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Problems of Chebyshev quadrature on sphere and circle

Discussion paper

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Abstract: The principal problem is to find optimal or nearly optimal $N$-tuples of nodes for Chebyshev quadrature on the unit sphere when $N$ is large. One tries to obtain such nodes as solutions to suitable extremal problems. In earlier work the author has described a complex-analytic method which would show that the extremal $N$-tuples provide good Chebyshev nodes. These are nodes such that, for constants $A, b, c > 0$, the quadrature remainder is $\leq Ae^{-b\sqrt{N}}$ for the polynomials of degree $\leq c\sqrt{N}$ and sup norm 1. However, in order to apply this method it is necessary to establish uniform distribution and good separation of the extremal points. In this note uniform distribution is proved with the aid of potential theory. It is plausible that one also has adequate separation. Indeed, there is such separation in the corresponding extremal problems for the unit circle which are considered at length.

Key words: Chebyshev quadrature, extremal problems for $N$-tuples, potential energy functions, separation of extremal points, spherical designs, uniform distribution.

1. Conjectures for the sphere.

Background information on the broader problem of finding good distributions of many points on a sphere may be found in Saff and Kuijlaars [6]. For Chebyshev quadrature on the unit sphere $S \subset \mathbb{R}^3$, one would like to find $N$-tuples

$$X_N = (x_1, \ldots, x_N) = (x_{N1}, \ldots, x_{NN}) \quad \text{of points on } S$$

(with $N \to \infty$) such that the quadrature remainder

$$R(f, X_N) = \int_S f(x) \frac{d\lambda(x)}{4\pi} - \frac{1}{N} \sum_{j=1}^N f(x_j)$$

(1.2)

(where $\lambda$ denotes Lebesgue area measure) is very small for all polynomials $f(x)$ up to relatively high degree, depending on $N$. We begin with the strongest conjecture.

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Conjecture 1.1. There exists a constant $c > 0$ such that for every $N$, there are $N$-tuples $Y_N \subset S$ with the property that $R(f, Y_N) = 0$ for all polynomials of degree $\leq c\sqrt{N}$.

Apart from the constant, the bound $c\sqrt{N}$ would be optimal, but the best known asymptotic bound is of the form $c^3\sqrt{N}$, cf. Korevaar and Meyers [3], Korevaar [4]. A related conjecture would be that there exist so-called spherical $p$-designs consisting of $N = \mathcal{O}(p^2)$ points. These are configurations of $N$ distinct points $x_j \in S$ for which formula (1.2) is polynomially exact to degree $p$.

It may be less difficult to prove

Conjecture 1.2. There exist positive constants $A, b, c$ such that for every $N$, there are $N$-tuples $Y_N \subset S$ with the property that

$$|R(f, Y_N)| \leq A e^{-b\sqrt{N}} \max |f|, \quad N = 1, 2, \ldots$$

(1.3)

for all polynomials $f(x) = f(x_1, x_2, x_3)$ of degree $\leq c\sqrt{N}$.

The difficulty is to find likely candidates for “good” $N$-tuples. In order to make methods of potential theory and complex analysis applicable, one may maximize or minimize suitable expressions involving $N$-tuples $X_N \subset S$. In this paper, we consider minimizing $N$-tuples $Y_N$ for one of the “potential-energy functions”

$$\Phi = \Phi(X_N, r) = \frac{1}{N^2} \sum_{j,k=1}^{N} \frac{1}{|x_j - r x_k|} = \frac{1}{N^2} \sum_{j,k=1}^{N} \frac{1}{\{(1 - r)^2 + r| x_j - x_k |^2 \}^{\frac{1}{2}}},$$

(1.4)

$$\Psi = \Psi(X_N, r) = \frac{1}{N^2} \sum_{j,k=1}^{N} \log \frac{1}{|x_j - r x_k|}, \quad r \in (0, 1).$$

The precise value of $r$ is not important. If we take $(1 - r)^2 = r$ the second case requires the maximization of

$$\prod_{j,k=1}^{N} (1 + r^2 - 2r x_j \cdot x_k) = \prod_{j,k=1}^{N} (1 + |x_j - x_k|^2), \quad X_N \subset S.$$

(1.5)

We will use potential theory to show that the points in extremizing $N$-tuples $Y_N$ are asymptotically equidistributed on $S$. That is, for every spherical cap $S_0 \subset S$, the number of points of $Y_N$ on $S_0$ is asymptotic to $\{\lambda(S_0)/4\pi\}N$ as $N \to \infty$, see Section 3. If we could
also prove good separation of the points in extremizing \(N\)-tuples we would be in business, see Theorem 1.6.

For the application of complex analysis it is important to have \(|x_j - x_k|^2\) in (1.4), (1.5) and not \(|x_j - x_k|\). The potential energies in (1.4) correspond to repulsive forces which go to 0 with the distance between particles. This makes it difficult to prove good separation of extremal points. If one had \(1 + |x_j - x_k|\) in (1.5), good separation would follow, cf. the related results in Dahlberg [1], Stolarsky [8] and see Section 8 for the case of the circle.

We consider the following conjectures, cf. Korevaar and Meyers [2], Korevaar [4] where related conjectures were stated.

**Conjecture 1.3.** The points in the extremizing \(N\)-tuples \(Y_N\) are well-distributed in the following sense. There are constants \(B\) such that every spherical cap \(S_0 = S_{N,0} \subset S\) of area \(B/N\) contains at least one point of \(Y_N\) when \(N \geq N_0(B)\).

A corresponding conjecture for the unit circle is proved in Section 7.

**Conjecture 1.4.** The points \(y_j = y_{Nj}\) in the extremizing \(N\)-tuples \(Y_N\) are well-separated, that is, there exists \(\delta > 0\) independent of \(N\) such that

\[
\min_{j \neq k} |y_{Nj} - y_{Nk}| \geq \frac{\delta}{\sqrt{N}}, \quad \forall N. \tag{1.6}
\]

For our purpose a weaker conjecture would suffice:

**Conjecture 1.5.** There is a constant \(\alpha > 0\) as follows. The extremizing \(N\)-tuples \(Y_N\) contain subsets \(Z_N\) such that each closed circular cone with vertex \(O\) and opening \(\pi\) (say) contains at least \(\alpha N\) points of \(Z_N\) which are well-separated.

**Theorem 1.6.** If the assertion on the extremizing \(N\)-tuples \(Y_N\) in one of the Conjectures 1.3 - 1.5 is correct, then so is the assertion on the quadrature remainder \(R(f, Y_N)\) in Conjecture 1.2.

A result of this type was proved by complex analysis in Korevaar [4]. To show the flavor of the proof we discuss a simpler analog for the circle in Section 10.
2. The equilibrium measure for our potentials.

We begin with the ‘classical’ potential energy for the potential

\[
\frac{1}{|x - ry|}, \quad r \in (0, 1)
\]

which corresponds to the function \( \Phi \) in (1.4).

