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## Some applications of hypergeometric shift operators

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### 1. Introduction

In this paper we will discuss some very natural consequences of the existence theorem of so called hypergeometric shift operators, which is proved in [O2] (Theorem 3.6). These shift operators play a role, as their name indicates, in the theory of generalized hypergeometric functions of several variables associated with root systems as developed in the papers [HO], [H], [O1] and [O2]. The above mentioned existence theorem is based on the main result of Heckman's paper [H], which states that a certain second order differential operator  $L$ , associated with a root system  $R$  and a multiplicity function  $\ell$  on  $R$ , has certain special eigenfunctions that are Nilsson class functions with a prescribed monodromy behaviour. The underlying principle in Heckman's result is, as far as we understand at this moment, the so called Riemann-Hilbert correspondence (in the form obtained by P. Deligne [D]).

In order to give the reader an idea what shift operators are we take a look at the simplest possible example, namely the case where  $R = BC_1$ . The general theory boils down to the theory of the ordinary hypergeometric function  $F(\alpha, \beta, \gamma; z)$  here, and an example of a shift operator is the operator  $\frac{d}{dz}$ , by virtue of the relation

$$\frac{d}{dz} F(\alpha, \beta, \gamma; z) = \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1; z).$$

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The idea to apply such operators in the theory of multivariable analogues of hypergeometric functions associated with root systems was used for the first time in Koornwinder's paper [K] (he introduced such an operator for the system  $BC_2$  and used it to study the Jacobi polynomials).

Let me describe briefly the results we will achieve in this paper:

(1.) (Section 4). Let  $R$  be a root system and  $\ell$  a Weyl group invariant function on  $R$ , with values in the nonnegative integers. The constant term of the Laurent polynomial  $\prod_{\alpha \in R} (1 - h^\alpha)^{\ell_\alpha}$  is equal to

$$\prod_{\alpha \in R} \frac{(|(\rho(\ell), \alpha^\vee) + \ell_\alpha + \frac{1}{2} \ell_{\frac{\alpha}{2}}|)!}{(|(\rho(\ell), \alpha^\vee) + \frac{1}{2} \ell_{\frac{\alpha}{2}}|)!}$$

where  $\rho(\ell) = \frac{1}{2} \sum_{\alpha \in R_+} \ell_\alpha \alpha$  and  $\alpha^\vee = \frac{2\alpha}{|\alpha|^2}$ . This was conjectured by Macdonald ([M1], Conjecture 2.3).

(2.) (Section 5). Let  $P(\mu, \ell; h) = \sum_{\nu \in C(\mu)} \Gamma_\nu(\mu, \ell) h^\nu$  be the Jacobi polynomial associated with  $R$ , normalized by the condition  $\Gamma_\mu(\mu, \ell) = 1$ . Then  $P(\mu, \ell; e) = \sum_{\nu \in C(\mu)} \Gamma_\nu(\mu, \ell) = 1/c(\mu - \rho(\ell), \ell)$ , where  $c$  is Harish Chandra's  $c$ -function. This is a special case of Conjecture 6.11 of [HO] (for a definition of Jacobi polynomials we refer the reader to [HO], Definition 3.13).

(3.) (Section 6). Let  $D(\ell; x) = \prod_{\alpha \in R_+} |\tilde{\alpha}(x)|^{\ell_\alpha}$ , where  $x \in (\mathbb{R} \cdot R)^* \equiv \mathfrak{a}$  and  $\tilde{\alpha} = \frac{\sqrt{2}\alpha}{|\alpha|}$ .

Let  $d\gamma$  denote the Gaussian measure on  $\mathfrak{a}$ :  $d\gamma(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}|x|^2} dx$ , where  $n$  equals the rank of  $R$  and  $dx$  is the Lebesgue measure on  $\mathfrak{a}$ . Then:

$$\int_{\mathfrak{a}} D(\ell; x) d\gamma(x) = \prod_{\alpha \in R_+} \frac{(\frac{1}{4} \ell_{\frac{\alpha}{2}} + \frac{1}{2} \ell_\alpha + \frac{1}{2} (\rho(\ell), \alpha^\vee))!}{(\frac{1}{4} \ell_{\frac{\alpha}{2}} + \frac{1}{2} (\rho(\ell), \alpha^\vee))!}.$$

This was also conjectured by Macdonald in his paper [M1], (Conjecture 6.1).

(4.) (Section 7). Let  $R$  be reduced now. Let  $I = \prod_{\alpha \in R_+} \alpha^2$  and consider  $I$  as an element of the polynomial algebra of  $W$ -invariants of  $R$ . The Bernstein-Sato polynomial  $b$  of  $I$  equals  $b(\ell) = \prod_{i=1}^n \prod_{j=1}^{d_i-1} \left( \ell + \frac{1}{2} + \frac{j}{d_i} \right)$ . This was a conjecture of Yano and Sekiguchi in their paper [YS] (Conjecture 5II). Moreover, we will actually construct a differential operator  $B$  such that  $BI^{\ell+1} = b(\ell)I^\ell$ .

We refer the reader to [M1] for an outstanding treatise on the conjectures 1) and 3). Note that, if  $R$  is reduced and  $\ell = \ell_\alpha \forall \alpha \in R, 1)$  is actually only the case  $q=1$  of a more general conjecture, the so called  $q$ -analogue of the constant term conjecture (see [M1], Conjecture 3.1). This conjecture is of course beyond

the scope of the shift operators we use here. However, Macdonald has developed a theory of  $q$ -analogues of Jacobi polynomials (see [M2]) and one could hope for existence of  $q$ -analogues of shift operators in this theory, which could possibly be used towards the solution of this generalization.

## 2. Notations and general concepts

Let  $\mathfrak{a}$  be an Euclidean space of dimension  $n$  provided with an inner product  $(\cdot, \cdot)$ , and let  $R \subset \mathfrak{a}^*$  be a root system, possibly non reduced, that spans  $\mathfrak{a}^*$ . Denote by  $W$  the group generated by the orthogonal reflections in the hyperplanes  $\alpha^\perp (\alpha \in R)$ , the so-called Weyl group associated with  $R$ . Write  $P$  for the weight lattice of  $R$ , so  $P = \{\lambda \in \mathfrak{a}^* \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \forall \alpha \in R\}$ , where  $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$  ( $\alpha^\vee$  is called a coroot; the set  $\{\alpha^\vee\}_{\alpha \in R}$  is again a root system, the coroot system), and write  $Q$  for the root lattice, thus  $Q = \mathbb{Z}.R$ . Let  $H$  be the complex torus characterized by the properties 1)  $\text{Lie}(H) = \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{a}$  and 2)  $\hat{H} = P$  is the character lattice of  $H = P$ . In other words:  $H \simeq \mathfrak{h}/M$  with  $M = \{X \in \mathfrak{h} \mid \lambda(X) \in 2\pi i \mathbb{Z} \forall \lambda \in P\}$ . Write  $H = AT$  for the ‘‘polar decomposition’’ of  $H$ ; here  $A$  is the split part of  $H$  and  $T$  the compact part. The natural map  $\mathfrak{h} \rightarrow H$  is denoted by ‘‘exp’’ and we will use the symbol  $h^\lambda$  for the character on  $H$  corresponding with  $\lambda \in P$  (so  $h^\lambda = e^{\lambda(X)}$  if  $X \in \mathfrak{h}$  such that  $\text{exp}(X) = h$ ). Choose a system of positive roots  $R_+$  in  $R$ . Let  $P_+$  be the corresponding set of dominant weights, i.e.  $P_+ = \{\lambda \in P \mid (\lambda, \alpha) \geq 0 \forall \alpha \in R_+\}$ . An exponential polynomial on  $H$  is an expression of the form  $\sum_{\lambda \in P} C_\lambda h^\lambda$ , where only a finite number of coefficients  $C_\lambda \in \mathbb{C}$  are unequal to zero. If  $\lambda_1, \dots, \lambda_n \in P_+$  is the set of fundamental weights we put

