widetilde{U}, \wedge^k s^*T^*M) \to \Gamma^\infty(U, \wedge^k F)$, $\omega \mapsto [\omega]$ is surjective and has

$$\mathcal{K}(\widetilde{U}) := \mathcal{J}(\widetilde{U})\Gamma^\infty(\widetilde{U}, \wedge^k s^*T^*M) + d_{\mathrm{rel}}\mathcal{J}(\widetilde{U}) \wedge \Gamma^\infty(\widetilde{U}, \wedge^{k-1} s^*T^*M)$$

contained in its kernel.

The sheaf $\Gamma^\infty_{\wedge^k F}$ is a $\mathcal{C}^\infty_{\Lambda_0}$-module sheaf, hence a soft sheaf. This entails surjectivity of $\theta^k_{\widetilde{U}}$. Assume that $\omega \in \Gamma^\infty(\widetilde{U}, \wedge^k s^*T^*M)$ is of the form $\omega = f\varrho$ for some $f \in \mathcal{J}(\widetilde{U})$ and $\varrho \in \Gamma^\infty(\widetilde{U}, \wedge^k s^*T^*M)$. Then

$$\theta^k_{\widetilde{U}}(\omega)_{(g,p)} = \theta^k_U(f\varrho)_{(g,p)} = f(q,p)\varrho_{(q,p)} = 0 \quad \text{for all } (g,p) \in U.$$

Now assume that $\omega = d_{\mathrm{rel}} f \wedge \varrho$ with f as before and $\varrho \in \Gamma^\infty(\widetilde{U}, \wedge^{k-1} s^*T^*M)$. To prove that $\theta^k_U(\omega) = 0$, it suffices to show that $\iota^*_{U^g_g}\omega = 0$ for all $g \in \mathrm{pr}_G(U)$. Fix some

$g \in \mathrm{pr}_G(U)$ and $p \in U_g^g$ and choose an open coordinate neighborhood $V \subset M$ with coordinates $(x_1, \ldots, x_d) : V \hookrightarrow \mathbb{R}^d$ such that $V \subset U_g$, $(x_{1|V^g}, \ldots, x_{k|V^g}) : V^g \hookrightarrow \mathbb{R}^k$ is a local coordinate system of M^g over V^g and such that V^g is the zero locus of the coordinate functions $(x_{k+1}, \ldots, x_d) : V \hookrightarrow \mathbb{R}^{d-k}$. After possibly shrinking V, there exists an open neighborhood O of g in G such that $O \times V \subset \widetilde{U}$. Extend the coordinate functions $(x_1, \ldots, x_d)$ to smooth functions on $O \times V$ constant along the fibers of the source map. Then we have

$$d_{\mathrm{rel}} f = \sum_{l=1}^{d} \frac{\partial f}{\partial x_l} dx_l.$$

Since $\frac{\partial f}{\partial x_l}(g, p) = 0$ for $p \in V^g$ and $1 \le l \le k$ and since $\iota_{V^g}^* dx_l = 0$ for $k < l \le d$, one gets

$$\iota_{V^g}^* \iota_{U_g^g}^* \omega = \iota_{V^g}^* (d_{\mathrm{rel}} f \wedge \varrho) = \sum_{l=1}^{d} \left(\iota_{V^g}^* \frac{\partial f}{\partial x_l} \right) (\iota_{V^g}^* dx_l) \wedge (\iota_{V^g}^* \varrho) = 0,$$

where, by slight abuse of notation, we have also used the symbol ι_{V^g} for the embedding $V^g \hookrightarrow U$, $p \mapsto (g, p)$. So $\iota_{U_g^g}^* \omega = 0$ and $\mathcal{K}(\widetilde{U})$ is in the kernel of $\theta_{\widetilde{U}}^k$. Hence $\theta_{\widetilde{U}}^k$ factors through some linear map

$$\Theta_U^k : \Omega_{\mathrm{rel}, \Lambda_0}^k(U) \to \Gamma^\infty(U, \wedge^k F).$$

This proves the claim. ∎

Remark 5.2. Conjecturally, the morphism Θ^k is an isomorphism, showing that the sheaf theoretic approach (A) and the differential geometric approach (B) above lead to the same definition of basic relative forms. Below, in Section 7, we prove this conjecture for the case of an $\mathbb{S}^1$-action. In the general case, this conjecture remains open.

Note that the image of the sheaf of horizontal relative k-forms under Θ^k coincides exactly with those families of forms $(\omega_g)_{g \in \mathrm{pr}_G(U)}$ fulfilling condition (i) above. Since G naturally acts on the generalized distribution F and Θ^k is obviously equivariant by construction, the original conditions by Brylinski are recovered now also in the differential geometric picture of relative forms.

Remark 5.3. In [3], Block and Getzler define a sheaf on G whose stalk at $g \in G$ is given by the space of G_g-equivariant differential forms on M^g. There are two differentials on this sheaf, d and ι, together constituting the equivariant differential $D := d + \iota$, which, under an HKR-type map correspond to the Hochschild and cyclic differential on the crossed product algebra $G \ltimes C^\infty(M)$. Taking cohomology with respect to ι only leads to a very similar definition of basic relative forms as above, however notice that the basic relative forms defined above form a sheaf over the quotient M/G, not the group G.

6. The group action case

In this section, we consider the action of a compact Lie group G on a complete bornological algebra A and then specialize to the case where A is the algebra of smooth functions on a smooth G-manifold M. More precisely, by a G-action on A one understands a map $\alpha : G \to \mathrm{Aut}(A)$ such that $\check{\alpha} : G \times A \to A$, $(g,a) \mapsto \alpha(g)(a) = g \cdot a$ is continuous in the natural locally convex topology induced by the bornology on A and such that $\alpha(G)$ is an equicontinuous set of continuous automorphisms of A. The other general assumption we always make is that the map $\check{\alpha} : G \times A \to A$ is smooth in the sense of [29] which means that each smooth curve in $G \times A$ is mapped to a smooth curve in A. These conditions are automatically guaranteed when G acts by diffeomorphisms on the manifold M and $A = \mathcal{C}^\infty(M)$. Under the assumptions made, the associated *smooth crossed product* $G \ltimes A$ is given by $\mathcal{C}^\infty(G, A)$ equipped with the product

$$(f_1 * f_2)(g) := \int_G f_1(h)\big(h \cdot f_2(h^{-1}g)\big)\, dh, \quad f_1, f_2 \in \mathcal{C}^\infty(G, A),\ g \in G. \tag{6.1}$$

6.1. The equivariant Hochschild complex

To compute the Hochschild homology of the smooth crossed product $G \ltimes A$, consider the bigraded vector space

$$C = \bigoplus_{p,q \geq 0} C_{p,q}, \quad \text{with} \quad C_{p,q} := \mathcal{C}^\infty(G^{(p+1)}, A^{\otimes(q+1)}).$$

There exists a bi-simplicial structure on C given by face maps $\delta_i^{\mathrm{v}} : C_{p,q} \to C_{p,q-1}$, $0 \leq i \leq q$ and $\delta_j^{\mathrm{h}} : C_{p,q} \to C_{p-1,q}$, $0 \leq j \leq p$ defined as follows. The vertical maps are given by

$$\delta_i^{\mathrm{v}}(F)(g_0, \ldots, g_p) := \begin{cases} b_{q,i}\big(F(g_0, \ldots, g_p)\big) & \text{for } 0 \leq i \leq q-1, \\ b_{q,q}^{(g_0 \cdots g_p)^{-1}}\big(F(g_0, \ldots, g_p)\big) & \text{for } i = q, \end{cases}$$

where the $b_{q,i}$ for $0 \leq i \leq q-1$ are the first $q-1$ simplicial face maps multiplying the i'th and $i+1$'th entry in $A^{\otimes(q+1)}$ underlying the Hochschild chain complex of A (see e.g. Appendix B.2), and $b_{q,q}^g$ is the g-twisted version of the last one:

$$b_{q,q}^g(a_0 \otimes \cdots \otimes a_q) := (g \cdot a_q)a_0 \otimes a_1 \otimes \cdots \otimes a_{q-1}, \quad a_0, \ldots, a_q \in A,\ g \in G.$$

The horizontal maps are defined by

$$\delta_j^{\mathrm{h}}(F)(g_0, \ldots, g_{p-1}) := \begin{cases} \int_G F(g_0, \ldots, g, g^{-1}g_j, \ldots g_{p-1})\, dg & \text{for } 0 \leq j \leq p-1, \\ \int_G g \cdot F(g^{-1}g_0, g_1, \ldots, g_{p-1}, g)\, dg & \text{for } j = p, \end{cases}$$

where, in the second line, g acts diagonally on $A^{\otimes(q+1)}$. The following observations now hold true.

(i). The diagonal complex $\mathrm{diag}(C_{\bullet,\bullet}) := \bigoplus_{k \geq 0} C_{k,k}$ equipped with the differential

$$d_{\mathrm{diag}} := \sum_i (-1)^i \delta_i^{\mathrm{h}} \delta_i^{\mathrm{v}}$$

is isomorphic to the Hochschild complex $C_k(G \ltimes A) = \mathcal{C}^\infty(G^{(k+1)}, A^{\otimes(k+1)})$ of the smooth crossed product algebra $G \ltimes A$ via the isomorphism

$$\overline{(-)} : \operatorname{diag}(C_{\bullet,\bullet}) \to C_\bullet(G \ltimes A),$$

$F \mapsto \bar{F}$ defined by

$$\begin{aligned}&\bar{F}(g_0, \dots, g_k)\\ &\quad := (g_k^{-1} \cdots g_0^{-1} \otimes g_k^{-1} \cdots g_1^{-1} \otimes \cdots \otimes g_k^{-1}) \cdot F(g_0, \dots, g_k), \quad F \in C_{k,k}, \end{aligned} \tag{6.2}$$

where the pre-factor on the right-hand side acts componentwise via the action of G on A.

(ii). The vertical differential δ^{v} in the total complex is given by a twisted version of the standard Hochschild complex of the algebra A. The horizontal differential δ^{h} in the q-th row can be interpreted as the Hochschild differential of the convolution algebra $\mathcal{C}^\infty(G)$ with values in the G-bimodule $\mathcal{C}^\infty(G, A^{\otimes(q+1)})$ with bimodule structure given by

$$(g \cdot f)(h) := g\big(f(g^{-1}h)\big), \quad (f \cdot g)(h) := f(hg), \quad f \in \mathcal{C}^\infty(G, A^{\otimes(q+1)}),\ g, h \in G.$$

The homology of this complex is isomorphic to the group homology of G with values in the adjoint module $\mathcal{C}^\infty(G, A^{\otimes(q+1)})_{\mathrm{ad}}$ given by $\mathcal{C}^\infty(G, A^{\otimes(q+1)})$ equipped with the diagonal action:

$$H_\bullet\big(\mathcal{C}^\infty(G), \mathcal{C}^\infty(G, A^{\otimes(q+1)})\big) \cong H_\bullet^{\mathrm{diff}}\big(G, \mathcal{C}^\infty(G, A^{\otimes(q+1)})_{\mathrm{ad}}\big).$$

Observe that the diagonal action commutes with the vertical face maps.

Lemma 6.1. *For all $g \in G$ and $f \in \mathcal{C}^\infty(G, A^{\otimes(q+1)})_{\mathrm{ad}}$, one has $\delta^{\mathrm{v}}(g \cdot f) = g \cdot \delta^{\mathrm{v}} f$.*

Proof. It suffices to show the claim for f of the form $f = a_0 \otimes \cdots \otimes a_q$, where $a_j : G \to A$ is smooth. For all $h \in G$, we can then write $f(h) = a_0(h) \otimes \cdots \otimes a_q(h)$, where $a_0(h), \dots, a_q(h)$ are elements in A. Now compute

$$\delta_q^{\mathrm{v}} f(h) = \big(h(a_q(h))\big)a_0(h) \otimes \cdots \otimes a_{q-1}(h)$$

and

$$\big(g \cdot \delta_q^{\mathrm{v}}(f)\big)(h) = \big(hg\big(a_q(g^{-1}hg)\big)\big)g\big(a_0(g^{-1}hg)\big) \otimes \cdots \otimes g\big(a_{q-1}(g^{-1}hg)\big).$$

On the other hand,

$$g \cdot f(h) = g\big(a_0(g^{-1}hg)\big) \otimes g\big(a_1(g^{-1}hg)\big) \otimes \cdots \otimes g\big(a_q(g^{-1}hg)\big)$$

and

$$\begin{aligned}&\delta_q^{\mathrm{v}}(g \cdot f)(h)\\ &\quad = \big(hg\big(a_q(g^{-1}hg)\big)\big)g\big(a_0(g^{-1}hg)\big) \otimes g\big(a_1(g^{-1}hg)\big) \otimes \cdots \otimes g\big(a_{q-1}(g^{-1}hg)\big).\end{aligned}$$

Hence one obtains $g \cdot \delta_q^{\mathrm{v}}(f) = \delta_q^{\mathrm{v}}(g \cdot f)$. The equalities $g \cdot \delta_i^{\mathrm{v}}(f) = \delta_i^{\mathrm{v}}(g \cdot f)$ for $0 \le i < q$ are obvious, so the claim follows. ∎

It is proved in [2] that the group homology complex computes the derived functor of taking coinvariants. For a compact Lie group, integration with respect to the Haar measure of volume 1 shows that coinvariant and invariant spaces are isomorphic, and therefore its group homology vanishes except for the zeroth degree. In our case, this gives

$$H_k^{\mathrm{diff}}\big(G, \mathcal{C}^\infty(G, A^{\otimes(q+1)})_{\mathrm{ad}}\big) = \begin{cases} \mathcal{C}^\infty(G, A^{\otimes(q+1)})_{\mathrm{ad}}^{\mathrm{inv}} & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

Observe that a smooth function $f : G \to A^{\otimes(q+1)}$ hereby is an element of the invariant space $\mathcal{C}^\infty(G, A^{\otimes(q+1)})_{\mathrm{ad}}^{\mathrm{inv}}$ if and only if $gf(g^{-1}hg) = f(h)$ for all $g, h \in G$. Note also that by the lemma the vertical differential δ^{v} maps $\mathcal{C}^\infty(G, A^{\otimes(q+1)})_{\mathrm{ad}}^{\mathrm{inv}}$ to $\mathcal{C}^\infty(G, A^{\otimes q})_{\mathrm{ad}}^{\mathrm{inv}}$.

(iii). Filtering the total complex by rows, we obtain a spectral sequence with E^1-terms

$$E^1_{0,q} \cong \mathcal{C}^\infty(G, A^{\otimes(q+1)})^{\mathrm{inv}}, \quad E^1_{p,q} = 0 \quad \text{for } p \geq 1.$$

The spectral sequence therefore collapses and the cohomology of the total complex is computed by the complex

$$C^G_\bullet(A) := \mathcal{C}^\infty(G, A^{\otimes(\bullet+1)})^{\mathrm{inv}}$$

equipped with the twisted Hochschild differential

$$(b^{\mathrm{tw}} f)(g) := \delta^{\mathrm{v}} f(g) := \sum_{i=0}^{q-1} (-1)^i b_{q,i}\big(f(g)\big) + (-1)^{q+1} b_{q,q}^{g^{-1}}\big(f(g)\big),$$
$$f \in \mathcal{C}^\infty(G, A^{\otimes(q+1)}), \quad g \in G.$$

This complex is called the *equivariant Hochschild complex* in [3].

(iv). By the Eilenberg–Zilber theorem, the diagonal complex is quasi-isomorphic to the total complex $\mathrm{Tot}(C_{\bullet,\bullet})$ with $\delta^{\mathrm{Tot}} := \delta^{\mathrm{h}} + \delta^{\mathrm{v}}$ where the horizontal and vertical differentials are given by the usual formulas $\delta^{\mathrm{h,v}} := \sum_i (-1)^i \delta_i^{\mathrm{h,v}}$. There is an explicit formula for the map $EZ : \mathrm{diag}(C_{\bullet,\bullet}) \to \mathrm{Tot}(C_{\bullet,\bullet})$ implementing this quasi-isomorphism.

Combining items (i)–(iv), above we conclude that the following holds.

Proposition 6.2. *Given a complete bornological algebra A with a smooth left G-action, the composition*

$$\widetilde{(-)} : C_\bullet(G \ltimes A) \xrightarrow{\overline{(-)}} \mathrm{diag}(C)_\bullet \xrightarrow{EZ} \mathrm{Tot}(C_{\bullet,\bullet}) \to C^G_\bullet(A)$$

is a quasi-isomorphism of complexes. The explicit formula is given by mapping a chain $F \in C_k(\mathcal{C}^\infty(G, A))$ to the equivariant Hochschild chain $\widetilde{F} \in C^G_k(A)$ defined by

$$\widetilde{F}(g) := \int_{G^k} (g^{-1} h_1 \cdots h_k \otimes 1 \otimes h_1 \otimes \cdots \otimes h_1 \cdots h_{k-1})$$
$$\times F(h_k^{-1} \cdots h_1^{-1} g, h_1, \ldots, h_k)\, dh_1 \cdots dh_k.$$

Remark 6.3. This result has originally been proved by Brylinski in [7, 8]. Observe that a right G-action β on an algebra A can be changed to a left G-action α on an algebra A by $\alpha(g)(a) := \beta(g^{-1})(a)$. Let A^{op} be the opposite algebra of A and assume that β defines a right G action on A^{op}. Use $A^{\mathrm{op}} \rtimes_\beta G$ to denote the (right) crossed product algebra defined by the right G action on A^{op}. Define a map $\Phi : G_\alpha \ltimes A \to A^{\mathrm{op}} \rtimes_\beta G$ by $\Phi(f)(g) := f(g^{-1})$. One directly checks the following identity,

$$\Phi(f_1 *_{G_\alpha \ltimes A} f_2) = \Phi(f_2) *_{A^{\mathrm{op}} \rtimes_\beta G} \Phi(f_1),$$

and concludes that the map Φ induces an isomorphism of algebras

$$G_\alpha \ltimes A \cong (A^{\mathrm{op}} \rtimes_\beta G)^{\mathrm{op}}.$$

Furthermore, notice that for a general algebra $\mathfrak{A}$, the algebra $\mathfrak{A} \otimes \mathfrak{A}^{\mathrm{op}}$ is naturally isomorphic to $\mathfrak{A}^{\mathrm{op}} \otimes \mathfrak{A}$ and therefore $HH_\bullet(\mathfrak{A}) \cong HH_\bullet(\mathfrak{A}^{\mathrm{op}})$ since the corresponding Bar resolutions coincide. Applying this observation to $(A^{\mathrm{op}} \rtimes_\beta G)^{\mathrm{op}}$, one concludes that

$$HH_\bullet(G_\alpha \ltimes A) \cong HH_\bullet(A^{\mathrm{op}} \rtimes_\beta G),$$

and that Proposition 6.2 holds also true for a smooth right G-action on an algebra A meaning that there is a quasi-isomorphism of chain complexes

$$\widehat{(-)} : C_\bullet(A \rtimes G) \to C_\bullet^G(A^{\mathrm{op}}).$$

Note that for a right G-action the convolution product on $\mathcal{C}^\infty(G, A)$ is given by

$$(f_1 * f_2)(g) := \int_G \big(f_1(h) \cdot (h^{-1}g)\big) f_2(h^{-1}g)\, dh, \quad f_1, f_2 \in \mathcal{C}^\infty(G, A),\ g \in G. \tag{6.3}$$

Throughout this paper, as it is more natural to have a left G-action on a manifold M, we will work with a right G-action on $\mathcal{C}^\infty(M)$.