**Theorem 2.1.** For probability measures \( \mu \) on \( S \) the potential energy

\[
I(\mu) = \int_{S \times S} \frac{1}{|x - ry|} \, d\mu(x) d\mu(y) \tag{2.2}
\]

is minimal if and only if

\[
\mu = \sigma = \frac{1}{4\pi} \lambda, \tag{2.3}
\]

the normalized area measure on \( S \).

**Proof.** On the sphere the potential (2.1) has a nice expansion in terms of Legendre polynomials:

\[
\frac{1}{|x - ry|} = \frac{1}{\sqrt{1 + r^2 - 2rx \cdot y}} = \sum_{m=0}^{\infty} P_m(x \cdot y) r^m. \tag{2.4}
\]

Note that \( L^2(S) \) is the direct sum of the pairwise orthogonal subspaces \( H_m \) of the spherical harmonics of order \( m \), cf. Stein and Weiss [7]. We will use the important representation

\[
Y_m(x) = Y_m(f, x) = (2m + 1) \int_S f(z) P_m(x \cdot z) d\sigma(z) \tag{2.5}
\]

for the orthogonal projection of \( f \in L^2(S) \) onto \( H_m \). Applying (2.4) and (2.5) to the spherical harmonic \( f(x) = P_m(x \cdot y) \), we find that for every probability measure \( \mu \) on \( S \)

\[
I(\mu) = \sum_{m=0}^{\infty} r^m \int_{S \times S} P_m(x \cdot y) d\mu(x) d\mu(y)
\]

\[
= \sum_{m=0}^{\infty} r^m \int_{S \times S} \left( (2m + 1) \int_S P_m(x \cdot z) P_m(z \cdot y) d\sigma(z) \right) d\mu(x) d\mu(y) \tag{2.6}
\]

\[
= 1 + \sum_{m=1}^{\infty} (2m + 1) \int_{S} \left( \int_{S} P_m(x \cdot z) d\mu(x) \right)^2 d\sigma(z) \geq 1.
\]

There is equality in (2.6) if and only

\[
\int_S P_m(x \cdot z) d\mu(x) = 0, \quad \forall m \geq 1 \text{ and } \forall z \in S. \tag{2.7}
\]

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Now for fixed $m$ the functions $P_m(x \cdot z)$, $z \in S$ span the $(2m + 1)$-dimensional linear space $H_m$, cf. (2.5). They are in particular orthogonal to $P_0(x \cdot z) \equiv 1$ relative to the measure $\sigma$, hence (2.7) holds for $\mu = \sigma$, so that $I(\sigma) = 1$. Thus for minimal $I(\mu)$ (2.7) implies that $\mu - \sigma$ is orthogonal to all spherical harmonics, including the constants. It follows that $\sigma$ is the unique minimizing probability measure (often denoted by $\omega$) for $I(\mu)$.

**Remark 2.2.** Does the preceding argument work for other potentials? Analysis shows that the only essential fact was that in (2.4) we had an expansion in terms of Legendre polynomials with *strictly positive* coefficients for $m \geq 1$. But such an expansion exists also for the potential $\log(1/|x - ry|)$ which corresponds to $\Psi$ in (1.4). Indeed, one has

$$- \frac{1}{2} \log(1 + r^2 - 2rx \cdot y) = - \frac{1}{2} \log(1 + r^2) - \frac{1}{2} \log(1 - \rho x \cdot y), \quad \rho = \frac{2r}{1 + r^2},$$

$$- \log(1 - \rho w) = \sum_{m=1}^{\infty} \frac{1}{m} \rho^m w^m,$$

and finally

$$w^m = c_{m,0}P_m(w) + c_{m,2}P_{m-2}(w) + \ldots + c_{m,2l}P_{m-2l}(w), \quad l = [m/2]$$

with positive coefficients $c_{m,2k}$, cf. Szegő [9], p. 387, exercise 63. Thus

$$\log \frac{1}{|x - ry|} = - \frac{1}{2} \log(1 + r^2) + A_0(\rho) + \sum_{m=1}^{\infty} A_m(\rho) P_m(x \cdot y), \quad \text{with } A_m(\rho) > 0.$$ 

The corresponding energy $I(\mu)$ is minimal if and only if $\mu = \sigma$ and

$$I(\sigma) = - \frac{1}{2} \log(1 + r^2) + A_0(\rho) = - \sum_{m=1}^{\infty} \frac{r^{2m}}{(2m - 1)2m(2m + 1)}.$$
3. Equidistribution of extremal points.

We next consider discrete probability measures \( \mu_N = \mu_{X_N} \) given by point masses or charges \( 1/N \) at the points \( x_j = x_{Nj} \) of an \( N \)-tuple \( X_N \subset S \). For the potential (2.1), naturally

\[
I(\mu_N) = \frac{1}{N^2} \sum_{j,k=1}^{N} \frac{1}{|x_j - rx_k|} > I(\sigma) = 1. \tag{3.1}
\]

This inequality will in particular hold for the \( N \)-th order “Fekete measures” \( \mu_N = \omega_N \) which minimize \( I(\mu_N) \) for fixed \( N \).

**Theorem 3.1.** For each \( N \), choose one Fekete measure \( \omega_N \) and denote its carrying \( N \)-tuple by \( Z_N \). Then the energies

\[
I(\omega_N) \text{ converge to } I(\sigma) = 1 \text{ as } N \to \infty. \tag{3.2}
\]

For \( N \to \infty \), the points \( \zeta_{N1}, \ldots, \zeta_{NN} \) of \( Z_N \) are asymptotically equidistributed on \( S \).

**Proof.** Choose any sequence of \( N \)-tuples \( X_N \) whose points \( x_{Nj} \) are asymptotically equidistributed on \( S \). Dividing \( S \) into pieces of equal area and small diameter, one finds that for any continuous function \( f(x) \) on \( S \),

\[
\frac{1}{N} \sum_{j=1}^{N} f(x_{Nj}) \to \int_S f(x) d\sigma(x) \quad \text{as } N \to \infty.
\]

One can apply the same argument to functions \( f(x, y) \) on \( S \times S \), hence

\[
I(\mu_N) = \frac{1}{N^2} \sum_{j,k=1}^{N} \frac{1}{|x_j - rx_k|} \to \int_{S \times S} \frac{1}{|x - ry|} d\sigma(x) d\sigma(y) = I(\sigma).
\]

Thus (3.2) follows from the inequalities \( I(\mu_N) \geq I(\omega_N) > I(\sigma) \).