$$z_j = \sum_{w \in W/W^{\lambda_j}} h^{w\lambda_j} \quad (j = 1, \dots, n),$$

where  $W^{\lambda_j}$  denotes the stabilizer of  $\lambda_j$  in  $W$ . The  $z_j$  are called the fundamental  $W$ -invariant exponential polynomials, and it is a well known fact (see for instance [B, p. 188]) that the algebra of all  $W$ -invariant exponential polynomials is equal to  $\mathbb{C}[z_1, \dots, z_n]$ . The Weyl denominator is by definition the following  $W$ -anti invariant function on  $H$ :

$$\Delta(h) = \prod_{\alpha \in R_+^0} (h^{-\frac{\alpha}{2}} - h^{\frac{\alpha}{2}}),$$

where  $R^0$  denotes the set of inmultiplicable roots in  $R$  (so  $\alpha \in R^0$  if and only if  $\alpha \in R$  and  $2\alpha \notin R$ ). The map

$$\begin{aligned} H &\rightarrow \mathbb{C}^n \\ h &\rightarrow (z_1(h), \dots, z_n(h)) \end{aligned} \tag{2.1}$$

is a  $W$ : 1 covering of  $\mathbb{C}^n$ , branched along the locus  $\{\Delta^2 = 0\}$  in  $\mathbb{C}^n$ . This locus is called the (global) discriminant of  $R$ . The map (2.1) gives an identification

$$W \setminus H^{\text{reg}} \xrightarrow{\sim} \mathbf{C}^n \setminus \{A^2 = 0\},$$

with  $H^{\text{reg}} = \{h \in H \mid h^\alpha \neq 1 \forall \alpha \in R\}$ .

Define  $\mathcal{K} \simeq \mathbf{C}^m$  as the vector space of all (complex valued)  $W$ -invariant functions on  $R$  ( $m$  equals the number of conjugacy classes of roots in  $R$ ). Elements of  $\mathcal{K}$  are called multiplicity functions on  $R$ . Introduce the following function on  $H$ :

$$\delta(\ell; h) = \prod_{\alpha \in R_+} (h^{-\frac{\alpha}{2}} - h^{\frac{\alpha}{2}})^{2\ell_\alpha}, \quad \ell \in \mathcal{K}.$$

(Note that  $\delta$  is, in general, multivalued). Now let  $\ell \in \mathcal{K}$  such that  $\ell_\alpha \geq 0 \forall \alpha \in R$ . We provide the space of  $W$ -invariant exponential polynomials with a Hermitian product:

$$(f, g)_\ell = \int_T f(t) \overline{g(t)} |\delta(\ell; t)| dt,$$

where  $dt$  is the Haar measure on  $T$  normalized such that  $\int_T dt = 1$ . Define the

Jacobi polynomials  $P(\mu, \ell; t)$  ( $\mu \in P_-$ ) associated with  $R$  and multiplicity  $\ell$  by means of the following properties:

$$(1.) P(\mu, \ell; t) = \sum_{\lambda \in C(\mu)} \Gamma_\lambda(\mu, \ell) t^\lambda \quad (\text{Here } C(\mu) \text{ is the convex hull of the orbit } W.\mu,$$

intersected with  $\mu + Q$ ) with  $\Gamma_\mu(\mu, \ell) = 1$  and  $\Gamma_{w\lambda}(\mu, \ell) = \Gamma_\lambda(\mu, \ell) \forall w \in W$ .

$$(2.) (P(\mu, \ell), P(\nu, \ell))_\ell = 0 \forall \nu \in P_- \text{ with } \nu > \mu \text{ (i.e. } \nu \in \mu + Q_+, \text{ with } Q_+ = \mathbf{Z}_+ \cdot R_+).$$

As remarked in ([H], Proposition 8.1) one may equally well replace (2) by the condition that  $P(\mu, \ell)$  is an eigenfunction of the differential operator  $L(\ell)$ :

$$L(\ell) P(\mu, \ell) = (\mu, \mu - 2\rho(\ell)) \cdot P(\mu, \ell),$$

where

$$\rho(\ell) = \frac{1}{2} \sum_{\alpha \in R_+} \ell_\alpha \alpha$$

and

$$L(\ell) = \sum_{i=1}^n \partial(X_i)^2 - \sum_{\alpha \in R_+} \ell_\alpha (1 + h^\alpha)(1 - h^\alpha)^{-1} \partial(X_\alpha).$$

(We use the symbol  $\partial(p)$  (for  $p \in \mathbf{C}[\mathfrak{h}^*]$ ) for the constant coefficient differential operator on  $H$  that corresponds with  $p$  by considering  $p$  as element of the symmetric algebra on  $\mathfrak{h}$ ;  $X_\alpha \in \mathfrak{h}$  is the vector such that  $(X_\alpha, Y) = \alpha(Y) \forall Y \in \mathfrak{h}$ ).

Introduce the following function

$$\begin{aligned} \check{c}: \mathfrak{h}^* \times \mathcal{K} &\rightarrow \mathbf{C} \\ (\lambda, \ell) &\rightarrow \prod_{\alpha \in R_+} \frac{\Gamma(-(\lambda, \alpha^\vee) + \frac{1}{2} \ell_\alpha)}{\Gamma(-(\lambda, \alpha^\vee) + \frac{1}{2} \ell_\alpha + \ell_\alpha)}. \end{aligned}$$

This function is closely related to Harish-Chandra's  $c$ -function:

$$c(\lambda, \kappa) = \frac{\tilde{c}(\lambda, \kappa)}{\tilde{c}(-\rho(\kappa), \kappa)}.$$

(Note that this definition of the  $c$ -function is a considerable simplification compared with our original definition (see [HO], Definition 6.4). We are indebted to Prof. Macdonald for suggesting this simplification to us). The  $c$ -function plays an important role in the theory of hypergeometric functions and, as we will see, in particular in this paper.

The following theorem is due to Heckman, and will be used in Sect. 4.

**2.1. Theorem.** ([H], Theorem 8.5). *The polynomials  $P(\mu, \kappa; t)$ ,  $\mu \in P_-$ , are orthogonal with respect to the inner product  $(\cdot, \cdot)_\kappa$ . Moreover*

$$(P(\mu, k), P(\mu, k))_\kappa = \lim_{\varepsilon \rightarrow 0} \frac{c(-\mu + \rho(\kappa) + \varepsilon, \kappa) c(-\rho(\kappa) + w_0 \varepsilon, \kappa)}{c(\rho(\kappa) + \varepsilon, \kappa) c(\bar{\mu} - \rho(\kappa) + w_0 \varepsilon, \kappa)} \cdot \int_T |\delta(\kappa, t)| dt$$

where  $w_0 \in W$  is the longest element and  $\bar{\mu} = -w_0 \mu$ .  $\square$

### 3. Properties of shift operators

We now arrive at the important notion of shift operator. A shift operator  $S$  with shift  $\ell \in \mathcal{X}$  is an element of  $\mathbf{C}[\mathcal{X}] \otimes \mathbf{A}_n$  ( $\mathbf{A}_n$  being the Weyl algebra of polynomial differential operators in the variables  $z_i$ ) which satisfies:

$$S(\kappa) \circ (L(\kappa) + (\rho(\kappa), \rho(\kappa))) = (L(\kappa + \ell) + (\rho(\kappa + \ell), \rho(\kappa + \ell))) \circ S(\kappa).$$

From this definition it is clear that  $S(\kappa)$  acts on the Jacobi polynomials as follows:

$$S(\kappa) P(\mu, \kappa) = \eta(S(\kappa)) (\mu - \rho(\kappa)) \cdot P(\mu + \rho(\ell), \kappa + \ell) \tag{3.1}$$

for some  $\eta(S(\kappa)) \in \mathbf{C}[\mathcal{X}] \otimes \mathbf{C}[\mathfrak{h}^*]$ . Note that  $\eta(S)$  determines  $S$  because the Jacobi polynomials form a  $\mathbf{C}$ -basis for  $\mathbf{C}[z_1, \dots, z_n]$ .