6.2. The G-manifold case

Let M be a manifold endowed with a smooth left G-action. Denote by $X = M/G$ the space of G-orbits in M and by $\pi : M \to X$ the canonical projection. We consider the action groupoid $\mathsf{G} = G \ltimes M \rightrightarrows M$ and the corresponding convolution sheaf $\mathcal{A} = \mathcal{A}_{G \ltimes M}$ over X. It is straightforward to check that in the case of $A = \mathcal{C}^\infty(M)$ the product defined by equation (6.3) coincides with the convolution product on $\mathcal{A}(M/G) \cong \mathcal{C}^\infty(G \ltimes M) \cong \mathcal{C}^\infty(G, A)$ given by equation (2.3). Hence $\mathcal{A}(M/G)$ coincides with $A \rtimes G$. According to Proposition 6.2 and Remark 6.3, we then have for each G-invariant open $V \subset M$ a quasi-isomorphism between Hochschild chain complexes

$$\widehat{(-)}_{|V/G} : C_\bullet\big(\mathcal{A}(V/G)\big) \to C_\bullet^G\big(\mathcal{C}^\infty(V)\big) \cong C_\bullet\big(\mathcal{C}^\infty(V), \mathcal{A}(V/G)\big)^{\mathrm{inv}}.$$

To compute the Hochschild homology $HH_\bullet(\mathcal{A}(V/G))$, it therefore suffices to determine the homology of the complex $C_\bullet(\mathcal{C}^\infty(V), \mathcal{A}(V/G))$ which we will consider in the following. Recall that $\mathcal{A}(V/G)$ is isomorphic as a bornological vector space to the completed tensor product $\mathcal{C}^\infty(G) \hat{\otimes} \mathcal{C}^\infty(V)$ and that $\mathcal{A}(V/G)$ carries the (twisted) $\mathcal{C}^\infty(V)$-bimodule structure

$$\mathcal{C}^\infty(V) \hat{\otimes} \mathcal{A}(V/G) \hat{\otimes} \mathcal{C}^\infty(V) \to \mathcal{A}(V/G),$$
$$f \otimes a \otimes f' \mapsto \big(G \times V \ni (g,v) \mapsto f(gv)a(g,v)f'(v) \in \mathbb{R}\big).$$

Since the bimodule structure is compatible with restrictions r_V^U for G-invariant open subsets $V \subset U \subset M$, one obtains a complex of presheaves which assigns to every open V/G with $V \subset M$ open and G-invariant the complex $C_\bullet(\mathcal{C}^\infty(V), \mathcal{A}(V/G))$. Sheafification gives rise to a sheaf complex which we denote by $\hat{\mathcal{C}}_\bullet(\mathcal{C}^\infty_M, \mathcal{A})$. Since

$$C_k\big(\mathcal{C}^\infty(V), \mathcal{A}(V/G)\big) \cong \mathcal{A}(V/G) \hat{\otimes}_{\mathcal{C}^\infty(V)} C_k\big(\mathcal{C}^\infty(V)\big) \cong \mathcal{C}^\infty(G) \hat{\otimes} \mathcal{C}^\infty(V)^{\hat{\otimes}(k+1)}$$

for all G-invariant open $V \subset M$ and $k \in \mathbb{N}$, the section spaces of this sheaf complex inherit the diagonal G action from (ii) above. Moreover, this action is compatible with restrictions, so $\hat{\mathcal{C}}_\bullet(\mathcal{C}^\infty_M, \mathcal{A})$ becomes a G-sheaf complex. We now have the following result.

Proposition 6.4. *Assume to be given a G-manifold M, let $\mathcal{A}$ be the convolution sheaf of the associated action groupoid $G \ltimes M \rightrightarrows M$ on the orbit space $X = M/G$, and put $A = \mathcal{A}(X)$. Then the chain map*

$$\varrho : C_\bullet\big(\mathcal{C}^\infty(M), A\big) \to \Gamma\big(X, \hat{\mathcal{C}}_\bullet(\mathcal{C}^\infty_M, \mathcal{A})\big), \quad c \mapsto ([c]_{\mathcal{O}})_{\mathcal{O}\in X}$$

which associates to every k-chain $c \in C_k(\mathcal{C}^\infty(M), A)$ the section $([c]_{\mathcal{O}})_{\mathcal{O}\in X}$, where $[c]_{\mathcal{O}}$ denotes the germ of c in the stalk $\hat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{C}^\infty_M, \mathcal{A})$, is an equivariant quasi-isomorphism.

Proof. Observe that the sheaves $\hat{\mathcal{C}}_k(\mathcal{C}^\infty_M, \mathcal{A})$ are fine and that

$$\varrho_0 : C_0\big(\mathcal{C}^\infty(M), A\big) \to \Gamma\big(X, \hat{\mathcal{C}}_0(\mathcal{C}^\infty_M, \mathcal{A})\big)$$

is the identity morphism. Using again the homotopies from Section 3.3, the proof that ϱ is a quasi-isomorphism is completely analogous to the one of Proposition 3.7, hence we skip the details. Equivariance of ϱ is immediate by definition. ∎

Next, we compare the sheaf complex $\hat{\mathcal{C}}_\bullet(\mathcal{C}^\infty_M, \mathcal{A})$ with the complex of relative forms by constructing a morphism of sheaf complexes between them.

Proposition 6.5. *Under the assumptions of the preceding proposition, define for each open G-invariant subset $V \subset M$ and $k \in \mathbb{N}$ a $\mathcal{C}^\infty(V/G)$-module map by*

$$\Phi_{k,V/G} : C_k\big(\mathcal{C}^\infty(V), \mathcal{A}(V/G)\big) \cong \mathcal{A}(V/G) \hat{\otimes} C_k\big(\mathcal{C}^\infty(V)\big) \to \Omega^k_{\mathrm{rel},\Lambda_0}\big(\Lambda_0(G \ltimes V)\big),$$
$$f_0 \otimes f_1 \otimes \cdots \otimes f_k \mapsto \big[f_0 d(s^*_{G\ltimes V} f_1) \wedge \cdots \wedge d(s^*_{G\ltimes V} f_k)\big]_{\Lambda_0}.$$

Then the $\Phi_{k,V/G}$ are the components of a morphism of sheaf complexes

$$\Phi_\bullet : \hat{\mathcal{C}}_\bullet(\mathcal{C}^\infty_M, \mathcal{A}) \to \pi_*(s_{|\Lambda_0})_* \Omega^\bullet_{\mathrm{rel},\Lambda_0},$$

where the differential on $\Omega^\bullet_{\mathrm{rel},\Lambda_0}$ is given by the zero differential. The image of a cycle under $\Phi_\bullet$ lies in the sheaf complex of horizontal relative forms $\Omega^\bullet_{\mathrm{hrel},\Lambda_0}$.

Proof. Let $f_0 \in \mathcal{A}(V/G)$ and $f_1, \ldots, f_k \in \mathcal{C}^\infty(V)$. Observe first that $\Phi_{k,V/G}(f_0 \otimes f_1 \otimes \cdots \otimes f_k)$ is a relative form indeed since $d(s^*_{G \ltimes V} f) \in \Omega^1_{G \ltimes V \to G}(G \ltimes M)$ for all $f \in \mathcal{C}^\infty(V)$. Now let $(g, p) \in \Lambda_0(G \ltimes V)$ and compute

$$\begin{aligned}
&\Phi_{k-1,V/G} b(f_0 \otimes f_1 \otimes \cdots \otimes f_k)(g, p) \\
&\quad = f_0(g, p) f_1(p) \big[d(s^*_{G \ltimes V} f_2) \wedge \cdots \wedge d(s^*_{G \ltimes V} f_k) \big]_{(g,p)} \\
&\qquad + \sum_{i=1}^{k-1} (-1)^i f_0(g, p) f_i(p) \big[d(s^*_{G \ltimes V} f_1) \wedge \cdots \wedge d(s^*_{G \ltimes V} f_{i-1}) \\
&\qquad\qquad \wedge d(s^*_{G \ltimes V} f_{i+1}) \wedge \cdots \wedge d(s^*_{G \ltimes V} f_k) \big]_{(g,p)} \\
&\qquad + \sum_{i=1}^{k-1} (-1)^i f_0(g, p) f_{i+1}(p) \big[d(s^*_{G \ltimes V} f_1) \wedge \cdots \wedge d(s^*_{G \ltimes V} f_i) \\
&\qquad\qquad \wedge d(s^*_{G \ltimes V} f_{i+2}) \wedge \cdots \wedge d(s^*_{G \ltimes V} f_k) \big]_{(g,p)} \\
&\qquad + (-1)^k f_k(gp) f_0(g, p) \big[d(s^*_{G \ltimes V} f_1) \wedge \cdots \wedge d(s^*_{G \ltimes V} f_{k-1}) \big]_{(g,p)} = 0.
\end{aligned}$$

Hence $\Phi_{\bullet,V/G}$ is a chain map in the sense that it intertwines the Hochschild boundary with the zero differential.

It remains to show that the image of $\Phi_{\bullet,V/G}$ is in the space of horizontal relative forms. To this end, assume for a moment that V is a G-invariant open ball around the origin in some Euclidean space $\mathbb{R}^n$ which is assumed to carry an orthogonal G-action. Consider the Connes–Koszul resolution of $\mathcal{C}^\infty(V)$ provided in (B.2). A chain map between the Connes–Koszul resolution and the Bar resolution of $\mathcal{C}^\infty(V)$ over the identity map $\mathrm{id}_{\mathcal{C}^\infty(V)}$ in degree 0 is given by the family of maps

$$\begin{aligned}
\Psi_{k,V} : \Gamma^\infty(V \times V, E_k) &\to B_k\big(\mathcal{C}^\infty(V)\big) = \mathcal{C}^\infty(V \times V) \mathbin{\hat{\otimes}} \mathcal{C}^\infty(V^k), \\
&\omega \mapsto \big((v, w, x_1, \ldots, x_k) \mapsto \omega_{(v,w)}\big(Y(x_1, w), \ldots, Y(x_k, w)\big)\big).
\end{aligned}$$

Tensoring the Connes–Koszul resolution of $\mathcal{C}^\infty(V)$ with $\mathcal{A}^\infty(V/G)$ results in the following complex:

$$\Omega^d_{G \ltimes V \to G}(V) \xrightarrow{i_{Y_{G \ltimes V}}} \cdots \xrightarrow{i_{Y_{G \ltimes V}}} \Omega^1_{G \ltimes V \to G}(V) \xrightarrow{i_{Y_{G \ltimes V}}} \mathcal{C}^\infty(G \ltimes V) \to 0, \tag{6.4}$$

where $Y_{G \ltimes V} : G \ltimes V \to s^* TV$ is defined by $Y_{G \ltimes V}(g, v) = v - gv$. The composition of $\mathrm{id}_{\mathcal{A}^\infty(V/G)} \mathbin{\hat{\otimes}} \Psi_{k,V}$ with $\Phi_{k,V/G}$ then is the map which associates to each relative form $\omega \in \Omega^k_{G \ltimes V \to G}(V)$ its restriction $[\omega]_{\Lambda_0}$ to the loop space. It therefore suffices to show that for $\omega \in \Omega^k_{G \ltimes V \to G}(V)$ with $i_{Y_{G \ltimes V}} \omega = 0$ the restriction to the loop space is a horizontal relative form. To verify this, let ξ be an element of the Lie algebra $\mathfrak{g}$ of G and again $(g, v) \in \Lambda_0(G \ltimes V)$. Then

$$0 = \frac{d}{dt} (i_{Y_{G \ltimes V}} \omega)_{(e^{-t\xi} g, v)} \Big|_{t=0} = (-i_{Y_{G \ltimes V}} i_{\xi_G} d^G \omega + i_{\xi_V} \omega)_{(g,v)} = (i_{\xi_V} \omega)_{(g,v)},$$

where d^G denotes the exterior differential with respect to G and ξ_G and ξ_V are the fundamental vector fields of ξ on G and V, respectively. So $i_{\xi_V}\omega \in \mathcal{J}(V)\Omega^{k-1}_{G\ltimes V\to G}(V)$, which means that $[\omega]_{\Lambda_0} \in \Omega^k_{\mathrm{hrel},\Lambda_0}(G \ltimes V)$. ∎

Proposition 6.6. *Let M be a G-manifold with only one isotropy type and assume that the orbit space M/G is connected. Then the following holds true.*

(1) *The quotient space M/G carries a unique structure of a smooth manifold such that $\pi : M \to M/G$ is a submersion.*

(2) *The loop space $\Lambda_0(G \ltimes M)$ is a smooth submanifold of $G \times M$.*

(3) *Let $p \in M$ be a point and $V_p \subset M$ a slice to the orbit through p that is*

 (SL1) *V_p is a G_p-invariant submanifold which is transverse to the orbit $\mathcal{O}_p := Gp$ at p,*

 (SL2) *$V := GV_p$ is an open neighborhood of the orbit $\mathcal{O}_p$ and V_p is closed in V,*

 (SL3) *there exists a G-equivariant diffeomorphism $\eta : N\mathcal{O}_p \to V$ mapping the normal space $N_p = T_pM/T_p\mathcal{O}_p$ onto V_p.*

 Then for every k, the map

$$\Psi_{k,V_p/G_p} : \Omega^k_{\mathrm{brel},\Lambda_0}\big(\Lambda_0(G \ltimes GV_p)\big) \to \Omega^k_{\mathrm{brel},\Lambda_0}\big(\Lambda_0(G_p \ltimes V_p)\big), \quad \omega \mapsto \omega_{|\Lambda_0(G_p\ltimes V_p)}$$

 is an isomorphism and the space of basic relative k-forms $\Omega^k_{\mathrm{brel},\Lambda_0}(\Lambda_0(G_p \ltimes V_p))$ coincides naturally with $\mathcal{C}^\infty(G_p)^{G_p} \mathbin{\hat{\otimes}} \Omega^k(V_p)$.

(4) *The chain map*

$$\Phi_{\bullet,M/G} : \big(C_\bullet\big(\mathcal{C}^\infty(M), \mathcal{A}(M/G)\big), b\big) \to \big(\Omega^\bullet_{\mathrm{hrel},\Lambda_0}\big(\Lambda_0(G \ltimes M)\big), 0\big)$$

 is a quasi-isomorphism when the graded module $\Omega^\bullet_{\mathrm{hrel},\Lambda_0}(\Lambda_0(G\ltimes M))$ is endowed with the zero differential.

Proof. *ad* (1). It is a well-known result about group actions on manifolds that under the assumptions made, the quotient space M/G carries a unique manifold structure such that $\pi : M \to M/G$ is a submersion; see e.g. [5, Section IV.3] or [37, Theorem 4.3.10].

ad (2). This has been proved in [18, Proposition 4.4]. Let us outline the argument since we need it for the following claims, too. By the assumptions made, there exists a compact subgroup $K \subset G$ such that every point of M has isotropy type (K). Let $p \in M$ be a point and G_p its isotropy group. Without loss of generality, we can assume that $G_p = K$. Let $V_p \subset M$ be a slice to the orbit $\mathcal{O}$ through p. The isotropy group of an element $q \in V_p$ then has to coincide with K, so $V_p^K = V_p$. Therefore, the map

$$\tau : G/K \times V_p \to M, \quad (gK, q) \mapsto gq$$

is a G-equivariant diffeomorphism onto a neighborhood of $\mathcal{O}$. Now choose a small enough open neighborhood of eK in G/K and a smooth section $\sigma : U \to G$ of the fiber bundle

$G \to G/K$. The map

$$\widetilde{\tau} : G \times U \times V_p \to G \times \tau(U \times V_p), \quad (h, gK, q) \mapsto \big(\sigma(gK)h\sigma(gK)^{-1}, \sigma(gK)q\big)$$

then is a diffeomorphism onto the open set $G \times \tau(U \times V_p)$ of $G \times M$. One observes that

$$\widetilde{\tau}(K \times U \times V_p) = \big(G \times \tau(U \times V_p)\big) \cap \Lambda_0(G \ltimes M),$$

which shows that $\Lambda_0(G \ltimes M)$ is a submanifold of $G \times M$, indeed.

ad (3). Put $K = G_p$ as before, let $N = GV_p$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Choose an Ad-invariant inner product on $\mathfrak{g}$ and let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Next choose for each $q \in N$ an element $h_q \in G$ such that $h_q q \in V_p$. Then

$$\pi^N : N \to \mathcal{O}_p, \quad q \mapsto h_q^{-1} p$$

is an equivariant fiber bundle. Let $TN \to N$ be the tangent bundle of the total space and $VN \to N$ the vertical bundle. Note that TN and VN inherit from N the equivariant bundle structures. Now put for $q \in N$

$$H_q N := \operatorname{span}\big\{\big(\operatorname{Ad}_{h_q^{-1}}(\xi)\big)_N(q) \in T_q N \mid \xi \in \mathfrak{m}\big\},$$

where ξ_N denotes the fundamental vector field of ξ on N. Then $HN \to N$ becomes an equivariant vector bundle complementary to $VN \to N$. Let $P^{\mathrm{v}} : TN \to VN$ be the corresponding fiberwise projection along HN. By construction, P^{v} is G-equivariant. After these preliminary considerations let $\omega \in \Omega^k_{\mathrm{brel},\Lambda_0}(\Lambda_0(G \ltimes GV_p))$. The restriction $\omega|_{\Lambda_0(K \ltimes V_p)}$ then is a basic relative form again, so $\Psi_{k,V_p/K}$ is well defined. Let us show that it is surjective. Assume that $\varrho \in \Omega^k_{\mathrm{brel},\Lambda_0}(\Lambda_0(K \ltimes V_p))$. We then put for $(g, q) \in \Lambda_0(G \ltimes N)$ and $X_1, \ldots, X_k \in T_q N$

$$\omega_{(g,q)}(X_1, \ldots, X_k) := \varrho_{(h_q g h_q^{-1}, h_q q)}\big(Th_q\big(P^{\mathrm{v}}(X_1)\big), \ldots, Th_q\big(P^{\mathrm{v}}(X_k)\big)\big), \tag{6.5}$$

where $Th : TN \to TN$ for $h \in G$ denotes the derivative of the action of h on N. Since Tk for $k \in K$ acts as identity on $TV_p \subset VN$, the value $\omega_{(g,q)}(X_1, \ldots, X_k)$ does not depend on the particular choice of a group element h_q such that $h_q q \in V_p$. Moreover, since for fixed $q_0 \in N$ one can find a small enough neighborhood U and choose h_q to depend smoothly on $q \in U$, ω is actually a smooth differential form on N. By construction, it is a relative form. If $X_l \in H_q N$ for some l, then $\omega_{(g,q)}(X_1, \ldots, X_k) = 0$ by definition. If $X_l = (\operatorname{Ad}_{h_q^{-1}}(\xi))_N(q)$ for some $\xi \in \mathfrak{k}$, then $P^{\mathrm{v}} X_l = X_l$ and $Th_q X_l(q) = \xi_N(h_q q)$ which entails by (6.5) that $\omega_{(g,q)}(X_1, \ldots, X_k) = 0$ again since ϱ is a horizontal form. So ω is a horizontal form. It remains to show that it is G-invariant. Let $h \in G$ and (g, q) and $X_1, \ldots, X_k$ as before. Then

$$\begin{aligned}
&\omega_{(hgh^{-1},hq)}(ThX_1, \ldots, ThX_k)\\
&\quad = \varrho_{(h_q g h_q^{-1}, h_q q)}\big(Th_q Th^{-1}\big(P^{\mathrm{v}}(ThX_1)\big), \ldots, Th_q Th^{-1}\big(P^{\mathrm{v}}(ThX_k)\big)\big)\\
&\quad = \omega_{(g,q)}(X_1, \ldots, X_k),
\end{aligned}$$

so ω is G-invariant and therefore a basic relative form. Hence $\Psi_{k,V_p/K}$ is surjective. To prove injectivity of $\Psi_{k,V_p/K}$ observe that if $\omega \in \Omega^k_{\mathrm{brel},\Lambda_0}(\Lambda_0(G \ltimes GV_p))$ and ϱ is the restriction $\omega|_{\Lambda_0(K\ltimes V_p)}$, then equation (6.5) holds true since ω is G-invariant and horizontal. But this implies that if $\omega|_{\Lambda_0(K\ltimes V_p)} = 0$, then ω must be 0 as well, so $\Psi_{k,V_p/K}$ is injective. It remains to show that

$$\Omega^k_{\mathrm{brel},\Lambda_0}\big(\Lambda_0(K \ltimes V_p)\big) \cong \mathcal{C}^\infty(K)^K \,\hat{\otimes}\, \Omega^k(V_p).$$

To this end, observe that $\Lambda_0(K \ltimes V_p) = K \times V_p$ since $V_p^K = V_p$ which in other words means that very K-orbit in V_p is a singleton. The claim now follows immediately.