We now write

\[
s_m^2 = s_{Nm}^2 = \int_S \left( \int_S P_m(x \cdot z) \, d\omega_N(x) \right)^2 d\sigma(z). \tag{3.3}
\]

Then by (2.6) and (3.2)

\[
I(\omega_N) - I(\sigma) = \sum_{m=1}^{\infty} (2m + 1) r^m s_m^2 \to 0 \quad \text{as } N \to \infty. \tag{3.4}
\]
We will show that (3.4) implies weak*-convergence $\omega_N \to \sigma$, that is,

$$\int_S f \, d\omega_N \to \int_S f \, d\sigma$$

(3.5)

for all continuous functions $f$ on $S$. For this it is sufficient to verify (3.5) for finite linear combinations of spherical harmonics,

$$f(x) = \sum_{m=0}^{M} Y_m(f, x),$$

(3.6)

since such functions lie dense in $C(S)$. In the latter case, cf. (1.2) and (2.5),

$$R(f, Z_N) = \int_S f(d\sigma - d\omega_N) = -\sum_{m=1}^{M} \int_S Y_m(x) d\omega_N(x)$$

$$= -\sum_{m=1}^{M} \int_S \left( (2m + 1) \int_S Y_m(z) P_m(x \cdot z) \, d\omega(z) \right) d\omega_N(x)$$

$$= -\int_S \sum_{m=1}^{M} \left( \sqrt{(2m + 1)r^{-m}Y_m(z)} \right) \cdot$$

$$\cdot \left( \sqrt{(2m + 1)r^m} \int_S P_m(x \cdot z) \, d\omega_N(x) \right) \, d\omega(z).$$

(3.7)

To estimate $|R(f, Z_N)|$ we apply Cauchy-Schwarz to the final inner product which has the form $(U, V) = \int_S \sum_{m=1}^{M} U_m V_m$. By (3.3) and (3.4) the result is

$$|R(f, Z_N)|^2 \leq \int_S \sum_{m=1}^{M} (2m + 1)r^{-m}|Y_m(z)|^2 \, d\omega(z) \cdot$$

$$\cdot \int_S \sum_{m=1}^{M} (2m + 1)r^m \left( \int_S P_m(x \cdot z) \, d\omega_N(x) \right)^2 \, d\sigma(z)$$

$$\leq (2M + 1)r^{-M} \sum_{m=1}^{M} \int_S |Y_m(z)|^2 \, d\sigma(z) \cdot \sum_{m=1}^{\infty} (2m + 1)r^m s_m^2$$

$$= (2M + 1)r^{-M} \int_S |f|^2 \, d\sigma \cdot \{I(\omega_N) - I(\sigma)\}.$$  

By (3.2) the final expression tends to 0 as $N \to \infty$, hence in view of (3.7), $\omega_N \to \sigma$ in weak* sense.

One may finally approximate the characteristic function $g$ of a spherical cap $S_0$ from above and below by continuous functions to conclude that (3.5) holds also for $f = g$. In other words, $\omega_N(S_0) \to \sigma(S_0)$. Thus the points $\zeta_j = \zeta_{Nj}$ in the $N$-tuples $Z_N$ are asymptotically equidistributed on the sphere.
4. Corresponding conjectures for the circle.

A major part of this note is devoted to analogs of the conjectures of Section 1 for the case of the unit circle \( C(0,1) \) in the plane \( \mathbb{R}^2 \cong \mathbb{C} \). Let \( X_N = (x_1, x_2, \ldots, x_N) \) and \( Y_N \) be \( N \)-tuples of points on the circle. It is convenient to set

\[
x_j = e^{it_j}, \quad t_1 \leq t_2 \leq \ldots \leq t_N < t_1 + 2\pi.
\]

(4.1)

**Conjecture 4.1.** Let the \( N \)-tuple \( Y_N = (y_1, \ldots, y_N) \) maximize

\[
\Pi = \Pi(X_N, r) = \prod_{j,k=1}^{N} \left\{ (1 - r)^2 + r|x_j - x_k|^2 \right\}
\]

\[
= \prod_{j,k=1}^{N} \{ 1 + r^2 - 2r \cos(t_j - t_k) \}, \quad x_j \in C(0,1),
\]

(4.2)

where \( r \in (0,1) \) is fixed. Then the points \( y_j = y_{NJ} \) are (i) well-distributed or even (ii) equidistant on the circle. In other words, for certain constants \( B \), every arc of the circle of length \( 2B/N \) contains at least one point of \( Y_N \) when \( N \) is large, or in case (ii), the points \( y_{NJ} \) are \( N \)-th roots of unity or rotated \( N \)-th roots of unity.

**Remark.** It is of course well-known that \( N \)-th roots of unity provide good (and in a sense, the best) nodes for Chebyshev quadrature on the circle, cf. Korevaar and Meyers [3]. We can prove that the maximizing \( N \)-tuples \( Y_N \) for (4.2) also are good nodes for Chebyshev quadrature, see Section 6. Unfortunately the proof makes use of the special properties of \( N \)-th roots of unity, hence it does not generalize to the case of the sphere. The quadrature result implies “good” distribution of the points \( y_{NJ} \) for any \( B > \pi \) (Section 7).

The second part of Conjecture 4.1 asserts maximal separation of the points in the maximizing \( N \)-tuples \( Y_N \).

**Discussion.** (i) One may think of

\[
\Psi = -\frac{1}{2N^2} \log \Pi = \frac{1}{N^2} \sum_{j,k=1}^{N} \log \frac{1}{\left\{ (1 - r)^2 + r|x_j - x_k|^2 \right\}^{\frac{1}{2}}}
\]

as the potential energy for a system of point charges \( 1/N \) at the points of \( X_N \). The generating potential associated with a unit charge at 0 would be

\[
\log \frac{1}{\left\{ (1 - r)^2 + r|x|^2 \right\}^{\frac{1}{2}}}.
\]

(4.3)
(Observe that it reduces to the classical logarithmic potential for the value \( r = 1 \) which we have excluded). Thus a unit charge at \( x \) would be subject to a force (minus the gradient of the potential) with components \( r x_1 / \{(1 - r)^2 + r|x|^2\} \), \( r x_2 / \{(1 - r)^2 + r|x|^2\} \). This is a repulsive force, roughly proportional to the distance when the distance is small. Compare this with the classical electrostatic force which is inversely proportional to the distance!

We have to minimize the potential energy (4.3).

(ii) The potential in (4.3) may be written as

\[
F(\cos t) = -\frac{1}{2} \log(1 + r^2 - 2r \cos t) \quad \text{with} \quad a_p = \frac{1}{p} r^p, \quad (4.5)
\]

compare the generating function for the Chebyshev polynomials (Szegő [9], Section 4.7).

To reformulate and extend the problem we consider “potentials” given by more general cosine series

\[
F(\cos t) = \sum_{p=1}^{\infty} a_p \cos pt \quad \text{with} \quad a_p > 0. \tag{4.6}
\]

Observe also that

\[
\sum_{j,k=1}^{N} \cos p(t_j - t_k) = \sum_{j,k} (\cos pt_j \cos pt_k + \sin pt_j \sin pt_k) = \left( \sum_{j} \cos pt_j \right)^2 + \left( \sum_{j} \sin pt_j \right)^2 = \left| \sum_{j=1}^{N} e^{ipt_j} \right|^2.
\]

The potential energy \( \Psi \) corresponding to (4.6) equals

\[
\Psi = \frac{1}{N^2} \sum_{j,k=1}^{N} F(\cos(t_j - t_k)) = \sum_{p=1}^{\infty} a_p \left( \frac{1}{N^2} \sum_{j,k=1}^{N} \cos p(t_j - t_k) \right) = \sum_{p=1}^{\infty} a_p |s_p|^2, \quad \text{where} \quad s_p = \frac{1}{N} \sum_{j=1}^{N} x_j^p = \frac{1}{N} \sum_{j=1}^{N} e^{ipt_j}. \tag{4.7}
\]

In particular Conjecture 4.1 is equivalent to

**Conjecture 4.2.** For \( N = 1, 2, \ldots \), let the \( N \)-tuple \( Y_N \) minimize

\[
\Psi = \sum_{p=1}^{\infty} a_p |s_p|^2 \quad \text{with} \quad a_p = \frac{1}{p} r^p, \quad r \in (0, 1). \tag{4.8}
\]
Then the points in the $N$-tuples are (i) well-distributed on the circle, or even (ii) equidistant.

