Chapter 1

Introduction

Statistical physics is the branch of physics which aims to describe the collective behavior of a large number of constituting pieces, typically particles. The precise, or near precise description of such a system (e.g. the trajectories of all the particles in an interacting classical system) is a hopelessly complicated task, and statistical physics does not aim for that. Its goal is to describe the system through average macroscopic quantities, such as temperature, density, magnetization, etc. To describe the intrinsic behavior of systems, correlations between different parts of the system are probed in the context of statistical physics. An example is how the correlation of local magnetization decays with the distance.

One important focus of statistical physical research is the description of phase transition: the phenomenon when some macroscopic quantity of the system undergoes a sudden change. It is observed that while usually the thermodynamic properties of the materials change smoothly, at certain values of the parameters, they undergo abrupt, often discontinuous changes. E.g. the density of the material changes drastically, or a magnet loses its magnetization. The phase transition happens at certain external conditions, e.g. at atmospheric pressure, the water boils at $T = 100^\circ C$. These external conditions are characterized by intensive thermodynamic quantities. Intensive thermodynamic quantities are the thermodynamic quantities which –instead of adding together– equalize when some interaction is introduced between two previously separate systems.

Two types of phase transitions are distinguished, first (or discontinuous) and second order (or continuous) phase transition. To describe the difference between them, first we introduce the concept of correlation length. Correlation length is an intrinsic length scale of the material, which can be observed, e.g. by scattering experiments. At distances less than the correlation
length, microscopic properties are correlated, while much further apart, they are not. Naturally, the correlation length depends on the external conditions, such as temperature, pressure, external magnetic field, etc. This property is used to characterize the phase transitions.

The correlation length behaves differently in the two types of phase transition. At the phase transition point, the correlation length either remains finite, or becomes effectively infinite.

In case of the first order phase transition, the correlation length remains finite. In this case, phase coexistence occurs, which means that two (or more) different phases of the material are in thermal equilibrium, e.g. water and steam are in equilibrium. This is in agreement with the finite correlation length, as one would expect that the different coexisting phases are uncorrelated.

For second order phase transition, the correlation length becomes infinite, correlations extend to the whole system. The system is said to be critical. In this case, the typical, exponentially decaying correlations disappear, and the correlations are ruled by power laws. The points in the phase diagram where a continuous transition takes place are called critical points. In the vicinity of the critical point, thermodynamic quantities are singular with simple non-integer powers. The scaling hypothesis further predicts simple algebraic relations between these exponents. As the whole system is correlated, one would expect the description at the critical points to be intrinsically difficult. Many approximation methods are expected to break down, as they are based on neglecting the correlations between components far apart (e.g. mean field theory).

Despite these difficulties, there are simplifications, which help to describe systems close to the critical point. It is observed that the critical exponents—the exponents appearing in the correlations and close to criticality—are independent of the microscopic details of the model. There are classes of systems with different Hamiltonians which seem to share the same exponents. These classes are based on a few properties of the Hamiltonian, like the dimensionality, the symmetry of the Hamiltonian.

Our main tool to probe the statistical properties of a system is the partition function. Define the partition function as follows

$$Z = \sum_s e^{-\beta H(s)}$$  \hspace{1cm} (1.1)

Here, the summation runs over all the possible states \(s\) of the system, \(H\) is the Hamiltonian of the system, and \(\beta = \frac{1}{T}\) is the inverse temperature (in appropriate units). As the states might be labeled by continuous index, the summation is rather formal here, and might denote integration. The partition function is the
fundamental object to describe systems in thermal equilibrium with some further restriction, such as a fixed number of degrees of freedom. The probability that the system is in state $s$ is proportional to the Boltzmann weight $e^{-\beta H(s)}$, and in this context, $Z$ acts as a normalization factor

$$P(s) = \frac{e^{-\beta H(s)}}{Z} \quad (1.2)$$

The partition function is a useful tool to derive thermodynamic quantities. The average thermodynamic value of some observable $A$ is computed as follows

$$\langle A \rangle = \frac{\sum_s A(s) e^{-\beta H(s)}}{Z} \quad (1.3)$$

Thermodynamic quantities follow as derivatives of the partition function. For example, the internal energy – the average energy of the system – is

$$U = \langle H \rangle = \frac{\sum_s H(s) e^{-\beta H(s)}}{Z} = -\frac{\partial}{\partial \beta} \log Z \quad (1.4)$$

$$= T^2 \frac{\partial}{\partial T} \log Z \quad (1.5)$$

Another example is the free energy $F$ of the system, which is given by the following equation in appropriate units

$$F = -T \log Z \quad (1.7)$$

One of the main challenges of statistical physics is the computation of the partition function. However, this is a tremendously difficult question for any nearly realistic system. In order to compute derivatives of $Z$, one needs analytic computations, which are even more hopeless. Nevertheless, perturbative methods are used to compute approximate results. Any of these perturbative methods which are somehow based on neglecting some of the correlations fail near critical point, which may lead one to think that computations are unmanageable here.