ad (4). By Theorem 3.3, it suffices to verify the claim for the case where $M = GV_p$, where p is a point and V_p a slice to the orbit $\mathcal{O}$ through p. As before let K be the isotropy G_p. By the slice theorem, there exists a K-equivariant diffeomorphism $\varphi : V_p \to \widetilde{V}_p \subset N_p\mathcal{O}$ onto an open zero neighborhood of the normal space $N_p\mathcal{O}$. Choose a K-invariant inner product on $N_p\mathcal{O}$ and a G-invariant inner product on the Lie algebra $\mathfrak{g}$. Again as before, let $\mathfrak{m}$ be the orthogonal complement of the Lie algebra $\mathfrak{k}$ in $\mathfrak{g}$. The inner product on $\mathfrak{g}$ induces a G-invariant Riemannian metric on G which then induces a G-invariant Riemannian metric on the homogeneous space G/K by the requirement that $G \to G/K$ is a Riemannian submersion. Now observe that the map $G/K \times V_p \to M$, $(gK, v) \mapsto gv$ is a G-invariant diffeomorphism, so we can identify M with $G/K \times V_p$. The chosen Riemannian metrics on G/K and V_p then induce a G-invariant metric on M. Since $\mathbb{C}$ is faithfully flat over $\mathbb{R}$, we can assume without loss of generality now that smooth functions and forms on M and $G \ltimes M$ are all complex valued, including elements of the convolution algebra. Let $e \in N_p\mathcal{O} \cong T_pV_p$ be a vector of unit length, and let Z be the vector field on M which maps every point to e (along the canonical parallel transport). Next choose a symmetric open neighborhood U of the diagonal of $G/K \times G/K$ such that for each pair $(gK, hK) \in U$ there is a unique $\xi \in \mathrm{Ad}_h(\mathfrak{m})$ such that $gK = \exp(\xi)hK$. Denote that ξ by $\exp^{-1}_{hK}(gK)$. Let $\chi : G/K \times G/K \to [0, 1]$ be a function with support contained in U and such that $\chi = 1$ on a neighborhood of the diagonal. Now define the vector field $Y : M \times M \to \mathrm{pr}_2^*(TM)$ by

$$\begin{aligned} Y\big((gK, v), (hK, w)\big) &= \chi(gK, hK)\big(\exp^{-1}_{hK}(gK), v - w\big) \\ &\quad + \sqrt{-1}\,\chi'(gK, hK) Z\big((gK, v), (hK, w)\big), \end{aligned}$$

where $\mathrm{pr}_2 : M \times M \to M$ is projection onto the second coordinate and where the smooth cut-off function $\chi' : G/K \times G/K \to [0, 1]$ vanishes on a neighborhood of the diagonal and is identical 1 on the locus where $\chi \neq 1$. Finally, put $E_k := \mathrm{pr}_2^*(\wedge^k T^*M)$. Then, by [11, Lemma 44], the complex

$$\Gamma^\infty(M \times M, E_{\dim M}) \xrightarrow{i_Y} \cdots \xrightarrow{i_Y} \Gamma^\infty(M \times M, E_1) \xrightarrow{i_Y} \mathcal{C}^\infty(M \times M) \to \mathcal{C}^\infty(M)$$

is a (topologically) projective resolution of $\mathcal{C}^\infty(M)$ as a $\mathcal{C}^\infty(M)$-bimodule. Tensoring this resolution with the convolution algebra $\mathcal{A}(G \ltimes M)$ gives the following complex of relative forms:

$$\Omega^{\dim M}_{G\ltimes M\to G}(G\ltimes M)\xrightarrow{i_{Y_G}}\cdots\xrightarrow{i_{Y_G}}\Omega^1_{G\ltimes M\to G}(G\ltimes M)\to\mathcal{C}^\infty(G\ltimes M),\tag{6.6}$$

where $Y_G : G\times M\to pr_2^*TM$ is the vector field

$$\big(g,(hK,v)\big)\mapsto\chi(ghK,hK)\big(\exp^{-1}_{hK}(ghK),0\big)+\sqrt{-1}\chi'(ghK,hK)Z\big((ghK,v),(hK,v)\big).$$

The vector field Y_G vanishes on $(g,(hK,v))$ if and only if $g\in hKh^{-1}$, that is if and only if $(g,(hK,v))\in\Lambda_0(G\ltimes M)$. We will use the parametric Koszul resolution (Proposition B.11) to show that the complex (6.6) is quasi-isomorphic to the complex of horizontal relative forms

$$\Omega^{\dim M}_{\mathrm{hrel},\Lambda_0}\big(\Lambda_0(G\ltimes M)\big)\xrightarrow{0}\cdots\xrightarrow{0}\Omega^1_{\mathrm{hrel},\Lambda_0}\big(\Lambda_0(G\ltimes M)\big)\xrightarrow{0}\mathcal{C}^\infty\big(\Lambda_0(G\ltimes M)\big).\tag{6.7}$$

This will then entail the claim. So it remains to show that (6.6) and (6.7) are quasi-isomorphic. We first consider the case where V_p consist just of a point. Then M coincides with the homogeneous space G/K and Y_G is a Euler-like vector field on its set of zeros

$$S=\big\{(g,hK)\in G\times G/K \mid g\in hKh^{-1}\big\}\subset M.$$

Note that S is a submanifold on M. That Y_G is Euler-like on S indeed follows from the equality

$$\frac{d}{dt}\exp^{-1}_{hK}\big(\exp(t\xi)ghK\big)\Big|_{t=0}=\frac{d}{dt}\exp^{-1}_{hK}\big(\exp(t\xi)hK\big)\Big|_{t=0}=\xi$$

for all $(g,hK)\in S$, $\xi\in\mathrm{Ad}_{gh}(\mathfrak{m})=\mathrm{Ad}_h(\mathfrak{m})$. Hence, by Proposition B.11, the complex

$$\Omega^{\dim G/K}_{G\ltimes G/K\to G}(G\ltimes G/K)\xrightarrow{i_{Y_G}}\cdots\xrightarrow{i_{Y_G}}\Omega^1_{G\ltimes G/K\to G}(G\ltimes G/K)\to\mathcal{C}^\infty(G\ltimes G/K)$$

is quasi-isomorphic to

$$0\to\cdots\to 0\to\mathcal{C}^\infty(S).$$

Since $\Omega^k_{\mathrm{hrel},\Lambda_0}(\Lambda_0(G\ltimes G/K))=0$ for $k\geq 1$, the claim follows in the case $V_p=\{p\}$. Now consider the case $M=G/K\times V_p$ with V_p an arbitrary manifold on which K acts trivially. Observe that in this situation

$$\Omega^k_{G\ltimes M\to G}(G\ltimes M)\cong\bigoplus_{0\leq l\leq k}\Omega^l_{G\ltimes G/K\to G}(G\ltimes G/K)\,\hat{\otimes}\,\Omega^{k-l}(V_p)$$

and that Y_G acts, near its zero set $S=\Lambda_0(G\ltimes M)$, only on the first components

$$\Omega^l_{G\ltimes G/K\to G}(G\ltimes G/K).$$

Hence the chain complex (6.6) is then quasi-isomorphic to the chain complex

$$\mathcal{C}^\infty\big(\Lambda_0(G\ltimes G/K)\big)\,\hat{\otimes}\,\Omega^\bullet(V_p)$$

with zero differential. But since

$$\Omega^k_{\mathrm{hrel},\Lambda_0}\big(\Lambda_0(G \ltimes M)\big) \cong \mathcal{C}^\infty\big(\Lambda_0(G \ltimes G/K)\big) \mathbin{\hat{\otimes}} \Omega^k(V_p),$$

the claim is now proved. ∎

Conjecture 6.7 (Brylinski [7, Proposition 3.4] and [8, p. 24, Proposition]). *Let M be G-manifold and regard $\Omega^\bullet_{\mathrm{hrel},\Lambda_0}(\Lambda_0(G \ltimes M))$ as a chain complex endowed with the zero differential. Then the chain map*

$$\Phi_{\bullet,M/G} : C_\bullet\big(\mathcal{C}^\infty(M), \mathcal{A}(M/G)\big) \to \Omega^\bullet_{\mathrm{hrel},\Lambda_0}\big(\Lambda_0(G \ltimes M)\big)$$

is a quasi-isomorphism.

Remark 6.8. Proposition 6.6 shows that Brylinski's conjecture holds true for G-manifolds having only one isotropy type. Corollary B.8 tells that Brylinski's conjecture is true for finite group actions. In the following section, we will verify it for circle actions.

7. The circle action case

7.1. Rotation in a plane

Let us consider the case of the natural $\mathbb{S}^1$-action on $\mathbb{R}^2$ by rotation. First, we describe the ideal sheaf $\mathcal{J} \subset \mathcal{C}^\infty_{\mathbb{S}^1 \ltimes \mathbb{R}^2}$ which consists of smooth functions on open sets of $\mathbb{S}^1 \times \mathbb{R}^2$ vanishing on $\Lambda_0(\mathbb{S}^1 \ltimes \mathbb{R}^2)$. To this end, denote by $x_j : \mathbb{S}^1 \times \mathbb{R}^2 \to \mathbb{R}$, $j = 1, 2$, the function given by the first, respectively second, Cartesian coordinate of $\mathbb{R}^2$ and by

$$\tau : \mathbb{S}^1 \setminus \{-1\} \times \mathbb{R}^2 \to (-\pi, \pi)$$

the function given by $(g, v) \mapsto \mathrm{Arg}(g)$. We denote by $r := \sqrt{x_1^2 + x_2^2}$ the radial coordinate and by $B_\varrho(v)$ the open disc of radius $\varrho > 0$ around a point $v \in \mathbb{R}^2$. Note that the loop space $\Lambda_0(\mathbb{S}^1 \ltimes \mathbb{R}^2)$ is the disjoint union of the strata $\{(1,0)\}$, $\{1\} \times (\mathbb{R}^2 \setminus \{0\})$, and $(\mathbb{S}^1 \setminus \{1\}) \times \{0\}$ and that the loop space is smooth outside the singular point $(1, 0)$.

Proposition 7.1. *Around the point $(1, 0)$, the vanishing ideal $\mathcal{J}((\mathbb{S}^1 \setminus \{-1\}) \times B_\varrho(0))$ consists of all smooth $f : (\mathbb{S}^1 \setminus \{-1\}) \times B_\varrho(0) \to \mathbb{R}$ which can be written in the form*

$$f = f_1 \tau x_1 + f_2 \tau x_2, \quad \text{where } f_1, f_2 \in \mathcal{C}^\infty\big((\mathbb{S}^1 \setminus \{-1\}) \times B_\varrho(0)\big). \tag{7.1}$$

Around the stratum $\{1\} \times (\mathbb{R}^2 \setminus \{0\})$, a function

$$f \in \mathcal{C}^\infty\big(\big(\mathbb{S}^1 \setminus \{-1\}\big) \times \big(\mathbb{R}^2 \setminus \{0\}\big)\big)$$

lies in the ideal $\mathcal{J}((\mathbb{S}^1 \setminus \{-1\}) \times (\mathbb{R}^2 \setminus \{0\}))$ if and only if f is of the form $h\tau$ for some $h \in \mathcal{C}^\infty((\mathbb{S}^1 \setminus \{-1\}) \times (\mathbb{R}^2 \setminus \{0\}))$. Finally, around the stratum $(\mathbb{S}^1 \setminus \{1\}) \times \{0\}$, a function $f \in \mathcal{C}^\infty((\mathbb{S}^1 \setminus \{1\}) \times \mathbb{R}^2)$ vanishes on $\Lambda_0(\mathbb{S}^1 \ltimes \mathbb{R}^2)$ if and only if it is of the form $f_1 x_1 + f_2 x_2$ with $f_1, f_2 \in \mathcal{C}^\infty((\mathbb{S}^1 \setminus \{1\}) \times \mathbb{R}^2)$.

Proof. Since the loop space is smooth at points of the strata $\{1\}\times(\mathbb{R}^2\setminus\{0\})$ and $(\mathbb{S}^1\setminus\{1\})\times\{0\}$, only the case where f is defined on a neighborhood of the singular point $(1,0)$ is non-trivial. So let us assume that

$$f \in \mathcal{C}^\infty\big((\mathbb{S}^1 \setminus \{-1\}) \times B_\varrho(0)\big)$$

vanishes on $\Lambda_0(\mathbb{S}^1 \ltimes \mathbb{R}^2)$. Using the coordinate functions, we can consider f as a function of $t \in (-\pi,\pi)$ and $x \in \mathbb{R}^2$. By the Malgrange preparation theorem, one then has an expansion

$$f(t,x) + t = c(t,x)\big(t + a_0(x)\big),$$

where c and a_0 are smooth and $a_0(0) = 0$. Since $t = c(t,0)t$ for all $t \in (-\pi,\pi)$, one has $c(t,0) = 1$. Putting $t = 0$ gives $0 = c(0,x)a_0(x)$ for all $x \in B_\varrho(0)$. Since $c(0,0) = 1$, one obtains $a_0(x) = 0$ for all x in a neighborhood of the origin. After possibly shrinking $B_\varrho(0)$, we can assume that $a_0 = 0$. Hence

$$f(t,x) = \big(c(t,x) - 1\big)t. \tag{7.2}$$

Taylor expansion of $c(t,x) - 1$ gives

$$c(t,x) - 1 = x_1 r_1(t,x) + x_2 r_2(t,x),$$

where

$$r_j(t,x) = \int_0^1 (1-s)\partial_j c(t,sx)\,ds, \quad j = 1,2.$$

Since the functions r_j are smooth, this expansion together with (7.2) entails (7.1). ∎

Lemma 7.2. *The vector fields*

$$Y = Y_{\mathbb{S}^1\ltimes\mathbb{R}^2} : \mathbb{S}^1 \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (g,x) \mapsto x - gx$$

and

$$Z = Z_{\mathbb{S}^1\ltimes\mathbb{R}^2} : \mathbb{S}^1 \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (g,x) \mapsto \frac{x + gx}{2}$$

have coordinate representations $Y = Y_1\frac{\partial}{\partial x_1} + Y_2\frac{\partial}{\partial x_2}$ *and* $Z = Z_1\frac{\partial}{\partial x_1} + Z_2\frac{\partial}{\partial x_2}$ *with coefficients given by*

$$Y_1 = x_1(1-\cos\tau) - x_2\sin\tau \quad \text{and} \quad Y_2 = x_2(1-\cos\tau) + x_1\sin\tau \tag{7.3}$$

respectively by

$$Z_1 = x_1(1+\cos\tau) + x_2\sin\tau \quad \text{and} \quad Z_2 = x_2(1+\cos\tau) - x_1\sin\tau. \tag{7.4}$$

Moreover, the vector fields Y and Z have square norms

$$\|Y\|^2 = 2r^2(1-\cos\tau) = r^2\tau^2(\xi\circ\tau) \quad \text{and} \quad \|Z\|^2 = 2r^2(1+\cos\tau), \tag{7.5}$$

where ξ is holomorphic with positive values over $(-\pi,\pi)$ and value 1 *at the origin.*

Proof. The representations

$$Y|_{(\mathbb{S}^1\setminus\{-1\})\times\mathbb{R}^2} = \big(x_1(1-\cos\tau) - x_2\sin\tau\big)\frac{\partial}{\partial x_1} + \big(x_2(1-\cos\tau) + x_1\sin\tau\big)\frac{\partial}{\partial x_2}$$

and

$$Z|_{(\mathbb{S}^1\setminus\{-1\})\times\mathbb{R}^2} = \big(x_1(1+\cos\tau) + x_2\sin\tau\big)\frac{\partial}{\partial x_1} + \big(x_2(1+\cos\tau) - x_1\sin\tau\big)\frac{\partial}{\partial x_2}$$

are immediate by definition of Y and Z and since $\mathbb{S}^1$ acts by rotation. Note that these formulas still hold true when extending τ to the whole circle by putting $\tau(-1) = \pi$. At $g = -1$, the extended τ is not continuous then, but compositions with the trigonometric functions cos and sin are smooth on $\mathbb{S}^1$. For the norms of Y and Z, one now obtains

$$\|Y\|^2 = x_1^2(1-\cos\tau)^2 + x_2^2\sin^2\tau + x_2^2(1-\cos\tau)^2 + x_1^2\sin^2\tau = 2r^2(1-\cos\tau)$$

and

$$\|Z\|^2 = x_1^2(1+\cos\tau)^2 + x_2^2\sin^2\tau + x_2^2(1+\cos\tau)^2 + x_1^2\sin^2\tau = 2r^2(1+\cos\tau).$$

By power series expansion of $1-\cos t$, one obtains the statement about ξ. ∎

Lemma 7.3. *For all open subsets U of the loop space $\Lambda_0 = \Lambda_0(\mathbb{S}^1 \ltimes \mathbb{R}^2)$ and all $k \in \mathbb{N}$, the map*

$$\Theta_U^k : \Omega^k_{\mathrm{rel},\Lambda_0}(U) \to \Gamma^\infty(U, \wedge^k F)$$

from Proposition 5.1 *is injective.*

Proof. Since $\Omega^0_{\mathrm{rel},\Lambda_0}(U) = \mathscr{C}^\infty(U) = \Gamma^\infty(U, \wedge^0 F)$ and $\Theta^0_U = \mathrm{id}$, we only need to prove the claim for $k \geq 1$. To this end, we have to show that for $\omega \in \Gamma^\infty(\widetilde{U}, \wedge^k s^*T^*M)$ with $[\omega]_F = 0$ the relation $[\omega]_{\Lambda_0} = 0$ holds true. Here, as before, $\widetilde{U} \subset \mathbb{S}^1 \times \mathbb{R}^2$ is an open subset such that $U = \widetilde{U} \cap \Lambda_0(\mathbb{S}^1 \ltimes \mathbb{R}^2)$. In other words, we have to show that each such ω has the form

$$\omega = \sum_{l\in L} f_l\omega_l + \sum_{j\in J} d_{\mathrm{rel}}h_j \wedge \eta_j,$$

where L, J are finite index sets,

$$f_l, h_j \in \mathcal{J}(\widetilde{U}), \quad \omega_l \in \Gamma^\infty(\widetilde{U}, \wedge^k s^*T^*M), \quad \text{and} \quad \eta_j \in \Gamma^\infty(\widetilde{U}, \wedge^{k-1}s^*T^*M).$$

Since the involved sheaves are fine, we need to show the claim only locally. So let $(g, v) \in \Lambda_0(\mathbb{S}^1 \ltimes \mathbb{R}^2)$. Choose $\varrho > 0$ and $\varepsilon > 0$ with $\varepsilon < \pi$ such that $0 \notin B_\varrho(v)$ if $v \neq 0$ and such that $e^{\sqrt{-1}t}g \neq 1$ for all t with $|t| < \varepsilon$ if $g \neq 1$. Let

$$\widetilde{U} = \big\{(e^{\sqrt{-1}t}g, w) \in \mathbb{S}^1 \times \mathbb{R}^2 \mid |t| < \varepsilon \text{ and } \|v - w\| < \varrho\big\}.$$

Using the coordinate maps τ, x_1, x_2, we now consider three cases.

Case 1*: $g = 1$ and $v = 0$.* Then $F_{(1,w)} = T_w^*\mathbb{R}^2$, hence $\omega_{(1,w)} = 0$ for all w such that $(1,w) \in \widetilde{U} \cap \Lambda_0$. Hence

$$\omega = \tau \sum_{1 \le i_1 < \cdots < i_k \le 2} \omega_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

with $\omega_{i_1,\ldots,i_k} \in \mathcal{C}^\infty(\widetilde{U})$. Now observe that $\tau x_j \in \mathcal{J}(\widetilde{U})$ for $j = 1,2$ and that $d_{\mathrm{rel}}(\tau x_j) = \tau dx_j$. Therefore, $\omega \in d_{\mathrm{rel}}\mathcal{J}(\widetilde{U}) \wedge \Gamma^\infty(\widetilde{U}, \wedge^{k-1} s^*T^*M)$.

Case 2*: $g \neq 1$ and $v = 0$.* Then $F_{(h,0)} = 0$ for all $h \in \mathbb{S}^1$ with $(h,0) \in \widetilde{U} \cap \Lambda_0$. Hence ω can be any k-form on $\widetilde{U}$. But over $\widetilde{U}$ one has $x_1, x_2 \in \mathcal{J}(\widetilde{U})$ which entails that

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le 2} \omega_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in d_{\mathrm{rel}}\mathcal{J}(\widetilde{U}) \wedge \Gamma^\infty(\widetilde{U}, \wedge^{k-1} s^*T^*M).$$

Case 3*: $g = 1$ and $v \neq 0$.* Then $F_{(1,w)} = T^*\mathbb{R}^2$ for all w such that $(1,w) \in \widetilde{U} \cap \Lambda_0$. Hence

$$\omega = \tau \sum_{1 \le i_1 < \cdots < i_k \le 2} \omega_{i_1,\ldots,i_k} dx_1 \wedge \cdots \wedge dx_{i_k}$$

with $\omega_{i_1,\ldots,i_k} \in \mathcal{C}^\infty(\widetilde{U})$. Since $\tau \in \mathcal{J}(\widetilde{U})$, one obtains $\omega \in \mathcal{J}(\widetilde{U})\Gamma^\infty(\widetilde{U}, \wedge^k s^*T^*M)$.

So in all three cases, ω is in the differential graded ideal

$$\mathcal{J}(\widetilde{U})\Gamma^\infty(\widetilde{U}, \wedge^k s^*T^*M) + d_{\mathrm{rel}}\mathcal{J}(\widetilde{U}) \wedge \Gamma^\infty(\widetilde{U}, \wedge^{k-1} s^*T^*M)$$

and $[\omega]_{\Lambda_0} = 0$. Hence Θ_U^k is injective. ∎

Lemma 7.4. *For every $\mathbb{S}^1$-invariant open $V \subset \mathbb{R}^2$, the restriction morphism*

$$[-]_{\Lambda_0} : \Omega^\bullet_{\mathbb{S}^1 \ltimes V \to \mathbb{S}^1}(\mathbb{S}^1 \ltimes V) \to \Omega^\bullet_{\mathrm{rel},\Lambda_0}\big(\Lambda_0(\mathbb{S}^1 \ltimes V)\big)$$

maps the space of cycles $Z_k(\Omega^\bullet_{\mathbb{S}^1 \ltimes V \to \mathbb{S}^1}(\mathbb{S}^1 \ltimes V), Y\lrcorner)$ onto the space $\Omega^k_{\mathrm{hrel},\Lambda_0}(\Lambda_0(\mathbb{S}^1 \ltimes V))$ of horizontal relative forms.