Remarks. KujiLaars [5], see Section 5 below, has shown that the “potential energy” $\Psi$ of (4.8) attains a local minimum for $N$-th roots of unity. This supports part (ii) of the conjecture. A proof for part (i) follows from Section 7.

Equidistant points on the circle are characterized by the condition

$$s_1 = s_2 = \ldots = s_{N-1} = 0.$$  

Indeed, for $|z| < 1$ and principal values of the logarithms, (4.9) is equivalent to each of the following relations:

$$\sum_{j=1}^{N} \log \left( 1 - \frac{z}{x_j} \right) = -\sum_{j} \sum_{p=1}^{\infty} \frac{1}{p} \frac{z^p}{x_j^p} = -\sum_{p=1}^{\infty} \frac{\bar{s}_p}{p} z^p = O(z^N) \quad \text{as} \quad z \to 0,$n

$$\prod \left( 1 - \frac{z}{x_j} \right) = 1 + O(z^N),$$

$$P(z) = \prod (z - x_j) = c_0 z^N + c_1 z^{N-1} + \ldots + c_N = \prod (-x_j) + O(z^N),$$

$$c_1 = \ldots = c_{N-1} = 0; \quad P(z) = z^N + c_N \quad \text{with} \quad |c_N| = 1.$$  

Related conjectures. There are other conjectures for $N$-tuples $X_N = (x_1, x_2, \ldots, x_N)$ on the unit circle which also have a physical interpretation. Would it be easier to verify one of those? The identities in the conjectures may be derived from Parseval’s theorem.

**Conjecture 4.3.** The sum

$$\tilde{\Psi} = \sum_{p=1}^{\infty} |s_p|^2 r^{2p-1} = \frac{1}{2\pi} \int_{C(0,r)} \left| \frac{1}{N} \sum_{j=1}^{N} \frac{1}{z - x_j} \right|^2 |dz|$$  

(with $0 < r < 1$ fixed) is minimal (only) for equidistant points $x_j$ on $C(0,1)$.

**Conjecture 4.4.** The sum

$$\Psi^* = \sum_{p=1}^{\infty} \frac{1}{2p^2} |s_p|^2 r^{2p} = \frac{1}{2\pi} \int_{C(0,r)} \left( \frac{1}{N} \sum_{j=1}^{N} \log |z - x_j| \right)^2 \left| \frac{dz}{z} \right|$$
is minimal (only) for equidistant points $x_j$ on $C(0,1)$

**Remark.** In terms of the representation $\Pi(z - x_j) = \sum_{k=0}^{N} c_k z^{N-k}$ one has

$$
H(X_N, r) = \frac{1}{2\pi} \int_{C(0,r)} \prod_{j=1}^{N} |z - x_j|^2 \left| \frac{dz}{z} \right| = \sum_{k=0}^{N} |c_k|^2 r^{2N-2k} \geq r^{2N} + 1.
$$

Here it is clear that the minimum is attained (only) for equidistant points $x_j$ on $C(0,1)$.

5. **Local minimum property in Conjecture 4.2.**

The following fact was observed by Kuijlaars [5].

**Theorem 5.1.** The “potential energy” $\Psi$ in Conjecture 4.2 attains a local minimum for $N$-th roots of unity. This minimum configuration is stable apart from rotations. The same is true for the sum $\Psi^*$ in Conjecture 4.4.

**Proof.** We compute the partial derivatives of $\Psi = \sum_{p=1}^{\infty} a_p |s_p|^2$ with $s_p$ as in (4.7):

$$
\frac{\partial \Psi}{\partial t_j} = -2 \sum_{p=1}^{\infty} p a_p \frac{1}{N^2} \sum_{k=1}^{N} \sin(p(t_j - t_k)).
$$

One readily verifies that they are equal to zero when $t_k = 2\pi k/N$.

The second order derivatives may be expressed in terms of the function

$$
G(t) = \sum_{p=1}^{\infty} b_p \cos pt \quad \text{with} \quad b_p = 2p^2 a_p.
$$

One thus finds that the Hessian matrix for $\Psi$ equals

$$
\mathcal{H}_N = \begin{bmatrix}
-\sum_{k\neq 1} G(t_1 - t_k) & G(t_1 - t_2) & \ldots & G(t_1 - t_N) \\
G(t_2 - t_1) & -\sum_{k\neq 2} G(t_2 - t_k) & \ldots & G(t_2 - t_N) \\
& \vdots & \ddots & \vdots \\
& G(t_N - t_1) & G(t_N - t_2) & \ldots & -\sum_{k\neq N} G(t_N - t_k)
\end{bmatrix}.
$$

We now take $t_k = 2\pi k/N$, $1 \leq k \leq N$. Then $\mathcal{H}_N$ may be written as

$$
\mathcal{H}_N = \begin{bmatrix}
-\sum_{j=1}^{N-1} G(t_j) & G(t_1) & \ldots & G(t_{N-1}) \\
G(t_1) & -\sum_{j=1}^{N-1} G(t_j) & \ldots & G(t_{N-2}) \\
& \vdots & \ddots & \vdots \\
& G(t_{N-1}) & G(t_{N-2}) & \ldots & -\sum_{j=1}^{N-1} G(t_j)
\end{bmatrix}.
$$

(5.2)
This symmetric matrix is a so-called \textit{circulant matrix}, hence (as easily verified), the eigenvectors have the form \((1, \zeta, \zeta^2, \ldots, \zeta^{N-1})^T\), where \(\zeta\) is an \(N\)-th root of unity. The (real) eigenvalue of \(\mathcal{H}_N\) corresponding to \(\zeta = e^{it_k}\) is

\[-\sum_{j=1}^{N-1} G(t_j) + \sum_{j=1}^{N-1} G(t_j)\zeta^j = -\sum_{j=1}^{N} G(t_j) + \sum_{j=1}^{N} G(t_j) \cos kt_j\]  \hspace{1cm} (5.3)

since \(\zeta^N = 1\) and \(jt_k = kt_j\). For \(\zeta = 1\) \((k = N)\) one obtains the eigenvalue 0 which corresponds to the eigenvector \((1,1,\ldots,1)^T\) of rotations. The following computation will show that the other eigenvalues are strictly positive.