In order to formulate the existence theorem we introduce the following notation. Let  $R = \bigsqcup_{i=1}^m C_i$  be the decomposition of  $R$  in conjugacy classes of roots.

Let  $e_i \in \mathcal{X}$  be the function defined by  $e_i(C_j) = \delta_{ij}$  (Kronecker's symbol). Write  $\mathcal{B}$  for the basis of  $\mathcal{X}$  consisting of the vectors  $\ell_i = e_i$  if  $2C_i \cap R = \emptyset$ , and  $\ell_i = (2e_i - e_j)$  if  $2C_i = C_j$ .

**3.1. Theorem (existence of shift operators).** ([Op2], Theorem 3.6). (1.) *There exist nontrivial shift operators if and only if  $\ell \in \mathbf{Z} \cdot \mathcal{B}$ .*

(2.) If  $\ell \in \mathbb{Z}_- \cdot \mathcal{B}$  there exists a shift operator  $G(\ell, \kappa)$  such that

$$\eta(G(\ell, \kappa))(\lambda) = \frac{\tilde{c}(\lambda, \kappa + \ell)}{\tilde{c}(\lambda, \kappa)}. \quad (3.2)$$

Moreover, every shift operator  $S(\kappa)$  with shift  $\ell$  can be decomposed like  $S(\kappa) = G(\ell, \kappa) \circ S'(\kappa)$ , where  $S'(\kappa)$  commutes with  $L(\kappa)$ .  $\square$

Among the properties that are derived in [Op1] we mention the following, which will be used in the subsequent sections of this paper

(A) For  $P \in \mathbb{A}_n$  we write  $P^*$  for the formal transpose of  $P$  (i.e.:  $(P_1 P_2)^* = P_2^* P_1^*$ ,  $z_j^* = z_j$  and  $\left(\frac{\partial}{\partial z_j}\right)^* = -\frac{\partial}{\partial z_j}$ ).

**3.2. Proposition.** ([Op1], Prop. 3.4, 3.5). Let  $\ell \in \mathbb{Z}_+ \cdot \mathcal{B}$ . The differential operator:

$$G(\ell, \kappa) = \delta(-\ell - \kappa + \frac{1}{2}e) \circ G^*(-\ell, \kappa + \ell) \circ \delta(\kappa - \frac{1}{2}e), \quad (3.3)$$

where  $e = \sum_{i: 2C_i \cap R = \emptyset} e_i$ , is a shift operator with shift  $\ell$ , and

$$\eta(G(\ell, \kappa))(\lambda) = \eta(G(-\ell, \kappa + \ell))(-\lambda) = \frac{\tilde{c}(-\lambda, \kappa)}{\tilde{c}(-\lambda, \kappa + \ell)}. \quad \square$$

Note that we may also write

$$G(\ell, \kappa) = \delta(-\ell - \kappa) \circ G^+(-\ell, \kappa + \ell) \circ \delta(\kappa), \quad (3.4)$$

where  $G^+$  denotes the formal transpose of  $G$  as operator on  $H$  (so  $f^+ = f$  for a function  $f$  on  $H$ , and  $\partial(x)^+ = \partial(-x)$ ).

(B) The degree of  $G(\ell, \kappa)$  ( $\ell \in \mathbb{Z}_+ \cdot \mathcal{B}$ ) is equal to  $\sum_{\alpha \in R_+} \ell_\alpha$ . The highest order part of  $G(\ell, \kappa)$  does not depend on  $\kappa$ , and for  $\ell = \ell_i$  such that  $2C_i \cap R = \emptyset$  this highest order part is equal to (on  $H$ ):  $\frac{1}{\delta(\frac{1}{2}\ell_i; h)} \prod_{\alpha \in C_i^+} \partial(X_\alpha)$  (because this expression is equal to  $G(\ell_i, 0)$  as one can check directly from the definition of shift operator).

(C) It is a well known fact that the polynomial  $W$ -invariants on  $\mathfrak{h}$  form a polynomial algebra  $\mathbb{C}[p_1, \dots, p_n]$ , where the  $p_i$  are homogeneous invariants of degree  $d_i$  (the  $d_i$  are determined by  $R$ ; they are called the primitive degrees of  $R$ ). Let  $R$  be a reduced root system now. The map

$$\begin{aligned} \mathfrak{h} &\rightarrow \mathbb{C}^n \\ x &\rightarrow (p_1(x), \dots, p_n(x)) \end{aligned}$$

is a  $|W|:1$  covering of  $\mathbb{C}^n$ , branched along the set  $I=0$ , where  $I = \prod_{\alpha \in R_+} \alpha^2$ . The locus  $I=0$  in  $\mathbb{C}^n$  is called the infinitesimal discriminant of  $R$ . Consider the

map  $\varphi$ , defined in a neighbourhood of the origin of  $\mathbb{C}^n$ , which makes the following diagram commute:

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\text{exp}} & H \\ \downarrow & & \downarrow \\ \mathbb{C}^n & \xrightarrow{\varphi} & \mathbb{C}^n \end{array}$$

It is easy to see that  $\varphi$  is a biholomorphic map from a neighbourhood of the origin to a neighbourhood of the identity element. So if  $P \in \mathbb{A}_n(z_i)$  we can write  $P = \sum_k f_k \left(\frac{\partial}{\partial p}\right)^k$  (where  $k$  denotes a multiindex), with  $f_k$  analytic in a neighbourhood of the origin. Let  $\varepsilon(P)$  denote the lowest homogeneous part of  $P$  with respect to the weighted Euler vectorfield  $E = \sum_i d_i p_i \frac{\partial}{\partial p_i}$ . We thus obtain a map:

$$\varepsilon: \mathbb{A}_n(z_1, \dots, z_n) \rightarrow \mathbb{A}_n(p_1, \dots, p_n).$$

It is not difficult to see that  $\varepsilon$  is actually the same map as defined in ([Op1], Sect. 4). The eigenvalue of  $\varepsilon(P)$  with respect to  $ad(E)$  is called the lowest homogeneous degree of  $P$  (denoted by l.h.d. ( $P$ )). From [Op1], Theorem 4.4 we have:

**3.3. Theorem.** (i) If  $\ell \in \mathbb{Z}_+ \cdot \mathcal{B}$  then l.h.d. ( $G(\ell, \ell)$ ) =  $-2 \sum_{\alpha \in R_+} \ell_\alpha$ .

(ii) If  $\ell \in \mathbb{Z}_- \cdot \mathcal{B}$  then l.h.d. ( $G(\ell, \ell)$ ) = 0.  $\square$

**3.4. Corollary.** Let  $\ell \in \mathbb{Z}_- \cdot \mathcal{B}$  and let  $f$  be a  $C^\infty$ -germ at  $e \in H$ , the identity element of  $H$ , such that  $G(\ell, \ell)f$  is again  $C^\infty$ -germ at  $e$ . Then:

$$G(\ell, \ell)f(e) = \eta(G(\ell, \ell))(-\rho(\ell)) \cdot P(\rho(\ell), \ell + \ell; e) \cdot f(e). \tag{3.5}$$

*Proof.* Theorem 3.3 (ii) implies (CT denoting constant term):

$$G(\ell, \ell)f(e) = CT(G(\ell, \ell))(e) \cdot f(e).$$

On the other hand, the function 1 is equal to the Jacobi polynomial  $P(0, \ell; h)$ . Hence

$$CT(G(\ell, \ell)) = G(\ell, \ell)(1) = G(\ell, \ell)(P(0, \ell)) = \eta(G(\ell, \ell))(-\rho(\ell)) \cdot P(\rho(\ell), \ell + \ell)$$

from which we conclude the validity of (3.5).  $\square$



**4. The constant term of a certain Laurent polynomial associated with a root system; proof of the constant term conjecture of I.G. Macdonald**

Let  $R$  be a (possibly non reduced) root system. Define

$$\sigma(\ell; h) = \prod_{\alpha \in R} (1 - h^\alpha)^{\ell_\alpha} \quad (\ell \in \mathcal{X}).$$