Proof. Since the sheaf $\Omega^\bullet_{\mathrm{hrel},\Lambda_0}$ is fine, it suffices to verify this claim for $V \subset \mathbb{R}^2$ of the form $V = B_\varrho(0)$ or $V = B_\varrho(0) \setminus \overline{B}_\sigma(0)$, where $0 < \sigma < \varrho$. So assume that $k = 1,2$ and $[\omega]_{\Lambda_0} \in \Omega^k_{\mathrm{hrel},\Lambda_0}(\Lambda_0(\mathbb{S}^1 \ltimes V))$ for some relative form $\omega \in \Omega^k_{\mathbb{S}^1 \ltimes V \to \mathbb{S}^1}(\mathbb{S}^1 \ltimes V)$. Now observe that

$$N_v^* = \mathbb{R}\, dr \quad \text{for all } v \in \mathbb{R}^2 \setminus \{0\},$$

where $dr = \frac{1}{r}(dx_1 + dx_2)$. Hence, $\omega|_{\{1\} \times V} = 0$ if $k = 2$ and $\omega|_{\{1\} \times (V \setminus \{0\})} = \varphi\, dr$ with $\varphi \in \mathcal{C}^\infty(V \setminus \{0\})$ if $k = 1$. Since the claim for $k = 2$ has just been proved, we assume from now on that $k = 1$. In Cartesian coordinates, $\omega = \omega_1 dx_1 + \omega_2 dx_2$ with $\omega_j \in \mathcal{C}^\infty(\mathbb{S}^1 \times (V \setminus \{0\}))$, $j = 1,2$. Comparing with the expansion in polar coordinates gives the following equality over $V \setminus \{0\}$

$$\omega_j(1,-) = \frac{\varphi}{r} x_i \quad \text{for } j = 1,2. \tag{7.6}$$

Note that if the origin is an element of V, then $\omega_{(1,0)} = 0$, hence $(\omega_j)_{(1,0)} = 0$, $j = 1, 2$. Choose a smooth cut-off function $\chi : \mathbb{S}^1 \to [0, 1]$ such that χ is equal to 1 near $\tau = 1$ and equal to 0 near $\tau = -1$. Now define the k-form $\widehat{\omega} \in \Omega^k(\mathbb{S}^1 \times V)$ by

$$\widehat{\omega}_{(g,x)} = \begin{cases} \frac{\chi(g)\varphi(x)}{\|Z(g,x)\|}\big\langle Z(g,x), -\big\rangle : \mathbb{R}^2 \to \mathbb{R} & \text{for } g \in \operatorname{supp}\chi \text{ and } x \in V \setminus \{0\}, \\ 0 & \text{for } g \in \mathbb{S}^1 \setminus \operatorname{supp}\chi \text{ or } x \in V \cap \{0\}. \end{cases}$$

where $\langle -, - \rangle$ is the Euclidean inner product on $\mathbb{R}^2$. It needs to be verified that $\widehat{\omega}$ is smooth on a neighborhood of $\mathbb{S}^1 \times \{0\}$ in case the origin is in V. To simplify notation, we denote the composition of a function $f : V \to \mathbb{R}$ with the projection $\mathbb{S}^1 \times V \to$ again by f and likewise for a function $\tilde{f} : \mathbb{S}^1 \to \mathbb{R}$. With this notational agreement, the formula for Z in (7.4) entails by (7.6) over $(\mathbb{S}^1 \setminus \{-1\}) \times (V \setminus \{0\})$

$$\begin{aligned} &\widehat{\omega}|_{(\mathbb{S}^1\setminus\{-1\})\times(V\setminus\{0\})} \\ &= \frac{\chi\varphi}{r\sqrt{2(1+\cos\tau)}}\Big(\big((1+\cos\tau)x_1 + \sin\tau x_2\big)dx_1 + \big((1+\cos\tau)x_2 - \sin\tau x_1\big)dx_2\Big) \\ &= \frac{\chi}{\sqrt{2(1+\cos\tau)}}\Big(\big((1+\cos\tau)\omega_1 + \sin\tau\omega_2\big)dx_1 + \big((1+\cos\tau)\omega_2 - \sin\tau\omega_1\big)dx_2\Big). \end{aligned}$$

The right-hand side can be extended by 0 to a smooth form on $\mathbb{S}^1 \times V$, hence $\widehat{\omega}$ is smooth. Moreover, the restriction of $\widehat{\omega}$ to $\{1\} \times V$ coincides with the restriction $\omega|_{\{1\}\times V}$. Finally, check that for $x \neq 0$ and $g \in \mathbb{S}^1 \setminus \{-1\}$

$$Y(g,x) \lrcorner \widehat{\omega}_{(g,x)} = \frac{\chi(g)\varphi(x)}{\|x + gx\|}\langle x + gx, x - gx\rangle = 0.$$

Hence $\widehat{\omega} \in Z_k(\Omega^\bullet_{\mathbb{S}^1\ltimes V\to\mathbb{S}^1}(\mathbb{S}^1 \ltimes V), Y\lrcorner)$ and $[\widehat{\omega}]_{\Lambda_0} = [\omega]_{\Lambda_0}$. ∎

Proposition 7.5. *For each $\mathbb{S}^1$-invariant open $V \subset \mathbb{R}^2$, the chain map*

$$[-]_{\Lambda_0} : \big(\Omega^\bullet_{\mathbb{S}^1\ltimes V\to\mathbb{S}^1}(\mathbb{S}^1 \ltimes V), Y\lrcorner\big) \to \big(\Omega^\bullet_{\mathrm{hrel},\Lambda_0}\big(\Lambda_0(\mathbb{S}^1 \ltimes V)\big), 0\big)$$

is a quasi-isomorphism.

Proof. It remains to prove that every $\omega \in Z_k(\Omega^\bullet_{\mathbb{S}^1\ltimes V\to\mathbb{S}^1}(\mathbb{S}^1 \ltimes V), Y\lrcorner)$ that satisfies the condition $[\omega]_{\Lambda_0} = 0$ is of the form $\omega = Y\lrcorner\eta$ for some $\eta \in \Omega^{k+1}_{\mathbb{S}^1\ltimes V\to\mathbb{S}^1}(\mathbb{S}^1 \ltimes V)$. Let us show this. We consider the three non-trivial cases $k = 0, 2, 1$ separately.

Case 1*:* $k = 0$. Then ω is a smooth function on $\mathbb{S}^1 \ltimes V$ vanishing on Λ_0. By Proposition 7.1, the function ω can be expanded over $\mathbb{S}^1 \setminus \{-1\} \times V$ in the form

$$\omega|_{\mathbb{S}^1\setminus\{-1\}\times V} = \omega_1\tau x_1 + \omega_2\tau x_2, \quad \text{where } \omega_1, \omega_2 \in \mathcal{C}^\infty\big(\mathbb{S}^1 \setminus \{-1\} \times V\big).$$

Moreover, the interior product of a form $\eta = \eta_1 dx_1 + \eta_2 dx_2 \in \Omega^1_{\mathbb{S}^1\ltimes V\to\mathbb{S}^1}(\mathbb{S}^1 \ltimes V)$ with the vector field Y has the form

$$Y\lrcorner\eta = Y_1\eta_1 + Y_2\eta_2 = \big(x_1(1-\cos\tau) - x_2\sin\tau\big)\eta_1 + \big(x_2(1-\cos\tau) - x_1\sin\tau\big)\eta_2.$$

This means that it suffices to find $\eta_1, \eta_2 \in \mathcal{C}^\infty(\mathbb{S}^1 \ltimes V)$ which solve the system of equations

$$\begin{aligned} \omega_1 \tau &= (1-\cos\tau)\eta_1 + (\sin\tau)\eta_2, \\ \omega_2 \tau &= -(\sin\tau)\eta_1 + (1-\cos\tau)\eta_2. \end{aligned} \tag{7.7}$$

The 1-form $\eta = \eta_1 dx_1 + \eta_2 dx_2$ will then satisfy $Y \lrcorner \eta = \omega$ which will prove the first case. The functions

$$\eta_1 = \frac{\tau(1-\cos\tau)}{(1-\cos\tau)^2 + \sin^2\tau}\omega_1 - \frac{\tau\sin\tau}{(1-\cos\tau)^2+\sin^2\tau}\omega_2 = \frac{\tau}{2}\omega_1 - \frac{\tau\sin\tau}{2(1-\cos\tau)}\omega_2,$$
$$\eta_2 = \frac{\tau\sin\tau}{(1-\cos\tau)^2+\sin^2\tau}\omega_1 + \frac{\tau(1-\cos\tau)}{(1-\cos\tau)^2+\sin^2\tau}\omega_2 = \frac{\tau\sin\tau}{2(1-\cos\tau)}\omega_1 + \frac{\tau}{2}\omega_2$$

now are well defined and smooth over $(\mathbb{S}^1 \times V) \setminus (\{1\} \times \mathbb{R}^2)$. They also solve (7.7). We are done when we can show that they can be extended smoothly to the whole domain $\mathbb{S}^1 \times V$. But this is clear since the function $(-\pi,\pi)\setminus\{0\} \to \mathbb{R}$, $t \mapsto \frac{t\sin t}{2(1-\cos t)}$ has a holomorphic extension near the origin as one verifies by power series expansion.

Case 2: $k = 2$. Let $\omega \in \Omega^2_{\mathbb{S}^1\ltimes V \to \mathbb{S}^1}(\mathbb{S}^1 \ltimes V)$ and $Y \lrcorner \omega = 0$. Then $\omega = \varphi dx_1 \wedge dx_2$ for some smooth function $\varphi \in \mathbb{S}^1 \ltimes V \to \mathbb{S}^1$. Now compute using (7.3)

$$\begin{aligned} 0 = Y \lrcorner \omega = \varphi \cdot (Y_1 - Y_2) &= \varphi \cdot \big(x_1(1-\cos\tau) - x_2\sin\tau - x_2(1-\cos\tau) - x_1\sin\tau\big) \\ &= \varphi \cdot (x_1 - x_2)\cdot(1-\cos\tau - \sin\tau). \end{aligned}$$

Hence $\varphi = 0$ and $\omega = 0$.

Case 3: $k = 1$. Observe that in this case ω can be written in the form $\omega = \omega_1 dx_1 + \omega_2 dx_2$ with $\omega_1, \omega_2 \in \mathcal{J}(\mathbb{S}^1 \times V) \subset \mathcal{C}^\infty(\mathbb{S}^1 \times V)$. By equation (7.1), $\omega_j|_{(\mathbb{S}^1\setminus\{-1\})\times V} = \tau\Omega_j$ for $j = 1, 2$ and functions $\Omega_j \in \mathcal{C}^\infty((\mathbb{S}^1 \setminus \{-1\}) \times V)$. The condition $Y \lrcorner \omega = 0$ implies that

$$Y_1\Omega_1 + Y_2\Omega_2 = Y_1\omega_1 + Y_2\omega_2 = 0. \tag{7.8}$$

Now define the function $\varphi : (\mathbb{S}^1 \times V) \setminus \Lambda_0 \to \mathbb{R}$ by $\varphi = \frac{1}{\|Y\|^2}(-Y_2\omega_1 + Y_1\omega_2)\big|_{(\mathbb{S}^1\times V)\setminus\Lambda_0}$. Since $\|Y\|^2 = 2r^2(1-\cos\tau)$, the vector field Y vanishes nowhere on $(\mathbb{S}^1 \times V) \setminus \Lambda_0$, so φ is well defined and smooth. By (7.8) one computes

$$\varphi(g,x) = \begin{cases} \frac{\omega_2}{Y_1}(g,x) & \text{if } g \neq 1, x \neq 0 \text{ and } Y_1(g,x) \neq 0, \\ \frac{-\omega_1}{Y_2}(g,x) & \text{if } g \neq 1, x \neq 0 \text{ and } Y_2(g,x) \neq 0. \end{cases}$$

Assume that φ can be extended smoothly to $\mathbb{S}^1 \times V$. Then $\eta = \varphi dx_1 \wedge dx_2$ is a smooth form on $\mathbb{S}^1 \times V$ which satisfies

$$Y \lrcorner \eta = \varphi(Y_1 dx_2 - Y_2 dx_1) = \omega.$$

So it remains to verify that φ can be smoothly extended to $\mathbb{S}^1 \times V$. To this end, we use the complex coordinate $z = x_1 + \sqrt{-1}x_2$ of V and introduce the complex valued function $\Omega = \Omega_1 + \sqrt{-1}\Omega_2$. Moreover, we define $y : \mathbb{S}^1 \times V \to \mathbb{C}$, $(g, z) \mapsto z - gz$. Then

$$y = (1 - e^{\sqrt{-1}\tau})z = Y_1 + \sqrt{-1}Y_2 \tag{7.9}$$

and, by equation (7.8),

$$\frac{1}{2}(y\overline{\Omega} + \bar{y}\Omega) = Y_1\Omega_1 + Y_2\Omega_2 = 0. \tag{7.10}$$

Next observe that $1 - e^{\sqrt{-1}\tau} = -\sqrt{-1}\tau(1 - \sqrt{-1}\tau(\zeta \circ \tau))$ for some holomorphic function $\zeta : \mathbb{C} \to \mathbb{C}$ which fulfills $\zeta(0) = \frac{1}{2}$. Then equation (7.10) entails

$$\big(1 - \sqrt{-1}\tau(\zeta \circ \tau)\big)z\overline{\Omega} = \big(1 + \sqrt{-1}\tau(\overline{\zeta} \circ \tau)\big)\bar{z}\Omega.$$

By power series expansion, it follows that $\frac{\partial \Omega}{\partial \bar{z}}|_{z=0} = 0$ for all $k \in \mathbb{N}$. Hence, by Taylor's theorem $\Omega = z\Phi$ for some smooth $\Phi : \mathbb{S}^1 \times V \to \mathbb{C}$. Since by Lemma 7.2 $\|Y\|^2 = r^2\tau^2(\xi \circ \tau)$ for some holomorphic function ξ not vanishing on $(-\pi, \pi)$, the following equalities hold over $(\mathbb{S}^1 \setminus \{\pm 1\}) \times (V \setminus \{0\})$:

$$\begin{aligned}
\varphi &= \frac{1}{\tau r^2(\xi \circ \tau)}(-Y_2\Omega_1 + Y_1\Omega_2) = \frac{\sqrt{-1}}{2\tau r^2(\xi \circ \tau)}(y\overline{\Omega} - \bar{y}\Omega) \\
&= \frac{1}{2r^2(\xi \circ \tau)}\Big(\big(1 - \sqrt{-1}\tau(\zeta \circ \tau)\big)z\bar{z}\overline{\Phi} + \big(1 + \sqrt{-1}\tau(\overline{\zeta} \circ \tau)\big)z\bar{z}\Phi\Big) \\
&= \frac{1}{(\xi \circ \tau)}\big(1 - \sqrt{-1}\tau(\zeta \circ \tau)\big)\overline{\Phi}\Big|_{(\mathbb{S}^1 \setminus \{\pm 1\}) \times (V \setminus \{0\})}.
\end{aligned}$$

Since the right-hand side has a smooth extension to $\mathbb{S}^1 \setminus \{-1\} \times V$, the function φ can be smoothly extended to $\mathbb{S}^1 \times V$ and the claim is proved. ∎

7.2. $\mathbb{S}^1$ rotation in $\mathbb{R}^{2m}$

In this subsection, we work with complex-valued functions, and differential forms over complex numbers. Since tensoring an $\mathbb{R}$-vector space with $\mathbb{C}$ is a faithfully flat functor, our results in this section still hold true for the algebra of real-valued functions.

We consider a linear representation of $\mathbb{S}^1$ on $\mathbb{R}^{2m}$. We identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$, and decompose $\mathbb{C}^m$ into the following two subspaces:

$$\mathbb{C}^m = V_0 \oplus V_1, \tag{7.11}$$

where V_0 is the subspace of $\mathbb{C}^m$ on which $\mathbb{S}^1$ acts trivially, and V_1 is the $\mathbb{S}^1$-invariant subspace of $\mathbb{C}^m$ orthogonal to V_0 with respect to an $\mathbb{S}^1$-invariant Hermitian metric on $\mathbb{C}^m$. Furthermore, V_1 is decomposed into irreducible unitary representations of $\mathbb{S}^1$, i.e.,

$$V_1 = \bigoplus_{j=1}^{t} \mathbb{C}_{w_j},$$

where $\mathbb{C}_{w_j}$ is an irreducible representation ρ_{w_j} of $\mathbb{S}^1$ with the weight $0 \neq w_j \in \mathbb{Z}$, i.e.,

$$\rho_{w_j}\big(\exp(2\pi\sqrt{-1})t\big)(z) := \exp(2w_j\pi\sqrt{-1}t)z.$$

We observe that $\mathcal{C}^\infty(\mathbb{C}^m) \rtimes \mathbb{S}^1$ is isomorphic to $(\mathcal{C}^\infty(V_0) \otimes \mathcal{C}^\infty(V_1)) \rtimes \mathbb{S}^1$. As $\mathbb{S}^1$ acts on V_0 trivially, we have

$$\mathcal{C}^\infty(\mathbb{C}^m) \rtimes \mathbb{S}^1 \cong \mathcal{C}^\infty(V_0) \otimes \big(\mathcal{C}^\infty(V_1) \rtimes \mathbb{S}^1\big).$$

The Künneth formula for the Hochschild homology [30, Theorem 4.2.5] gives

$$HH_\bullet\big(\mathcal{C}^\infty(\mathbb{C}^m) \rtimes \mathbb{S}^1\big) = HH_\bullet\big(\mathcal{C}^\infty(V_0)\big) \otimes HH_\bullet\big(\mathcal{C}^\infty(V_1) \rtimes \mathbb{S}^1\big).$$

The Connes–Hochschild–Kostant–Rosenberg theorem asserts that $HH_\bullet(\mathcal{C}^\infty(V_0))$ is isomorphic to $\Omega^\bullet(V_0)$. Hence, we have reduced the computation of $HH_\bullet(\mathcal{C}^\infty(\mathbb{C}^m) \rtimes \mathbb{S}^1)$ to that of $HH_\bullet(\mathcal{C}^\infty(V_1) \rtimes \mathbb{S}^1)$. Without loss of generality, we assume in the remainder of this subsection that $\mathbb{C}^m = V_1$, i.e.,

$$\mathbb{C}^m = \bigoplus_{j=1}^m \mathbb{C}_{w_j}, \quad 0 \neq w_j \in \mathbb{Z}.$$

Let w be the lowest common multiplier of $w_1, \ldots, w_m$. We observe that for $t \in [0, 1)$, if $t \neq \frac{j}{w}$, $j = 0, \ldots, w-1$, the fixed point subspace of t is $\{0\}$; if $t = \frac{j}{w}$, the fixed point subspace of t is

$$\mathbb{C}_{w_{k_1}} \oplus \cdots \oplus \mathbb{C}_{w_{k_l}},$$

for $w_{k_1}, \ldots, w_{k_l}$ that w divides $jw_{k_1}, \ldots, jw_{k_l}$. Hence the loop space $\Lambda_0(\mathbb{S}^1 \ltimes \mathbb{C}^m)$ has the following form:

$$\begin{aligned}\Lambda_0(\mathbb{S}^1 \ltimes \mathbb{C}^m) = \big\{\big(&\exp(2\pi\sqrt{-1}t), (0, \ldots, z_{w_{k_1}}, \ldots, z_{w_{k_l}}, 0, \ldots)\big) \mid \\ &(0, \ldots, z_{w_{k_1}}, \ldots, z_{w_{k_l}}, 0, \ldots) \in \mathbb{C}^m,\ tw_{k_1}, \ldots, tw_{k_l} \in \mathbb{Z}w\big\}.\end{aligned}$$

Let $\sigma : \Lambda_0(\mathbb{S}^1 \ltimes \mathbb{C}^m) \to \mathbb{S}^1$ be the projection onto the first factor. Following Proposition 6.5 and equation (6.4), the Hochschild homology of $\mathcal{C}^\infty(\mathbb{C}^m) \rtimes \mathbb{S}^1$ is computed by the $\mathbb{S}^1$-invariant part of the cohomology of the following Koszul-type complex:

$$\begin{aligned}\Omega^{2m}_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}(\mathbb{S}^1 \ltimes \mathbb{C}^m) \xrightarrow{i_{Y_{\mathbb{S}^1\ltimes\mathbb{C}^m}}} \cdots \xrightarrow{i_{Y_{\mathbb{S}^1\ltimes\mathbb{C}^m}}} \Omega^1_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}(\mathbb{S}^1 \ltimes \mathbb{C}^m) \\ \xrightarrow{i_{Y_{\mathbb{S}^1\ltimes\mathbb{C}^m}}} \mathcal{C}^\infty(\mathbb{S}^1 \ltimes \mathbb{C}^m) \to 0,\end{aligned} \tag{7.12}$$

where $Y_{\mathbb{S}^1\ltimes\mathbb{C}^m} : \mathbb{S}^1 \ltimes \mathbb{C}^m \to s^*T\mathbb{C}^m$ is defined by $Y_{\mathbb{S}^1\ltimes\mathbb{C}^m}(g, v) = v - gv$. Below, we sometimes abuse notation by denoting $Y_{\mathbb{S}^1\ltimes\mathbb{C}^m}$ by Y. Fix a choice of coordinates $(z_1, \ldots, z_m)$ for $z_j \in \mathbb{C}_{w_j}$. The vector field $Y := Y_{\mathbb{S}^1\ltimes\mathbb{C}^m}(\exp(2\pi\sqrt{-1}t), z)$ is written as

$$\begin{aligned}Y &:= Y_{\mathbb{S}^1\ltimes\mathbb{C}^m}\big(\exp(2\pi\sqrt{-1}t), z\big) \\ &= \sum_{k=1}^m \big(\exp(2\pi\sqrt{-1}w_k t) - 1\big) z_k \frac{\partial}{\partial z_k} + \big(\exp(-2\pi\sqrt{-1}w_k t) - 1\big)\bar{z}_k \frac{\partial}{\partial \bar{z}_k}.\end{aligned}$$

Define an analytic function $a(z)$ on $\mathbb{C}$ by

$$a(z) := \frac{\exp(2\pi\sqrt{-1}z) - 1}{z}.$$

Then we have

$$\exp(2\pi\sqrt{-1}w_k t) - 1 = w_k t a(w_k t),$$
$$\exp(-2\pi\sqrt{-1}w_k t) - 1 = w_k t \bar{a}(w_k t).$$

Observe that for $t \in \mathbb{R}$, $a(t) = \bar{a}(t)$, and $a(t) \neq 0$ for all t sufficiently close to 0. For a sufficiently small ε, the vector field Y on $(-\varepsilon, \varepsilon) \times \mathbb{C}^m$ is of the following form:

$$Y = t \sum_{k=1}^{m} w_k \left(a(w_k t) z_k \frac{\partial}{\partial z_k} + \overline{a(w_k t)} \bar{z}_k \frac{\partial}{\partial \bar{z}_k} \right).$$

This leads to the following property of the vector field Y.