Since \(\sum_1^N \cos pt_j = 0\) if \(p \not\equiv 0\) \((N)\) and \(= N\) if \(p \equiv 0\) \((N)\), one finds that

\[
\sum_{j=1}^{N} G(t_j) = N \sum_{q=1}^{\infty} b_{qN},
\]

\[
\sum_{j=1}^{N} G(t_j) \cos kt_j = \sum_{p=1}^{\infty} b_{p} \sum_{j=1}^{N} \frac{1}{2} \{\cos (p + k)t_j + \cos (p - k)t_j\} = N \sum_{q=1}^{\infty} \frac{1}{2} (b_{qN-k} + b_{qN-(N-k)}), \quad 1 \leq k \leq N - 1. \]  \hspace{1cm} (5.4)

The final sums are clearly larger than the first if the sequence \(\{b_p\}\) is \textit{decreasing}. This is the case for \(\Psi^*\) in Conjecture 4.4 where \(b_p = r^{2p}\). In the case of Conjecture 4.2 one has \(b_p = 2pr^p\). Here the proof may be completed by explicit summation of the series in (5.4).
6. A quadrature result for the circle.

We do not know if the extremizing $N$-tuples in Conjectures 4.1 – 4.4 consist of equidistant points, but we can show that they are fairly good $N$-tuples for Chebyshev quadrature on the circle.

**Theorem 6.1.** Let the $N$-tuples $Y_N$ be extremizing in one of these Conjectures. Then there are positive constants $A, b, c$ (where $c < 1$ may be taken close to 1) such that

$$|R(f, Y_N)| = \left| \int_{C(0,1)} f(x) \frac{d\lambda(x)}{2\pi} - \frac{1}{N} \sum_{j=1}^{N} f(y_j) \right| \leq A e^{-bN} \max_{C(0,1)} |f|, \quad N = 1, 2, \ldots \quad (6.1)$$

for all polynomials $f(x) = f(x_1, x_2)$ of degree $\leq cN$.

**Proof.** We focus on $N$-tuples $Y_N$ which maximize $\Pi$ in (4.2) or minimize $\Psi$ in (4.8) for some fixed $r \in (0, 1)$. The restriction to the circle of a polynomial $f(x_1, x_2)$ of degree $\leq M$ can be represented in the form

$$f(\cos t, \sin t) = \sum_{p=-M}^{M} c_p e^{ipt}. \quad (6.2)$$

Let us set

$$y_j = e^{it_j} \quad \text{and} \quad \frac{1}{N} \sum_{j=1}^{N} e^{ipt_j} = \tilde{s}_p. \quad (6.3)$$

Then

$$R(f, Y_N) = c_0 - \frac{1}{N} \sum_{j=1}^{N} \sum_{p=-M}^{M} c_p e^{ipt_j} = -\sum_{p=-M}^{M} c_p \tilde{s}_p, \quad (6.4)$$

where $\sum'$ means that we exclude the term with $p = 0$.

By Parseval’s theorem

$$\sum_{p=-M}^{M} |c_p|^2 = \frac{1}{2\pi} \int_{0}^{2\pi} |f(\cos t, \sin t)|^2 dt \leq \max_{C(0,1)} |f|^2. \quad (6.5)$$

We next estimate $\sum_{p=-M}^{M} |\tilde{s}_p|^2 = 2 \sum_{p=1}^{M} |\tilde{s}_p|^2$. One has

$$\sum_{1}^{M} |\tilde{s}_p|^2 \leq \frac{M}{rM} \sum_{1}^{M} \frac{1}{p} |\tilde{s}_p|^2 r^p, \quad 13$$
\[
\sum_{1}^{M} \frac{1}{p} |s_p|^2 r^p \leq \min_{X_N} \sum_{p=1}^{\infty} \frac{1}{p} |s_p|^2 r^p \leq \sum_{q=1}^{\infty} \frac{1}{qN} r^{qN} < \frac{1}{N} \frac{r^N}{1 - r^N}.
\]

In the last line we have used the fact that \(\min_{X_N} \Psi\) is majorized by the value of \(\Psi\) for \(N\)-th roots of unity. Taking \(M = [cN]\) with \(c \in (0, 1)\) and combining the above inequalities, we obtain

\[
\sum_{-M}^{M} |s_p|^2 < 2 \frac{M}{r^M} \frac{1}{N} \frac{r^N}{1 - r^N} \leq 2 \frac{c}{1 - r} r^{(1 - c)N}. \tag{6.6}
\]

We finally apply Cauchy-Schwarz to the sum for \(R(f, Y_N)\) in (6.4). Thus by (6.5) and (6.6)

\[
|R(f, Y_N)| = |\sum_{-M}^{M} c_p \bar{s}_p| \leq \sqrt{\frac{2c}{1 - r} r^{\frac{1}{2}(1 - c)N} \max_{C(0, 1)} |f|} \tag{6.7}
\]

for all polynomials \(f(x)\) of degree \(\leq cN\).

7. Good distribution of extremal points on the circle.

**Theorem 7.1.** Suppose that for \(N = 1, 2, \ldots\), the \(N\)-tuples \(Y_N \subset C(0, 1)\) are extremizing in one of the Conjectures 4.1 - 4.4. More generally, suppose that the \(Y_N\) are good \(N\)-tuples for Chebyshev quadrature in the sense that (6.1) holds for all polynomials \(f(x)\) of degree \(\leq cN\) where \(c \in (0, 1)\). Then the points in the \(N\)-tuples are well-distributed on the circle. In fact, for any constant \(B > \pi/c\) and for \(N \geq N_0(B)\), every arc of length \(2B/N\) contains at least one point of \(Y_N\).

**Proof.** Let the \(N\)-tuples \(Y_N\) satisfy (6.1) and choose \(B > \pi/c\). For simplicity we denote the points of \(Y_N\) by \(e^{it_j}, j = 1, 2, \ldots, N\). Because one can rotate, it will be enough to show that the arc of the circle \(x = e^{it}\) with \(-B/N < t < B/N\) contains at least one point \(e^{it_j}\) for every large \(N\).

