Diverse methods for integrable models
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Chapter 2

Spin-1 current in the dilute $O(n = 1)$ loop model

2.1 Introduction of loop models

The study of exactly solvable lattice models dates back to Ising and Bethe [4,6]. Loop models are two-dimensional integrable models, with non-local structure, based on loop representations of various algebras, including the Temperley-Lieb algebra, the dilute Temperley-Lieb algebra, and the Brauer algebra [7–9]. Loop models are defined by fixing the possible decorations on the tiles of the lattice, and imposing matching rules for neighboring tiles. In this non-technical introduction, we focus on the bulk properties of the model on a square lattice, leaving the boundary considerations for later.

The dense $O(n)$ loop model has been introduced in [7]. The model describes the statistical ensemble of densely packed, non-intersecting paths which either form closed loops or are attached to the boundary (given that we consider the model with appropriate boundary conditions). Every closed loop carries a statistical weight $n$, the loop fugacity. The dense $O(n)$ loop model has two possible faces

A typical configuration is in fig. 2.1. The model is closely related to the celebrated six-vertex (6V) model [10]. Historically, the six-vertex model was formed as a two-dimensional toy-model for the residual entropy of water ice: Ice (the $I_h$ phase of ice) has a crystal
structure of stacked layers of tessellating hexagons. The vertices of the hexagons are occupied by the oxygen atoms. These planes and hexagons form a tetrahedral diamond lattice, with four oxygen atoms around every oxygen atom. Hydrogen atoms form bonds between the oxygen atoms. Four bonds correspond to every oxygen atom which are occupied by four hydrogen atoms. Energetically the most suitable is if two of these hydrogen atoms are closer to the oxygen, and two are farther, locally imitating the structure of the $H_2O$ water molecule. This is the origin for the residual entropy of ice: Even at zero temperature, there are many microstates corresponding to the different arrangements of hydrogen atoms. Six-vertex model is a two-dimensional version of the problem formulated above: Consider a square lattice, with degrees of freedom living on the bonds of the lattice. Every bond is occupied by an arrow. The arrows satisfy the ice-rule: There are two arrows pointing in and two pointing out with respect to any vertex. Hence the name of the model, there are six possible arrow configurations around any vertex (fig. 2.2). The number of possible configurations and hence the residual entropy of the 6V model with periodic boundary conditions has been computed by Lieb [10].

The mapping between the 6V model and the dense $O(n)$ model is realized by an intermediate step, a dense directed loop model. In the directed loop model all loops have an orientation. The loop model is recovered by summing over the orientations, and the vertex model is obtained by summing over the loop configurations consistent with the orientation of the edges of the loop. An other interpretation of the six-vertex model is a spin-$\frac{1}{2}$ model, the arrows corresponding to the spins. This gives a possibility to generalize the model to higher spin, still
2.1. INTRODUCTION OF LOOP MODELS

Figure 2.2: The six possible configurations of the six-vertex model.

...satisfying the generalized version of the ice rule. The generalization to spin-1 gives rise to the nineteen vertex (19V) models: Every edge is either empty or occupied by an arrow pointing in or out. There are three integrable versions of 19V model: the Faddeev-Zamolodchikov (FZ) [11], the Izergin-Korepin (IK) [12] and the Bazhanov-Shadrikov model [13]. The Faddeev-Zamolodchikov model is constructed by fusion from the 6V model: Two spin-$\frac{1}{2}$ are fused, and projected to the spin-1 (triplet) representation. The other two models are constructed independently, not by fusion.

Dilute $O(n)$ loop models on square lattice have been introduced equivalent to the Izergin-Korepin 19V model [14, 15]. Again via a directed loop model this vertex model can be mapped onto a dilute loop model. The mapping is such that integrability is preserved, and the loop fugacity $n$ becomes a free parameter, $-2 \leq n \leq 2$. The dilute loop model is defined as the statistical ensemble of the following nine plaquettes

Based on the same scheme, the Faddeev-Zamolodchikov 19V model is mapped to a loop model, with an extra plaquette of crossing loops, and with $n = 2$.

Since their introduction, loop models are the subject of growing interest. One of the most fruitful connection is with combinatorics. One of the early signs of the connection of integrable models to counting of certain objects was the discovery that $n \times n$ 6V configurations with domain wall boundary condition (DWBC) counts the number of $n \times n$ Alternating Sign Matrices (ASMs) [16–20]. Similar observations has been made in a number of cases [21, 22]. The most prominent result is the Razumov-Stroganov conjecture (Cantini-Sportiello theorem), which relates the groundstate elements of the dense $O(1)$ loop model to the counting of fully packed loop configurations (FPLs) [23]. The conjecture was proven by purely combinatorial methods [24]. The search for proof by algebraic methods led to new useful techniques [25,26]. Our approach to the computation of correlators is based on these methods.

Loop models form a natural basis for discretely holomorphic operators. Discrete holomorpicity is a lattice generalization of holomorphic functions: lattice contour...
integrals of the function are expected to be zero. Discrete holomorphicity has been used to prove the conformal invariance of the Ising model [27], and to prove that the growth constant of the number of self-avoiding-walks on the honeycomb lattice is $\sqrt{2 + \sqrt{2}}$ [28, 30]. Surprisingly, prescribing a certain parafermionic observable to be discretely holomorphic results in the Boltzmann weights of the model to be Yang-Baxter integrable. This connection has been found in several models [31–36], and also for boundary cases [37].

In this chapter our main focus is to present our computation of a certain parafermionic observable, in the context of dilute $O(n=1)$ loop model on a strip with open boundary conditions. The strip is infinite vertically, and has finite width horizontally. The open boundary conditions allow the loops to connect to the boundaries. We compute the spin-1 boundary to boundary current on the infinite strip, with finite width. Every path, connecting the left and right boundary, carries an equal unit of current from the left boundary to the right boundary. Closed loops and paths connected only to one of the boundaries do not carry any current. In this chapter, we compute the mean current density induced by the statistics of the paths. We introduce the observable $F^{(x_1, x_2)}$, as the mean current between the points $x_1$ and $x_2$

$$F^{(x_1, x_2)} = \sum_{C \in \Gamma} P(C) N_C^{(x_1, x_2)} \text{sign}_{(x_1, x_2)}^{(x_1, x_2)}$$ (2.1)

Here, $\Gamma$ is the set of all configurations, $N_C^{(x_1, x_2)}$ is the number of paths passing in between the points $x_1$ and $x_2$ from the left to the right boundary, $P(C)$ is the ensemble probability of the configuration, and $\text{sign}_{(x_1, x_2)}^{(x_1, x_2)}$ is $+1$ if $x_1$ lies in the region above the paths, and $-1$ if it lies below, as shown in fig. [2.3]. Due to the non intersecting nature of the loop model, the quantities $N_C^{(x_1, x_2)}$ and $\text{sign}_{(x_1, x_2)}^{(x_1, x_2)}$ are well defined. This observable is discretely holomorphic, antisymmetric, $F^{(x_1, x_2)} = -F^{(x_2, x_1)}$, and additive, $F^{(x_1, x_3)} = F^{(x_1, x_2)} + F^{(x_2, x_3)}$.

Up to an overall phase factor, $F$ is the $s = 1$ special case of the more general arbitrary $s$ spin case

$$\tilde{F}^{s,(x_1, x_2)} = \sum_{C \in \Gamma} P(C) N_C^{(x_1, x_2)} e^{is \phi(C)}$$ (2.2)

where $\phi(C)$ is the winding angle between the starting direction on the left boundary and the $x_1, x_2$ line.

Our result [1] is in close connection with the result of de Gier, Nienhuis and Ponsaing [38]. They computed the same spin-1 current for the dense $O(1)$ model, while here we compute it for the dilute $O(1)$ model.
2.1. INTRODUCTION OF LOOP MODELS

Figure 2.3: The spin-1 property of the current: different paths contribute with different signs.

The continuum limit of such expressions was studied by Gorin and Panova [39]. They studied the $N \to \infty$ homogeneous limit of certain classes of symmetric polynomials.

This chapter is organized as follows: In section 2.2 we introduce the dilute $O(n)$ lattice models, with various boundary conditions. In section 2.3 we describe the integrable version of the model with open boundary conditions. This defines the $R$-matrix for generic $n$. In section 2.3.1 we fix the value of the loop fugacity to $n = 1$, and derive the Boltzmann weights (the elements of the $R$-matrix) for this case. In section 2.4 we introduce the vector space, where the transfer matrix (section 2.5) acts. We present our main computation tool, the quantum Knizhnik-Zamolodchikov equations in section 2.6. We show our main result on the current in section 2.7. The recursion relations needed for proving our statements are introduced in section 2.8. We discuss the symmetry structure of our expressions in section 2.9. Note that we do not prove all the observed symmetries, we use some assumptions in our final results. In section 2.10 we prove the main result under the aforementioned technical assumptions.

The model, we discuss is equivalent to a percolation model in a certain limit, the details of this mapping is in appendix A.2. We put some more technical calculations in the appendix.
2.2 Dilute $O(n)$ loop model

Consider a square lattice of width $L$ and infinite height. Each tile of the lattice is decorated randomly by one of the nine plaquettes

\[
\begin{array}{c}
\text{b}_1 & \text{b}_2 & \text{b}_3 & \text{b}_4 & \text{b}_5 & \text{b}_6 & \text{b}_7 & \text{b}_8 & \text{b}_9 \\
\end{array}
\]

The decoration is subject to the restriction, that the formed paths have to be continuous. They form closed loops, or end on the boundaries, if the boundary conditions allow. Because adjacent tiles are thus not chosen independently, their Boltzmann weights can not be interpreted as probabilities. Closed loops carry a statistical weight $n$.

On the $L \times \infty$ geometry, we can introduce different boundary conditions. Here we discuss five different cases: two periodic, one closed and two open boundary conditions. In the case of the periodic boundary conditions, the continuity condition is extended to the left and right boundary, i.e. the right side of the right boundary has to have the same occupancy as the left side of the left boundary, and the configuration is considered, as these two sides are connected (fig. 2.4a). The vector spaces on which the transfer matrix act depend on the boundary conditions.

In the case of closed and open boundary conditions, the model is considered on an $L \times \infty$ strip, with left and right boundary. The interaction with the left and right boundary is described by the following plaquettes, respectively

\[
\begin{array}{c}
\text{l}_1 & \text{l}_2 & \text{l}_3 & \text{l}_4 & \text{l}_5 \\
\end{array}
\]

\[
\begin{array}{c}
\text{r}_1 & \text{r}_2 & \text{r}_3 & \text{r}_4 & \text{r}_5 \\
\end{array}
\]
The closed boundary condition involves only the \( \{ l_1, l_2, r_1, r_2 \} \) plaquettes, hence the choice is completely determined by the occupancy. Consequently their statistical weight is considered to be identically one. The loops cannot connect to the boundary, hence only loops are formed.