Lemma 7.6. *The vector field* $Y : \mathbb{S}^1 \times \mathbb{C}^m \to \mathbb{C}^m$, $(g, z) \mapsto z - gz$ *has a coordinate representation* $Y = \sum_{k=1}^{m} Y^k z_k \frac{\partial}{\partial z_k} + \overline{Y}^k \bar{z}_k \frac{\partial}{\partial \bar{z}_k}$ *with coefficients given by*

$$Y^k\big(\exp(2\pi\sqrt{-1}t)\big) = \exp(2\pi\sqrt{-1}w_k t) - 1.$$

Set $w = \operatorname{lcm}(w_1, \ldots, w_m)$. *There exists an* $\varepsilon > 0$ *such that*

- *if* $t_0 = \frac{j}{w}$, *for* $0 \leq j < w$, *on* $(\frac{j}{w} - \varepsilon, \frac{j}{w} + \varepsilon)$, Y^k *is of the following form:*

$$Y^k\big(\exp(2\pi\sqrt{-1}t)\big) = w_k\left(t - \frac{j}{w}\right) a\left(w_k\left(t - \frac{j}{w}\right)\right), \quad \textit{for } w_k j \in \mathbb{Z}w,$$

  where $a(w_k(t - \frac{j}{w})) \neq 0$ *for all* $t \in (\frac{j}{w} - \varepsilon, \frac{j}{w} + \varepsilon)$,
- *for* k *with* $w_k j \notin \mathbb{Z}w$, $Y^k(\exp(2\pi\sqrt{-1}t)) \neq 0$ *for all* $t \in (\frac{j}{w} - \varepsilon, \frac{j}{w} + \varepsilon)$,
- *if* $t_0 \neq \frac{j}{w}$, $Y^k(\exp(2\pi\sqrt{-1}t)) \neq 0$ *for all* $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.

The next lemma provides a local expression for the vanishing ideal $\mathcal{J}$ of the loop space $\Lambda_0(\mathbb{S}^1 \ltimes \mathbb{C}^m)$ associated to the $\mathbb{S}^1$-action on $\mathbb{C}^m$ defined by equation (7.11). We write $B_\varrho(Z_0) \subset \mathbb{C}^m$ for the open ball of radius $\varrho > 0$ around $Z_0 \in \mathbb{C}^m$.

Lemma 7.7. *The vanishing ideal* $\mathcal{J}$ *of* $\Lambda_0(\mathbb{S}^1 \ltimes \mathbb{C}^m)$ *has the following local form. For each* $(\exp 2\pi\sqrt{-1}t_0, Z_0) \in \mathbb{S}^1 \times \mathbb{C}^m$, *there exist* $\varepsilon, \varrho > 0$ *such that*

- *if* $t_0 = \frac{j}{w}$, $Z_0 = 0$, *then the vanishing ideal* $\mathcal{J}((\frac{j}{w} - \varepsilon, \frac{j}{w} + \varepsilon) \times B_\varrho(0))$ *consists of all smooth functions* $f \in \mathcal{C}^\infty((\frac{j}{w} - \varepsilon, \frac{j}{w} + \varepsilon) \times B_\varrho(0))$ *which can be written in the form*

$$f = \left(t - \frac{j}{w}\right) \sum_{k, w_k j \in w\mathbb{Z}} (z_k f_k + \bar{z}_k g_k) + \sum_{k, w_k j \notin w\mathbb{Z}} (z_k f_k + \bar{z}_k g_k),$$

  with $f_k, g_k \in \mathcal{C}^\infty((\frac{j}{w} - \varepsilon, \frac{j}{w} + \varepsilon) \times B_\varrho(0))$,

- *if* $t_0 = \frac{j}{w}$*,* $Z_0 \neq 0$ *with* $\exp(2\pi\sqrt{-1}\frac{j}{w})Z = Z$*, then the vanishing ideal* $\mathcal{J}((\frac{j}{w}-\varepsilon, \frac{j}{w}+\varepsilon) \times B_\varrho(Z))$ *consists of all smooth functions* $f \in \mathcal{C}^\infty((\frac{j}{w}-\varepsilon, \frac{j}{w}+\varepsilon) \times B_\varrho(Z))$ *which can be written in the form*

$$f = \left(t - \frac{j}{w}\right) f + \sum_{k, w_k j \notin w\mathbb{Z}} (z_k f_k + \bar{z}_k g_k),$$

for $f, f_k, g_k \in \mathcal{C}^\infty((\frac{j}{w}-\varepsilon, \frac{j}{w}+\varepsilon) \times B_\varrho(Z))$*,*

- *if* $t_0 \neq \frac{j}{w}$*,* $Z_0 = 0$*, then the vanishing ideal* $\mathcal{J}((t_0-\varepsilon, t_0+\varepsilon) \times B_\varrho(0))$ *consists of all smooth functions* $f \in \mathcal{C}^\infty((t_0-\varepsilon, t_0+\varepsilon) \times B_\varrho(0))$ *which can be written in the form*

$$f = \sum_{k=1}^{m} (z_k f_k + \bar{z}_k g_k),$$

for $f_k, g_k \in \mathcal{C}^\infty((t_0-\varepsilon, t_0+\varepsilon) \times B_\varrho(0))$*.*

Proof. We will prove the case around the most singular point $(1, 0) \in \mathbb{S}^1 \times \mathbb{C}^m$. A similar proof works for the other points. We leave the details to the reader.

For $(1, 0) \in \mathbb{S}^1 \times \mathbb{C}^m$, choose a sufficiently small $\varepsilon > 0$ such that there is no other point in the interval $(-\varepsilon, \varepsilon)$ of the form $\frac{j}{w}$ for an integer $0 < j < w$. We identify $(-\varepsilon, \varepsilon)$ with a neighborhood of 1 in $\mathbb{S}^1$ via the exponential map. For a positive ϱ, the loop space $\Lambda_0(\mathbb{S}^1 \ltimes \mathbb{C}^m)$ in $(-\varepsilon, \varepsilon) \times B_\varrho(0)$ is of the form

$$\Lambda_0(\mathbb{S}^1 \times \mathbb{C}^m)_{(0,0)} = \big\{(0, z) \mid z \in B_\varrho(0)\big\} \cup \big\{(t, 0)\big\}.$$

A smooth function f on $(-\varepsilon, \varepsilon) \times B_\varrho(0)$ belongs to $\mathcal{J}((-\varepsilon, \varepsilon) \times B_\varrho(0))$ if and only if

$$f(0, z) = f(t, 0) = 0.$$

We consider f as a function of $t \in (-\varepsilon, \varepsilon)$. By the Malgrange preparation theorem, we have the expansion

$$f(t, z) + t = c(t, z)\big(t + a_0(z)\big),$$

where $c(t, z)$ and $a_0(z)$ are smooth and $a_0(0) = 0$. Since $t = c(t, 0)t$ for all $t \in (-\varepsilon, \varepsilon)$, $c(t, 0) = 1$. Putting $t = 0$ gives $0 = c(0, z)a_0(z)$ for all $z \in B_\varrho(0)$. Recall that $c(0, 0) = 1$. Therefore, $a_0(z) = 0$ for all z in a neighborhood of 0. After possibly shrinking ϱ, we can assume that $a_0(z) = 0$ on $B_\rho(0)$. Hence, we conclude that

$$f(t, z) = t\big(c(t, z) - 1\big).$$

Taking the parametric Taylor expansion of $c(t, z) - 1$ gives

$$c(t, z) - 1 = \sum_{j=1}^{m} z_j f_j(t, z) + \bar{z}_j g_j(t, z),$$

where f_j and g_j are smooth functions on $(-\varepsilon, \varepsilon) \times B_\varrho(0)$. ∎

In the following, we compute the cohomology of the complex (7.12). We observe that the complex $(\Omega^\bullet_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}(\mathbb{S}^1\ltimes\mathbb{C}^m), i_Y)$ for $Y := Y_{\mathbb{S}^1\ltimes\mathbb{C}^m}$ forms a sheaf of complexes over $\mathbb{S}^1$ via the map $\sigma : \Lambda_0(\mathbb{S}^1\ltimes\mathbb{C}^m)\to\mathbb{S}^1$. Accordingly, we compute the cohomology $(\Omega^\bullet_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}(\mathbb{S}^1\ltimes\mathbb{C}^m), i_Y)$ as a sheaf over $\mathbb{S}^1$.

Proposition 7.8. *For all open subsets U of the loop space $\Lambda_0 = \Lambda_0(\mathbb{S}^1\ltimes\mathbb{C}^m)$ and all $k\in\mathbb{N}$, the map*

$$\Theta^k_U : \Omega^k_{\mathrm{rel},\Lambda_0}(U)\to\Gamma^\infty(U,\wedge^k F)$$

from Proposition 5.1 *is injective.*

Proof. We will prove the case around the most singular point $(1,0)\in\mathbb{S}^1\times\mathbb{C}^m$. A similar proof works for the other points. We leave the detail to the reader.

Recall that we show in Lemma 7.7 that near $(1,0)$, the vanishing ideal $\mathcal{J}((-\varepsilon,\varepsilon)\times B_\varrho(0))$ for a sufficiently small $\varepsilon>0$ and a ball $B_\varrho(0)\subset\mathbb{C}^m$ centered at 0 with a sufficiently small radius $\varrho>0$ consists of all smooth functions $f\in\mathcal{C}^\infty((-\varepsilon,\varepsilon)\times B_\varrho(0))$ which can be written in the form

$$f = t\sum_{k=1}^m (z_k f_k + \bar z_k g_k),$$

for $f_k, g_k\in\mathcal{C}^\infty((-\varepsilon,\varepsilon)\times B_\varrho(0))$. Recall that by definition, $\Omega^p_{\mathrm{rel},\Lambda_0}((-\varepsilon,\varepsilon)\times B_\varrho(0))$ is the quotient

$$\Omega^p_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}\big((-\varepsilon,\varepsilon)\times B_\varrho(0)\big)/\mathcal{J}\Omega^p_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1} + d\mathcal{J}\wedge\Omega^p_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}\big((-\varepsilon,\varepsilon)\times B_\varrho(0)\big).$$

In the following, we will describe $\Omega^p_{\mathrm{rel},\Lambda_0}((-\varepsilon,\varepsilon)\times B_\varrho(0))$ in more details and, for ease of notation, will use the symbols $\Omega^p_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}$ and $\Omega^p_{\mathrm{rel},\Lambda_0}$ to stand for the spaces $\Omega^p_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}((-\varepsilon,\varepsilon)\times B_\varrho(0))$ and $\Omega^p_{\mathrm{rel},\Lambda_0}((-\varepsilon,\varepsilon)\times B_\varrho(0))$, respectively, and $\mathcal{J}$ for the vanishing ideal $\mathcal{J}((-\varepsilon,\varepsilon)\times B_\varrho(0))$.

In degree $p=0$, $\Omega^0_{\mathrm{rel},\Lambda_0}$ coincides with the quotient of $\mathcal{C}^\infty((-\varepsilon,\varepsilon)\times B_\varrho(0))$ by $\mathcal{J}((-\varepsilon,\varepsilon)\times B_\varrho(0))$.

In degree $p=1$, we know by Lemma 7.7 that $d\mathcal{J}$ consists of 1-forms which can be expressed as follows:

$$t\sum_{k=1}^m (f_k dz_k + g_k d\bar z_k),\qquad f_k, g_k\in\mathcal{C}^\infty\big((-\varepsilon,\varepsilon)\times B_\varrho(0)\big).$$

Hence, $d\mathcal{J}$ is of the form $t\Omega^1_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}$, which contains $\mathcal{J}\Omega^1_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}$. Note that if $(0,z)\in\mathbb{S}^1\times\mathbb{C}^m$, then $F_{(0,z)}$ coincides with $T^*_z\mathbb{C}^m$. For $\omega = \sum_{k=1}^m f_k dz_k + g_k d\bar z_k\in\Omega^1_{\mathrm{rel},\Lambda_0}$, if $\Theta(\omega)=0$, then

$$f_k(0,z) = g_k(0,z) = 0\quad\text{for } 1\le k\le m.$$

Therefore, taking the parametric Taylor expansion of f_k, g_k at $(0,z)$, we have that there are $\tilde f_k$ and $\tilde g_k$ in $\mathcal{C}^\infty((-\varepsilon,\varepsilon)\times B_\varrho(0))$ such that $f_k = t\tilde f_k$ and $g_k = t\tilde g_k$. Hence, $\omega = t\sum_{k=1}^m \tilde f_k dz_k + \tilde g_k d\bar z_k\in d\mathcal{J}$ and $[\omega]=0$ in $\Omega^1_{\mathrm{rel},\Lambda_0}$.

In degree $p > 1$, the above description of $\Omega^1_{\mathrm{rel},\Lambda_0}$ generalizes with the above expression for $d\mathcal{J}$. As $\Omega^k_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}$ is of the form

$$\sum_j dz_j \wedge \Omega^{k-1}_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1} + d\bar{z}_j \wedge \Omega^{k-1}_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1},$$

we conclude that $d\mathcal{J} \wedge \Omega^{k-1}_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}$ can be identified as $t\,\Omega^k_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}$, which contains $\mathcal{J}\Omega^{k-1}_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}$ as a subspace.

We notice that at $(0,z)\in\mathbb{S}^1\times\mathbb{C}^m$, $\wedge^k F_{(0,z)}$ is $\wedge^k T^*_{(0,z)}\mathbb{C}^m$. For $\omega = \sum_{I,J} f_{I,J}\,dz_{I_1}\wedge \cdots \wedge dz_{I_s}\wedge d\bar{z}_{J_{s+1}}\wedge\cdots\wedge d\bar{z}_{J_k}$, with $1\le I_1<\cdots<I_s\le m$ and $1\le J_{s+1}<\cdots<J_k\le m$, if $\Theta(\omega)=0$, we then get $f_{I,J}(0,z)=0$ for all I,J. And we can conclude from the Taylor expansion that there exists $\tilde{f}_{I,J}$ such that $f_{I,J}=t\,\tilde{f}_{I,J}$, and

$$\omega = t\sum_{I,J}\tilde{f}_{I,J}\,dz_{I_1}\wedge\cdots\wedge dz_{I_s}\wedge d\bar{z}_{J_{s+1}}\wedge\cdots\wedge d\bar{z}_{J_k}$$

which is an element in $d\mathcal{J}\wedge\Omega^{k-1}_{\mathbb{S}^1\ltimes\mathbb{C}^m\to\mathbb{S}^1}$. Therefore, $[\omega]=0$ in $\Omega^k_{\mathrm{rel},\Lambda_0}$ and the proof is complete. ∎

Proposition 7.9. *For each $\mathbb{S}^1$-invariant open $V\subset\mathbb{C}^m$ the chain map*

$$\mathfrak{R}: \big(\Omega^\bullet_{\mathbb{S}^1\ltimes V\to\mathbb{S}^1}(\mathbb{S}^1\ltimes V), Y\lrcorner\big)\to\big(\Omega^\bullet_{\mathrm{hrel},\Lambda_0}\big(\Lambda_0(\mathbb{S}^1\ltimes V)\big),0\big)$$

given by restriction, $\mathfrak{R}(\omega):=[\omega]_{\Lambda_0}$, is a quasi-isomorphism.

Proof. We consider both sides as sheaves over $\mathbb{S}^1$, and prove that $\mathfrak{R}$ is a quasi-isomorphism of sheaves over $\mathbb{S}^1$. Since both sheaves are fine, it is sufficient to prove the quasi-isomorphism $\mathfrak{R}$ at each stalk; cf. [45, Section 6.8, Theorem 9]. We split our proof into two parts according to the point t_0 in $\mathbb{S}^1$:

(1) at $\exp(2\pi\sqrt{-1}t_0)$ with $t_0\neq\frac{j}{w}$ for $0\le j<w$ and $t\in[0,1)$,

(2) at $\exp(2\pi\sqrt{-1}\frac{j}{w})$ for $0\le j<w$.

Case (1). We prove that

$$\mathfrak{R}_{\exp(2\pi\sqrt{-1}t_0)}: \big(\Omega^\bullet_{\mathbb{S}^1\ltimes V\to\mathbb{S}^1,\exp(2\pi\sqrt{-1}t_0)}(\mathbb{S}^1\ltimes V), Y\lrcorner\big) \to \Omega^\bullet_{\mathrm{hrel},\Lambda_0,\exp(2\pi\sqrt{-1}t_0)}\big(\Lambda_0(\mathbb{S}^1\ltimes V)\big)$$

is a quasi-isomorphism for $t_0\neq\frac{j}{w}$ for $0\le j<w$ and $t_0\in[0,1)$. It is crucial to observe that for a sufficiently small $\varepsilon>0$, on $(t_0-\varepsilon,t_0+\varepsilon)\times\mathbb{C}^m$, the vector field Y is of the form

$$Y=\sum_{j=1}^m\big(\exp(2\pi\sqrt{-1}w_jt)-1\big)z_j\frac{\partial}{\partial z_j}+\big(\exp(-2\pi\sqrt{-1}w_jt)-1\big)z_j\frac{\partial}{\partial\bar{z}_j}.$$

Observe that the vector field Y vanishes exactly at $(t,0)$. Moreover,

$$\big(\Omega^\bullet_{\mathbb{S}^1\ltimes V\to\mathbb{S}^1,\exp(2\pi\sqrt{-1}t_0)}\big((t_0-\varepsilon,t_0+\varepsilon)\times\mathbb{C}^m\big),Y\lrcorner\big)$$

is a smooth family of generalized Koszul complexes over $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. Using Proposition B.10, its cohomology can be computed as

$$H^k\big(\Omega^\bullet_{\mathbb{S}^1 \ltimes V \to \mathbb{S}^1, \exp(2\pi\sqrt{-1}t_0)}\big((t_0 - \varepsilon, t_0 + \varepsilon) \times \mathbb{C}^m\big), Y \lrcorner\big) = \begin{cases} \mathcal{C}^\infty\big(t_0 - \varepsilon, t_0 + \varepsilon\big), & k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

At the same time, for every t in $(t_0 - \varepsilon, t_0 + \varepsilon)$, the fixed point of $\exp(2\pi\sqrt{-1}t)$ is 0 in $\mathbb{C}^m$. Therefore, the complex $\Omega^\bullet_{\mathrm{hrel},\Lambda_0}((t_0 - \varepsilon, t_0 + \varepsilon) \times \mathbb{C}^m)$ which coincides with $\Gamma^\infty((t_0 - \varepsilon, t_0 + \varepsilon) \times \{0\}, \wedge^\bullet F)$ is given as follows:

$$\Gamma^\infty\big((t_0 - \varepsilon, t_0 + \varepsilon) \times \{0\}, \wedge^k F\big) = \begin{cases} \mathcal{C}^\infty\big(t_0 - \varepsilon, t_0 + \varepsilon\big) & \text{for } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

From the above computation, it is straightforward to conclude that $\mathfrak{R}_{\exp(2\pi\sqrt{-1}t_0)}$ is a quasi-isomorphism.