In [M1] Macdonald conjectured that, if  $\ell_\alpha \in \mathbb{Z}_+, \forall \alpha \in R$ , the constant term of the Laurent polynomial  $\sigma(\ell)$  should be equal to:

$$\prod_{\alpha \in R} \frac{(|(\rho(\ell), \alpha^\vee) + \ell_\alpha + \frac{1}{2} \ell_{\frac{\alpha}{2}}|)!}{(|\rho(\ell), \alpha^\vee) + \frac{1}{2} \ell_{\frac{\alpha}{2}}|)!}$$

([M1], Conjecture 2.3). This conjecture is a generalization of a conjecture of Dyson (the case  $R = A_n$  in Macdonald's conjecture), which was already proved at that time by Gunson [Gu] and Wilson [W] (see I. Good [G] for a short, elegant proof). Macdonald gives a proof of his conjecture if  $R = BC_n$  or  $D_n$ , by means of a clever change of variables in Selberg's integral formula (see [S] and [M1], Sect. 2). Finally, it is shown in [M1] that the formula holds for arbitrary  $R$  if  $\ell = \epsilon$  or  $2\epsilon$  (recall that  $\epsilon = \sum_{i; 2C_i \cap R = \emptyset} \epsilon_i$ ) ([M1], Sect. 2 and 3).

Observe that, if  $\ell_\alpha \in \mathbb{Z}_+, \forall \alpha \in R$ , the constant term of  $\sigma(\ell)$  is equal to the integral  $\int_T \sigma(\ell; t) dt$ . Also note that, if  $\ell \in \mathcal{X}$  arbitrary,  $\sigma(\ell; t) = \prod_{\alpha \in R_+} |t^{\frac{\alpha}{2}} - t^{-\frac{\alpha}{2}}|^{2\ell_\alpha}$  ( $= |\delta(\ell; t)|$  if  $\ell$  is real), so  $\int_T \sigma(\ell; t) dt$  makes sense  $\forall \ell \in \mathcal{X}$  with  $\text{Re}(\ell_\alpha) \geq 0 \forall \alpha \in R$ .

Moreover,  $\int_T \sigma(\ell; t) dt$  depends analytically on  $\ell$  ( $\text{Re}(\ell_\alpha) > 0$ ). With the aid of shift operators it is easy to prove Macdonald's constant term conjecture in full generality:

**4.1. Theorem.** *Let  $R$  be a possibly non reduced root system, and  $\ell \in \mathcal{X}$  such that  $\text{Re}(\ell_\alpha) \geq 0 \forall \alpha \in R$ . Then*

$$\int_T \sigma(\ell; t) dt = \prod_{\alpha \in R_+} \frac{\Gamma((\rho(\ell), \alpha^\vee) + \ell_\alpha + \frac{1}{2} \ell_{\frac{\alpha}{2}} + 1) \Gamma((\rho(\ell), \alpha^\vee) - \ell_\alpha - \frac{1}{2} \ell_{\frac{\alpha}{2}} + 1)}{\Gamma((\rho(\ell), \alpha^\vee) + \frac{1}{2} \ell_{\frac{\alpha}{2}} + 1) \Gamma((\rho(\ell), \alpha^\vee) - \frac{1}{2} \ell_{\frac{\alpha}{2}} + 1)}. \quad (4.1)$$

*Proof.* Take  $\ell_i \in \mathcal{B}$  and choose  $\mu \in P_-$  arbitrarily. Consider the following identity, which obviously holds if  $\ell_\alpha \in \mathbb{R}_+$  is sufficiently positive ( $\forall \alpha \in R$ ) (see Sect. 3, property A))

$$\begin{aligned} & (G(-\ell_i, \ell + \ell_i) P(\mu + \rho(\ell_i), \ell + \ell_i), P(\mu, \ell))_\ell \\ &= (-1)^{\sum_{\alpha \in \mathcal{K}} \langle \ell_i, \alpha \rangle} (P(\mu + \rho(\ell_i), \ell + \ell_i), G(\ell_i, \ell) P(\mu, \ell))_{\ell + \ell_i}. \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \frac{(P(\mu + \rho(\ell_i), \ell + \ell_i), P(\mu + \rho(\ell_i), \ell + \ell_i)_{\ell + \ell_i})}{(P(\mu, \ell), P(\mu, \ell))_{\ell}} \\ &= (-1)^{\sum_{\alpha \in R_+} \langle \ell_i, \alpha \rangle} \frac{\eta(G(-\ell_i, \ell + \ell_i))(\mu - \rho(\ell))}{\eta(G(\ell_i, \ell))(\bar{\mu} - \rho(\ell))}. \end{aligned}$$

If we use formula (3.2) and (3.3) of Sect. 3 and Theorem 2.1 this becomes:

$$\frac{\int_T \sigma(\ell + \ell_i; t) dt}{\int_T \sigma(\ell; t) dt} = (-1)^{\sum_{\alpha \in R_+} \langle \ell_i, \alpha \rangle} \lim_{\varepsilon \rightarrow 0} \frac{c(\rho(\ell + \ell_i) + \varepsilon, \ell + \ell_i)}{c(\rho(\ell) + \varepsilon, \ell)} \quad (4.2)$$

(so the dependence on  $\mu$  cancels as should!).

The function

$$c(\rho(\ell) + \varepsilon, \ell) = \prod_{\alpha \in R_+} \frac{\Gamma(-(\rho(\ell), \alpha^\vee) - (\varepsilon, \alpha^\vee) + \frac{1}{2} \ell_{\frac{\alpha}{2}}) \Gamma((\rho(\ell), \alpha^\vee) + \frac{1}{2} \ell_{\frac{\alpha}{2}} + \ell_\alpha)}{\Gamma(-(\rho(\ell), \alpha^\vee) - (\varepsilon, \alpha^\vee) + \frac{1}{2} \ell_{\frac{\alpha}{2}} + \ell_\alpha) \Gamma((\rho(\ell), \alpha^\vee) + \frac{1}{2} \ell_{\frac{\alpha}{2}})}$$

can be factorized as  $f(\varepsilon, \ell) \cdot g(\varepsilon, \ell)$  where  $f(\varepsilon, \ell)$  is holomorphic on  $\text{Re}(\ell) > 0$  and  $g(\varepsilon, \ell)$  is a meromorphic function which satisfies  $g(\varepsilon, \ell + \ell_i) = (-1)^{\sum_{\alpha \in R_+} \langle \ell_i, \alpha \rangle} \cdot g(\varepsilon, \ell)$ , by means of the formula  $\Gamma(z) \Gamma(1-z) = \pi / \sin \pi z$ . For instance, take

$$f(\varepsilon, \ell) = \prod_{\alpha \in R_+} \frac{\Gamma((\rho(\ell), \alpha^\vee) + (\varepsilon, \alpha^\vee) - \frac{1}{2} \ell_{\frac{\alpha}{2}} - \ell_\alpha + 1) \Gamma((\rho(\ell), \alpha^\vee) + \frac{1}{2} \ell_{\frac{\alpha}{2}} + \ell_\alpha)}{\Gamma((\rho(\ell), \alpha^\vee) + (\varepsilon, \alpha^\vee) - \frac{1}{2} \ell_{\frac{\alpha}{2}} + 1) \Gamma((\rho(\ell), \alpha^\vee) + \frac{1}{2} \ell_{\frac{\alpha}{2}})},$$

and

$$g(\varepsilon, \ell) = \prod_{\alpha \in R_+} \frac{\sin \pi((\rho(\ell), \alpha^\vee) + (\varepsilon, \alpha^\vee) - \frac{1}{2} \ell_{\frac{\alpha}{2}} - \ell_\alpha)}{\sin \pi((\rho(\ell), \alpha^\vee) + (\varepsilon, \alpha^\vee) - \frac{1}{2} \ell_{\frac{\alpha}{2}})}.$$