The first open boundary condition involves \( \{ l_1, l_2, l_3, r_1, r_2, r_3 \} \), the second one all the ten plaquettes \( \{ l_1, \ldots, l_5, r_1, \ldots, r_5 \} \). The main difference between the first and second open boundary condition is the number of global parameters: The first open boundary condition for the integrable version of the model allows larger freedom for the loop weights, while for the second one, all the loop weight have to be equal \[37\]. As our main interest involves the second open boundary case, we will refer as restricted open boundary condition and open boundary condition to the first and second one, respectively. Typical configurations with different boundary conditions are in fig. 2.4.

The statistical weight of a configuration \( C \) is given by the product of the weights of the constituent plaquettes and the weight of the loops. Every loop carries a weight, and depending on the boundary conditions, different type of loops carry different weights. We call the two periodic boundary conditions as PBC with puncture and PBC without puncture. The difference between these two cases is whether we topologically distinguish loops winding around the cylinder from contractible loops or not. For the case without puncture, there is only one type of loop, which carries the weight \( n \). For the case with puncture there are two kind of loops, contractible and non-contractible, with loop weight \( n \) and \( b \), respectively. For closed boundary conditions, there are only contractible loops, with weight \( n \). For restricted open boundary condition, there are the most possible different loops. As the loops connected to the boundary only through \( l_3 \) and \( r_3 \), the boundary loops have certain "parity" properties. The topmost connection to the boundary is either at the top or bottom of the boundary plaquette, and the bottom connection is either at the bottom or the top, respectively (possibly on the same boundary plaquette, for the first case). There are four types of loops stretching between the left and right boundary, according to their connection on the left and right side: on both side, they can be connected via the top or bottom part of the boundary plaquette. These gives eight distinguishable boundary loops, two-two on the left and right boundary, and four connected to both. The largest freedom which can be consistent with exactly solvable version of the model distinguishes these eight types of boundary loops with weights \( n_1, \ldots, n_8 \), and the contractible loop with weight \( n \).

For the open boundary conditions, such an abundance of free parameters does not generally permit exact solution. As loops attached to the boundary, we can distinguish three types of non-contractible loops: loops attached to the left boundary, loops attached to the right boundary, and loops stretching between...
CHAPTER 2. CURRENT IN DILUTE LOOP MODEL

Figure 2.4: Typical configurations with different boundary conditions at $L = 10$. 

(a) Periodic boundary conditions (both) 
(b) Closed boundary conditions 
(c) Restricted open boundary conditions 
(d) Open boundary conditions
2.3. INHOMOGENEOUS WEIGHTS

the two boundaries. These carry the weights \( n_1, n_2, n_3 \), respectively. The statistical weight of any configuration with any boundary condition is given by the product of the weights of the plaquettes constituting the configuration, multiplied by the weights of the loops formed in the configuration. Here, we show the explicit expression for a configuration \( C \) with all the five boundary condition

\[
P_{\text{w/o punct.}}^{(\text{PBC})}(C) = \left( \prod_{i=1}^{9} b_i^{\#b_i \text{ plaquettes}} \right)^{n \# \text{contr. loops}} \quad (2.3a)
\]

\[
P_{\text{w/ punct.}}^{(\text{PBC})}(C) = \left( \prod_{i=1}^{9} b_i^{\#b_i \text{ plaquettes}} \right)^{n \# \text{contr. loops} n_1 \# \text{non-contr. loops}} \quad (2.3b)
\]

\[
P^{\text{CBC}}(C) = \left( \prod_{i=1}^{9} b_i^{\#b_i \text{ plaquettes}} \right)^{n \# \text{contr. loops}} \quad (2.3c)
\]

\[
P^{(\text{OBC})}_{\text{restricted}}(C) = \left( \prod_{i=1}^{3} b_i^{\#b_i \text{ plaquettes}} \right) \left( \prod_{i=1}^{8} l_i^{\#l_i \text{ plaquettes}} \right) \times \left( \prod_{i=1}^{3} r_i^{\#r_i \text{ plaquettes}} \right) \left( \prod_{i=1}^{5} n_i^{\#n_i \text{ type loops}} \right)^{n \# \text{contr. loops}} \quad (2.3d)
\]

\[
P_{\text{OBC}}(C) = \left( \prod_{i=1}^{9} b_i^{\#b_i \text{ plaquettes}} \right) \left( \prod_{i=1}^{5} l_i^{\#l_i \text{ plaquettes}} \right) \times \left( \prod_{i=1}^{5} r_i^{\#r_i \text{ plaquettes}} \right) \left( \prod_{i=1}^{5} \right)^{n_1 \# \text{left bndry loops} n_2 \# \text{right bndry loops} n_3 \# \text{stretching loops} \# \text{contr. loops}} \quad (2.3e)
\]

Note that for the closed boundary conditions, the boundary plaquettes do not carry any statistical weight, as they are entirely prescribed by the decoration of the normal plaquettes. By this, we defined the homogeneous dilute \( O(n) \) loop model with different boundary conditions, i.e. with two periodic, closed, restricted open and open boundary conditions. It is possible to define further versions of the model, e.g. with one boundary, on a half-plane, but our main focus is on the model with finite width.

### 2.3 Inhomogeneous weights

According to the standard technique in exactly solvable models \cite{40}, we introduce inhomogeneous weights corresponding to the tiles.
Figure 2.5: Rapidities for the open boundary case: $u_1, \ldots, u_L$ and $v_i$ are the rapidities. The $\zeta_l$ and $\zeta_r$ are further parameters of the left and right $K$-matrices. As later we see, for many occasions we can threat them as rapidities, hence we will call them boundary rapidities. (More precisely, we express the $\zeta$’s in the boundary rapidities.)
As our main interest is the model with open boundary conditions, we focus on this case, the other cases are similar. In order to do so, we introduce rapidities—also called spectral parameters—flowing through the sites, as in fig. 2.5. The rapidities are complex parameters, associated with directed rapidity lines. Two rapidity lines crossing each other correspond to a square tile in the bulk. From the boundary, the lines ”bounce back”, while the rapidity changes sign. We make the weights of the tiles to be the function of the corresponding rapidities. The integrable version of the model is defined by the following \( R \)-matrix

\[
R(u,v) \equiv R(u-v) = \begin{pmatrix}
    u & v \\
    v & u
\end{pmatrix}
\]

where the inhomogeneous weights are the following

\[
\begin{align*}
    r_1(x) &= \sin(2\lambda) \sin(3\lambda - x) \\
    r_2(x) &= \sin(2\lambda) \sin(x) \\
    r_3(x) &= \sin(x) \sin(3\lambda - x) \\
    r_4(x) &= \sin(x) \sin(3\lambda - x) + \sin(2\lambda) \sin(3\lambda) \\
    r_5(x) &= \sin(2\lambda - x) \sin(3\lambda - x) \\
    r_6(x) &= -\sin(x) \sin(\lambda - x)
\end{align*}
\]

Here \( \lambda \) is a global parameter, related to the loop fugacity \( n \)

\[
n = -2 \cos(4\lambda)
\]

The previously defined \( R \)-matrix can be regarded in two different way: as a Boltzmann weight of the tile, or as an operator. To define it as an operator, we introduce some more objects later. We consider the model with open boundary conditions, and define the inhomogeneous \( K \)-matrix as follows

\[
K_1(u, \zeta) = \zeta \begin{pmatrix}
    u \\
    -u
\end{pmatrix} = k_1^l(u, \zeta) \begin{pmatrix}
    1 \\
    1
\end{pmatrix} + k_2^l(u, \zeta) - (2.7)
\]
2.3 \textbf{Current in Dilute Loop Model}

\begin{align*}
K_r(u, \zeta) &= k_r^1(u, \zeta) \left(\left\langle \begin{array}{c}
\mathcal{R}1(u, \zeta) + \mathcal{R}2(u, \zeta)
\end{array}\right\rangle\right) + k_r^2(u, \zeta) + k_r^3(u, \zeta) + k_r^4(u, \zeta) \tag{2.8}
\end{align*}

The weights of the left \( K \)-matrix elements are

\begin{align*}
k_1^l(u, \zeta) &= \zeta \sin(2\lambda) \sin(2u) \tag{2.9a} \\
k_2^l(u, \zeta) &= 2 \cos \lambda \sin \left( \frac{3}{2} \lambda + u \right) - n_1 \zeta^2 \sin \left( \frac{1}{2} \lambda + u \right) \sin \left( \frac{3}{2} \lambda - u \right) \tag{2.9b} \\
k_3^l(u, \zeta) &= -\zeta^2 \sin(2\lambda) \sin(3u) \sin \left( \frac{1}{2} \lambda - u \right) \tag{2.9c} \\
k_4^l(u, \zeta) &= \sin \left( \frac{3}{2} \lambda - u \right) \left( 2 \cos \lambda - n_1 \zeta^2 \sin^2 \left( \frac{1}{2} \lambda - u \right) \right) \tag{2.9d}
\end{align*}

The weight of the right \( K \)-matrix elements are related to the left one by the following equation

\begin{align*}
k_i^r(u, \zeta) &= k_i^l(-u, \zeta), \quad i = 1 \ldots 4, \tag{2.10}
\end{align*}

where the \( n_1 \to n_2 \) substitution is also understood due to the different loop weights. By these definition, we defined the inhomogeneous dilute \( O(n) \) model with open boundary conditions. The \( K \)-matrix weights for the restricted open boundary conditions are in [37].