Case (2). We prove that at $\exp(2\pi\sqrt{-1}\frac{j}{w})$, the morphism $\mathfrak{R}_{\exp(2\pi\sqrt{-1}\frac{j}{w})}$ is a quasi-isomorphism. Following Lemma 7.6, we write the vector field Y as a sum of two components:

$$\begin{aligned} Y &= Y_1 + Y_2, \\ Y_1 &= \sum_{k, kj \notin w\mathbb{Z}} Y^k z_k \frac{\partial}{\partial z_k} + \overline{Y}^k \bar{z}_k \frac{\partial}{\partial \bar{z}_k}, \\ Y_2 &= \left(t - \frac{j}{w}\right) \sum_{k, kj \in w\mathbb{Z}} w_k \left(a_k z_k \frac{\partial}{\partial z_k} + \bar{a}_k \bar{z}_k \frac{\partial}{\partial z_k}\right), \end{aligned}$$

where $a_k = a(w_k(t - \frac{j}{w}))$. Define $\widetilde{Y}_2$ to be $\sum_{k, kj \in w\mathbb{Z}} w_k(a_k z_k \frac{\partial}{\partial z_k} + \bar{a}_k \bar{z}_k \frac{\partial}{\partial z_k})$. Then we have the following expression for Y:

$$Y = Y_1 + \left(t - \frac{j}{w}\right)\widetilde{Y}_2.$$

Accordingly, we can decompose $\mathbb{C}^m$ as a direct sum of two subspaces, that is write $\mathbb{C}^m = S_1 \oplus S_2$ with

$$S_1 := \bigoplus_{k, kj \notin w\mathbb{Z}} \mathbb{C}_{w_k}, \quad S_2 := \bigoplus_{k, kj \in w\mathbb{Z}} \mathbb{C}_{w_k}.$$

Both S_1 and S_2 are equipped with $\mathbb{S}^1$-actions such that the above decomposition of $\mathbb{C}^m$ is $\mathbb{S}^1$-equivariant. As our argument is local, we can assume to work with an open set V, which is of the product form $V = V_1 \times V_2$ such that V_1 (resp. V_2) is an $\mathbb{S}^1$-invariant neighborhood of 0 in S_1 (and S_2).

We consider

$$\left(\Omega^{\bullet}_{\mathbb{S}^1\ltimes V_l\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_l\right), i_{Y_l}\right)\quad \text{for } l=1,2.$$

Observe that each complex $\Omega^{\bullet}_{\mathbb{S}^1\ltimes V_l\to\mathbb{S}^1}((\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)\times V_l)$ is a $\mathcal{C}^\infty(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)$-module, and their tensor product over the algebra $\mathcal{C}^\infty(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)$ defines a bicomplex

$$\Omega^{p}_{\mathbb{S}^1\ltimes V_1\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_1\right)\otimes_{\mathcal{C}^\infty(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)}\Omega^{q}_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_2\right)$$

with $i_{Y_1}\otimes 1$ being the horizontal differential and $1\otimes i_{Y_2}$ being the vertical one. The total complex of this double complex is exactly

$$\Omega^{\bullet}_{\mathbb{S}^1\ltimes V\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V\right)$$

with the differential $i_Y=i_{Y_1}\otimes 1+1\otimes i_{Y_2}$. The E_1-page of the spectral sequence associated to the bicomplex

$$\Omega^{\bullet}_{\mathbb{S}^1\ltimes V_1\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_1\right)\otimes_{\mathcal{C}^\infty(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)}\Omega^{\bullet}_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_2\right)$$

is

$$H^{\bullet}\left(\Omega^{\bullet}_{\mathbb{S}^1\ltimes V_1\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_1\right), i_{Y_1}\right)\otimes_{\mathcal{C}^\infty(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)}\Omega^{q}_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_2\right),$$

with the differential $1\otimes i_{Y_2}$. We observe that Y_1 vanishes only at 0 for every fixed t. Therefore, $(\Omega^{\bullet}_{\mathbb{S}^1\ltimes V_1\to\mathbb{S}^1}((\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)\times V_1), i_{Y_1})$ is a smooth family of generalized Koszul complexes. Its cohomology is computed by Proposition B.10 as follows:

$$H^{\bullet}\left(\Omega^{\bullet}_{\mathbb{S}^1\ltimes V_1\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_1\right), i_{Y_1}\right)=\begin{cases}\mathcal{C}^\infty(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon) & \bullet=0,\\ 0 & \bullet\neq 0.\end{cases}$$

Therefore, we get the following expression of $E_1^{p,q}$:

$$E_1^{p,q}=\begin{cases}\Omega^{q}_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}\big((\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)\times V_2\big), & p=0,\\ 0, & p\neq 0.\end{cases}$$

Next we compute the cohomology of $(E_1^{0,q}, i_{Y_2})$. Recall by Lemma 7.6 that Y_2 has the form $Y_2=(t-\frac{j}{w})\widetilde{Y}_2$, where $\widetilde{Y}_2$ vanishes exactly at 0 for every fixed $t\in(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)$. At degree q, we notice that if an element $\omega\in\Omega^{q}_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}((\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)\times V_2)$ belongs to $\ker(i_{Y_2})$, then $(t-\frac{j}{w})i_{\widetilde{Y}_2}\omega=0$. Thus, ω belongs to $\ker(i_{\widetilde{Y}_2})$. Hence, we have reached the equation

$$\ker(i_{Y_2})=\ker(i_{\widetilde{Y}_2}).$$

It is also easy to check that

$$i_{Y_2}\Omega^{q+1}_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_2\right)=\left(t-\frac{j}{w}\right)i_{\tilde{Y}_2}\Omega^{q+1}_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_2\right).$$

We conclude that the quotient $\ker(i_{Y_2})/i_{Y_2}\Omega^{q+1}_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}((\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)\times V_2)$ is isomorphic to

$$\ker(i_{\tilde{Y}_2})/\left(t-\frac{j}{w}\right)i_{\tilde{Y}_2}\Omega^{q+1}_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_2\right).$$

Recall that the cohomology of $(\Omega^\bullet_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}((\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)\times V_2),i_{\tilde{Y}_2})$ is computed as follows:

$$H^q\left(\Omega^\bullet_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_2\right),i_{\tilde{Y}_2}\right)=\begin{cases}\mathcal{C}^\infty(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon), & q=0,\\ 0, & q\neq 0.\end{cases}$$

Therefore, for all q, we conclude that

$$i_{\tilde{Y}_2}\Omega^{q+1}_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_2\right)=\ker(i_{\tilde{Y}_2}),$$

and the quotient $\ker(i_{Y_2})/i_{Y_2}\Omega^{q+1}_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}((\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)\times V_2)$ is isomorphic to

$$\ker(i_{\tilde{Y}_2})/\left(t-\frac{j}{2}\right)\ker(i_{\tilde{Y}_2}).$$

As the E_2 page has only nonzero component when $p=0$, the spectral sequence collapses at the E^2 page, and we conclude that the cohomology of the total complex, which is the cohomology of $\Omega^\bullet_{\mathbb{S}^1\ltimes V\to\mathbb{S}^1}((\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)\times V)$ with the differential $i_{Y_1}\otimes 1+1\otimes i_{Y_2}$, is equal to the quotient

$$\ker(i_{\tilde{Y}_2})/\left(t-\frac{j}{2}\right)\ker(i_{\tilde{Y}_2})$$

for the contraction $i_{\tilde{Y}_2}$ on $\Omega^\bullet_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}((\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)\times V_2)$.

We now prove that the morphism

$$\mathfrak{R}:\left(\Omega^q_{\mathbb{S}^1\ltimes V\to\mathbb{S}^1}(\mathbb{S}^1\ltimes V),Y\lrcorner\right)\to\left(\Omega^\bullet_{\mathrm{hrel},\Lambda_0}\left(\Lambda_0(\mathbb{S}^1\ltimes V)\right),0\right)$$

is a quasi-isomorphism. The above discussion and description of $\Lambda_0((\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon)\times V)$ reduces us to prove that the morphism

$$\begin{aligned}\mathfrak{R}_2:&\left(\Omega^\bullet_{\mathbb{S}^1\ltimes V_2\to\mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_2\right),Y_2\lrcorner\right)\\ &\to\left(\Omega^\bullet_{\mathrm{hrel},\Lambda_0}\left(\Lambda_0\left(\left(\frac{j}{w}-\varepsilon,\frac{j}{w}+\varepsilon\right)\times V_2\right)\right),0\right)\end{aligned}$$

is a quasi-isomorphism. We prove this by examination of $\mathfrak{R}_2$ in degree q. Hereby, we will work with $\wedge^\bullet F$ as its smooth section space is isomorphic to $\Omega^\bullet_{\mathrm{rel},\Lambda_0}$ by Proposition 7.8.

Case $q \geq 1$. Recall that $\Gamma^\infty((\frac{j}{w}-\varepsilon, \frac{j}{w}+\varepsilon) \times V_2, \wedge^q F)$ is $\wedge^q F_{(\frac{j}{w}, z)}$. We observe that the vector field $\widetilde{Y}_2$ at $t = \frac{j}{w}$ coincides with the fundamental vector field of the $\mathbb{S}^1$ action on V_2. Hence, if $\phi \in \wedge^q F_{(\frac{j}{w}, z)}$ is horizontal, ϕ satisfies the equation $i_{\widetilde{Y}_2(\frac{j}{w}, z)}\phi = 0$. As the cohomology of the $(\Omega^\bullet(V_2), i_{\widetilde{Y}_2(0,z)})$ at degree q vanishes, there is a degree $q+1$ form $\psi \in \Omega^\bullet(V_2)$ such that $i_{\widetilde{Y}_2(\frac{j}{w}, z)}\psi = \phi$. Define $\omega \in \Omega^\bullet_{\mathbb{S}^1 \ltimes V_2 \to \mathbb{S}^1}((\frac{j}{w}-\varepsilon, \frac{j}{w}+\varepsilon) \times V_2)$ by $\omega := i_{\widetilde{Y}_2}\psi$, where ψ is viewed as an element in $\Omega^\bullet_{\mathbb{S}^1 \ltimes V_2 \to \mathbb{S}^1}((\frac{j}{w}-\varepsilon, \frac{j}{w}+\varepsilon) \times V_2)$ constant along the t direction. Then we can easily check that ω belongs to the kernel of $i_{\widetilde{Y}_2}$ and $\mathfrak{R}_2(\psi) = \phi$. We conclude that $\mathfrak{R}_2$ is surjective.

For the injectivity of $\mathfrak{R}_2$, we suppose that $\omega \in \ker(i_{\widetilde{Y}_2})$. Hence,

$$\mathfrak{R}_2(\omega)\left(\frac{j}{w}, z\right) = \omega\left(\frac{j}{w}, z\right) = 0.$$

Then by the parametrized Taylor expansion, we can find a form

$$\widetilde{\omega} \in \Omega^\bullet_{\mathbb{S}^1 \ltimes V_2 \to \mathbb{S}^1}\left(\left(\frac{j}{w}-\varepsilon, \frac{j}{w}+\varepsilon\right) \times V_2\right)$$

such that $\omega = (t - \frac{j}{w})\widetilde{\omega}$. As $0 = i_{\widetilde{Y}_2}\omega = (t-\frac{j}{w}) i_{\widetilde{Y}_2}\widetilde{\omega}_2, i_{\widetilde{Y}_2}\widetilde{\omega} = 0$. Hence $\omega = (t-\frac{j}{w})\widetilde{\omega}$ belongs to $(t-\frac{j}{w})\ker i_{\widetilde{Y}_2}$, and $[\omega]$ is zero in the cohomology of i_{Y_2}.

Case $q = 0$. Recall that $\widetilde{Y}_2$ is of the form

$$\sum_k w_k\left(a\left(w_k\left(t-\frac{j}{w}\right)\right)z_k\frac{\partial}{\partial z_k} + \bar{a}\left(w_k\left(t-\frac{j}{w}\right)\right)\right)\bar{z}_k\frac{\partial}{\partial \bar{z}_k},$$

where $a(w_k(t-\frac{j}{w})) \neq 0$ for all $t \in (\frac{j}{w}-\varepsilon, \frac{j}{w}+\varepsilon)$. Therefore, the space $(t-\frac{j}{w})i_{\widetilde{Y}_2}$ is of the form

$$\left(t-\frac{j}{w}\right)\sum_k z_k f_k + \bar{z}_k g_k,$$

which is exactly the vanishing ideal $\mathcal{J}((\frac{j}{w}-\varepsilon, \frac{j}{w}+\varepsilon) \times V_2)$. This shows that the cohomology of $(\Omega^\bullet_{\mathbb{S}^1 \ltimes V_2 \to \mathbb{S}^1}((\frac{j}{w}-\varepsilon, \frac{j}{w}+\varepsilon) \times V_2), Y_2 \lrcorner)$ at degree 0 coincides with $\mathcal{C}^\infty(\Lambda_0(\mathbb{S}^1 \ltimes V_2))|_{(\frac{j}{w}-\varepsilon, \frac{j}{w}+\varepsilon) \times V_2}$. One concludes that $\mathfrak{R}_2$ is an isomorphism in degree 0, and the proof is complete ∎

7.3. Stitching it all together

We are now in a position to prove Conjecture 6.7 in the case of circle actions:

Theorem 7.10. *Let M be an $\mathbb{S}^1$-manifold and regard $\Omega^\bullet_{\mathrm{hrel},\Lambda_0}(\Lambda_0(\mathbb{S}^1 \ltimes M))$ as a chain complex endowed with the zero differential. Then the chain map*

$$\Phi_{\bullet, M/\mathbb{S}^1} : C_\bullet(\mathcal{C}^\infty(M), \mathcal{A}(M/\mathbb{S}^1)) \to \Omega^\bullet_{\mathrm{hrel},\Lambda_0}(\Lambda_0(\mathbb{S}^1 \ltimes M))$$

is a quasi-isomorphism.

Proof. Since $\Phi_{\bullet,M/\mathbb{S}^1}$ is the global sections of a morphism of fine sheaves on $M/\mathbb{S}^1$, it suffices to prove that

$$\Phi_\bullet : \widehat{\mathcal{C}}_\bullet(\mathcal{C}^\infty_M, \mathcal{A}) \to \pi_*(s_{|\Lambda_0})_*\Omega^\bullet_{\mathrm{rel},\Lambda_0}$$

is a quasi-isomorphism of sheaf complexes, i.e., that the induced map on the stalks $\Phi_{\bullet,\mathcal{O}}$ is a quasi-isomorphism for all orbits $\mathcal{O} \subset M$. By properness of the action, the isotropy group is a compact subgroup of $\mathbb{S}^1$, leaving us with two cases:

(i) when the isotropy subgroup $\Gamma_x \subset \mathbb{S}^1$ of a point $x \in \mathbb{S}^1$ is a finite group, this follows from the (proof of) Corollary B.8. Indeed, it is easily checked that the morphism $\Phi_\bullet$ is the composition of the morphism of Proposition 4.5 reducing to the local model, with the one inducing the isomorphism of Corollary B.8;

(ii) when the isotropy group is $\mathbb{S}^1$ itself, Proposition 7.9 entails that $\Phi_{\bullet,\mathcal{O}}$ is a quasi-isomorphism.

This finishes the proof. ■

A. Tools from singularity theory

A.1. Differentiable stratified spaces

Assume that $X \subset \mathbb{R}^n$ is a locally closed subspace that is the intersection of an open and a closed subset of the ambient $\mathbb{R}^n$. The sheaf $\mathcal{C}^\infty_X$ of smooth functions on X then is defined as the quotient sheaf $\mathcal{C}^\infty_U/\mathcal{J}_{X,U}$, where $U \subset \mathbb{R}^n$ open is chosen such that $X \subset U$ is relatively closed, $\mathcal{C}^\infty_U$ is the sheaf of smooth functions on U, and $\mathcal{J}_{X,U}$ the ideal sheaf of smooth functions on open subsets of U vanishing on X. Note that $\mathcal{C}^\infty_X$ does not depend on the particular choice of the ambient open subset $U \subset \mathbb{R}^n$.

Definition A.1. A commutative locally ringed space $(A, \mathcal{O})$ is called an *affine differentiable space* if there is a closed subset $X \subset \mathbb{R}^n$ and an isomorphism of ringed spaces $(f, F) : (A, \mathcal{O}) \to (X, \mathcal{C}^\infty_X)$.

By a *differentiable stratified space*, we understand a commutative locally ringed space $(X, \mathcal{C}^\infty)$ consisting of a separable locally compact topological Hausdorff space X equipped with a stratification $\mathcal{S}$ on X in the sense of Mather [31] (cf. also [37, Section 1.2]) and a sheaf $\mathcal{C}^\infty$ of commutative local $\mathbb{C}$-rings on X such that for every point $x \in X$ there is an open neighborhood U together with $\varphi_1, \dots, \varphi_n \in \mathcal{C}^\infty(U)$ having the following properties:

(DS1) the map $\varphi : U \to \mathbb{R}^n$, $y \mapsto (\varphi_1(y), \dots, \varphi_n(y))$ is a homeomorphism onto a locally closed subset $\widetilde{U} := \varphi(U) \subset \mathbb{R}^n$ and induces an isomorphism of ringed spaces $\varphi : (U, \mathcal{C}^\infty_{|U}) \to (\widetilde{U}, \mathcal{C}^\infty_{\widetilde{U}})$;

(DS2) the map φ endows $(U, \mathcal{C}^\infty_{|U})$ with the structure of an affine differentiable space which means that $(\varphi, \varphi^*) : (U, \mathcal{C}^\infty_{|U}) \to (\widetilde{U}, \mathcal{C}^\infty_{\widetilde{U}})$ is an isomorphism of ringed spaces, where $\mathcal{C}^\infty_{\widetilde{U}}$ denotes the sheaf of smooth functions on $\widetilde{U}$ as defined above;

(DS3) for each stratum $S \subset U$, the map $\varphi_{|S\cap U}$ is a diffeomorphism of $S \cap U$ onto a submanifold $\varphi(S \cap U) \subset \mathbb{R}^n$.

A map $\varphi : U \to \mathbb{R}^n$ fulfilling the axioms (DS1) to (DS3) is often called a *singular chart* of X (cf. [37, Section 1.3]).

A differentiable stratified space is in particular a reduced differentiable space in the sense of Spallek [44] or Gonzáles–de Salas [34]. Moreover, differentiable stratified spaces defined as above coincide with the stratified spaces with smooth structure as in [37].

Proposition A.2 (cf. [37, Theorem 1.3.13]). *The structure sheaf of a differentiable stratified space is fine.*

To formulate the next result, we introduce the commutative ringed space $(\mathbb{R}^\infty, \mathcal{C}^\infty_{\mathbb{R}^\infty})$. It is defined as the limit of the direct system of ringed spaces $((\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n}), \iota_{nm})_{n,m\in\mathbb{N},n\le m}$, where $\iota_{nm} : \mathbb{R}^n \hookrightarrow \mathbb{R}^m$ is the embedding given by

$$\iota_{nm}(v_1, \dots, v_n) = (v_1, \dots, v_n, 0, \dots, 0).$$

Note that for each open set $U \subset \mathbb{R}^\infty$ the section space $\mathcal{C}^\infty_{\mathbb{R}^\infty}(U)$ coincides with the inverse limit of the projective system of nuclear Fréchet algebras $(\mathcal{C}^\infty_{\mathbb{R}^n}(U \cap \mathbb{R}^n), \iota^*_{nm})_{n,m\in\mathbb{N},n\le m}$. Hence the $\mathcal{C}^\infty_{\mathbb{R}^\infty}(U)$ and, in particular, $\mathcal{C}^\infty_{\mathbb{R}^\infty}(\mathbb{R}^\infty)$ are nuclear Fréchet algebras by [48, Proposition 50.1].