Put  $f(\ell) = \lim_{\varepsilon \rightarrow 0} f(\varepsilon, \ell)$ . So (4.2) takes the form:

$$\int_T \sigma(\ell + \ell_i; t) dt / \int_T \sigma(\ell; t) dt = f(\ell + \ell_i) / f(\ell),$$

or equivalently, the function

$$h(\ell) = \int_T \sigma(\ell; t) dt / f(\ell)$$

is periodic on  $\mathcal{K}$  with period lattice  $\mathcal{B}$ . Consequently,  $h(\mathcal{K})$  is an entire function (because it is certainly analytic on  $\operatorname{Re}(\mathcal{K}) > 0$ ). From Stirling's asymptotic expansion formula

$$\Gamma(z) \sim e^{-z} z^z \left( \frac{2\pi}{z} \right)^{\frac{1}{2}} \left[ 1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots \right] \quad (|\arg z| < \pi) \quad (4.3)$$

we see that, if  $\mathcal{K} \in \mathcal{K}$  fixed such that  $\mathcal{K}_\alpha \in \mathbb{R}_+$ , and  $z \in \mathbb{C}$  a complex indeterminate:

$$f(z, \mathcal{K}) \sim C^z \left( 1 + 0 \left( \frac{1}{z} \right) \right) \quad (4.4)$$

for some  $C \in \mathbb{R}_{>0}$ . On the other hand,

$$\left| \int_T \sigma(z, \mathcal{K}; t) dt \right| \leq \int_T \sigma(\operatorname{Re}(z), \mathcal{K}; t) dt. \quad (4.5)$$

If we choose  $\mathcal{K}$  such that  $\mathcal{K}_\alpha \in \mathbb{N} \forall \alpha \in \mathbb{R}$ , then  $z \rightarrow h(z, \mathcal{K})$  is periodic, and thus bounded as a consequence of (4.4) and (4.5). Therefore, by Liouville's theorem,  $h(\mathcal{K})$  is a constant:

$$\int_T \sigma(\mathcal{K}; t) dt = a \cdot f(\mathcal{K})$$

for some  $a \in \mathbb{C}$ .

We determine  $a$  as follows:

$$1 = \lim_{\varepsilon \downarrow 0} \int_T \sigma(\varepsilon \mathcal{K}; t) dt = a \cdot \lim_{\varepsilon \downarrow 0} f(\varepsilon \mathcal{K}) = a \cdot \prod_{\alpha \in R_+} \frac{((\rho(\mathcal{K}), \alpha^\vee) + \frac{1}{2} \mathcal{K}_{\frac{\alpha}{2}})}{((\rho(\mathcal{K}), \alpha^\vee) + \frac{1}{2} \mathcal{K}_{\frac{\alpha}{2}} + \mathcal{K}_\alpha)}.$$

Hence

$$a = \prod_{\alpha \in R_+} \frac{((\rho(\mathcal{K}), \alpha^\vee) + \frac{1}{2} \mathcal{K}_{\frac{\alpha}{2}} + \mathcal{K}_\alpha)}{((\rho(\mathcal{K}), \alpha^\vee) + \frac{1}{2} \mathcal{K}_{\frac{\alpha}{2}})} = |W|. \quad (4.6)$$

The last equality is obtained in the following way: if we put  $\mathcal{K} = e = \sum_{i: 2C_i \cap R = \emptyset} e_i$

then  $\forall \alpha \in R^0$ ,  $(\rho(\mathcal{K}), \alpha^\vee) = (\rho(e), \alpha^\vee) = ht(\alpha^\vee)$ , the height of  $\alpha^\vee$  in the root system  $(R^0)^\vee$ . So we have:  $a = \prod_{\alpha \in R_+^0} \frac{ht(\alpha^\vee) + 1}{ht(\alpha^\vee)}$ . In order to see that this is equal to  $|W|$

we take the limit  $t \rightarrow 1$  in the following identity of Macdonald, Bott and Solomon (see for instance [C], Sect. 9.4 and 10.2):

$$\sum_{w \in W} t^{l(w)} = \prod_{\alpha \in R_+^0} \left( \frac{t^{ht(\alpha^\vee) + 1} - 1}{t^{ht(\alpha^\vee)} - 1} \right) = \prod_{i=1}^n \left( \frac{t^{d_i} - 1}{t - 1} \right)$$

where  $l(w)$  denotes the length of  $w \in W$  and  $d_1, \dots, d_n$  are the primitive degrees of  $R^0$ .  $\square$

### 5. Evaluation of Jacobi polynomials at the identity element

In [HO] we conjectured that the value at the identity element  $e \in H$  of the hypergeometric function associated with the root system  $R$  should be 1, in analogy with the group case ([HO], Conjecture 6.11). In this section we will show that the existence of shift operators implies a partial affirmation of this conjecture. (For a definition of  $F(\lambda, \ell; h)$ , the hypergeometric function, we refer the reader to [HO], Sect. 6 or [H], Sect. 7).

**5.1. Theorem.**  $F(\lambda, \ell; e) = 1$  if  $\ell \in \mathbb{Z}_+ \cdot \mathcal{B}$ .

*Proof.* From the determination of the constant  $a = |W|$  in the proof of Theorem 4.1 (see formula (4.6)) we see that

$$\lim_{\varepsilon \downarrow 0} \tilde{c}(-\rho(\varepsilon \ell), \varepsilon \ell) = |W|.$$

Therefore it is obvious that

$$F(\lambda, 0; e) = 1 \forall \lambda \in \mathfrak{h}^*.$$

Suppose that  $F(\lambda, \ell; e) = 1$  for all  $\lambda \in \mathfrak{h}^*$  and  $\ell = \sum_{i=1}^m n_i \ell_i$  with  $n_i \in \mathbb{Z}_+$  and  $\sum_i n_i \leq N$ .

Take  $\ell \in \mathbb{Z}_+ \cdot \mathcal{B}$  with  $\ell = \sum_{i=1}^m n'_i \ell'_i$  and  $\sum_i n'_i = N + 1$ . Let  $i \in \{1, \dots, m\}$  such that  $n'_i > 0$ . Then:

$$G(-\ell_i, \ell) F(\lambda, \ell; h) = \eta(G(-\ell_i, \ell))(\lambda) \frac{c(\lambda, \ell)}{c(\lambda, \ell - \ell_i)} F(\lambda, (\ell - \ell_i); h). \quad (5.1)$$

This formula (5.1) is the generalization of formula (3.1) to the case of arbitrary hypergeometric functions (instead of Jacobi polynomials), and can be derived from the definition of shift operator in a completely similar fashion (recall that, in general,  $P(\mu, \ell) = (1/c(\mu - \rho(\ell), \ell)) \cdot F(\mu - \rho(\ell), \ell)$  if  $\mu \in P_-$  (see [H], Sect. 8)). Now evaluate formula (5.1) at  $h = e$ , and use Corollary 3.4:

$$\begin{aligned} \eta(G(-\ell_i, \ell))(-\rho(\ell)) \frac{c(-\rho(\ell), \ell)}{c(-\rho(\ell), \ell - \ell_i)} F(-\rho(\ell), \ell - \ell_i; e) \cdot F(\lambda, \ell; e) \\ = \eta(G(-\ell_i, \ell))(\lambda) \frac{c(\lambda, \ell)}{c(\lambda, \ell - \ell_i)} F(\lambda, \ell - \ell_i; e). \end{aligned}$$

Thus, from the induction hypothesis:

$$\begin{aligned} F(\lambda, \ell; e) &= \frac{\eta(G(-\ell_i, \ell))(\lambda)}{\eta(G(-\ell_i, \ell))(-\rho(\ell))} \cdot \frac{c(\lambda, \ell)}{c(\lambda, \ell - \ell_i)} \cdot \frac{c(-\rho(\ell), \ell - \ell_i)}{c(-\rho(\ell), \ell)} \\ &= \frac{1}{c(-\rho(\ell), \ell)} = 1. \quad \square \end{aligned}$$