We will use two auxiliary polynomials of degree \(\leq M\) where \(M = [cN]\) and \(N\) is relatively large. The restrictions of our polynomials to the unit circle \(x = e^{it}\) are given by the trigonometric polynomials

\[
F(t) = c_1 \frac{1 + \cos Mt}{(\cos \frac{\pi}{M} - \cos t)^2}, \quad G(t) = F(t)(\cos \frac{\pi}{M} - \cos t). \tag{7.1}
\]
Both functions attain their maximum absolute value for \( t = 0 \). We choose \( c_1 = c_1(M) > 0 \) such that \( F(0) = 1 \) and define \( c_2 = |G(0)| \). For large \( M \) one then has

\[
c_1 \approx \frac{\pi^4}{8M^4}, \quad c_2 = \max|G(t)| \approx \frac{\pi^2}{2M^2}.
\]  

(7.2)

To compute the integrals of \( F \) and \( G \) we use Chebyshev quadrature at the equidistant points on the circle given by \( t = (2k - 1)\pi/M, \, 1 \leq k \leq M \). The result is

\[
\int_0^{2\pi} F(t) \frac{dt}{2\pi} = \frac{1}{M} \sum_{k=1}^{M} F\left(\frac{(2k - 1)\pi}{M}\right) = \frac{2}{M} F\left(\frac{\pi}{M}\right) \approx \frac{\pi^2}{8M}, \quad \int_0^{2\pi} G(t) \frac{dt}{2\pi} = 0.
\]  

(7.3)

Combining (7.3), (2.2) and (6.1) we find

\[
\frac{1}{N} \sum_{j=1}^{N} F(t_j) > \frac{1}{M} - Ae^{-bN}, \quad \frac{1}{N} \left| \sum_{j=1}^{N} G(t_j) \right| < Ae^{-bN} \frac{5}{M^2}.
\]  

(7.4)

Suppose now that none of the points \( t_j \) lies on the interval \( -(1 + \delta)\pi/M, (1 + \delta)\pi/M \) (modulo \( 2\pi \)) where \( \delta \) is a small positive number such that \( (1 + \delta)\pi/c < B \). Then

\[
G(t_j) \geq F(t_j)\left(\cos \frac{\pi}{M} - \cos \frac{(1 + \delta)\pi}{M}\right) \geq \frac{\delta \pi^2}{M^2} F(t_j), \quad \forall \, j.
\]

Hence by (7.4)

\[
\frac{1}{N} \sum_{j=1}^{N} G(t_j) \geq \frac{\delta \pi^2}{M^2} \frac{1}{N} \sum_{j=1}^{M} F(t_j) \geq \frac{\delta \pi^2}{M^2} \left( \frac{1}{M} - Ae^{-bN} \right).
\]  

(7.5)

In combination with the second inequality (7.4) there results

\[
\frac{\delta \pi^2}{M^2} \left( \frac{1}{M} - Ae^{-bN} \right) < Ae^{-bN} \frac{5}{M^2} \quad \text{or} \quad A \left( 1 + \frac{5}{\delta \pi^2} \right) M > e^{bN}.
\]  

(7.6)

However, since \( M = [cN] \) this is impossible when \( N \) is sufficiently large.

This contradiction proves that for every large \( N \), there is at least one point \( t_j \) for which

\(-B/N < t_j < B/N \) (modulo \( 2\pi \)).

**Corollary of the preceding proofs.** For extremizing \( N \)-tuples \( Y_N \) in Conjecture 4.1 or 4.2 the final statement in Theorem 7.1 may be strengthened as follows. There is a constant \( D = D(r) \) such that every (open) arc of \( C(0,1) \) of length

\[
\frac{2\pi}{N} \left( 1 + \frac{D \log N}{N} \right)
\]  

(7.7)
contains at least one point of $Y_N$ when $N$ is large. Thus if one covers the circle by contiguous half-open arcs of length (7.7) and one possibly smaller arc, then at most $D \log N + 1$ of the arcs contain more than one point of $Y_N$. This observation gives additional support to part (ii) of Conjecture 4.1.

**Proof.** For $M = \lfloor N - C \log N \rfloor$ with $C \log(1/r) = 4$, inequalities (6.6) and (6.7) show that

$$|R(f, Y_N)| \leq \frac{A}{N^2} \max |f|, \quad A = \sqrt{\frac{2}{1 - r}}$$

for all polynomials $f$ of degree $\leq M$. Replacing $A e^{-bN}$ in the preceding proof by $A/N^2$ and taking $\delta = (\log N)/N$, (7.6) becomes

$$A \left( 1 + \frac{5N}{\pi^2 \log N} \right) M > N^2.$$

For large $N$ this gives a contradiction which proves the corollary with any $D > C + 1$.

**8. Convexity argument for extremal points.**

We will again consider $N$-tuples of points

$$X_N : x_j = e^{it_j} \in C(0, 1) \quad \text{with} \quad t_1 \leq t_2 \leq \ldots \leq t_N < t_1 + 2\pi. \quad (8.1)$$

Proposition 8.2 below implies the following. If the $N$-tuple $X_N$ maximizes

$$\prod_{j,k=1}^{N} (1 + |x_j - x_k|), \quad (8.2)$$

then the points $x_j$ are equidistant. This result gives some more support to part (ii) of Conjecture 4.1.

Indeed, the associated potential is now $-\log(1 + |x|)$, with corresponding repulsive force $\approx 1$ at short distances. The repulsion is stronger than in the case of the potential $-\log(1 + |x|^2)$ (cf. Section 4), although not as strong as for the classical potential $-\log |x|$.

The result for (8.2) is not suitable for the application of complex methods because the function $\log(1 + |v|)$ is not analytic for $v = 0$.

A proof may be derived from Jensen’s inequality for convex functions:
Inequality 8.1. Let $f : [a, b] \rightarrow \mathbb{R} \cup +\infty$ be strictly convex and let $u_1, \ldots, u_N$ be any $N$-tuple of points on $[a, b]$. Then

$$f \left( \frac{1}{N} \sum_{j=1}^{N} u_j \right) \leq \frac{1}{N} \sum_{j=1}^{N} f(u_j),$$  \hspace{1cm} (8.3)

with equality if and only if all $u_j$’s are equal.

[The value of the function at the center of mass of the points $u_j$ is $\leq$ the average of the function values at the points $u_j$.]

Proposition 8.2. Let $g(v) \geq 0$ be such that $f(u) = -\log g(2 \sin \frac{1}{2}u)$ is strictly convex for $0 < u < 2\pi$. Then

$$\hat{\Pi} = \prod_{j,k=1 \atop j \neq k}^{N} g(|x_j - x_k|), \quad x_j \in C(0, 1)$$  \hspace{1cm} (8.4)

is maximal if and only if the points $x_j$ are equidistant.

Proof. In addition to the notation of (8.1) it is convenient to define

$$t_{N+j} = t_j + 2\pi, \quad x_{N+j} = x_j.$$  

Observing that $|x_{j+k} - x_j| = 2 \sin \frac{1}{2}(t_{j+k} - t_j)$ for $1 \leq k \leq N - 1$, one then obtains from (8.4) that

$$- \log \Phi = \sum_{k=1}^{N-1} \sum_{j=1}^{N} - \log g(|x_{j+k} - x_j|) = \sum_{k=1}^{N-1} \sum_{j=1}^{N} f(t_{j+k} - t_j).$$  \hspace{1cm} (8.5)