Proposition A.3. *For every differentiable stratified space* $(X, \mathcal{C}^\infty)$*, there exists a proper embedding* $\varphi : (X, \mathcal{C}^\infty) \hookrightarrow (\mathbb{R}^\infty, \mathcal{C}^\infty_{\mathbb{R}^\infty})$.

Proof. Since X is separable and locally compact, there exists a compact exhaustion, that is, a family $(K_k)_{k\in\mathbb{N}}$ of compact subsets $K_k \subset X$ such that $K_k \subset K^\circ_{k+1}$ for all $k \in \mathbb{N}$ and such that $\bigcup_{k\in\mathbb{N}} K_k = X$. By [37, Lemma 1.3.17], there then exists an inductively embedding atlas that is a family $(\varphi_k)_{k\in\mathbb{N}}$ of singular charts $\varphi_k : K^\circ_{k+1} \to \mathbb{R}^{n_k}$ together with a family $(U_k)_{k\in\mathbb{N}}$ of relatively compact open subsets $U_k \Subset K^\circ_{k+1}$ such that $K_k \subset U_k$ and $\varphi_{k+1}|_{U_k} = \iota_{n_k n_{k+1}} \circ \varphi_k|_{U_k}$ for all $k \in \mathbb{N}$. Now define $\varphi : X \to \mathbb{R}^\infty$ by $\varphi(x) = \varphi_k(x)$ whenever $x \in U_k$. Then φ is well defined and an embedding by construction. By a straightforward partition of unity argument, one constructs a smooth function $\psi : X \to \mathbb{R}$ such that $\psi(x) \ge k$ for all $x \in K_{k+1} \setminus K^\circ_k$. The embedding $(\varphi, \psi) : X \to \mathbb{R}^\infty \times \mathbb{R} \cong \mathbb{R}^\infty$ then is proper, and the claim is proved. ∎

Corollary A.4. *Let* $(X, \mathcal{C}^\infty)$ *be a differential stratified space. Then there exists a complete metric* $d : X \times X \to \mathbb{R}$ *such that* $d^2 \in \mathcal{C}^\infty(X \times X)$.

Proof. The Euclidean inner product $\langle -, - \rangle_{\mathbb{R}^n}$ extends uniquely to an inner product $\langle -, - \rangle_{\mathbb{R}^\infty}$ on $\mathbb{R}^\infty$ such that $\langle j_n(x), j_n(y) \rangle_{\mathbb{R}^\infty} = \langle x, y \rangle_{\mathbb{R}^n}$ for all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$, where $j_n : \mathbb{R}^n \hookrightarrow \mathbb{R}^\infty$ is the canonical embedding $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0, \dots)$. The associated metric $d_{\mathbb{R}^\infty} : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}$, $(x, y) \mapsto \sqrt{\langle x - y, x - y \rangle_{\mathbb{R}^\infty}}$ then is related to the Euclidean metric $d_{\mathbb{R}^n}$ by $d_{\mathbb{R}^\infty}(j_n(x), j_n(y)) = d_{\mathbb{R}^n}(x, y)$ for $x, y \in \mathbb{R}^n$. Now choose a proper embedding $X \hookrightarrow \mathbb{R}^\infty$ and denote the restriction of $d_{\mathbb{R}^\infty}$ to X by d. By construction, d^2 then is smooth. Moreover, d is a complete metric since the embedding is proper and each of the metrics $d_{\mathbb{R}^n}$ is complete. ■

B. The cyclic homology of bornological algebras

B.1. Bornological vector spaces and tensor products

We recall some basic notions from the theory of bornological vector spaces and their tensor products. For details we refer to [27] and [33, Chapter 1].

Definition B.1 (cf. [27, Chapter I, 1:1 Definition]). By a *bornology* on a set X, one understands a set $\mathcal{B}$ of subset of X such that the following conditions hold true:

(BS) $\mathcal{B}$ is a covering of X, $\mathcal{B}$ is hereditary under inclusions, and $\mathcal{B}$ is stable under finite unions.

A map $f : X \to Y$ from a set X with bornology $\mathcal{B}$ to a set Y carrying a bornology $\mathcal{D}$ is called *bounded*, if the following is satisfied:

(BM) the map f preserves the bornologies, i.e., $f(B) \in \mathcal{D}$ for all $B \in \mathcal{B}$.

If V is a vector space over $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$, a bornology $\mathcal{B}$ is called a *convex vector bornology* on V if the following additional properties hold true:

(BV) the bornology $\mathcal{B}$ is stable under addition, under scalar multiplication, under forming balanced hulls, and finally under forming convex hulls.

A set together with a bornology is called a *bornological set*, and a vector space with a convex vector bornology is called a *bornological vector space*. For clarity, we sometimes denote a bornological vector space as a pair $(V, \mathcal{B})$, where V is the underlying vector space, and $\mathcal{B}$ is the corresponding convex vector bornology.

A bornological vector space $(V, \mathcal{B})$ is called *separated* if the condition (S) below is satisfied. If in addition condition (C) holds true as well, $(V, \mathcal{B})$ is called *complete*.

(S) The subspace $\{0\}$ is the only bounded subvector space of V.

(C) Every bounded set is contained in a bounded completant disk, where by a *completant disk* one understands a non-empty balanced convex subset $D \subset V$ such that the space V_D spanned by D and semi-normed by the gauge of D is a Banach space.

As for the category of topological vector spaces, there exist functors of separation and completion within the category of bornological vector spaces; see e.g. [32, Sections A.1.1 and A.1.3].

Example B.2. Let V be a locally convex topological vector space. The *von Neumann bornology* on V consists of all (von Neumann) bounded subsets of V, i.e., all $B \subset V$ that are absorbed by every 0-neighborhood. One immediately checks that the von Neumann bornology is a convex vector bornology on V. We sometimes denote this bornology by $\mathcal{B}_{\mathrm{vN}}$.

Definition B.3. The *bornological tensor product* of two bornological vector spaces $(V_1, \mathscr{B}_1)$ and $(V_2, \mathscr{B}_2)$ is defined as the algebraic tensor product $V_1 \otimes V_2$ endowed with the smallest bornology on $V_1 \otimes V_2$ containing all the tensor product sets $B_1 \otimes B_2$, where $B_1 \in \mathscr{B}_1$ and $B_2 \in \mathscr{B}_2$. The completion of the bornological tensor product is denoted by $V_1 \hat{\otimes} V_2$ and called the *bornological tensor product*.

Similarly to the topological case, the bornological tensor product satisfies a universal property. The proof is straightforward.

Proposition B.4. *Given two bornological vector spaces* $(V_1, \mathscr{B}_1)$ *and* $(V_2, \mathscr{B}_2)$*, the bornological tensor product* $(V_1 \otimes V_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$ *together with the canonical bounded map* $V_1 \times V_2 \to V_1 \otimes V_2$ *satisfies the following universal property: for each bornological vector space* $(W, \mathcal{B})$ *and bounded bilinear map* $\lambda : V_1 \times V_2 \to W$*, there exists a unique bounded linear map* $\overline{\lambda} : V_1 \otimes V_2 \to W$ *making the diagram*

$$\begin{array}{ccc} V_1 \times V_2 & \xrightarrow{\lambda} & W \\ \downarrow & \nearrow_{\overline{\lambda}} & \\ V_1 \otimes V_2 & & \end{array}$$

commute.

Remark B.5. Since tensor products of topological vector spaces are also needed in this paper, let us briefly recall that the completed projective (resp. inductive) topological tensor product $\hat{\otimes}_\pi$ (resp. $\hat{\otimes}_\iota$) can be defined as the (up to isomorphism) unique bifunctor on the category of complete locally convex topological vector spaces which is universal with respect to jointly (resp. separately) continuous bilinear maps with values in complete locally convex topological vector spaces. For Fréchet spaces, the completed projective and completed inductive tensor products coincide, since separately continuous bilinear maps on Fréchet spaces are automatically jointly continuous. See [23, 33] for details.

B.2. The Hochschild chain complex

In this section, we recall the construction of the Hochschild bicomplex associated to a possibly non-unital complete bornological algebra A. To this end, note first that the space of Hochschild k-chains $C_k(A) := A^{\hat{\otimes}(k+1)}$ is defined using the completed bornological tensor product $\hat{\otimes}$. Together with the face maps

$$\begin{aligned} b_{k,i} &: C_k(A) \to C_{k-1}(A), \\ a_0 \otimes \cdots \otimes a_k &\mapsto \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k & \text{for } 0 \le i < k, \\ a_k a_0 \otimes \cdots \otimes a_{k-1} & \text{for } i = k \end{cases} \end{aligned}$$

and the cyclic operators

$$t_k : C_k(A) \to C_k(A), \quad a_0 \otimes \cdots \otimes a_k \mapsto (-1)^k a_k \otimes a_0 \otimes \cdots \otimes a_{k-1},$$

the graded linear space of Hochschild chains $C_\bullet(A) := (C_k(A))_{k\in\mathbb{N}}$ then becomes a pre-cyclic object; see e.g. [30, Sections 1.1 and 2.5] for details. This means that the following commutation relations are satisfied:

$$\begin{aligned} b_{k-1,i} b_{k,j} &= b_{k-1,j-1} b_{k,i} \quad \text{for } 0 \le i < j \le k, \\ b_{k,i} t_k &= \begin{cases} -t_{k-1} b_{k,i-1} & \text{for } 1 \le i \le k, \\ (-1)^k b_{k,k} & \text{for } i = 0, \end{cases} \\ t_k^{k+1} &= 1. \end{aligned}$$

From the pre-cyclic structure, one obtains two boundary maps, namely the one of the Bar complex $b' : C_k(A) \to C_{k-1}(A)$, $b' := \sum_{i=0}^{k-1} (-1)^i b_{k,i}$ and the Hochschild boundary $b : C_k(A) \to C_{k-1}(A)$, $b := b' + (-1)^k b_{k,k}$. The commutation relations for the face maps $b_{k,i}$ immediately entail that $b^2 = (b')^2 = 0$. This gives rise to the two-column bicomplex

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ C_2(A) & \xleftarrow{1-t} & C_2(A) \\ b \downarrow & & \downarrow -b' \\ C_1(A) & \xleftarrow{1-t} & C_1(A) \\ b \downarrow & & \downarrow -b' \\ C_0(A) & \xleftarrow{1-t} & C_0(A). \end{array}$$

We will denote this two-column bicomplex by $C_{\bullet,\bullet}(A)^{\{2\}}$. By definition, the homology of its total complex is the Hochschild homology

$$HH_\bullet(A) := H_\bullet\big(\mathrm{Tot}_\bullet(C_{\bullet,\bullet}(A)^{\{2\}})\big). \tag{B.1}$$

In case A is a unital complete bornological algebra and M a unitary complete bornological A-bimodule, or more generally if A is H-unital and M H-unitary as explained in [52, Section 9], then the Hochschild homology $HH_\bullet(A, M)$ of A in the bimodule M is given by the homology of the chain complex $C_\bullet(A, M) = M \,\hat{\otimes}_{A \hat{\otimes} A}\, C_\bullet(A)$ endowed with the induced Hochschild boundary b.

B.3. A twisted version of the Connes–Hochschild–Kostant–Rosenberg theorem

The classical theorem by Hochschild–Kostant–Rosenberg identifies the Hochschild homology of the algebra of regular functions on a smooth affine variety with the graded module of Kähler forms of that algebra [26]. In his seminal paper [11], Connes proved that for compact smooth manifolds an analogous result holds true which means that the (continuous) Hochschild homology of the algebra of smooth functions on a manifold coincides naturally with the complex of differential forms over the manifold (see [36] for the non-compact case of that result). We will refer to this result as the Connes–Hochschild–Kostant–Rosenberg theorem. In the following, we prove a twisted version which originally goes back to [9, Lemma 5.2]. Here we provide an alternative proof which is closer to Connes' proof of the manifold version of the Hochschild–Kostant–Rosenberg theorem; cf. also [6].

Assume that h is an orthogonal transformation acting on some Euclidean space $\mathbb{R}^d$. Let V be an open ball around the origin of $\mathbb{R}^d$. Then we denote by ${}^h\mathcal{C}^\infty(V)$ the space $\mathcal{C}^\infty(V)$ with the h-twisted $\mathcal{C}^\infty(V)$-bimodule structure

$$\begin{gathered}\mathcal{C}^\infty(V) \mathbin{\hat{\otimes}} {}^h\mathcal{C}^\infty(V) \mathbin{\hat{\otimes}} \mathcal{C}^\infty(V) \to {}^h\mathcal{C}^\infty(V),\\ f \otimes a \otimes f' \mapsto \big(V \ni v \mapsto f(hv)a(v)f'(v) \in \mathbb{R}\big).\end{gathered}$$

In the following, we compute the *twisted* Hochschild homology $H_\bullet(\mathcal{C}^\infty(V), {}^h\mathcal{C}^\infty(V))$ which by definition is the homology of the chain complex $C_\bullet(\mathcal{C}^\infty(V), {}^h\mathcal{C}^\infty(V))$. Denote by $\langle -, - \rangle$ the Euclidean inner product on $\mathbb{R}^d$. By the orthogonality assumption, $\langle -, - \rangle$ is G-invariant, hence so is V. Recall that for every topological projective resolution $R_\bullet \to \mathcal{C}^\infty(V)$ of $\mathcal{C}^\infty(V)$ as a $\mathcal{C}^\infty(V)$-bimodule, the Hochschild homology groups $H_k(\mathcal{C}^\infty(V), {}^h\mathcal{C}^\infty(V))$ are naturally isomorphic to the homology groups $H_k(R_\bullet, {}^h\mathcal{C}^\infty(V))$; see [25]. Recall further that a topological projective resolution of the $\mathcal{C}^\infty(V)$-bimodule $\mathcal{C}^\infty(V)$ is given by the Connes–Koszul resolution [11, p. 127ff]

$$\Gamma^\infty(V \times V, E_d) \xrightarrow{i_Y} \cdots \xrightarrow{i_Y} \Gamma^\infty(V \times V, E_1) \xrightarrow{i_Y} \mathcal{C}^\infty(V \times V) \to \mathcal{C}^\infty(V) \to 0, \quad \text{(B.2)}$$

where E_k is the pull-back bundle $\mathrm{pr}_2^*(\Lambda^k T^*\mathbb{R}^d)$ along the projection $\mathrm{pr}_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $(v, w) \mapsto w$, and i_Y denotes contraction with the vector field $Y : V \times V \to \mathrm{pr}_2^*(T\mathbb{R}^d)$, $(v, w) \mapsto w - v$. By tensoring the Connes–Koszul resolution with ${}^h\mathcal{C}^\infty(V)$, one obtains the chain complex

$$\Omega^d(V) \xrightarrow{i_{Y_h}} \cdots \xrightarrow{i_{Y_h}} \Omega^1(V) \xrightarrow{i_{Y_h}} \mathcal{C}^\infty(V) \to 0, \quad \text{(B.3)}$$

where the vector field $Y_h : V \to T\mathbb{R}^d$ is given by $Y_h(v) = v - hv$. Denote by V^h the fixed point set of h in V, let $\iota_h : V^h \hookrightarrow V$ be the canonical embedding, and $\pi_h : V \to V^h$ the restriction of the orthogonal projection onto the fixed point space $(\mathbb{R}^d)^h$. One obtains the following commutative diagram:

$$\begin{array}{ccccccc}
\Omega^d(V) & \xrightarrow{i_{Y_h}} \cdots \xrightarrow{i_{Y_h}} & \Omega^1(V) & \xrightarrow{i_{Y_h}} & \mathcal{C}^\infty(V) \\
\downarrow{\scriptstyle \iota_h^*} & & \downarrow{\scriptstyle \iota_h^*} & & \downarrow{\scriptstyle \iota_h^*} \\
\Omega^d(V^h) & \xrightarrow{0} \cdots \xrightarrow{0} & \Omega^1(V^h) & \xrightarrow{0} & \mathcal{C}^\infty(V^h) \\
\downarrow{\scriptstyle \pi_h^*} & & \downarrow{\scriptstyle \pi_h^*} & & \downarrow{\scriptstyle \pi_h^*} \\
\Omega^d(V) & \xrightarrow{i_{Y_h}} \cdots \xrightarrow{i_{Y_h}} & \Omega^1(V) & \xrightarrow{i_{Y_h}} & \mathcal{C}^\infty(V).
\end{array} \quad \text{(B.4)}$$

Proposition B.6. *The chain maps ι_h^* and π_h^* are quasi-isomorphisms.*

Proof. Since the restriction of the vector field Y_h to V^h vanishes, the diagram (B.4) commutes, and the ι_h^* and π_h^* are chain maps indeed. Let W be the orthogonal complement of $(\mathbb{R}^d)^h$ in $\mathbb{R}^d$, $m = \dim W$, and $\pi_W := \mathrm{id}_V - \pi_h$ the orthogonal projection onto W. Since the h-action on W is orthogonal and has as only fixed point the origin, there exists an orthonormal basis $w_1, \ldots, w_m$ of W, a natural $l \leq \frac{m}{2}$, and $\theta_1, \ldots, \theta_l \in (-\pi, \pi) \setminus \{0\}$ such that the following holds:

$$hw_k = \begin{cases} \cos\theta_i\, w_{2i-1} + \sin\theta_i\, w_{2i} & \text{if } k = 2i-1 \text{ with } i \leq l, \\ -\sin\theta_i\, w_{2i-1} + \cos\theta_i\, w_{2i} & \text{if } k = 2i \text{ with } i \leq l, \\ -w_k & \text{if } 2l < k \leq m. \end{cases}$$

Denote by $\varphi_t : \mathbb{R}^d \to \mathbb{R}^d$, $t \in \mathbb{R}$ the flow of the complete vector field Y_h or in other words the solution of the initial value problem $\frac{d}{dt}\varphi_t = (\mathrm{id}_V - h)\varphi_t$, $\varphi_0 = \mathrm{id}_V$. Then $\varphi_t v = v$ for all $v \in (\mathbb{R}^d)^h$, and

$$\varphi_t(w_k) = \begin{cases} e^{(1-\cos\theta_i)t}\big(\cos(t\sin\theta_i)w_{2i-1} + \sin(t\sin\theta_i)w_{2i}\big), & \text{if } k = 2i-1 \text{ with } i \leq l, \\ e^{(1-\cos\theta_i)t}\big(-\sin(t\sin\theta_i)w_{2i-1} + \cos(t\sin\theta_i)w_{2i}\big), & \text{if } k = 2i \text{ with } i \leq l, \\ e^{2t} w_k, & \text{if } 2l < k \leq m. \end{cases} \tag{B.5}$$

Now let $v_1, \ldots, v_n$ be a basis of V^h, and denote by $v^1, \ldots, v^n, w^1, \ldots, w^m$ the basis of V' dual to $v_1, \ldots, v_n, w_1, \ldots, w_m$. Then every k-form ω on V is the sum of monomials $dv^{i_1} \wedge \cdots \wedge dv^{i_l} \wedge \omega_{i_1,\ldots,i_l}$, where

$$1 \leq i_1 < \cdots < i_l \leq n \quad \text{and} \quad \omega_{i_1,\ldots,i_l} = i_{v_{i_1}\wedge\cdots\wedge v_{i_l}}\omega \in \Gamma^\infty(\pi_W^*\Lambda^{k-l}T^*W).$$

Let d_W be the restriction of the exterior differential to $\Gamma^\infty(\pi_W^*\Lambda^\bullet T^*W)$ and define $S : \Omega^k(V) \to \Omega^{k+1}(V)$ by its action on the monomials:

$$S\omega = \sum_{l=0}^{k} \sum_{1\leq i_1<\cdots<i_l\leq n} dv^{i_1} \wedge \cdots \wedge dv^{i_l} \wedge \int_{-\infty}^{0} \varphi_t^*(d_W\omega_{i_1,\ldots,i_l})\, dt.$$