**5.2. Corollary.**  $P(\mu, \ell; e) = \frac{1}{c(\mu - \rho(\ell), \ell)} \forall \mu \in P_-, \ell \in \mathcal{K}$ .

*Proof.*  $P(\mu, \ell; e) = \sum_{\nu \in C(\mu)} \Gamma_\nu(\mu, \ell)$  is a rational function in  $\ell$ . Therefore, this corollary follows from Theorem 5.1 and the remark that  $\mathbb{Z}_+ \cdot \mathcal{B}$  is Zariski dense in  $\mathcal{K}$ .  $\square$

## 6. Proof of I.G. Macdonald's generalization of a conjecture of M.L. Mehta

The following formula was also conjectured by Macdonald in his paper [M1] (Conjecture 6.1)

$$\int_{\mathfrak{a}} D(\ell; x) d\gamma(x) = \prod_{\alpha \in R_+} \frac{(\frac{1}{2}\ell_\alpha + \frac{1}{4}\ell_{\frac{\alpha}{2}} + \frac{1}{2}(\rho(\ell), \alpha^\vee))!}{(\frac{1}{4}\ell_{\frac{\alpha}{2}} + \frac{1}{2}(\rho(\ell), \alpha^\vee))!}. \quad (6.1)$$

Here  $R$  is any (possibly non reduced) root system and  $\ell \in \mathcal{K}$  such that  $\operatorname{Re}(\ell_\alpha) \geq 0 \forall \alpha \in R$ , and  $D(\ell; x) = \prod_{\alpha \in R_+} \left| \frac{\sqrt{2}}{|\alpha|} \alpha(x) \right|^{\ell_\alpha} = \prod_{\alpha \in R_+} |\tilde{\alpha}(x)|^{\ell_\alpha}$ . The measure  $d\gamma(x)$  denotes the Gaussian measure on  $\mathfrak{a}$ , i.e.  $d\gamma(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}|x|^2} dx$ , where  $dx$  is the ordinary Lebesgue measure on  $\mathfrak{a}$ . One might think of this formula as the Lie algebra counterpart of formula (4.1), and this idea leads to a proof of (6.1) if  $\ell$  corresponds to the multiplicity of a symmetric space with reduced root system  $R$  (see [M1], Sect. 6.b). Formula (6.1) is Macdonald's "root system generalization" of an older conjecture due to Mehta (the case  $R = A_n$ ), which can be proved using Selberg's integral formula ([M1], Sect. 4). In ([M1], Sect. 6) Macdonald establishes formula (6.1) for all other classical root systems, again by making use of Selberg's integral formula.

The shift operators enable us to prove (6.1) in a uniform way, as we will see in Theorem 6.4. First of all, let us formulate two facts (Lemma 6.1 and Theorem 6.3) that are needed in the course of the proof of Theorem 6.4.

**6.1. Lemma.** *Let  $i \in \{1, \dots, m\}$  such that  $2C_i \cap R = \emptyset$ . Then (recall that  $\varepsilon$  denotes the lowest homogeneous part):*

$$\varepsilon(G(\ell_i, \ell)) e^{-\frac{1}{2}|x|^2} = \left( \prod_{\alpha \in C_{i+}} \frac{2}{|\alpha|^2} \right) e^{-\frac{1}{2}|x|^2}. \quad (6.2)$$

*Proof.* From Sect. 3B) and C) we know that  $\varepsilon(G(\ell_i, \ell)) \in \mathbb{A}_n(p_1, \dots, p_n)$  (the  $p_i$  being a set of fundamental polynomial  $W$ -invariants), and this operator has degree  $\sum_{\alpha \in R_+} (\ell_i)_\alpha$ , while its homogeneous degree (with respect to the weighted Euler vector field  $E = \sum_i d_i p_i \frac{\partial}{\partial p_i}$ ) equals  $-2 \sum_{\alpha \in R_+} (\ell_i)_\alpha$ . Choose  $p_1 = -\frac{|x|^2}{2}$ . The

only summands of  $\varepsilon(G(\ell_i, \kappa))$  that do not annihilate the function  $e^{p_1}$  are those of the form  $p^\alpha \left( \frac{\partial}{\partial p_1} \right)^\beta$  (with  $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_+)^n$  a multiindex and  $\beta \in \mathbb{Z}_+$ ). But

$$\deg \left( p^\alpha \left( \frac{\partial}{\partial p_1} \right)^\beta \right) = \beta \leq \sum_{\alpha \in R_+} (\ell_i)_\alpha \quad (6.3)$$

and

$$\text{h.d.} \left( p^\alpha \left( \frac{\partial}{\partial p_1} \right)^\beta \right) = \sum_{i=1}^n d_i \alpha_i - 2\beta = -2 \sum_{\alpha \in R_+} (\ell_i)_\alpha. \quad (6.4)$$

These relations imply that  $\beta = \sum_{\alpha \in R_+} (\ell_i)_\alpha$  and  $\alpha = 0$ , which shows that  $e^{p_1}$  is indeed an eigenfunction of  $\varepsilon(G(\ell_i, \kappa))$ . In order to determine the eigenvalue, observe that  $f(\kappa) \left( \frac{\partial}{\partial p_1} \right)_{\alpha \in R_+}^{\sum (\ell_i)_\alpha}$  occurs, for some  $f(\kappa) \in \mathbb{C}[\mathcal{X}]$ , as a summand of the highest order part of  $\varepsilon(G(\ell_i, \kappa))$ . The highest order part of  $\varepsilon(G(\ell_i, \kappa))$  does not depend on  $\kappa$ , as we remarked in Sect. 3B). Consequently,  $f(\kappa)$  is a constant,  $f$  say, and we may determine this eigenvalue by calculating  $\varepsilon(G(\ell_i, 0))(e^{p_1})$ . According to Sect. 3B) we have:  $\varepsilon(G(\ell_i, 0)) = \frac{(-1)^{|\mathcal{C}_i|+1}}{\prod_{\alpha \in \mathcal{C}_i} (\alpha)} \prod_{\beta \in \mathcal{C}_i} \partial(X_{\beta^\vee})$ . Hence:

$$f = e^{-p_1} \frac{(-1)^{|\mathcal{C}_i|+1}}{\prod_{\alpha \in \mathcal{C}_i} \alpha} \prod_{\beta \in \mathcal{C}_i} \partial(X_{\beta^\vee})(e^{p_1}) = (-1)^{|\mathcal{C}_i|+1} \frac{\prod_{\alpha \in \mathcal{C}_i} (-\alpha^\vee)}{\prod_{\alpha \in \mathcal{C}_i} \alpha} = \prod_{\alpha \in \mathcal{C}_i} \frac{2}{|\alpha|^2}.$$

**6.2. Remark.** If  $C_i$  is a conjugacy class such that  $2C_i = C_j \subset R$  then one can show that (if  $\ell_i = 2e_j - e_j$ ):

$$\varepsilon(G(\ell_i, \kappa)) e^{-\frac{1}{2}|x|^2} = \left( \prod_{\alpha \in \mathcal{C}_i} \frac{8}{|\alpha|^2} \right) e^{-\frac{1}{2}|x|^2}.$$

This calculation is similar to the calculation above, but more complicated because there is not such a simple formula for  $G(\ell_i, 0)$  available in this case. Moreover, this result is not necessary for the proof of Theorem 6.4.  $\square$

**6.3. Theorem.** (Carlson, see [T] Theorem 5.81). *Let  $f$  be a function of one complex variable  $z$ , and suppose that*

(i)  *$f$  is analytic and bounded on  $\text{Re}(z) \geq 0$ ,*

(ii)  *$f(z) = 0$  for  $z = 0, 1, 2, \dots$*

*Then  $f(z) = 0$  identically.  $\square$*

After these preparations we can prove

**6.4. Theorem.** *Let  $R$  be a root system and  $\kappa \in \mathcal{X}$  such that  $\text{Re}(\kappa_\alpha) \geq 0 \forall \alpha \in R$ . Then:*

$$\int_{\mathfrak{a}} D(\kappa; x) d\gamma(x) = \prod_{\alpha \in R_+} \frac{(\frac{1}{2}\kappa_\alpha + \frac{1}{4}\kappa_{\frac{\alpha}{2}} + \frac{1}{2}(\rho(\kappa), \alpha^\vee))!}{(\frac{1}{4}\kappa_{\frac{\alpha}{2}} + \frac{1}{2}(\rho(\kappa), \alpha^\vee))!}.$$