\section{Specialization of loop weights to \( n = 1 \)}

Our main interest is the dilute \( O(n) \) loop model with open boundary conditions, when all the loop fugacities equal 1: \( n = n_1 = n_2 = n_3 = 1 \). In this section we derive the Boltzmann weights for this case, in multiplicative notation. The weights shown in the previous section (eqs. (2.5a) to (2.5f), (2.9a) to (2.9d) and (2.10)) are in the additive notation: the \( R \)-matrix is a function of the
difference of the two rapidities. We introduce the multiplicative notation by introducing the following new variables

\[ q = e^{i\lambda} \]
\[ z = e^{ix} \]

In the new variables, the eqs. (2.5a) to (2.5f) weights are the following (after multiplying by a common factor \(4q^5z^2\))

\[ r_1(z) = (q^4 - 1) (q^6 - z^2) \frac{z}{z} \]  \hspace{1cm} (2.12a)
\[ r_2(z) = (q^4 - 1) (z^2 - 1) z q^3 \]  \hspace{1cm} (2.12b)
\[ r_3(z) = (z^2 - 1) (q^6 - z^2) q^2 \]  \hspace{1cm} (2.12c)
\[ r_4(z) = (z^2 - 1) (q^6 - z^2) q^2 + (q^4 - 1) (q^6 - 1) z^2 \]  \hspace{1cm} (2.12d)
\[ r_5(z) = (q^4 - z^2) (q^6 - z^2) \]  \hspace{1cm} (2.12e)
\[ r_6(z) = - (z^2 - 1) (q^2 - z^2) q^4 \]  \hspace{1cm} (2.12f)

The loop weight \(n\) is related to the new parameter \(q\) as

\[ n = -2 \cos 4\lambda = -q^4 - q^{-4} \]  \hspace{1cm} (2.13)

Setting the loop fugacity to \(n = 1\) means that \(q\) takes one of the following values

\[ q_1^{(\pm)} = e^{\pm i\pi/6} \]  \hspace{1cm} (2.14a)
\[ q_2^{(\pm)} = e^{\pm i\pi/3} \]  \hspace{1cm} (2.14b)
\[ q_3^{(\pm)} = e^{\pm 2i\pi/3} \]  \hspace{1cm} (2.14c)
\[ q_4^{(\pm)} = e^{\pm 5i\pi/6} \]  \hspace{1cm} (2.14d)

Out of these solutions, we will consider \(q_2^{(\pm)}\). Using the property that \((q_2^{(\pm)})^3 = -1\), the weights in this case simplify to the following (writing \(q\) instead of \(q_2^{(\pm)}\))

\[ r_1(z) = - (1 + q) (1 - z^2) z \]  \hspace{1cm} (2.15a)
\[ r_2(z) = - (1 + q) (1 - z^2) \frac{z}{z} \]  \hspace{1cm} (2.15b)
\[ r_3(z) = - (1 - z^2) (1 - z^2) q^2 \]  \hspace{1cm} (2.15c)
\[ r_4(z) = - (1 - z^2) (1 - z^2) q^2 \]  \hspace{1cm} (2.15d)
\[ r_5(z) = (-q - z^2) (1 - z^2) \]  \hspace{1cm} (2.15e)
\[ r_6(z) = (1 - z^2) \left( q^2 - z^2 \right) (-q) \]  

(2.15f)

The solution \( q_3^{(\pm)} \) gives the same weights, as here. The expressions are related to the previous expressions by a \( q \rightarrow -q \) change, and \( q_3^{(\pm)} = -q_2^{(\mp)} \). Consequently the two minus factors cancel each other, resulting in the same weights. After factorizing a common factor \( (1 - z^2) \), we get the following weights

\[
\begin{align*}
  r_1(z) &= r_2(z) = -(1 + q) z & (2.16a) \\
  r_3(z) &= r_4(z) = -(1 - z^2) q^2 & (2.16b) \\
  r_5(z) &= -q - z^2 & (2.16c) \\
  r_6(z) &= (q^2 - z^2) (-q) & (2.16d)
\end{align*}
\]

This defines the \( R \)-matrix for \( n = 1 \), in Regime I. The \( K \)-matrix is also computable similarly.

We will work in a different convention, defined by the following \( R \)-matrix for \( n = 1 \), in Regime I, in multiplicative notation

\[
R(z,w) \equiv R \left( \frac{w}{z} \right) = \begin{array}{c}
\text{z} \\
\text{w}
\end{array}
\]

\[
W_1 \left( \frac{w}{z} \right) \left( \begin{array}{c}
\text{z} \\
\text{w}
\end{array} \right) + W_2 \left( \frac{w}{z} \right) \left( \begin{array}{c}
\text{z} \\
\text{w}
\end{array} \right) + W_3 \left( \frac{w}{z} \right) \left( \begin{array}{c}
\text{z} \\
\text{w}
\end{array} \right),
\]

(2.17)

where the weights are defined as

\[
\begin{align*}
  W_1(z) &= -1 + z^2, & (2.18a) \\
  W_2(z) &= (q + q^2) z, & (2.18b) \\
  W_3(z) &= q^2 + qz^2, & (2.18c) \\
  W_m(z) &= -q - q^2 z^2. & (2.18d)
\end{align*}
\]

As one can see, this convention is related to the previous one by a \( z \rightarrow z^{-1} \) transformation and an overall factor \(-q\).

We define the \( K \)-matrix for this case as follows

\[
K(z,z_B) = z_B \begin{array}{c}
\text{z} \\
\text{z}^{-1}
\end{array} = K_{id}^I(z,z_B) \left( \begin{array}{c}
\text{z} \\
\text{z}
\end{array} \right) +
\]

(2.19)
2.3. INHOMOGENEOUS WEIGHTS

\[ K_{m}^{l}(z, z_B) \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) + K_{1}^{l}(z, z_B) \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) \] ,

\[ K_{r}(z, z_B) = \frac{z}{z_B} = K_{id}^{T}(z, z_B) \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) + \] (2.20)

\[ K_{m}^{r}(z, z_B) \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) + K_{1}^{r}(z, z_B) \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) , \]

with the following weights

\[ K_{id}^{l}(z, z_B) = k^{2} \left( z^{-1} \right) x^{2} (z_B) - 1 , \] (2.21a)

\[ K_{m}^{l}(z, z_B) = x^{2} (z_B) k \left( z^{-1} \right) \left( k(z) - k \left( z^{-1} \right) \right) , \] (2.21b)

\[ K_{1}^{l}(z, z_B) = x(z_B) \left( k(z) - k \left( z^{-1} \right) \right) , \] (2.21c)

\[ K_{id}^{r}(z, z_B) = 1 - k^{2} \left( z \right) x^{2} (z_B) , \] (2.21d)

\[ K_{m}^{r}(z, z_B) = x^{2} (z_B) k(z) \left( k(z) - k \left( z^{-1} \right) \right) , \] (2.21e)

\[ K_{1}^{r}(z, z_B) = x(z_B) \left( k(z) - k \left( z^{-1} \right) \right) , \] (2.21f)

\[ k(z) = qz - z^{-1} , \] (2.21g)

\[ x(z_B) = q \frac{z_B}{z_B^2 - 1} . \] (2.21h)

This \( K \)-matrix is also gauge equivalent with the generic one, the precise derivation is in appendix A.1.

In the followings we use the \( R \) and \( K \)-matrix weights defined in eqs. (2.18a) to (2.18d) and (2.21a) to (2.21f). The \( R \) and \( K \)-matrix are stochastic operators with the following weights

\[ W_{R}(z, w) \equiv W_{R} \left( \frac{w}{z} \right) = -1 + (q + q^{2}) + z^{2} , \] (2.22)

\[ W_{K_{1}}(z, z_B) = (k(z)x(z_B) - 1) \left( 1 + k(z^{-1})x(z_B) \right) , \] (2.23)

\[ W_{K_{r}}(z, z_B) = \left( 1 - k(z^{-1})x(z_B) \right) \left( 1 + k(z)x(z_B) \right) \] (2.24)
Figure 2.6: The mapping from loop configurations to dilute link patterns. The two outermost points –connected with dotted line to the others– represent the two boundaries. Here the image of the mapping is $|\cdot(\cdot)|$

As the $R$ and $K$-matrices are not defined yet as operators, the meaning of these normalizations will be clearer in the next section.

### 2.4 Vector space of link patterns

In this section, we introduce the vector space of link patterns, in which the $R$ and $K$-matrices act as stochastic operators. Based on this description, we present the equations which are satisfied by the inhomogeneous model. The $R$ and $K$-matrices act on the vector space as elements of the two boundary representation of the dilute Temperley-Lieb algebra (For a brief overview, see [9]).

We introduce $dLP_L$, the set of dilute link patterns of size $L$, the possible connectivities on a half-infinite strip with width $L \in \mathbb{N}$. The set is built up as follows: Consider $L$ sites. Every site is either occupied or empty. An occupied site is connected to the left or right boundary, or to an other occupied site. As the underlying loop model is such that the paths are non-intersecting, the chords representing the connectivities are also non-intersecting. Consequently, every dilute loop configuration on the half-infinite strip with width $L$ corresponds to one dilute link pattern. There are finitely many dilute link patterns, and infinitely many loop configurations, hence many (in fact infinitely many) loop configurations correspond to one dilute link pattern. An example of such a mapping is in fig. 2.6. The set $dLP_L$ is in bijection with $L$ long strings of the
2.4. VECTOR SPACE OF LINK PATTERNS

Figure 2.7: The elements of $dLP_{L=2}$. The two outermost points represent the boundaries, the inner ones the sites. The top row: $|\bullet\bullet\rangle$ (empty element), $|\bullet\rangle$, $|(\bullet), |()\rangle$, $|(()\rangle$ (fully nested element). The bottom row: $|()\bullet\rangle$, $|\bullet\rangle$, $|()\rangle$, $|()\rangle$ (fully nested element).

characters (, • and ), consequently $dLP_L$ contains $3^L$ elements. The bijection is the following: • represents an empty site, occupied sites are represented by the parentheses. As far as the parentheses are well nested, the matching parentheses represent connected edges. The unmatched right or left parentheses represent edges that are connected to the left or right boundary, respectively. The mapping for all the elements of $dLP_{L=2}$ are in fig. 2.7. There are three elements which have names because of their importance: The $|\bullet\bullet\rangle$ is the empty element, the $|()\rangle\rangle$ and $|()\rangle\rangle\rangle$ are the fully nested elements.

We define the vector space of link patterns $V_L$, as the vector space over $\mathbb{C}$ which is spanned by the elements of the $dLP_L$ dilute link patterns: $V_L = \text{span}(dLP_L)$. The result of the action of an $R$- or $K$-matrix on an element of $dLP_L$, typically denoted as $|\pi\rangle$, is a linear combination of link patterns that results by attaching the graphical representation of the operators below the picture of the link pattern, and keeping only the terms in which the occupancy matches. Left and right $K$-matrices act only on the connectivity of the left-most and right-most site, while for $R$-matrices, two consecutive sites are specified, where it acts. It acts, as it is tilted by 45 degree. If an $R$-matrix acts on the two consecutive sites $i, i+1$, we denote it in the subscript: $R_{i,i+1}$. Some examples of such mapping are in fig. 2.8. Note that every loop closed by this operation is a new loop which has to be considered. Consequently, the configuration is multiplied by $n, n_1, n_2$ or $n_3$, depending on the type of newly formed loop. As all these loop weights are 1 in our case, we do not have to take this into account.

The $R$ and $K$-matrices are stochastic operators on $V_L$, with weights (2.22) to (2.24). To compute the correct weights, $R$-matrices ($K$-matrices) should be considered as they map from the two (one) sites, where the rapidities enter to the two (one), where they exit. Relations involving $R$ and $K$-matrices can be
Figure 2.8: Some examples of $R$, and $K$-matrix elements acting on dilute link patterns.

represented by drawings. The directed red lines always represent rapidity flows. A crossing of two rapidity flows is an $R$-matrix, its weight is computed respecting the direction of lines. We introduce lines for the boundary rapidities too. It is helpful to treat rapidities and boundary rapidities in a more uniform way. By this, we give new pictorial representation for the $K$-matrices.

The reflection of a rapidity on a boundary represents a $K$-matrix. The order of the operators is prescribed by the direction of the rapidities. The rapidity lines uniquely determine the drawing, hence we draw tiles only in some of the pictures.