Thus by inequality (8.3) with $u_j = t_{j+k} - t_j \in [0, 2\pi]$,

$$\frac{1}{N} \sum_{j=1}^{N} f(t_{j+k} - t_j) \geq f \left( \frac{1}{N} \sum_{j=1}^{N} (t_{j+k} - t_j) \right),$$  \hspace{1cm} (8.6)

with equality if and only if $(t_{j+k} - t_j)$ is independent of $j$. Now

$$\sum_{j=1}^{N} (t_{j+k} - t_j) = t_{1+k} + \ldots + t_{N} + t_{N+1} + \ldots + t_{N+k}$$

$$- (t_1 + \ldots + t_k + t_{k+1} + \ldots + t_N) = k \cdot 2\pi.$$
Hence by (8.6)
\[ \sum_{j=1}^{N} f(t_{j+k} - t_j) \geq N \int \left( \frac{2\pi k}{N} \right), \quad k = 1, \ldots, N - 1, \quad (8.7) \]
with equality if and only if
\[ t_{j+k} - t_j = \frac{2\pi k}{N}, \quad j = 1, \ldots, N. \quad (8.8) \]

Summing (8.7) over \( 1 \leq k \leq N - 1 \), we conclude that
\[ -\log \hat{H} \geq N \sum_{k=1}^{N-1} f \left( \frac{2\pi k}{N} \right), \quad (8.9) \]
with equality if and only if (8.8) holds for \( 1 \leq k \leq N - 1 \), that is, if and only if
\[ t_{j+1} - t_j = \frac{2\pi}{N} \quad \text{for all } j. \]

Remarks. In the case of (8.2) one has
\[ g(v) = 1 + |v|, \quad f(u) = -\log(1 + 2 \sin \frac{1}{2} u) \quad (0 \leq u \leq 2\pi); \]
differentiation shows that \( f''(u) > 0 \). The classical “Fekete extreme points” for the circle correspond to the case \( g(v) = |v| \).

In the case of Conjecture 4.1 one has \( g(v) = (1 - r)^2 + rv^2 \) and \( f(u) = -\log(1 + r^2 - 2r \cos u) \): no convexity.
9. Equidistribution of extremal points on the circle.

Let $x = e^{it}$ and $y = e^{iu}$ be in $C(0,1)$, so that $x \cdot y = \cos(t-u)$. For $r \in (0,1)$ we again consider the first potential of Section 4,

$$
\log \frac{1}{|x - ry|} = -\frac{1}{2} \log(1 + r^2 - 2rx \cdot y) = \sum_{p=1}^{\infty} \frac{1}{p} r^p \cos p(t - u). \tag{9.1}
$$

(Here we may also introduce more general coefficients $a_p > 0$.)

For a probability measure $\mu$ on the circle, the potential energy $I(\mu)$ is defined by

$$
I(\mu) = \int_{\mathbb{C} \times \mathbb{C}} \log \frac{1}{|x - ry|} d\mu(x) d\mu(y).
$$

Abusing the notation we will replace $\mu(x)$ by $\mu(t)$, etc. Then by (9.1)

$$
I(\mu) = \sum_{p=1}^{\infty} \frac{1}{p} r^p \int_{[0,2\pi) \times [0,2\pi)} \cos p(t - u) \, d\mu(t) d\mu(u)
$$

$$
= \sum_{p=1}^{\infty} \frac{1}{p} r^p \left( \int_0^{2\pi} \cos pt \, d\mu(t) \int_0^{2\pi} \cos pu \, d\mu(u) + \int_0^{2\pi} \sin pt \, d\mu(t) \int_0^{2\pi} \sin pu \, d\mu(u) \right)
$$

$$
= \sum_{p=1}^{\infty} \frac{1}{p} r^p \left[ \left( \int_0^{2\pi} \cos pt \, d\mu(t) \right)^2 + \left( \int_0^{2\pi} \sin pt \, d\mu(t) \right)^2 \right] \geq 0. \tag{9.2}
$$

Equality will hold if and only if all the Fourier coefficients of $d\mu(t)$ with $p \geq 1$ are equal to zero, hence, if and only if

$$
d\mu(t) = \frac{1}{2\pi} dt = d\omega(t), \tag{9.3}
$$

the classical equilibrium measure for the circle.

Suppose now that we place charges $1/N$ at the points $x_j$ of an $N$-tuple $X_N \subset C(0,1)$. This configuration defines a discrete probability measure $\mu_N$ on $C(0,1)$. For minimal potential energy $I(\mu_N)$, the point charges define an $N$-th discrete equilibrium measure (“Fekete measure”) $\omega_N$. We denote its carrying $N$-tuple by $Y_N$ with $y_j = e^{it_j}$ and write $(1/N) \sum_{j=1}^{N} e^{ipt_j} = s_p$.

If the points in the $N$-tuples $X_N$ are asymptotically equidistributed on $C(0,1)$, then

$$
I(\mu_N) = \frac{1}{N^2} \sum_{j,k=1}^{N} \log \frac{1}{|x_j - r x_k|} \to \int_{\mathbb{C} \times \mathbb{C}} \log \frac{1}{|x - ry|} d\omega(x) d\omega(y) = I(\omega) = 0.
$$

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Hence by (4.8) and the minimizing property of $\omega_N$,

$$
I(\omega_N) = \sum_{p=1}^{\infty} \frac{1}{p} r^p |s_p|^2 \leq I(\mu_N) \rightarrow I(\omega) = 0. \quad (9.4)
$$

We will verify that (9.4) implies that the discrete equilibrium measures $\omega_N$ are weak* convergent to the classical equilibrium measure $\omega$.

Indeed, for trigonometric polynomials $f(t) = \sum_{-M}^{M} c_p e^{int}$ formula (9.4) and an argument as in Section 6 show that

$$
R(f, Y_N) = \int_0^{2\pi} f(t) d(\omega - \omega_N)(t) = - \sum_{-M}^{M} c_p s_p \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.
$$

By approximation,

$$
\int_0^{2\pi} f \, d\omega_N \rightarrow \int_0^{2\pi} f \, d\omega \quad \text{as} \quad N \rightarrow \infty \quad (9.5)
$$

for every periodic continuous function $f(t)$.

Finally approximating the characteristic function of an “arc” or interval $\gamma$ from above and below by continuous functions, one concludes that $\omega_N(\gamma) \rightarrow \omega(\gamma)$. That is, the points $y_j = y_{Nj}$ in the $N$-tuples $Y_N$ are asymptotically equidistributed on the circle.
10. A quadrature result via complex analysis.

To illustrate the use of complex analysis we continue with the extremal points on the circle
of Conjecture 4.1 or 4.2. The following result gives no new information, but a similar
method of proof assures good Chebyshev quadrature on the sphere provided there are
enough well-separated extremal points.

Theorem 10.1. Suppose that there is a constant $\alpha > 0$ (independent of $N$) such that
the maximizing $N$-tuples $Z_N \subset C(0,1)$ for (4.2) contain subsets $W_N$ which have at least
$\alpha N$ distinct points on each of three contiguous arcs of length $\frac{2}{3} \pi$. Then the $N$-tuples $Z_N$
are good $N$-tuples for Chebyshev quadrature in the sense of Theorem 6.1.

Proof. (i) It is convenient to denote the points of $Z_N$ by $\zeta_j = \zeta_{N,j} = (\xi_j, \eta_j)$ and to
replace vectors $x$ and $x_k$ by $\hat{x} = (x, y)$ and $\hat{x}_k = (x_k, y_k)$. We let $\mu_N$ and $\nu_N$ be the
measures defined by charges $1/N$ at the points of $X_N$ and $Z_N$, respectively.