*Proof.* Consider identity (3.4) on  $A$ :

$$G^+(\ell_i, \ell - \ell_i) \circ \delta(\ell) = \delta(\ell - \ell_i) \circ G(-\ell_i, \ell)$$

and let this act on the function  $1 \equiv P(0, \ell)$ :

$$G^+(\ell_i, \ell - \ell_i)(\delta(\ell)) = \eta(G(-\ell_i, \ell))(-\rho(\ell))P(-\rho(\ell_i), \ell - \ell_i) \cdot \delta(\ell - \ell_i).$$

Now take the lowest homogeneous part of this equation (see Sect. 3C)):

$$\varepsilon(G^+(\ell_i, \ell - \ell_i))\Pi(2\ell) = \eta(G(-\ell_i, \ell))(-\rho(\ell))P(-\rho(\ell_i), \ell - \ell_i; e) \cdot \Pi(2(\ell - \ell_i))$$

where  $\Pi(\ell; x) = \prod_{\alpha \in R_+} |\alpha(x)|^{\ell_\alpha}$  (so in this formula we assume that  $\ell_\alpha \in \mathbb{R}_+$ , sufficiently positive. Note that the map  $\varepsilon$  we used here is an extension to the localization of  $\mathbb{A}_n(z_1, \dots, z_n)$  at  $\Delta^2 \neq 0$  of the map we defined in Sect. 3C)). Use the expression (3.2) for  $\eta(G(-\ell_i, \ell))$  and Cor. 5.2 to write this as follows:

$$\varepsilon(G^+(\ell_i, \ell - \ell_i))\Pi(2\ell) = \frac{\tilde{c}(-\rho(\ell - \ell_i), \ell - \ell_i)}{\tilde{c}(\rho(\ell), \ell)} \cdot \Pi(2(\ell - \ell_i)).$$

Observe that  $\varepsilon(G^+) = \varepsilon(G)^+$ , where  $+$  denotes the formal transpose on  $A$  and  $a$  respectively. Hence, if  $\ell$  is real and sufficiently positive, we have:

$$\begin{aligned} & \int_a D(2(\ell - \ell_i); x) e^{-\frac{1}{2}|x|^2} (2\pi)^{-\frac{n}{2}} dx \\ &= \prod_{\alpha \in R_+} \frac{2^{(\ell - \ell_i)_\alpha}}{|\alpha|^{2(\ell - \ell_i)_\alpha}} \int_a \Pi(2(\ell - \ell_i); x) e^{-\frac{1}{2}|x|^2} (2\pi)^{-\frac{n}{2}} dx \\ &= \prod_{\alpha \in R_+} \frac{2^{(\ell - \ell_i)_\alpha}}{|\alpha|^{2(\ell - \ell_i)_\alpha}} \int_a \frac{\tilde{c}(-\rho(\ell), \ell)}{\tilde{c}(-\rho(\ell - \ell_i), \ell - \ell_i)} \\ & \quad \varepsilon(G(\ell_i, \ell - \ell_i))^+ (\Pi(2\ell; x)) e^{-\frac{1}{2}|x|^2} (2\pi)^{-\frac{n}{2}} dx \\ &= \prod_{\alpha \in R_+} \frac{2^{(\ell - \ell_i)_\alpha}}{|\alpha|^{2(\ell - \ell_i)_\alpha}} \int_a \frac{\tilde{c}(-\rho(\ell), \ell)}{\tilde{c}(-\rho(\ell - \ell_i), \ell - \ell_i)} \\ & \quad \Pi(2\ell; x) \varepsilon(G(\ell_i, \ell - \ell_i)) e^{-\frac{1}{2}|x|^2} (2\pi)^{-\frac{n}{2}} dx. \end{aligned} \tag{6.5}$$

At this point we assume that  $i \in \{1, \dots, m\}$  is such that  $2C_i \cap R = \emptyset$ , and we apply Lemma 6.1. Formula (6.5) becomes:

$$\int_a D(2(\ell - \ell_i); x) d\gamma(x) = \frac{\tilde{c}(-\rho(\ell), \ell)}{\tilde{c}(-\rho(\ell - \ell_i), \ell - \ell_i)} \int_a D(2\ell; x) d\gamma(x),$$

where we assume  $2C_i \cap R = \emptyset$ . Therefore, if  $\ell \in \sum_{i=1}^m \mathbb{Z}_+ \cdot e_i$ , we have

$$\int_a D(2\ell; x) d\gamma(x) = \frac{a(\ell)}{\tilde{c}(\rho(\ell), \ell)}, \tag{6.6}$$

where  $a(\kappa)$  is a function that (only) depends on  $\kappa_\alpha$  with  $2\alpha \in R$ . However, it is obvious that the left hand side of this equation is in fact a function only depending on  $\kappa_\alpha + \kappa_{\frac{\alpha}{2}}$ , and it is an easy calculation to see that  $f(\kappa) = \frac{1}{\tilde{c}(-\rho(\kappa), \kappa)}$  has the property that  $f(z \cdot e_i) = f(z \cdot e_j) \forall z \in \mathbb{C}$ , if  $2C_i = C_j$ . So  $a(\kappa)$  is a constant, also denoted by  $a$ .

The arguments so far establish the theorem (up to the constant  $a$ ) for  $\kappa \in \sum_{i=1}^m \mathbb{Z}_+ \cdot e_i$ . In the remaining part of the proof we will see how this implies the general case  $\text{Re}(\kappa_\alpha) \geq 0$ . Consider the following functions of the complex variable  $z$ : (fix  $\kappa \in \sum_i \mathbb{R}_{>0} \cdot e_i$ )

$$\begin{aligned} \varphi(z) &= \int_a D(2z\kappa; x) d\gamma(x) = \int_a D(2\kappa; x)^z d\gamma(x) \\ \Psi(z) &= \frac{1}{c(-2z\rho(\kappa), z\kappa)} = \prod_{\alpha \in R_+} \frac{\Gamma(z(\kappa_\alpha + \frac{1}{2}\kappa_{\frac{\alpha}{2}} + (\rho(\kappa), \alpha^\vee)))}{\Gamma(z(\frac{1}{2}\kappa_{\frac{\alpha}{2}} + (\rho(\kappa), \alpha^\vee)))}. \end{aligned}$$

First we study the asymptotic behaviour of these two functions. From its explicit expression, it is clear that (use Stirling's asymptotic expansion for  $\Gamma(z)$  (see (4.3)))

$$\Psi(z) \sim (N \cdot z)^{\sum_{\alpha \in R_+} (\kappa_\alpha) \cdot z}$$

for a certain  $N \in \mathbb{R}_{>0}$  and  $|z|$  large (with  $\text{Re } z > 0$ ).