With these prescriptions, the $R$-matrix satisfies the following equations. (In the figures, for legibility, we omit the prefactors. The equations hold for $n = 1$, for
2.4. VECTOR SPACE OF LINK PATTERNS

generic \( n \), the same equations hold with different normalization.)

- the inversion/unitary relation:

\[
R(z_2, z_1)R(z_1, z_2) = W_R(z_2, z_1)W_R(z_1, z_2) \cdot \text{id} \quad (2.25)
\]

- the crossing relation:

\[
R(z, w) = -\left(\frac{w}{z}\right)^2 R^{\text{rot}}(-w, z) \quad (2.26)
\]

As the \( R \)-matrix only depends on the ratio \( \frac{w}{z} \), an equivalent form of the crossing relation is

\[
R(z, w) = -\left(\frac{w}{z}\right)^2 R^{\text{rot}}(w, -z). \quad (2.27)
\]

- the Yang-Baxter equation (this equation holds for generic \( n \) without modification):

\[
R_{23}(u, v)R_{12}(u, w)R_{23}(v, w) = R_{12}(v, w)R_{23}(u, w)R_{12}(u, v) \quad (2.28)
\]
The $K$-matrix satisfies a similar set of equations. Similarly, we omit prefactors in the figures. These equations also generalize to generic $n$ with minor modifications in the normalizations. The equations are the following:

- the boundary inversion/unitary relation:
  - left boundary:
    \[ K_l(w^{-1}, z_B)K_l(w, z_B) = W_{K_l}(w^{-1}, z_B)W_{K_l}(w, z_B) \cdot \text{id} \quad (2.29) \]
  - right boundary:
    \[ K_r(w^{-1}, z_B)K_r(w, z_B) = W_{K_r}(w^{-1}, z_B)W_{K_r}(w, z_B) \cdot \text{id} \quad (2.30) \]
• the boundary crossing relation:

\[ K_l(w, z_B) = -K_r(-w^{-1}, -z_B) \]  \hspace{1cm} (2.31)

• the reflection equation:

- left boundary:

\[ R(v^{-1}, u^{-1})K_l(v, z_B)R(u^{-1}, v)K_l(u, z_B) = K_l(u, z_B)R(v^{-1}, u)K_l(v, z_B)R(u, v) \] \hspace{1cm} (2.32)
We consider the right reflection equation with a rapidity flowing in the opposite direction. With our definitions of the left and right $K$-matrices, this is only possible with a boundary rapidity which changes orientation. Note the direction of rapidities and boundary rapidities. The following equation holds

\[ R(v^{-1}, u^{-1})K_r(u, z_B)R(u, v^{-1})K_r(v, z_B) = K_r(v, z_B)R(v, u^{-1})K_r(u, z_B)R(u, v) \]  

\[ (2.33) \]
We restore the continuity of the orientation of the boundary rapidity line, by introducing the boundary-crossed left $K$-matrix. It is introduced by reversing the orientation of the boundary rapidity. The boundary-crossed $K$-matrix is as follows

$$K_{l}^{b,\text{reversed}}(w, -z_B) = K_{r}(w, z_B).$$

This definition leads to the following, preferred form of eq. (2.34) which restores the continuity of the boundary rapidity flow

$$R(u, v^{-1})K_{l}^{b,\text{reversed}}(u, z_B)R(v, u^{-1})K_{r}(v, z_B) =$$

$$= K_{r}(v, z_B)R(u, v)K_{l}^{b,\text{reversed}}(u, z_B)R(v, u^{-1}).$$

(2.36)
We would like to give some comments regarding the boundary rapidity. The boundary rapidity is a free parameter in the $K$-matrix of the model. It is not clear, why would we introduce a corresponding rapidity. As it turns out, this $K$-matrix is constructed from the closed boundary $K$-matrix, with the insertion of a rapidity line. The boundary rapidity corresponds to the rapidity of this inserted line. This computation is shown in details in appendix A.6. This explains, why a factor $-1$ relates the normal and the boundary-crossed $K$-matrix: The (2.26) crossing relation introduces the factor $-1$. The boundary-crossed right $K$-matrix can be introduced analogously. We introduce the boundary-crossed $K$-matrix in order to define the transfer matrix.
2.5. DOUBLE ROW TRANSFER MATRIX

2.5 Double row transfer matrix

We define the double row transfer matrix ($T$-matrix) as follows

\[
T_L(w, z_0, z_1, \ldots, z_L, z_{L+1}) = \quad \begin{array}{cccc}
  z_0 & z_1 & z_2 & \ldots & \ldots & z_{L-1} & z_L & z_{L+1} \\
  \downarrow & \downarrow & \ddots & \ddots & \ddots & \downarrow & \downarrow & \downarrow \\
  w & \ldots & \ldots & \ldots & \ldots & w^{-1} & \ldots & \ldots
\end{array}
\]

(2.37)

Note that a boundary-crossed left $K$-matrix is on the right side. By standard method, using the Yang-Baxter equation, the inversion relation, and the reflection equation, we can prove that such double row transfer matrices form a one-parameter family of commuting matrices [41]

\[
[T_L(w_1, z_0, z_1, \ldots, z_L, z_{L+1}), T_L(w_2, z_0, z_1, \ldots, z_L, z_{L+1})] = 0 .
\]

(2.38)

The transfer matrix is a stochastic operator on $V_L$. The weight is defined by the $R$ and $K$-matrices, constituting the $T$-matrix. A matrix element of the transfer matrix corresponds to a mapping from one element of $dLP_L$ to another. Its value is the sum of the weights of all the possible $T$-matrix configurations (the $R$ and $K$-matrix content of the transfer matrix) which maps the preimage link pattern to the image link pattern. As certain maps cannot be realized by one $T$-matrix configuration, some matrix elements are zero. The matrix element of the transfer matrix are Laurent polynomials in the rapidities $z_1, \ldots, z_L$ and the boundary rapidities $z_0, z_{L+1}$.

Take a half-infinite configuration, with finite width $L$ and infinite height upward, with an edge at the bottom. The states of the edge map to the elements of $V_L$ (fig. 2.6). We act on $V_L$ by the transfer matrix, if we add the graphical representation of the $T$-matrix to the bottom, and consider the new edge. In fact, loosely speaking, the half infinite configuration can be considered as a tower of stacked $T$-matrices. The (unnormalized) probability distribution of the link patterns is given by the groundstate eigenvector of the $T$-matrix which we will denote by $|\Psi(z_0, z_1, \ldots, z_L, z_{L+1})\rangle$. The groundstate eigenvector is the eigenvector corresponding to the largest eigenvalue. The entries of the groundstate eigenvector are Laurent polynomials in $z_0, \ldots, z_{L+1}$, and due to the (2.38) commutation property, independent of the $w$ spectral parameter. The existence and uniqueness of such a vector is provided by the Perron-Frobenius
The eigenvalue equation is the following

\[ T(w, z_0, \ldots, z_{L+1}) |\Psi(z_0, \ldots, z_{L+1})\rangle = N(w, z_0, \ldots, z_{L+1}) |\Psi(z_0, \ldots, z_{L+1})\rangle. \] (2.39)

Note that the eigenvector is not a function of \( w \), which we call auxiliary rapidity. Here, \( N \) is the normalization of the \( T \)-matrix

\[ N(w, z_0, \ldots, z_{L+1}) = W_{K_i}(w, z_0)W_{K_i}(w^{-1}, -z_{L+1}) \prod_{i=1}^{L} W_R(w, z_i)W_R(z_i, w^{-1}) + \] \[ \tilde{W}_{K_i}(w, z_0)\tilde{W}_{K_i}(w^{-1}, -z_{L+1}) \prod_{i=1}^{L} \tilde{W}_R(w, z_i)\tilde{W}_R(z_i, w^{-1}), \] (2.40)

where \( W_R \) and \( W_{K_i} \) are defined in eqs. (2.22) and (2.23), and \( \tilde{W}_R \) and \( \tilde{W}_{K_i} \) are the following functions

\[ \tilde{W}_R(z_1, z_2) = W_1(z_1, z_2) - W_t(z_1, z_2), \] (2.41a)
\[ \tilde{W}_{K_i}(z, z_B) = K_{id}(z, z_B) + K_m(z, z_B) - K_1(z, z_B). \] (2.41b)

The derivation for \( N \) is in appendix A.3.

The groundstate vector is

\[ |\Psi_L(z_0, \ldots, z_{L+1})\rangle = \sum_{\pi \in dLP_L} \psi_\pi(z_0, \ldots, z_{L+1}) |\pi\rangle. \] (2.42)

This is a vector in \( V_L \), with Laurent polynomial entries \( \psi_\pi(z_0, \ldots, z_{L+1}) \), where \( |\pi\rangle \in dLP_L \) denotes the basis vector. We use the following shortened notation: \( |\psi_\pi\rangle \equiv \psi_\pi |\pi\rangle \).

In order to compute quantities on the full strip, we introduce \( dLP_L^* \) and \( V_L^* \), the dual basis of \( dLP_L \) and the dual vector space of \( V_L \). The dual basis \( dLP_L^* \) consist of link patterns in the downward direction. The dual vector space is spanned by the dual basis over \( \mathbb{C} \): \( V_L^* = \text{span}(dLP_L^*) \). We define the scalar product of \( \langle \alpha | \in dLP_L^* \) and \( |\beta\rangle \in dLP_L \)

\[ \langle \alpha | \beta \rangle = \begin{cases} 1 & \text{if the two link patterns match respecting the occupation,} \\ 0 & \text{otherwise.} \end{cases} \] (2.43)
2.5. DOUBLE ROW TRANSFER MATRIX

In an analogous fashion, we can build up the probabilistic picture for the dual vector space with a dual transfer-matrix.

Scalar products of vectors and dual vectors are considered as probabilities of certain configurations on the full strip. The dual transfer matrix acts "upward". In order to express the eigenvector of $dLP_L^*$, we relate the transfer matrices of $dLP_L$ and $dLP_L^*$

$$\begin{align*}
\langle \Psi | &= \sum_{\pi \in dLP_L^*} \psi_\pi^* \langle \pi | \equiv \sum_{\pi \in dLP_L^*} \langle \psi_\pi |.
\end{align*}$$

Equation (2.44) relates the groundstate and dual groundstate elements

$$\begin{align*}
\psi_\alpha(z_0, \ldots, z_{L+1}) &= \psi_\alpha^*(z_{L+1}, \ldots, z_0), \\
\langle \alpha |\beta \rangle &= \langle \alpha |\beta \rangle.
\end{align*}$$

The transfer matrix of the dual space has a unique largest eigenvector, the dual groundstate

$$\langle \psi | = \sum_{\pi \in dLP_L^*} \psi_\pi^* \langle \pi | \equiv \sum_{\pi \in dLP_L^*} \langle \psi_\pi |.$$
\[ Z_{f,s}(z_0, \ldots, z_{L+1}) = \langle \Psi(z_0, \ldots, z_{L+1})|\Psi(z_0, \ldots, z_{L+1}) \rangle = \sum_{\alpha \in dLP_L} \psi_\alpha(z_0, \ldots, z_{L+1})^* \psi_\alpha(z_0, \ldots, z_{L+1})^* \psi_\beta(z_{L+1}, \ldots, z_0), \quad (2.50) \]

where we sum up to \( \alpha, \beta \in dLP_L \) link patterns with matching occupation. These functions are normalization factors for the unnormalized probabilities.