Note that the integral is well defined since $\varphi_t(V) \subset V$ for all $t \leq 0$ by equation (B.5). Observe that $\varphi_{t*}Y_h = Y_h$ by construction of φ_t and that the fibers of the projection π_h are left invariant by φ_t. Hence one concludes by Cartan's magic formula

$$\begin{aligned} &(S i_{Y_h} + i_{Y_h} S)\omega \\ &= \sum_{l=0}^{k} \sum_{1\leq i_1<\cdots<i_l\leq n} dv^{i_1} \wedge \cdots \wedge dv^{i_l} \wedge \int_{-\infty}^{0} (d_W i_{Y_h} + i_{Y_h} d_W)\varphi_t^*\omega_{i_1,\ldots,i_l}\, dt \\ &= \sum_{l=0}^{k} \sum_{1\leq i_1<\cdots<i_l\leq n} dv^{i_1} \wedge \cdots \wedge dv^{i_l} \wedge \int_{-\infty}^{0} \mathcal{L}_{Y_h}\varphi_t^*\omega_{i_1,\ldots,i_l}\, dt \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{k} \sum_{1 \le i_1 < \cdots < i_l \le n} dv^{i_1} \wedge \cdots \wedge dv^{i_l} \wedge \int_{-\infty}^{0} \frac{d}{dt} \varphi_t^* \omega_{i_1,\ldots,i_l}\, dt \\
&= \sum_{l=0}^{k-1} \sum_{1 \le i_1 < \cdots < i_l \le n} dv^{i_1} \wedge \cdots \wedge dv^{i_l} \wedge \omega_{i_1,\ldots,i_l} \\
&\quad + \sum_{1 \le i_1 < \cdots < i_k \le n} dv^{i_1} \wedge \cdots \wedge dv^{i_k} \wedge (\omega_{i_1,\ldots,i_k} - \pi_h^* \iota_h^* \omega_{i_1,\ldots,i_k}) \\
&= \omega - \pi_h^* \iota_h^* \omega. \qquad \text{(B.6)}
\end{aligned}
$$

To verify the second last equality observe that the $\omega_{i_1,\ldots,i_k}$ are smooth functions which satisfy

$$\lim_{t \to -\infty} \varphi_t^* \omega_{i_1,\ldots,i_k} = \pi_h^* \iota_h^* \omega_{i_1,\ldots,i_k}.$$

Thus equation (B.6) proves the claim. ∎

The proposition entails the following twisted version of the Connes–Hochschild–Kostant–Rosenberg theorem:

Theorem B.7. *Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be an orthogonal linear transformation and $V \subset \mathbb{R}^d$ an open ball around the origin. Then the Hochschild homology $H_\bullet(\mathcal{C}^\infty(V), {}^h\mathcal{C}^\infty(V))$ is naturally isomorphic to $\Omega^\bullet(V^h)$, where V^h is the fixed point manifold of h in V. A quasi-isomorphism inducing this identification is given by*

$$
\begin{aligned}
&C_k\big(\mathcal{C}^\infty(V), {}^h\mathcal{C}^\infty(V)\big) \to \Omega^k(V^h), \\
&\qquad f_0 \otimes f_1 \otimes \cdots \otimes f_k \mapsto f_{0|V^h}\, d f_{1|V^h} \wedge \cdots \wedge d f_{k|V^h}.
\end{aligned}
$$

Proof. As explained above, the homology of the chain complex (B.3) coincides naturally with the Hochschild homology $H_\bullet(\mathcal{C}^\infty(V), {}^h\mathcal{C}^\infty(V))$. By commutativity of diagram (B.4) and by Proposition B.6, the chain complex (B.3) is quasi-isomorphic to the chain complex

$$\Omega^d(V^h) \xrightarrow{0} \cdots \xrightarrow{0} \Omega^1(V^h) \xrightarrow{0} \mathcal{C}^\infty(V^h) \to 0,$$

hence $H_k(\mathcal{C}^\infty(V), {}^h\mathcal{C}^\infty(V))$ can be identified with $\Omega^k(V^h)$ for all k. The explicit form of the quasi-isomorphism is obtained by composing the quasi-isomorphism ι_h^* with the quasi-isomorphism between the Hochschild chain complex of the twisted module ${}^h\mathcal{C}^\infty(V)$ and the chain complex (B.3). ∎

Next we consider a finite subgroup Γ of the orthogonal group $\mathrm{O}(\mathbb{R}^d)$. Let $V \subset \mathbb{R}^d$ be an open ball around the origin that is invariant with respect to the Γ action on $\mathbb{R}^d$. We can apply the quasi-isomorphism

$$\widehat{(-)} : C_\bullet\big(\mathcal{C}^\infty(V) \rtimes \Gamma\big) \to C_\bullet^\Gamma\big(\mathcal{C}^\infty(V)\big) \cong C_\bullet\big(\mathcal{C}^\infty(V), \mathcal{C}^\infty(V) \rtimes \Gamma\big)^\Gamma$$

from Section 6.2 to identify $HH_\bullet(\mathcal{C}^\infty(V) \rtimes \Gamma)$ with the homology of the complex $C_\bullet^\Gamma(\mathcal{C}^\infty(V))$. Since Γ is a finite group, the crossed product algebra $\mathcal{C}^\infty(V) \rtimes \Gamma$ can be identified with the direct sum $\bigoplus_{\gamma \in \Gamma} \mathcal{C}^\infty(V)$ endowed with the convolution product and the twisted $\mathcal{C}^\infty(V)$-bimodule structure. Hence the homology of

$$C^\Gamma_\bullet\big(\mathcal{C}^\infty(V)\big) \cong C_\bullet\big(\mathcal{C}^\infty(V), \mathcal{C}^\infty(V) \rtimes \Gamma\big)^\Gamma$$

is computed as the invariant part of the direct sum $\bigoplus_{\gamma\in\Gamma} H_\bullet(\mathcal{C}^\infty(V), {}^\gamma\mathcal{C}^\infty(V))$. As a corollary to Theorem B.7, we thus obtain the following computation of the Hochschild homology of $\mathcal{C}^\infty(V) \rtimes \Gamma$:

Corollary B.8. *The Hochschild homology $HH_\bullet(\mathcal{C}^\infty(V) \rtimes \Gamma)$ is naturally isomorphic to*

$$\Big(\bigoplus_{\gamma\in\Gamma} \Omega^\bullet(V^\gamma)\Big)^\Gamma = \bigoplus_{\langle\gamma\rangle\in\mathrm{Conj}(\Gamma)} \Omega^\bullet(V^\gamma)^{\mathrm{Z}(\gamma)},$$

where Γ acts on the disjoint union $\coprod_{\gamma\in\Gamma} V^\gamma$ by $\gamma'(\gamma, x) = (\gamma'\gamma(\gamma')^{-1}, \gamma' x)$, $\mathrm{Conj}(\gamma)$ *denotes the set of conjugacy classes of Γ, and $\mathrm{Z}(\gamma)$ is the centralizer of an element $\gamma \in \Gamma$.*

Remark B.9. In the case of a smooth affine algebraic variety, Corollary B.8 has been proved in [6, Theorem 2.19]. The cyclic homology theory of finite group actions on manifolds has been considered in other work as well. Early ideas can be traced back to the work by Burghelea [10], Feigin–Tsygan [21], Baum–Connes [1], and also Wassermann [49]. As already mentioned, Brylinski–Nistor [9] provided the first full proof of a twisted Connes–Hochschild–Kostant–Rosenberg theorem for finite group actions. Getzler–Jones constructed in [22] an isomorphism on the level of cyclic homology theory for crossed product algebras of discrete group actions. In [35], Nistor examines the localization of periodic cyclic homology of crossed products by algebraic groups at maximal ideals of the algebra of class functions on the group. As already pointed out before, Block–Getzler introduced in [3] a Cartan model for the cyclic homology of the crossed product algebra of a Lie group action on a manifold and derived from it a quasi-isomorphism which they call equivariant Hochschild–Kostant–Rosenberg map. More recently, Ponge [42, 43] constructed a quasi-isomorphism of "twisted" mixed complexes from which the above can be derived as well.

Let us end with a generalization of Proposition B.6 which will serve as a useful tool in our computations. Observe that in the complex (B.3) the vector field Y_h can be extended to be a more general linear vector field $Y_H : \mathbb{R}^n \to T\mathbb{R}^d$ of the form $Y_H(v) = H(v) \in T_v\mathbb{R}^d$ where $H : \mathbb{R}^d \to \mathbb{R}^d$ is a diagonalizable linear map. A construction similar to the homotopy operator S in the proof of Proposition B.6 (see also [49]) computes the homology of $(\Omega^\bullet(V), i_{Y_H})$ to be $(\Omega^\bullet(V^H), 0)$ where $V^H = \ker(H)$. Furthermore, if $H : S \to \mathrm{End}(\mathbb{R}^d)$ is a smooth family of diagonalizable linear operators parametrized by a smooth manifold S, H is called regular if H satisfies the following properties:

(1) the kernel $\ker(H) := \{\ker(H(s))\}_{s\in S} \subset S \times \mathbb{R}^d$ is a smooth subbundle of the trivial vector bundle $S \times \mathbb{R}^d$;

(2) near every $s_0 \in S$, there is a local frame of $S \times \mathbb{R}^d$ on a neighborhood U_{s_0} of s_0 in S consisting of $\xi_1, \dots, \xi_d$ such that

- the collection $\{\xi_1, \dots, \xi_k\}$ is a local frame of the subbundle $\ker(H)$ on U_{s_0},

- for every $j = k + 1, \dots, d$, there is a smooth eigenfunction $\lambda_j(s)$ defined on U_{s_0} satisfying $H(s)\xi_j(s) = \lambda_j(s)\xi_j(s)$ and $\lambda_j(s) \neq 0$ for all $s \in U_{s_0}$.

The proof of Proposition B.6 generalizes to the following result.

Proposition B.10. *Let $H : S \to \mathrm{End}(\mathbb{R}^d)$ be a smooth family of diagonalizable linear operators parametrized by a smooth manifold S. Assume that H is regular. Let*

$$i_{\ker(H)} : \ker(H) \to S \times \mathbb{R}^d$$

be the canonical embedding, and $\Omega^\bullet(\ker(H))$ the restriction of $\mathcal{C}^\infty(S, \Omega^\bullet(V))$ to $\ker(H)$ along $i_{\ker(H)}$. Then the restriction map $R_{\ker(H)} : (\mathcal{C}^\infty(S, \Omega^\bullet(V)), i_{Y_H}) \to (\Omega^\bullet(\ker(H)), 0)$ is a quasi-isomorphism.

In a certain sense, the following proposition is a variant of the preceding one. To formulate our final result recall that by a Euler-like vector field for an embedded smooth manifold $S \hookrightarrow M$, one understands a vector field $Y : M \to TM$ such that S is the zero set of Y and such that for each $f \in \mathcal{C}^\infty(M)$ vanishing on S the function $Yf - f$ vanishes to second order on S; cf. [24, Definition 1.1].

Proposition B.11. *Let M be a smooth manifold of dimension d, $S \hookrightarrow M$ an embedded submanifold, and $Y : M \to TM$ a smooth vector field which is Euler-like with respect to S. Then the complex*

$$\Omega^d(M) \xrightarrow{i_Y} \cdots \xrightarrow{i_Y} \Omega^1(M) \xrightarrow{i_Y} \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(S) \to 0 \tag{B.7}$$

is exact and called the parametrized Koszul resolution *of $\mathcal{C}^\infty(S)$.*

Proof. The claim is an immediate consequence of the Koszul resolution as for example stated in [49, Section V]. ∎

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References

[1] P. Baum and A. Connes, Chern character for discrete groups. In *A fête of topology*, pp. 163–232, Academic Press, Boston, MA, 1988 Zbl 0656.55005 MR 928402

[2] P. Blanc, Cohomologie différentiable et changement de groupes. *Astérisque* **124–125** (1985), 113–130 Zbl 0563.57023

[3] J. Block and E. Getzler, Equivariant cyclic homology and equivariant differential forms. *Ann. Sci. École Norm. Sup. (4)* **27** (1994), no. 4, 493–527 Zbl 0849.55008 MR 1290397

[4] J.-P. Brasselet and M. J. Pflaum, On the homology of algebras of Whitney functions over subanalytic sets. *Ann. of Math. (2)* **167** (2008), no. 1, 1–52 Zbl 1153.32010 MR 2373151

[5] G. E. Bredon, *Introduction to compact transformation groups*. Pure Appl. Math. 46, Academic Press, New York, 1972 Zbl 0246.57017 MR 0413144

[6] J. Brodzki, S. Dave, and V. Nistor, The periodic cyclic homology of crossed products of finite type algebras. *Adv. Math.* **306** (2017), 494–523 Zbl 1405.16010 MR 3581309

[7] J.-L. Brylinski, Algebras associated with group actions and their homology. 1987, Brown University preprint

[8] J.-L. Brylinski, Cyclic homology and equivariant theories. *Ann. Inst. Fourier (Grenoble)* **37** (1987), no. 4, 15–28 Zbl 0625.55003 MR 927388

[9] J.-L. Brylinski and V. Nistor, Cyclic cohomology of étale groupoids. *K-Theory* **8** (1994), no. 4, 341–365 Zbl 0812.19003 MR 1300545

[10] D. Burghelea, The cyclic homology of the group rings. *Comment. Math. Helv.* **60** (1985), no. 3, 354–365 Zbl 0595.16022 MR 814144

[11] A. Connes, Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.* (1985), no. 62, 257–360 MR 823176

[12] A. Connes, *Noncommutative geometry*. Academic Press, San Diego, CA, 1994 Zbl 0818.46076 MR 1303779

[13] M. Crainic, Cyclic cohomology of étale groupoids: the general case. *K-Theory* **17** (1999), no. 4, 319–362 Zbl 0937.19007 MR 1706117

[14] M. Crainic and I. Moerdijk, Foliation groupoids and their cyclic homology. *Adv. Math.* **157** (2001), no. 2, 177–197 Zbl 0989.22010 MR 1813430

[15] M. Crainic and I. Struchiner, On the linearization theorem for proper Lie groupoids. *Ann. Sci. Éc. Norm. Supér. (4)* **46** (2013), no. 5, 723–746 Zbl 1286.53080 MR 3185351

[16] M. del Hoyo and R. L. Fernandes, Riemannian metrics on Lie groupoids. *J. Reine Angew. Math.* **735** (2018), 143–173 Zbl 1385.53076 MR 3757473

[17] L. D. Drager, J. M. Lee, E. Park, and K. Richardson, Smooth distributions are finitely generated. *Ann. Global Anal. Geom.* **41** (2012), no. 3, 357–369 Zbl 1243.57018 MR 2886203

[18] C. Farsi, M. J. Pflaum, and C. Seaton, Stratifications of inertia spaces of compact Lie group actions. *J. Singul.* **13** (2015), 107–140 Zbl 1364.58002 MR 3343617

[19] C. Farsi, M. J. Pflaum, and C. Seaton, Differentiable stratified groupoids and a de Rham theorem for inertia spaces. 2016, arXiv:1511.00371

[20] B. L. Feĭgin and B. L. Tsygan, Additive K-theory. In *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, pp. 67–209, Lecture Notes in Math. 1289, Springer, Berlin, 1987 Zbl 0635.18008 MR 923136

[21] B. L. Feĭgin and B. L. Tsygan, Cyclic homology of algebras with quadratic relations, universal enveloping algebras and group algebras. In *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, pp. 210–239, Lecture Notes in Math. 1289, Springer, Berlin, 1987 Zbl 0635.18009 MR 923137

[22] E. Getzler and J. D. S. Jones, The cyclic homology of crossed product algebras. *J. Reine Angew. Math.* **445** (1993), 161–174 Zbl 0795.46052 MR 1244971

[23] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.* **16** (1955), 140 pp. Zbl 0064.35501 MR 75539

[24] A. R. Haj Saeedi Sadegh and N. Higson, Euler-like vector fields, deformation spaces and manifolds with filtered structure. *Doc. Math.* **23** (2018), 293–325 Zbl 1393.57018 MR 3846057

[25] A. Y. Helemskii, *The homology of Banach and topological algebras. Translated from the Russian by Alan West*. Math. Appl. (Soviet Ser.) 41, Kluwer Academic Publishers Group, Dordrecht, 1989 MR 1093462

[26] G. Hochschild, B. Kostant, and A. Rosenberg, Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.* **102** (1962), 383–408 Zbl 0102.27701 MR 142598

[27] H. Hogbe-Nlend, *Bornologies and functional analysis. Introductory course on the theory of duality topology-bornology and its use in functional analysis. Translated from the french by V. B. Moscatelli*. North-Holland Math. Stud. 26, North-Holland Publishing, Amsterdam, 1977 Zbl 0359.46004 MR 0500064

[28] M. Jotz, T. S. Ratiu, and J. Śniatycki, Singular reduction of Dirac structures. *Trans. Amer. Math. Soc.* **363** (2011), no. 6, 2967–3013 Zbl 1385.70039 MR 2775795

[29] A. Kriegl and P. W. Michor, *The convenient setting of global analysis*. Math. Surveys Monogr. 53, American Mathematical Society, Providence, RI, 1997 Zbl 0889.58001 MR 1471480

[30] J.-L. Loday, *Cyclic homology*. 2nd edn., Grundlehren Math. Wiss. 301, Springer, Berlin, 1998 Zbl 0885.18007 MR 1600246

[31] J. N. Mather, Stratifications and mappings. In *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pp. 195–232, Academic Press, New York, 1973 Zbl 0286.58003 MR 0368064

[32] R. Meyer, *Analytic cyclic homology*. Ph.D. thesis, Universität Münster, 1999, arXiv:math/9906205

[33] R. Meyer, *Local and analytic cyclic homology*. EMS Tracts Math. 3, EMS Press, Zürich, 2007 Zbl 1134.46001 MR 2337277

[34] J. A. Navarro González and J. B. Sancho de Salas, C^∞*-differentiable spaces*. Lecture Notes in Math. 1824, Springer, Berlin, 2003 Zbl 1039.58001 MR 2030583

[35] V. Nistor, Cyclic cohomology of crossed products by algebraic groups. *Invent. Math.* **112** (1993), no. 3, 615–638 Zbl 0799.46078 MR 1218326

[36] M. J. Pflaum, On continuous Hochschild homology and cohomology groups. *Lett. Math. Phys.* **44** (1998), no. 1, 43–51 Zbl 0943.46043 MR 1623762

[37] M. J. Pflaum, *Analytic and geometric study of stratified spaces*. Lecture Notes in Math. 1768, Springer, Berlin, 2001 Zbl 0988.58003 MR 1869601

[38] M. J. Pflaum, H. Posthuma, and X. Tang, Cyclic cocycles on deformation quantizations and higher index theorems. *Adv. Math.* **223** (2010), no. 6, 1958–2021 Zbl 1190.53087 MR 2601006

[39] M. J. Pflaum, H. Posthuma, and X. Tang, Geometry of orbit spaces of proper Lie groupoids. *J. Reine Angew. Math.* **694** (2014), 49–84 Zbl 1297.53022 MR 3259039

[40] M. J. Pflaum, H. Posthuma, and X. Tang, The localized longitudinal index theorem for Lie groupoids and the van Est map. *Adv. Math.* **270** (2015), 223–262 Zbl 1408.58016 MR 3286536

[41] M. J. Pflaum, H. Posthuma, and X. Tang, The Grauert–Grothendieck complex on differentiable spaces and a sheaf complex of Brylinski. *Methods Appl. Anal.* **24** (2017), no. 2, 321–332 Zbl 1380.32012 MR 3746621

[42] R. Ponge, The cyclic homology of crossed-product algebras, II. *C. R. Math. Acad. Sci. Paris* **355** (2017), no. 6, 623–627 Zbl 1380.46055 MR 3661544

[43] R. Ponge, Cyclic homology and group actions. *J. Geom. Phys.* **123** (2018), 30–52 Zbl 1468.19003 MR 3724774

[44] K. Spallek, Differenzierbare Räume. *Math. Ann.* **180** (1969), 269–296 Zbl 0169.52901 MR 261035

[45] E. H. Spanier, *Algebraic topology. Corrected reprint of the 1966 original*. Springer, New York, 1995 Zbl 0810.55001 MR 1325242

[46] P. Stefan, Integrability of systems of vector fields. *J. London Math. Soc. (2)* **21** (1980), no. 3, 544–556 Zbl 0432.58002 MR 577729

[47] H. J. Sussmann, Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.* **180** (1973), 171–188 Zbl 0274.58002 MR 321133

[48] F. Trèves, *Topological vector spaces, distributions and kernels*. Academic Press, New York, 1967 Zbl 0171.10402 MR 0225131

[49] A. Wassermann, Cyclic cohomology of algebras of smooth functions on orbifolds. In *Operator algebras and applications, Vol. 1*, pp. 229–244, London Math. Soc. Lecture Note Ser. 135, Cambridge Univ. Press, Cambridge, 1988 Zbl 0693.55015 MR 996447

[50] C. Weibel, Cyclic homology for schemes. *Proc. Amer. Math. Soc.* **124** (1996), no. 6, 1655–1662 Zbl 0855.19002 MR 1277141

[51] A. Weinstein, Linearization of regular proper groupoids. *J. Inst. Math. Jussieu* **1** (2002), no. 3, 493–511 Zbl 1043.58009 MR 1956059

[52] M. Wodzicki, Excision in cyclic homology and in rational algebraic K-theory. *Ann. of Math. (2)* **129** (1989), no. 3, 591–639 Zbl 0689.16013 MR 997314

[53] N. T. Zung, Proper groupoids and momentum maps: Linearization, affinity, and convexity. *Ann. Sci. École Norm. Sup. (4)* **39** (2006), no. 5, 841–869 Zbl 1163.22001 MR 2292634

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