The potential associated with $\nu_N$ then takes the form

$$ U(\hat{x}) = U_N(\hat{x}) = U(\hat{x}, Z_N) = \frac{1}{N} \sum_{j=1}^{N} \log \frac{1}{|\zeta_j - r \hat{x}|} $$

$$ = -\frac{1}{2N} \sum_{j=1}^{N} \log \{1 + r^2 - 2r(\xi_j x + \eta_j y)\} \quad \text{(if } \hat{x} \in C(0,1)). \tag{10.1} $$

We now use the fact that for fixed $N$ the energy

$$ I(\mu_N) = -\frac{1}{2N^2} \sum_{j,k=1}^{N} \log \{1 + r^2 - 2r(x_j x_k + y_j y_k)\}, \quad (x_j, y_j) \in C(0,1) $$

is minimal for $(x_j, y_j) = (\xi_j, \eta_j), j = 1, 2, \ldots, N$. Writing $y_k = \sqrt{1 - x_k^2}$ with an appro-
 priate value of the square root, we differentiate with respect to $x_k$ to conclude that

$$ \sum_{j=1}^{N} \frac{x_j - y_j x_k / y_k}{1 + r^2 - 2r(x_j x_k + y_j y_k)} $$

is equal to 0 for $(x_i, y_i) = (\xi_i, \eta_i), i = 1, 2, \ldots, N$ at least when $\eta_i \neq 0$. For the application
below we write the resulting equations in the form

$$ W'(z) = \frac{r}{N} \sum_{j=1}^{N} \frac{\xi_j - \eta_j z / \sqrt{1 - z^2}}{1 + r^2 - 2r(\xi_j z + \eta_j \sqrt{1 - z^2})} = 0 \tag{10.2} $$

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for $z = \xi_1, \ldots, \xi_N$.

(ii) By coordinate rotation one may assume that there are $\geq \alpha N$ distinct points $\zeta_j = \zeta_{Nj}$ on the arc $\tilde{x} = e^{it}$ given by $\frac{1}{6} \pi \leq t \leq \frac{5}{6} \pi$. Then one has (10.2) for at least $\alpha N$ distinct points $z = \xi_k = \xi_{Nk}$ on the interval $I = [-\frac{1}{2} \sqrt{3}, \frac{1}{2} \sqrt{3}]$ with $\sqrt{1 - z^2} = \eta_k > 0$. We now introduce \textit{analytic functions} $W(z) = W_N(z)$ with appropriate domain $D \subset \mathbb{C}$ whose derivatives $W'(z) = W'_N(z)$ are bounded by a constant independent of $N$ and vanish at the real points $\xi_k = \xi_{Nk}$ we just mentioned. In a sense $W$ will be the complexified potential $U$:

$$W(z) = W_N(z) = -\frac{1}{2N} \sum_{j=1}^{N} \log \{1 + r^2 - 2r(\xi_j z + \eta_j \sqrt{1 - z^2})\}, \quad (10.3)$$

Taking $\frac{3}{4} < R < 1$ we let $D$ be the domain determined by the inequalities

$$|z|^2 < R, \quad (|z|^2 + |1 - z^2|)^{\frac{1}{2}} < \left(\frac{1 + r^2}{4r}\right). \quad (10.4)$$

Thus $I \subset D$ and for $z \in D$,

$$|\sqrt{1 - z^2}| > \sqrt{1 - R}, \quad 2r|\xi_j z + \eta_j \sqrt{1 - z^2}| < (1 + r^2)/2, \quad \forall j.$$ 

In formula (10.3) for $W(z)$ on $D$ we now take the principal values of the square root and the logarithm. By a simple calculation, cf. (10.2),

$$|W'(z)| = |W'_N(z)| < M = \frac{2r}{(1 - r)^2} \left(1 + \sqrt{\frac{R}{1 - R}}\right) \text{ throughout } D.$$ 

Now $W'$ has at least $\alpha N$ zeros on the compact subset $I \subset D$. Then by complex analysis (see (iv) below) there is a positive number $b$ independent of $N$ such that

$$|W'_N(x)| < Me^{-\alpha x} \text{ everywhere on } I.$$ 

It follows that the oscillation (maximum – minimum) of $W(x)$ on $I$ is less than $\sqrt{3}Me^{-\alpha x}$. For the potential $U(\tilde{x})$ this means that its oscillation on the arc $\gamma \subset C(0,1)$ above $I$ is less than $\sqrt{3}Me^{-\alpha x}$. By our hypotheses and rotation, the oscillation of $U(\tilde{x})$ on $C(0,1)$ will then surely be less than $3Me^{-\alpha x}$. 

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(iii) Observe that $U(\hat{x})$ is harmonic on the closed unit disc in $\mathbb{R}^2$, cf. (10.1), hence its average over $C(0,1)$ is equal to $U(0) = 0$. Thus by the preceding

$$|U(\hat{x})| < 3M e^{-b\alpha N} \text{ on } C(0,1).$$

(10.5)

The exponential smallness of $U(\hat{x}, Z_N)$ implies the same smallness of the corresponding energy

$$I(\nu_N) = \frac{1}{N^2} \sum_{j,k=1}^{N} \log \frac{1}{|\zeta_j - r\zeta_k|}$$

which is just an average of values of the potential on $C(0,1)$. The earlier argument (cf. Sections 6 and 4) then gives a quadrature result as in (6.1).

(iv) We have used the following auxiliary result from complex analysis:

**Proposition 10.2.** Let $D \subset \mathbb{C}$ be a (simply connected) domain and let $E_0$ and $E$ be compact subsets. Then there is a constant $b > 0$ depending only on $D$, $E_0$ and $E$ such that for any analytic function $f$ on $D$ which is in absolute value bounded by 1,

$$\max_E |f| \leq e^{-bn},$$

where $n$ is the number of zeros of $f$ in $E_0$.

A proof may be derived from the following lemma by conformal mapping:

**Lemma 10.3.** Let $f$ be analytic on the unit disc $\Delta$ and bounded by 1 on $\Delta$. Suppose that $f$ has $n$ zeros $z_1, \ldots, z_n$ with $|z_k| = r_k \leq s < 1$ for each $k$. Let $B(z)$ be the corresponding Blaschke product

$$B(z) = \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}.$$ 

Then $|f(z)| \leq |B(z)|$ on $\Delta$ and for $|z| \leq r < 1$,

$$|B(z)| \leq \prod_{k=1}^{n} \frac{r + r_k}{1 + rr_k} \leq \left( \frac{r + s}{1 + rs} \right)^n.$$ 

**Remark.** In the case of the sphere (Conjecture 1.5 and Theorem 1.6) one needs more than just distinct extremal points. To get an analog to Proposition 10.2 for analytic functions
\(f(z)\) of two (or more) complex variables, one needs \(n\) well-separated points in the zero set of \(f\) on \(E_0\), cf. Korevaar [4].

REFERENCES


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