### 2.6 The quantum Knizhnik – Zamolodchikov equations

Based on the properties of \( R \) and \( K \)-matrices, the transfer matrix satisfies the following equations (suppressing irrelevant notation)

\[
\begin{align*}
R_{i,i+1}(z_i, z_{i+1}) T(\ldots z_{i}, z_{i+1} \ldots) &= T(\ldots z_{i+1}, z_{i} \ldots) R_{i,i+1}(z_i, z_{i+1}), \quad (2.51a) \\
K_l(z_1, z_0) T(z_1, \ldots) &= T(z_1^{-1}, \ldots) K_l(z_1, z_0), \quad (2.51b) \\
K_r(z_L, z_{L+1}) T(\ldots, z_L) &= T(\ldots, z_L^{-1}) K_r(z_L, z_{L+1}). \quad (2.51c)
\end{align*}
\]

Act with both sides on \(|\Psi_L\rangle\), and the \( q \)-Knizhnik-Zamolodchikov equations follows \[25\,44\,45\] for the dilute \( O(1) \) model, with open boundaries

\[
\begin{align*}
R_{i,i+1}(z_i, z_{i+1}) |\Psi(\ldots z_i, z_{i+1} \ldots)\rangle &= W_R(z_i, z_{i+1}) |\Psi(\ldots z_{i+1}, z_i \ldots)\rangle, \quad (2.52a) \\
K_l(z_1, z_0) |\Psi(z_0, z_1, \ldots)\rangle &= W_{K_l}(z_1, z_0) |\Psi(z_0, z_1^{-1}, \ldots)\rangle, \quad (2.52b) \\
K_r(z_L, z_{L+1}) |\Psi(\ldots z_L, z_{L+1})\rangle &= W_{K_r}(z_L, z_{L+1}) |\Psi(\ldots z_L^{-1}, z_{L+1})\rangle, \quad (2.52c)
\end{align*}
\]

where the prefactors \( W \)'s follow from the stochasticity and the normalization of the \( R \) and \( K \)-matrices.

We used the \( q \)KZ equations to compute the groundstate vector of the transfer matrix for \( L = 1, 2, 3 \) system sizes. Starting from the fully nested elements, the full groundstate is computable, by the repeated application of the \( q \)KZ equations. The algorithm to generate the full groundstate vector from the fully nested elements is described in details in \[46\]. Further details are in \[25\,47\]. Explicit groundstate elements and partition sum for \( L = 1 \) are presented in appendix A.4.
2.7 Definition of spin-1 current and main result

In this section we define precisely the spin-1 boundary to boundary current in the dilute O(1) model, and we give our main result, the closed expression to it. In eq. (2.1), we introduced the spin-1 boundary to boundary current

\[ F^{(x_1,x_2)} = \sum_{C \in \Gamma} P(C) N_C^{(x_1,x_2)} \text{sign}_C^{(x_1,x_2)} \]

This current is the signed and weighted sum of all the full strip configurations which contain boundary to boundary paths passing through between points \(x_1\) and \(x_2\). \(N_C^{(x_1,x_2)}\) is the number of paths passing through, \(\text{sign}_C^{(x_1,x_2)}\) is either +1 or -1, if \(x_1\) is in the region above or below the path, respectively. \(P(C)\) is the ensemble probability of the configuration. After introducing inhomogeneous weights, \(P(C)\) and \(F^{(x_1,x_2)}\) become a function of the rapidities. There are two cases to consider, all other expressions follow from the additivity. These two cases are when \(x_1\) and \(x_2\) are on two adjacent vertices of the square lattice, and either separated by a horizontal or a vertical line. Denote the position of the \(x_1\) and \(x_2\) markers by horizontal and vertical lattice indices. Denote the horizontal and vertically separated case by \(X\) and \(Y\), respectively

\[ X^{(k)} = F^{((k,i),(k+1,i))} \]  
\[ Y^{(k)} = F^{((k,j),(k,j+1))} \]  

After introducing the rapidities, \(X\) and \(Y\) become a function of them. As \(T\)-matrices with different auxiliary rapidity commute, the \(X\) current cannot depend on them, and the \(Y\) current depends on the auxiliary rapidity between the points \((k, j)\) and \((k, j + 1)\)

\[ X^{(k)} = X^{(k)}(z_0, \ldots, z_{L+1}), \]  
\[ Y^{(k)} = Y^{(k)}(w, z_0, \ldots, z_{L+1}). \]  

The currents are computable by the introduction of operators \(\hat{X}^{(k)}\) and \(\hat{Y}^{(k)}\), as the following expectation values

\[ X^{(k)}(z_0, \ldots, z_{L+1}) = \frac{\langle \Psi(z_0, \ldots, z_{L+1}) \left| \hat{X}^{(k)} \right| \Psi(z_0, \ldots, z_{L+1}) \rangle}{\langle \Psi(z_0, \ldots, z_{L+1}) \left| \Psi(z_0, \ldots, z_{L+1}) \right\rangle}, \]  
\[ Y^{(k)}(w, z_0, \ldots, z_{L+1}) = \frac{\langle \Psi(z_0 \ldots z_{L+1}) \left| \hat{Y}^{(k)}(w, z_0 \ldots z_{L+1}) \right| \Psi(z_0 \ldots z_{L+1}) \rangle}{\langle \Psi(z_0 \ldots z_{L+1}) \left| T(w, z_0 \ldots z_{L+1}) \right| \Psi(z_0 \ldots z_{L+1}) \rangle}. \]
Here $\hat{X}^{(k)}$ is a matrix, acting on $dLP_L$. Its matrix elements are 0 and $\pm 1$. It has the $\pm 1$ nonzero elements between $\langle \alpha | \in dLP_L^*$ and $| \beta \rangle \in dLP_L$ elements, if the $\langle \alpha | \beta \rangle$ link pattern configuration contains a boundary to boundary path on the $k$th site. The sign depends on weather position $k$ or $k + 1$ is above the path. $\hat{Y}^{(k)}$ has a more complicated structure, as only the two link patterns do not tell information about the horizontal connectivity. $\hat{Y}^{(k)}$ is a modified $T$-matrix. An element of $\hat{Y}^{(k)}$ is a weighted sum—with weights 0 and $\pm 1$—of all $T$-matrix configuration with the proper occupancy on the top and bottom. The weight is $\pm 1$ if a proper boundary to boundary path is formed with that $T$-matrix configuration. The $\pm 1$ sign is chosen according to the current.

### 2.7.1 Main result

Our main result is an explicit closed expression for the $X$ and $Y$ spin-1 boundary to boundary current in the inhomogeneous dilute $O(1)$ model, defined on a strip, infinite in vertical direction (both upward and downward), with a finite width $L$. To present the expression, first we introduce some auxiliary functions. We use the standard definition of the elementary symmetric functions

$$e_k(z_1, \ldots, z_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} z_{i_1} \cdots z_{i_k} \text{ for } 1 \leq k \leq n \text{, otherwise } 0 . \quad (2.56)$$

We introduce elementary symmetric functions over $z_i$ and $z_{i-1}$

$$E_i(z_0, z_1, \ldots, z_L, z_{L+1}) = e_i(z_0, z_1 \ldots z_L, z_{L+1}, z_0^{-1}, z_1^{-1} \ldots z_L^{-1}, z_{L+1}^{-1}) . \quad (2.57)$$

Note that these symmetric polynomials take the same value for indices $i$ and $L + 1 - i$

$$E_i(z_0, z_1, \ldots, z_L, z_{L+1}) = E_{L+1-i}(z_0, z_1, \ldots, z_L, z_{L+1}) . \quad (2.58)$$

Define the following auxiliary polynomial

$$P_L^p(z_0, z_1, \ldots, z_{L+1}) = \frac{i^{L+1}}{2(q-q^{-1})} \left( \prod_{j=0}^{L+1} \frac{(qz_j+i)(q^{-1}z_j-i)}{z_j} - \prod_{j=0}^{L+1} \frac{(q^{-1}z_j+i)(qz_j-i)}{z_j} \right) . \quad (2.59)$$

With these definitions, Garbali and Nienhuis found closed expressions for the empty element and the partition sums (in [46][47], where alternative definitions are also presented)

$$\psi_{L,EE}(z_0, \ldots, z_{L+1}) = \frac{\det_{1 \leq i,j \leq L+1} (E_{3j-2i} - E_{3j+2i-4(L+2)})}{P_L^p(z_0, \ldots, z_{L+1})} , \quad (2.60)$$
2.8. RECURSION RELATIONS

\[
Z_{L, \text{h.s.}}(z_0, \ldots, z_{L+1}) = 2^L \psi_{L, EE}(z_0, \ldots, z_{L+1}) , \quad (2.61)
\]

\[
Z_{L, \text{f.s.}}(z_0, \ldots, z_{L+1}) = 2^L (\psi_{L, EE}(z_0, \ldots, z_{L+1}))^2 . \quad (2.62)
\]

In [1], under some technical assumption, we prove that on an inhomogeneous lattice of width \( L \)
\[
X_L^{(i)}(z_0, \ldots, z_{L+1}) = \frac{1 - 2q}{2} \left( \frac{1}{z_i} - 1 \right) \frac{1}{E_1(z_0, \ldots, z_{L+1})} , \quad (2.63)
\]

\[
Y_L^{(i)}(w, z_0, \ldots, z_{L+1}) = 3 (-1)^{L+1} \left( w - \frac{1}{w} \right)^2 \frac{w^{2(L+2)}}{W_Y(w, z_0, \ldots, z_{L+1})} \times
\]

\[
\times \frac{\psi_{L+2, EE}(w, -w, z_0, \ldots, z_{L+1})}{E_1(z_0, \ldots, z_{L+1}) \psi_{L, EE}(z_0, \ldots, z_{L+1})} . \quad (2.64)
\]

Note that \( Y_L^{(i)} \) is independent of the index \( i \). \( W_Y \) is an auxiliary function, similar
to the normalization \( N \) of the \( T \)-matrix
\[
W_Y(w, z_0, \ldots, z_{L+1}) = \prod_{i=0}^{L+1} W_R(z_i, w)W_R(w^{-1}, z_i) + \prod_{i=0}^{L+1} \tilde{W}_R(z_i, w)\tilde{W}_R(w^{-1}, z_i) . \quad (2.65)
\]

Notice that \( W_Y \) is a symmetrized version of \( N \). Instead of the normalization of
the \( K \)-matrix, it contains \( R \)-matrix normalization, belonging to the boundary
rapidities. \( X \) has the following equivalent form
\[
X_L^{(i)}(z_0, \ldots, z_{L+1}) = \frac{1 - 2q}{2} z_i \frac{\partial}{\partial z_i} \log E_1(z_0, \ldots, z_{L+1}) . \quad (2.66)
\]

2.8 Recursion relations

In this section, we derive recursion equations, relating systems with sizes \( L \) and \( L - 1 \). For a special ratio of two consecutive rapidities, the possible configurations
on the two sites are restricted in such a way that they effectively act as a single site. By setting the left (right) boundary rapidity and the first (last) rapidity to
the special ratio, the same relation holds. This allows us to write down recursion equations, relating systems with system sizes \( L \) and \( L - 1 \).
At the special ratio of the two rapidities, the polynomial weights of the forbidden link patterns are zero. If the link pattern is allowed, the polynomial weight
factorizes into a product of a symmetric prefactor and the polynomial weight of the one size smaller system.
This kind of equations show up in integrable quantum field theories, usually referred as fusion equation and boundary fusion equation [48].