As for  $\varphi$ , if  $z \in \mathbb{R}_{>0}$  then

$$\varphi(z) = z^{\left(\sum_{\alpha \in R_+} (\kappa_\alpha) \cdot z + \frac{1}{2}n\right)} \int_a (D(2\kappa; y) \cdot e^{-\frac{1}{2}|y|^2})^z e^{-\frac{1}{2}|y|^2 \cdot z} (2\pi)^{-\frac{n}{2}} dy$$

and this identity holds for all  $z$  with  $\text{Re } z > 0$  because of analyticity in  $z$ . The function  $D(2\kappa; y) \cdot e^{-\frac{1}{2}|y|^2}$  is bounded on  $a$ , and we choose  $N' \in \mathbb{R}_{>0}$  such that  $D(2\kappa; y) e^{-\frac{1}{2}|y|^2} < N'$  on  $a$ . Hence:

$$|\varphi(z)| \leq |z|^{\left(\sum_{\alpha \in R_+} (\kappa_\alpha) \cdot z + \frac{1}{2}n\right)} \left| |(N')^z \right| \left| \frac{1}{(\text{Re } z)} \right|^{\frac{n}{2}} \int_a e^{-\frac{1}{2}|t|^2} (2\pi)^{-\frac{n}{2}} dt$$

So on the half plane  $\text{Re}(z) \geq 1$  we have

$$|\varphi(z)| \leq |(N'' z)^{\left(\sum_{\alpha \in R_+} (\kappa_\alpha) \cdot z + \frac{1}{2}n\right)}|$$

for some  $N'' \in \mathbb{R}_{>0}$ . Thus, for a suitable constant  $N''' \in \mathbb{R}_{>0}$ , the function

$$\sigma(z) = (N''' z)^{-\left(\sum_{\alpha \in R_+} (\kappa_\alpha) \cdot z - \frac{1}{2}n\right)} [\varphi(z) - a\psi(z)]$$



is analytic and bounded on  $\text{Re } z \geq 1$ . If we take  $\ell \in \sum_i \mathbb{N} e_i$  then, by virtue of (6.6),  $\sigma(z) = 0$  for  $z \in \mathbb{N}$ . Consequently, by Carlson's theorem (Theorem 6.3), the equality  $\varphi(z) = a\psi(z)$  holds whenever  $\ell \in \sum_i \mathbb{N} e_i$ . This, in turn, implies immediately that:

$$\int_a D(2\ell; x) d\gamma(x) = \frac{a}{c(-\rho(\ell), \ell)}, \forall \ell \in \mathcal{K} \quad \text{with } \text{Re}(\ell_\alpha) \geq 0 \forall \alpha \in R$$

If we take the same type of limit as in the proof of Theorem 4.1 (see formula (4.6)) we get:

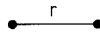
$$a = |W| = \prod_{\alpha \in R_+} \frac{(\ell_\alpha + \frac{1}{2}\ell_{\frac{\alpha}{2}} + (\rho(\ell), \alpha^\vee))}{(\frac{1}{2}\ell_{\frac{\alpha}{2}} + (\rho(\ell), \alpha^\vee))}.$$

This completes the proof of Theorem 6.4.  $\square$

6.5. *Remark.* Macdonald also generalized Mehta's conjecture to the case of an arbitrary finite reflection group on  $\mathfrak{a}$  (see [M1], Conjecture 5.1). From the classification of finite reflection groups (see [B]) we know that, apart from the Weyl groups or crystallographic finite reflection groups, there exist the following non crystallographic types:  $H_3$  and  $H_4$ , with Coxeter diagrams



respectively, and the dihedral groups  $I_{2r}(r=5, r \geq 7)$  with Coxeter diagram



Our methods do not apply to these non crystallographic groups, but the dihedral case can be verified by means of a direct calculation (see [M1], Sect. 5). Summarizing, Conjecture 5.1 of [M1] remains unproved for the groups  $H_3, H_4$  only.  $\square$

**7. The Bernstein-Sato polynomial of the infinitesimal discriminant; proof of a conjecture of T. Yano and J. Sekiguchi**

The following theorem was conjectured by Yano and Sekiguchi in their paper [Y.S]. In this section we will show that this theorem is an easy consequence of the previous section of this paper.

**7.1. Theorem.** Let  $R$  be a reduced root system and put  $I = \prod_{\alpha \in R_+} \alpha^2$ . Interpreted as an element of the polynomial algebra  $\mathbb{C}[p_1, \dots, p_n]$  of polynomial  $W$ -invariants, the Bernstein Sato polynomial  $b$  of  $I$  equals  $b(\ell) = \prod_{i=1}^n \prod_{j=1}^{d_i-1} \left( \ell + \frac{1}{2} + \frac{j}{d_i} \right)$ , where the  $d_i$  are the primitive degrees of  $R$ .

*Proof.* We derive from formula (3.3) that

$$\varepsilon(G(1, \ell - 1))^* I^{\ell - \frac{1}{2}} = \frac{\tilde{c}(-\rho(\ell - 1), \ell - 1)}{\tilde{c}(-\rho(\ell), \ell)} I^{\ell - \frac{3}{2}}, \quad (7.1)$$

just as we did in the proof of Theorem 6.4. Here we identify  $\mathbb{R}_e \subset \mathcal{K}$  with  $\mathbb{R}$  by sending  $e$  to 1. The expression  $\frac{1}{\tilde{c}(-\rho(\ell), \ell)}$  reduces, in this special case of equal multiplicities for all roots, to the expression

$$\prod_{i=1}^n \frac{\Gamma(d_i \ell)}{\Gamma(\ell)}.$$

Hence, if we use Gauss' multiplication formula, (7.1) becomes:

$$\varepsilon(G(1, \ell + \frac{1}{2}))^* I^{\ell + 1} = |\mathcal{W}| \prod_{i=1}^n \prod_{j=1}^{d_i - 1} (d_i(\ell + \frac{1}{2}) + j) I^{\ell}.$$

It is easy to see that  $G(1, \ell + \frac{1}{2})^* = (-1)^{|R^+|} G(1, \frac{1}{2} - \ell)$ , so

$$\varepsilon(G(1, \frac{1}{2} - \ell)) I^{\ell + 1} = (-1)^{|R^+|} |\mathcal{W}| \prod_{i=1}^n \prod_{j=1}^{d_i - 1} (d_i(\ell + \frac{1}{2}) + j) I^{\ell}. \quad (7.2)$$

Define  $\tilde{b}(\ell) = \prod_{i=1}^n \prod_{j=1}^{d_i - 1} (d_i(\ell + \frac{1}{2}) + j)$ . If  $b(\ell)$  denotes the  $b$ -function of  $I$ , the relation (7.2) implies that

$$b | \tilde{b} \quad (7.3)$$

In order to prove that  $\tilde{b}$  is actually minimal (so is equal to  $b$ ) introduce the following notations: if  $f$  is a meromorphic function on  $\mathbb{C}$ , we define the following multiplicity function

$$m(f): \mathbb{C} \rightarrow \mathbb{Z}$$

$$z \rightarrow n \quad \text{if } f(w) = \sum_{k=n}^{\infty} a_k (w - z)^k \quad \text{with } a_n \neq 0.$$

For any polynomial  $P \in \mathbb{C}[z]$  we define

$$m_{\Sigma}(P): \mathbb{C} \rightarrow \mathbb{Z}$$

$$z \rightarrow \sum_{n \in \mathbb{Z}_+} m\left(\frac{1}{P}\right)(z + n).$$

Note that  $m(P)(z) = m_{\Sigma}(P)(z + 1) - m_{\Sigma}(P)(z)$ .

So (7.3) implies

$$m_{\Sigma}(b) \geq m_{\Sigma}(\tilde{b}). \quad (7.4)$$

But from the Bernstein equation

$$BI^{\ell+1} = b(\ell)I^{\ell} \text{ (for some } B \in \mathbb{C}[\mathcal{K}] \otimes \mathbb{A}_n(p_1, \dots, p_n))$$

it is obvious that

$$m\left(\int_{\alpha} I^{\ell+\frac{1}{2}}(x) d\gamma(x)\right) \geq m_{\Sigma}(b).$$

On the other hand, from the explicit formula (6.1) for  $\int_{\alpha} I^{\ell}(x) d\gamma(x)$  we see that

$$m\left(\int_{\alpha} I^{\ell+\frac{1}{2}}(x) d\gamma(x)\right) = m_{\Sigma}(\tilde{b}).$$

Hence  $m_{\Sigma}(\tilde{b}) = m_{\Sigma}(b)$  and thus  $b$  and  $\tilde{b}$  are equal up to a multiplicative constant. This completes the proof of Theorem 7.1.  $\square$

*7.2. Remark.* Yano and Sekiguchi actually conjectured that the formula for the  $b$ -function of the infinitesimal discriminant should hold for non-crystallographic finite reflection groups as well. Unfortunately, our methods are restricted to the Weyl group cases (see also Remark 6.5).  $\square$

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