However, the idea of universality shines a light on this problem: In order to compute the universal properties of certain material, one does not need to consider the actual material, but equivalently might look at a much simpler model in the same universality class. This leads to a different approach to the same problem: Which are those simple systems where the partition function is
exactly computable, and what can we learn from these systems? This question leads us to the two-dimensional lattice models. Two-dimensional lattice models are simple statistical physical models, where the degrees of freedom live on a two-dimensional lattice, e.g. spins on the vertices, or on the bonds. Usually the degrees of freedom take values from a finite set, and in general, these models are highly oversimplified compared to real world systems. Exactly solvable – here meaning that the partition function is computed exactly – models were found mainly in one and two spatial dimensions. Accepting the assumption that our world has three spatial dimensions, lower dimensional systems are generally less interesting: they might be considered as non-physical. However, this is not completely the case. In recent experimental settings, effectively one and two-dimensional systems are created with tunable parameters, relating these result to actual physical experiments. Also, these models are very valuable in understanding concepts and ideas in statistical physics. And let me quote a final motivation from Baxter: they are relevant and they can be solved, so why not do so and see what they tell us?

Systems that are exactly at critical point exhibit scale invariance. Due to the infinite correlation length and the power-law decay, the system is without any macroscopic length-scale, so as long as we stay above the length-scale of atomic distances, the system is scale invariant. In case of a local Hamiltonian, when the interaction is short ranged, the scale invariance leads to the much stronger conformal invariance. Based on conformal symmetry, and using the toolbox of the naturally arising quantum field theory (QFT), conformal field theory (CFT) is constructed. Conformal field theories describe statistical systems exactly at their critical point; however, there are other motivations to study them. Conformal field theories are most rich in two dimension, hence giving further reason to study two-dimensional lattice models.

One of the main method of computing the partition function of lattice models is based on the transfer matrix. Informally speaking, the transfer matrix –or $T$-matrix for short– is an operator, which represents a row of the lattice model, and the full lattice is built up, as $T$-matrices are multiplied with each other, corresponding to the graphical picture of building up the lattice row-by-row. Considering an $N \times M$ square lattice, the partition function is as follows

$$Z = \text{Tr} \ T_M^N$$

where $T_M$ is the transfer matrix describing one row of the $N \times M$ system. For sufficiently large $N$, or in the $N \to \infty$ limit, in case of a unique largest eigenvalue $\Lambda$, the following approximate expression is used for the partition function

$$Z = \text{Tr} \ T_M^N \approx \Lambda^N$$
As we see, using the transfer matrix method, the search for the partition function simplifies to the search of the largest eigenvalue of $T$. There are numerous quantities which are computed in the context of $T$-matrix method. In this thesis, we consider two lattice models, the dilute $O(n)$ loop model and the Restricted Solid-On-Solid (RSOS) model.

In case of the dilute $O(n)$ loop model, the degrees of freedom are not local variables on the bonds or on the vertices, but are rather loops restricted to certain paths by the underlying lattice. The non-locality is inherent in the model, as statistical weight $n$ is associated to the non-local loops. Nevertheless, we use techniques based on the previous approach, to compute a certain current in the context of dilute $O(n = 1)$ loop model.

The other lattice model we consider is the Restricted Solid-On-Solid model, which is a height model, originally developed to describe the fluctuating interface between different phases. In the context of the RSOS model, we probe the continuum limit, and the conformal field theory in the continuum limit of the model. As this is a well known result for the RSOS models, we consider fused versions of the RSOS model. The fusion is a procedure to define new lattice models from already existing ones. We consider the continuum limit of fused RSOS models, and identify the continuum CFTs.

Two-dimensional lattice models in statistical physics are closely related to one-dimensional quantum chains. The study of one-dimensional quantum chains started when Bethe solved the one-dimensional spin-$\frac{1}{2}$ Heisenberg spin chain – also known as spin-$\frac{1}{2}$ XXX spin chain– in 1931 [4]. He gave the solution by a method, which later became known as the coordinate Bethe ansatz. Bethe gave the multiparticle wavefunction in the terms of symmetrized plane waves. The momenta associated to the waves satisfy certain conditions, known as Bethe equations. This method solves exactly the eigenvalue problem of the spin chain in the following sense: For the spin-$\frac{1}{2}$ spin chain, assuming $N$ sites, the Hilbert-space is $2^N$ dimensional. In principle, as the Hamiltonian is a finite matrix, there is no problem to diagonalize it. However, due to the exponential increase in the dimension, with the increasing system size, this approach quickly breaks down. Bethe ansatz reformulates the question of diagonalizing the $2^N \times 2^N$ matrix in terms of $O(N)$ coupled nonlinear algebraic equations. The method was extended by the Leningrad school, introducing the algebraic Bethe ansatz (for a good review, and further literature, see [5]). The algebraic Bethe ansatz clearly shows the connection to two-dimensional lattice models. The idea of algebraic Bethe ansatz is to work with the so called monodromy matrix $\mathcal{T}$, and express the wavefunction in algebraic terms, instead of the complicated real-space wavefunction. The monodromy matrix is closely related to the transfer
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matrix

\[ T = \text{Tr}_0 \mathcal{T} \]

where the partial trace is taken over the so called auxiliary space. Pictorially, the transfer matrix corresponds to a finite, approximate time evolution of the system. The connection to the lattice models is straightforward, as the two transfer matrices coincide. Also, typically the Hamiltonian of a quantum chain is the logarithmic derivative of the transfer matrix of the corresponding lattice model.

The method of Bethe ansatz was extended to include more than one species of particles by the nested Bethe ansatz (which has both coordinate and algebraic version). In this thesis, we consider a quantum chain model, and solve it by nested coordinate Bethe ansatz. The model is formulated in terms of spinless fermions; however the fermion number is not conserved, and the Bethe ansatz is done in other degrees of freedom.