### 2.8.1 Fusion equation

The $R$-matrix factorizes into the product of two ”triangle operators”, if we set the two rapidities to a ratio $q^2$, e.g. set them to $zq^{-1}$ and $zq$

$$R (zq^{-1}, zq) = (-1 - q) M \cdot S = (-1 - q) \left( \triangle + \triangle + \triangle + \triangle \right) \cdot \left( \triangle + \triangle + \triangle + \triangle \right)$$

$$\text{(2.67)}$$

$$zq^{-1} \quad zq$$

$$zq \quad \triangle$$

$$zq^{-1} \quad zq$$

With the help of the operator $M$, the following fusion equation holds

$$R_i(zq, w)R_{i+1}(zq^{-1}, w)M_i = 2 \frac{(w - z)(w + z)}{z^2} M_i R_i(z, w).$$

$$\text{(2.68)}$$

This equation is derived by Garbali and Nienhuis [46,47], and we give a detailed proof in appendix A.5. This equation should also be understood as the previous equations: the statistical weight of both configurations are the same, or equivalently, their action on $V_L$ is the same. Intuitively, one can say that setting the two consecutive rapidities to the special ratio $q^2$, we can use the fusion equation from row to row which effectively decreases the system size by one. In
2.8. RECURSION RELATIONS

fact, the ”triangle operators” are intertwiners between \( dLP_L \) and \( dLP_{L-1} \). The fusion equation relates the transfer matrix of size \( L \) and \( L-1 \)

\[
M_i T_L(\ldots, z_{i-1}, zq, zq^{-1}, z_{i+2}, \ldots) = T_{L-1}(\ldots, z_{i-1}, z, z_{i+2}, \ldots)M_i .
\] (2.69)

Act by both sides on the \( |\Psi_L(\ldots, zq, zq^{-1}, \ldots)\rangle \) eigenvector of \( T_L \)

\[
NM_i |\Psi_L(\ldots, zq, zq^{-1}, \ldots)\rangle =
= T_{L-1}(\ldots, z, \ldots) (M_i |\Psi_L(\ldots, zq, zq^{-1}, \ldots)\rangle) ,
\] (2.70)

which means using the uniqueness of the eigenvector of \( T_{L-1} \)

\[
M_i |\Psi_L(\ldots, z_{i-1}, zq, zq^{-1}, z_{i+2}, \ldots)\rangle =
= F(z; z_0, \ldots, z_{i-1}, z_{i+2}, \ldots, z_{L+1}) |\Psi_{L-1}(\ldots, z_{i-1}, z, z_{i+2}, \ldots)\rangle .
\] (2.71)

Here \( F(z; z_0, \ldots, z_{i-1}, z_{i+2}, \ldots, z_{L+1}) \) is a proportionality factor \(^{[46]}\)

\[
F(z; z_1, \ldots, z_n) = \prod_{j=1}^{n} E_1(z, z_j) = \prod_{j=1}^{n} \frac{(1 + z_j z)(z_j + z)}{z_j z} .
\] (2.72)

The mapping of a \( dLP_L \) link configuration to a \( dLP_{L-1} \) configuration depends on the local configuration on sites \( i, i+1 \). There are five different local configurations: ( ), ••, •|, |•, and ||. Here we use the notation with parentheses, with the addition that | denotes a link connected somewhere outside the two sites. Elements with the local configuration || are mapped to 0 (the corresponding groundstate weight become 0). The full list of mapping is the following

- **Mapping to empty site:**

  \[
  M : ( ) \rightarrow •
  \]

- **Mapping to one occupied, one empty site:**

  \[
  M : •• \rightarrow •
  \]

- **Disappearing elements:**

  \[
  M : || \rightarrow 0
  \]
There is a simple argument for the existence of the recursion relation. Consider the following $qKZ$ equation

$$R_{i,i+1} (zq^{-1}, zq) \left| \Psi_L (\ldots zq^{-1}, zq \ldots) \right\rangle = \left| \Psi_L (\ldots zq, zq^{-1} \ldots) \right\rangle . \quad (2.73)$$

On the r.h.s., the groundstate vector is set to the recursion ratio. On the l.h.s., $R_{i,i+1} (zq^{-1}, zq)$ acts on the state. As at this value, the $R$ matrix factorizes into the two triangle operators, $W_2 (zq^{-1}, zq) = 0$. This sets the probability of any element with $||$ local configuration to 0, and by removing the bottom triangle, we get the mapping from $dLP_L$ link configurations to $dLP_{L-1}$ configurations.

### 2.8.2 Boundary fusion equation

The fusion equation has a boundary version, involving the boundary rapidity and the first or last rapidity. Setting $z_0 = zq$, $z_1 = zq^{-1}$, or $z_L = zq$, $z_{L+1} = zq^{-1}$ effectively decreases the size of the system by one. The reasoning is basically identical for the left and right boundary, so here we present it only for the left side.

Setting $z_0 = zq$, $z_1 = zq^{-1}$, the left $K$-matrix factorizes into an upper and a lower triangle

$$K_l (zq^{-1}, zq) = \frac{-1 + 2q + z^2 + qz^2}{-1 + q + z^2} L_l \cdot U_l =$$

$$= \frac{-1 + 2q + z^2 + qz^2}{-1 + q + z^2} \left( \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array} \right) \cdot \left( \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array} \right) \quad (2.74)$$
Using the operator $U$, the boundary reflection holds

\[ U_l R_1 \left( w^{-1}, zq^{-1} \right) K_1(w, zq) R_1 \left( zq^{-1}, w \right) = \]
\[ = \frac{q(w + z)(-1 + wz)w^2(1 + z^2)^2}{z^2(-w1 + qw - qz + qw^2z - qz^2)} K_1(w, z) U_l \] (2.75)

This is proved by combining the (2.68) fusion equation and the construction of the $K$-matrix described in appendix [A.6]. Consequently, the following intertwining equation holds

\[ L_l T_L(w, zq, zq^{-1}, z_2, \ldots) = T_{L-1}(w, z, z_2 \ldots)L_l . \] (2.76)
Similarly, as before, $L_l$ effectively decreases the system size by one

$$L_l |\Psi_L(zq, zq^{-1}, z_2, \ldots)\rangle = F(z, z_2, \ldots, z_{L+1}) |\Psi_{L-1}(z, z_2, \ldots)\rangle,$$

(2.77)

where $F(z, z_0, \ldots, z_{i-1}, z_{i+2}, \ldots, z_{L+1})$ is the same proportionality factor as in eq. (2.72). This recursion relation decreases the system size by erasing the first site. The rule for the mapping is the following

- Elements with erased first site:
  $$L_l : |\ldots\rangle \rightarrow |\ldots\rangle$$
  $$|\ldots\rangle \rightarrow |\ldots\rangle$$

- Disappearing element:
  $$L_l : |\ldots\rangle \rightarrow 0$$

A completely analogous derivation holds for the right boundary, with an analogous $K_r (zq, zq^{-1}) \sim L_r U_r$ right boundary fusion equation. As a consequence, based on the fusion and boundary fusion equations, regarding the fusion properties, we can treat all the rapidities and boundary rapidities on the same footing.

The fusion equations leads to the following recursion relations, satisfied by the partition sums (without the rapidities not participating in the fusion)

$$Z_{L,h.s.} (z_{i-1}, zq, zq^{-1}, z_{i+2}) = 2 \left( \prod_{k \neq i, i+1} E_1(z, z_k) \right) Z_{L-1,h.s.} (z_{i-1}, z, z_{i+2}),$$

(2.78)

$$Z_{L,f.s.} (z_{i-1}, zq, zq^{-1}, z_{i+2}) = 2 \left( \prod_{k \neq i, i+1} E_1^2(z, z_k) \right) Z_{L-1,f.s.} (z_{i-1}, z, z_{i+2}).$$

(2.79)

The prefactor 2 is a consequence of the mappings realized by $M, L_l$, and $L_r$: Every image in $dLP_{L-1}$ has two preimage. The $F^2$ proportionality factor for $Z_{L-1,f.s.}$ is the result of $Z_{f.s.}$ being a sum of products of two groundstate elements. The recursion acts on the dual space $dLP_L^*$ in the flipped order, hence the recursion only takes place after the action with $M$ to project out the extra site.
2.8.3 The missing recursion by size two

In the previous section, we introduced a recursion by system size one. For generic \( n \), there are two recursions, with increment one and two in the size of the system. The recursion by system size two is missing in the case \( n = 1 \).

These two recursions are based on the zeros of the unitarity/inversion relation (eq. (2.25)). The inversion relation for general \( n \), in additive notation is as follows

\[
R(u)R(-u) = \sin(u - 3\lambda) \sin(u - 2\lambda) \sin(u + 2\lambda) \sin(u + 3\lambda) \tag{2.80}
\]

This equation in the multiplicative notation is

\[
R(z)R(z^{-1}) = \frac{1}{16z^2q^{10}} (z^2 - q^6) (z^2 - q^4) (z^2q^4 - 1) (z^2q^6 - 1) \tag{2.81}
\]

As we take the loop weight to \( n = 1 \), the factor \( (z^2 - 1) \) is one that all \( R \)-matrix elements have in common (and we divide it out), so it is not the result of a projection of the \( R \)-operator. As for \( n = 1, q^3 = -1 \), this leads to the following equation

\[
R(z)R(z^{-1}) = \frac{1}{16z^2q^{10}} (z^2 - q^4) (z^2q^4 - 1) \tag{2.82}
\]

As one can see, this equation does not have zeros at \( z = q^{\pm 3} \), and the other possibility is covered by the discussed recursion.

2.9 Symmetries

The (2.63, 2.64) expressions for the \( X^{(k)} \) and \( Y^{(k)} \) exhibit a number of symmetries, most arising from the unitarity relations and the \( q \)KZ and boundary \( q \)KZ equations.

\( X^{(k)} \) and \( Y^{(k)} \) are symmetric under \( z_i \leftrightarrow z_j \) (for \( X^{(k)}, i, j \neq k \)), and under \( z_i \rightarrow z_i^{-1} \) (for \( X^{(k)}, i \neq k \)). We prove these symmetries in the following steps.

(i) First, we prove that \( X^{(k)} \) and \( Y^{(k)} \) are symmetric under \( z_i \leftrightarrow z_j \) for certain sets of indices, namely

\[
X^{(k)}: \quad 1 \leq i, j \leq k - 1 \text{ or } k + 1 \leq i, j \leq L
\]

\[
Y^{(k)}: \quad 1 \leq i, j \leq k - 1 \text{ or } k \leq i, j \leq L
\]
We did not find a way to extend the symmetry to the boundary rapidities (with indices 0 and $L+1$), we checked these cases for small system analytically and numerically: We computed the $X^{(k)}$ current analytically for $L = 1, 2, 3$, and found it symmetric in all the variables. We computed the $Y^{(k)}$ current analytically for $L = 0, 1$ and found it symmetric, and numerically checked the symmetry for the boundary rapidity for $L = 2, 3$.

(ii) Using this, we prove the $z_i \rightarrow z_i^{-1}$ symmetry, for $X^{(k)}$, $0 \leq i \leq L + 1$, $i \neq k$, and for $Y^{(k)}$, $0 \leq i \leq L + 1$.

(iii) We use the $z_i \rightarrow z_i^{-1}$ inversion symmetry as an ingredient to prove the full symmetry of $Y^{(k)}_L$ under $z_i \leftrightarrow z_j$ without restriction on the indices.

We did not find a proof for the full symmetry of $X^{(k)}_L$, so in the proof of the main result for $X$, we use it as a necessary assumption. As the symmetry for the boundary rapidities is also an assumption, our statements hold up to these assumption.

2.9.1 Partial symmetry of $X$ and $Y$

We prove the symmetry of $X^{(k)}$ and $Y^{(k)}$ under $z_i \leftrightarrow z_j$, given that $i, j < k$ or $k < i, j$. The proof is based on the (2.25) unitarity relation and the (2.52a) qKZ equation. The proof is the same for both expressions, so here we present the one for $X$. Use the operator form of $X$ (suppressing irrelevant notations and normalization)

$$X^{(k)}(z_i, z_{i+1}) \simeq \langle \Psi(z_i, z_{i+1}) | \hat{X}^{(k)} | \Psi(z_i, z_{i+1}) \rangle \simeq \langle \Psi(z_i, z_{i+1}) | \hat{X}^{(k)} R_{i,i+1}(z_{i+1}, z_i) R_{i,i+1}(z_i, z_{i+1}) | \Psi(z_i, z_{i+1}) \rangle \simeq \langle \Psi(z_i, z_{i+1}) | R_{i,i+1}(z_{i+1}, z_i) \hat{X}^{(k)} | \Psi(z_{i+1}, z_i) \rangle \simeq \langle \Psi(z_{i+1}, z_i) | \hat{X}^{(k)} | \Psi(z_{i+1}, z_i) \rangle \simeq X^{(k)}(z_{i+1}, z_i) . \quad (2.83)$$

The $R$-matrix and the $\hat{X}^{(k)}$ operators commute, if $i+1 < k$ or $k < i$. By repeating this procedure, we extend the symmetry to $z_1, \ldots, z_{k-1}$ and $z_{k+1}, \ldots, z_L$. We did not find a way to extend the proof to the $z_0, z_{L+1}$ boundary rapidities. The same proof with the same flaw applies for $Y^{(k)}$, the only difference is in the restriction for the indices, i.e. for $Y^{(k)}$, $z_i \leftrightarrow z_{i+1}$, $i + 1 \leq k$ or $k < i$.

To give evidence of the symmetry involving the bulk rapidities, we made the following analytic and numerical check: We checked analytically the expression for $X$ with $L = 1, 2, 3$, and for $Y$ with $L = 0, 1$. For larger system sizes, we made numerical checks, namely for $L = 2, 3$. Due to the largely increasing terms especially in $Y$ in these expressions, these checks are strong evidences
2.9. SYMMETRIES

supporting our assumption.

The symmetry of \( X \) and \( Y \) under \( z_i \rightarrow z_{i-1} \) is based on the (2.29, 2.30) unitarity relations of the \( K \)-matrix and on the (2.52b, 2.52c) boundary \( qKZ \) equations. The proof is again the same for \( X^{(k)} \) and \( Y^{(k)} \), so we present it for \( X^{(k)} \). Derive the inversion symmetry for \( z_1 \), using the unitarity of the left \( K \)-matrix (assume \( k \neq 1 \))

\[
X^{(k)}(z_1) \simeq \langle \Psi(z_1) | \hat{X}^{(k)} | \Psi(z_1) \rangle \\
\simeq \langle \Psi(z_1) | \hat{X}^{(k)} K_l(z_1^{-1}, z_0) K_l(z_1, z_0) | \Psi(z_1) \rangle \\
\simeq \langle \Psi(z_1) | K_l(z_1^{-1}, z_0) \hat{X}^{(k)} | \Psi(z_1^{-1}) \rangle \\
\simeq \langle \Psi(z_1^{-1}) | \hat{X}^{(k)} | \Psi(z_1^{-1}) \rangle \simeq X^{(k)}(z_1^{-1}). \tag{2.84}
\]

The proof for the \( z_L \rightarrow z_L^{-1} \) symmetry goes the same way, using the unitarity of the right \( K \)-matrix. The inversion symmetry extends to the other rapidities on the two sides of the position of the current by the consecutive application of \( z_i \leftrightarrow z_j \) symmetry. By this we extend the inversion symmetry to \( 0 \leq i \leq k - 1 \) and \( k + 1 \leq i \leq L + 1 \) for \( X^{(k)} \), and for all \( 0 \leq i \leq L + 1 \) for \( Y^{(k)} \).

2.9.2 Full symmetry of \( Y \)

In this section, we show that \( Y_L^{(k)} \) is independent of the position \( k \), and symmetric in the variables \( z_0, \ldots, z_{L+1} \).

Define \( p \) as a path going from the left boundary to the right boundary. The path is defined as a set of \( K \) and \( R \)-matrix elements which constitute the line connecting the two boundaries. Regard two paths to be different, if the line connecting the two boundaries are the same, but there is difference between the content of the \( K \) and \( R \)-matrix elements, as on fig. 2.9. By this definition, we identify configurations which only differ in their position, and which are related by a vertical translation. The weight of the \( p \) path is the weight of the constituting matrices. Since \( Y \) depends only on one auxiliary rapidity, we set all the auxiliary rapidities to the same value \( w \). Denote the weight of \( p \) by \( \Omega_p(w, z_0 \ldots z_{L+1}) \).

The set of all paths, \( P \) is a union of two disjoint sets, \( P_T \) and \( P_B \). \( P_T \) contains the paths starting from the top of the left \( K \)-matrix, \( P_B \) contains the ones starting from the bottom of it. Every path, \( p \in P_T \) is in bijection with a path \( \tilde{p} \in P_B \), by a horizontal mirroring, as in fig. 2.10. By the properties of the \( R \) and \( K \)-matrices, it is easy to see that \( \Omega_p(w, z_0 \ldots z_{L+1}) = \Omega_{\tilde{p}}(w, z_0^{-1} \ldots z_{L+1}^{-1}) \). We introduce \( m_{p, k, x} \), where \( p \) denotes the path, \( 0 \leq k \leq L \) the horizontal position of \( Y_L^{(k)} \), and
Figure 2.9: These two paths are not equivalent.

$x \in T, B$ is stands for 'top' or 'bottom'. Define $m_{p,k,x}$ as the signed crossing of the path $p$ at horizontal line $k$ on the 'top' or 'bottom' section, i.e. at the top or bottom of the double row transfer matrix (See in fig. 2.11). Signed crossing means that if the line crosses from left to right, it counts as 1, if from right to left, it counts as $-1$. Since a path is crossing once more from left to right then to right to left, $m_{p,k,T} + m_{p,k,B} = 1$ (as in fig. 2.11).

By the mirroring, the crossing from the top of the path $p$ maps to the crossing to the bottom of the path $\tilde{p}$, and vice versa. This means that $m_{p,k,x} = m_{\tilde{p},k,\bar{x}}$, where $\bar{T} = B, \bar{B} = T$. Combining these features, it is clear that $m_{p,k,x} + m_{\tilde{p},k,\bar{x}} = 1$. Denote by $Y_{L}^{(k,T)}$ and $Y_{L}^{(k,B)}$ the current through the top and the bottom of the $T$-matrix, respectively. By this definitions, the $Y$ current is given by

$$Y_{L}^{(k,x)}(w, z_0, \ldots, z_{L+1}) = \sum_{p \in P_T} m_{p,k,x} \Omega_{p}(w, z_0, \ldots, z_{L+1}) + \sum_{\tilde{p} \in P_B} m_{\tilde{p},k,\bar{x}} \Omega_{\tilde{p}}(w, z_0, \ldots, z_{L+1})$$

$$= \sum_{p \in P_T} m_{p,k,x} \Omega_{p}(w, z_0, \ldots, z_{L+1}) + m_{\tilde{p},k,\bar{x}} \Omega_{p}(w, z_0^{-1}, \ldots, z_{L+1}^{-1}) \ . \ (2.85)$$

Using that $Y$ is symmetric under $z_i \rightarrow z_i^{-1}$, we consider the following construction

$$Y_{L}^{(k,x)}(w, z_0, \ldots, z_{L+1}) = \frac{1}{2} \left( Y_{L}^{(k,x)}(w, \{z_i\}) + Y_{L}^{(k,x)}(w, \{z_i^{-1}\}) \right)$$
2.9. SYMMETRIES

Figure 2.10: Path $p$ and the mirrored path $\tilde{p}$.

\[
\frac{1}{2} \left( \sum_{p \in P_T} m_{p,k,x} \Omega_p (w, \{ z_i \}) + m_{\tilde{p},k,x} \Omega_p (w, \{ z_i^{-1} \}) + \sum_{p \in P_T} m_{p,k,x} \Omega_p (w, \{ z_i^{-1} \}) + m_{\tilde{p},k,x} \Omega_p (w, \{ z_i \}) \right) = (2.86)
\]

\[
= \frac{1}{2} \left( \sum_{p \in P_T} (m_{p,k,x} + m_{\tilde{p},k,x}) \Omega_p (w, \{ z_i \}) + \sum_{p \in P_T} (m_{p,k,x} + m_{\tilde{p},k,x}) \Omega_p (w, \{ z_i^{-1} \}) \right) =
\]

\[
= \frac{1}{2} \sum_{p \in P_T} \Omega_p (w, \{ z_i \}) + \Omega_p (w, \{ z_i^{-1} \}).
\]

It thus follows that each path in $P_T$ contribute to $Y$, by the average of the weights of $p$ and $\tilde{p}$. It is also clear from this reasoning that $Y^{(k)}$ is independent of $k$. This means that there is no further restriction on its symmetries, so it is symmetric in $z_i$, and under $z_i \rightarrow z_i^{-1}, \forall i$. 

2.10 Proof of the main result

In order to prove our main result, we utilize the symmetries and the recursion relations.

The $X$ and $Y$ currents satisfy the following recursion relations

$$X_L^{(i)} (\ldots, z_{j-1}, zq, zq^{-1}, z_{j+2}, \ldots) = X_{L-1}^{(i)} (\ldots, z_{j-1}, z, z_{j+2}, \ldots), \quad \forall i \neq j, j+1,$$

$$(2.87a)$$

$$Y_L (\ldots, z_{j-1}, zq, zq^{-1}, z_{j+2}, \ldots) = Y_{L-1} (\ldots, z_{j-1}, z, z_{j+2}, \ldots), \quad \forall i \neq j, j+1.$$

$$(2.87b)$$

Note that depending on the relative position of $j$ compared to $i$, the actual position of the current might change, however, we misuse the $X^{(i)}$ notation for both cases. The symmetry property extends to non-adjacent rapidities.

To prove our main result, we list the recursion relations for the unnormalized expressions $X_{u.n.}$ and $Y_{u.n.}$, and use that to prove the recursion relation for the normalized cases.

2.10.1 Proof for the $Y$ current

In this section we prove that $Y$ is indeed in the form.

We define the unnormalized version of $Y$ as follows

$$Y^{(i), u.n.}_L = \langle \Psi_L | \hat{Y}^{(i)}_L | \Psi_L \rangle = \sum_{\alpha, \beta, \gamma} (-1)^{\text{sign}(\alpha, \beta, \gamma)} \psi_\alpha T_\beta \psi_\gamma.$$

$$(2.88)$$
Here, $\alpha \in dLP_L$, $\gamma \in dLP^*_L$, and $T_\beta$ is a $T$-matrix configuration which provides the necessary tiling to form a path through the top row of the $T$-matrix, at position $i$. The $\text{sign}(\alpha, \beta, \gamma) = \pm 1$ term encodes the spin-1 property. The relation between the normalized and unnormalized $Y$ current is

$$Y^{(i)}_L = \frac{Y^{(i), \text{u.n.}}_L}{Z_{L,f.s.,N_L}} ,$$

(2.89)

where $N_L$ is the $T$-matrix normalization.

The (2.87b) and (2.79) recursion relations for $Y$ and $Z_{f.s.}$ lead to the following recursion relation for the unnormalized $Y$ current

$$\left. \frac{Y^{(i), \text{u.n.}}_L}{W_{T,L}} \right|_{z_j = z q, z_{j+1} = z q^{-1}} = 2 \left( \prod_{k \neq j,j+1} E_1^2(z, z_k) \right) \frac{Y^{(i), \text{u.n.}}_{L-1}}{W_{T,L-1}} .$$

(2.90)

As $Y$ is fully symmetric, and $N$ is not fully symmetric, $Y^{\text{u.n.}}$ is not symmetric. To exploit the symmetric properties, we introduce an auxiliary Laurent polynomial function which is symmetric by construction

$$\tilde{Y}^{\text{u.n.}}_L = Y^{\text{u.n.}}_L W_{Y,L} N_L = Y_L W_Y Z_{f.s.} .$$

(2.91)

$W_Y$ is the quantity defined in eq. (2.65) which satisfies the following recursion relation

$$W_{Y,L}(\ldots, z q, z q^{-1}, \ldots) = -w^2 E_1(w, z) E_1(-w, z) W_{Y,L-1}(\ldots, z, \ldots) .$$

(2.92)

These lead to the following recursion relation for the symmetrized, unnormalized current $\tilde{Y}^{\text{u.n.}}_L$.

$$\tilde{Y}^{\text{u.n.}}_L(z_i = z q \ldots z_j = z q^{-1}) = -2w^2 E_1(w, z) E_1(-w, z) \prod_{k \neq i,j} E_1^2(z, z_k) \tilde{Y}^{\text{u.n.}}_{L-1}(z) .$$

(2.93)

As $Y$, the normalized current is fully symmetric in the rapidities $z_0, \ldots, z_{L+1}$, any two of them set to the recursion ratio leads to a recursion relation. The recursion relation holds for two additional value, following from the $z_i \rightarrow z_{i-1}$ symmetry. The complete set of recursions involving two chosen rapidities $z_i, z_j$ is the following

$$Y_L(z_i = z q \ldots z_j = z q^{-1}) = Y_{L-1}(\ldots z \ldots) ,$$

(2.94a)

$$Y_L(z_i = z q^{-1} \ldots z_j = z q) = Y_{L-1}(\ldots z \ldots) ,$$

(2.94b)
Involving two fixed variables leads to 4 recursion relations. Involving one fixed variable and leaving the other one freely chosen leads to 4\((L + 1)\) recursion relations, relating \(Y_L\) and \(Y_{L-1}\), fully exploiting the symmetry in the variables and the inversion symmetry.

We calculated \(\tilde{Y}^{u.n.}\) for \(L = 0, 1\), giving starting elements for the following conjectured form

\[
\tilde{Y}^{u.n.}(w, z_0, \ldots, z_{L+1}) = 2^L 3(-1)^{L+1} w^{2(L+2)} \left(w - \frac{1}{w}\right)^2 \times \\
\frac{\psi_{L,EE}(z_0, \ldots, z_{L+1}) \psi_{L+2,EE}(w, -w, z_0, \ldots, z_{L+1})}{E_1(z_0, \ldots, z_{L+1})}.
\]

(2.95)

The degree width of this expression is \(4(L + 1) - 1\) for size \(L\) which means that the recursion fully fixes \(\tilde{Y}^{u.n.}\) for any system size, as a Lagrange interpolation. Based on eq. (2.91), we get \(Y\) for any system size. This proves that the \(Y\) current has the proposed \((2.64)\) form

\[
Y^{(i)}_L(w, z_0, \ldots, z_{L+1}) = 3(-1)^{L+1} \left(w - \frac{1}{w}\right)^2 \\
\frac{w^{2(L+2)}}{W_Y(w, z_0, \ldots, z_{L+1})} \frac{\psi_{L+2,EE}(w, -w, z_0, \ldots, z_{L+1})}{E_1(z_0, \ldots, z_{L+1}) \psi_{L,EE}(z_0, \ldots, z_{L+1})}.
\]

By this, based on the recursion relation, under the assumption of the symmetry of the bulk and boundary rapidities, we proved that the unique solution which satisfies eq. (2.87b) with the computed starting element, is indeed eq. (2.64).

2.10.2 Proof for the \(X\) current

In this section, under some technical assumption, we prove the \((2.63)\) form of the \(X\) current. We use the same method as for the \(Y\) current.

We define the unnormalized version of the \(X\) current as follows

\[
X^{(i), u.n.}_L = \langle \Psi_L | \hat{X}^{(i)}_L | \Psi_L \rangle = \sum_{\alpha, \beta} (-1)^{\text{sign}(\alpha, \beta)} \psi_{\alpha}^* \psi_{\beta},
\]

(2.96)

where \(\alpha \in dLP_L\), \(\beta \in dLP^*_L\) are such that \(\langle \alpha | \beta \rangle\) link pattern forms a boundary to boundary path through the \(i\)th site. The factor \(\text{sign}(\alpha, \beta) = \pm 1\) is chosen
according to the direction of the path. The normalized and unnormalized \( X \) currents are related as follows

\[
X_L^{(i)} = \frac{X_L^{(i),u.n.}}{Z_{L,f.s.}}. \tag{2.97}
\]

As the partition sum \( Z_{L,f.s.} \) is fully symmetric, \( X_L^{(i)} \) and \( X_L^{(i),u.n.} \) share the same symmetry properties. The recursion relation for the unnormalized \( X_L^{(i),u.n.} \) current is

\[
X_L^{(i),u.n.}(z_j = zq \ldots z_k = zq^{-1}) = 2 \prod_{n \neq j,k} E_1^2(z, z_n) X_{L-1}^{(i),u.n.}(z). \tag{2.98}
\]

We computed this quantity explicitly for \( L = 1, 2, 3 \) which lead us to the following conjectured form

\[
X_L^{(i),u.n.}(z_0, \ldots, z_{L+1}) = (2 - q)2^{L-1} \left( z_i - \frac{1}{z_i} \right) \frac{\psi_{L,EE}^2(z_0, \ldots, z_{L+1})}{E_1(z_0, \ldots, z_L)}. \tag{2.99}
\]

The computation for \( L = 1 \) is in appendix A.4. Based on the recursion properties of \( \psi_{L,EE} \) and \( E_1 \), this form clearly satisfies eq. (2.87a). To prove the uniqueness of the solution, we assume the symmetry in all the variables expect \( z_i \). The degree width of \( X_L^{(i),u.n.} \) in any \( z_j \neq z_i \) rapidity is \( 4L - 1 \). Based on a similar counting as in the \( Y \), there are \( 4L \) recursion relations relating systems \( L \) and \( L - 1 \). All the arguments hold as for \( Y \), only the number of variables is smaller by one (since \( X^i \) is not symmetric in \( z_i \), but all the other \( z \)). By this, we see that under the aforementioned assumption, we found the unique solution for \( X \) presented in eq. (2.63)

\[
X_L^{(i)}(z_0, \ldots, z_{L+1}) = \frac{(1-2q)}{2} \left( z_i - \frac{1}{z_i} \right) \frac{1}{E_1(z_0, \ldots, z_{L+1})}. \tag{2.63}
\]

2.11 Conclusion

In this chapter, we showed our calculation on a boundary to boundary spin-1 current in the context of the dilute \( O(n = 1) \) model on a strip, with open boundary conditions. We introduced the generic \( O(n) \) model with different boundary conditions, and then specialized the loop weight to \( n = 1 \). This specializations allows us to do analytic computation on finite size systems. Due to the additivity of the current, it was sufficient to focus on two cases, which we denoted by \( X \) and \( Y \). We have conducted the computation for the inhomogeneous
case, i.e. our expressions are symmetric rational functions in the rapidities and boundary rapidities of the model. We have found a construction for the open boundary from closed boundary case, with the insertion of a rapidity line. This construction explains many observed properties of our model, however it is not sufficient to prove the symmetry of the current in the bulk and boundary rapidity. We also proved the fusion equation. From the fusion equation, and construction of the open boundary case, the boundary recursion relation follows, as a corollary. Using similar methods to the one of Gorin and Panova, the homogeneous continuum limit of the current expressions is a topic of further interest.