Diverse methods for integrable models
Fehér, G.

Citation for published version (APA):
Fehér, G. (2017). Diverse methods for integrable models

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
Chapter 3

One-dimensional sums and finitized characters of $2 \times 2$ fused RSOS models

3.1 Introduction

Minimal model conformal field theories (CFTs) and certain statistical lattice models, RSOS models were introduced around the same time, and soon the connection was also established. An infinite family of conformal field theories was introduced by Belavin-Polyakov-Zamolodchikov \[49\], when they introduced the minimal model conformal field theories $\mathcal{M}(m, m')$, with central charge

$$c = c^{m, m'} = 1 - \frac{6(m- m')^2}{mm'}, \quad 2 \leq m < m', \quad m, m' \text{ coprime.} \quad (3.1)$$

In the same year, Andrews-Baxter-Forrester \[50\] introduced an infinite family of two-dimensional exactly solvable lattice models, the Restricted Solid-on-Solid models, parametrized by integer $m'$, denoted by RSOS($m' - 1, m'$) by defining the infinite family of $R$-matrices, satisfying the Yang-Baxter equation. The result has been generalized in \[51\] to introduce RSOS($m, m'$) models, depending on two integer parameters $m, m'$. RSOS($m, m'$) models are height models on a lattice, with integer heights range from 1, \ldots, $m' - 1$.

It was shown in \[52\] by Friedan-Qiu-Schenker that unitarity –the conservation of probability in scattering processes– requires $m = m' - 1$ for the $\mathcal{M}(m, m')$
minimal models. Based on critical exponents, Huse could identify the universality class of the RSOS\((m' - 1, m')\) models as the universality class of multicritical Ising-model [53], hence in the continuum limit, they realize the unitary minimal models \(M(m' - 1, m')\). The generic RSOS\((m, m')\) models have been identified as the lattice realizations of the non-unitary \(M(m, m')\) models \([54, 55]\). The minimal models include some of the most prominent studied models, including the Ising model \(M(3, 4)\) \([6, 56, 57]\), the tricritical Ising model \(M(4, 5)\) \([58–61]\) and the Lee-Yang model \(M(2, 5)\) \([62–65]\).

By the method of fusion, new exactly solvable models are defined, by taking an \(n \times n\) (or more generally \(n \times m\)) block of \(R\)-matrices, and regarding this block as the new \(R\)-matrix for the model \([66]\). This was carried out for RSOS models in \([67]\), defining the \(n \times n\) fused RSOS models \(RSOS(m, m')_{n \times n}\).

On the CFT side, the corresponding field theories are expected to be the higher fusion level minimal models \(M(M, M', n')\), with integer fusion level \(n' \in \mathbb{N}\), and fractional fusion level \(k \in \mathbb{Q}\), which are constructed as GKO cosets \([68]\)

\[
\text{COSET}(k, n') : \frac{(A_{1}^{(1)})_{k} \oplus (A_{1}^{(1)})_{n'}}{(A_{1}^{(1)})_{k+n'}}
\]

\[
k = \frac{n'M}{M' - M} - 2, \quad \gcd \left( \frac{M' - M}{n'}, M' \right) = 1
\]

Setting \(n' = 1\) gives the usual minimal models, \(M(M, M', 1) = M(M, M')\), \(n' = 2\) gives the superconformal minimal models \([69]\). The identification is known for the unitary cases \([70–72]\). The parameters of the CFT corresponding to the unitary RSOS\((m' - 1, m)_{n \times n}\) are given by \(n' = n, (M, M') = (m' - n, m')\).

This chapter is based on results which are part of a larger project. The ultimate goal of the project is to identify the continuum limit of the non-unitary fused RSOS\((m, m')_{n \times n}\) models, for all values of \((m, m')\) and the fusion level \(n\). This program has started by Tartaglia and Pearce in \([73]\), where \(n \times n\) fused RSOS models were probed for values of the crossing parameter \(0 < \lambda = \frac{(m' - m)\pi}{m} < \frac{\pi}{n}\).

It was conjectured that the RSOS\((m, m')_{n \times n}\) models are lattice realizations of \(M(M', n')\) models with \((M, M') = (nm - (n - 1)m', m')\) and \(n = n'\) in the interval \(0 < \lambda < \pi/n\).

The identification of the CFT follows from the correspondence between the one-dimensional sums of the RSOS model and the branching functions of the CFT. One-dimensional sums are partition functions of the RSOS model in the low temperature, critical limit, where the model is effectively one-dimensional. They are computed in the framework of the corner transfer matrix (CTM) method, and approximate the characters of the CFT. Tartaglia and Pearce carried out explicit calculations for \(n = 2, 3\), which supports their general conjecture.
Here, based on these earlier works, we identify the CFTs for \( n = 2, \pi/2 < \lambda < \pi \). In the continuum limit, the RSOS(\( m, m' \))\(_{2\times2}\) models are described by the \( M(m, m', 1) = M(m, m') \) minimal models, for the crossing parameter \( \pi/2 < \lambda < \pi \).

Following a similar method, we compute the local energy functions of the model, i.e. the exponents of the Boltzmann weights in the low temperature limit. Based on this, one-dimensional sums are computed. The one-dimensional sums are identified with finitized characters, up to the leading power. The conjectured correspondence can be proven as follows. One-dimensional sums satisfy certain recursion relations and initial conditions. Based on small system results, closed expressions can be conjectured for one-dimensional sums. Knowing these and the groundstate energies (the leading powers of the finitized characters), it can be proven that the closed expressions satisfy the recursion relations with the initial conditions. This program was used in the \( n = 1 \) unfused case in \([51]\). However, we are not able to replicate this proof for the \( n = 2 \) case, as we are missing the groundstate energies and certain one-dimensional sums which do not coincide with characters.

In their paper of 2007, Jacob and Matthieu studied one-dimensional sums of certain paths with half-integer steps \([74]\). Their one-dimensional sums are not based on a lattice model. However, they reproduce the Virasoro characters of the non-unitary minimal models \( M(m, 2m + 1) \), for \( m \geq 2, m \in \mathbb{N} \). By a bijection between the JM paths and the fused RSOS paths, we show that these one-dimensional sums are reproduced by the fused RSOS(\( m, 2m + 1 \))\(_{2\times2}\) models. Thus we find the underlying lattice model for their one-dimensional sums. This is an interesting discovery in its own right.

This chapter organized as follows: In section 3.2 we introduce minimal models and higher-level coset model conformal field theories, in section 3.3 we define both unfused and fused (section 3.3.1) RSOS models. In sections 3.3.2 to 3.3.5 we build up the necessary machinery for making the correspondence with the conformal field theory. In section 3.4 we give the identification with the Jacob-Mathieu theory. In section 3.5 we describe the conjectured finitized characters (section 3.5.1) from \([73]\) and our result (section 3.5.2) reported in \([2]\).

### 3.2 Minimal models and higher-level minimal coset models

In this section we give a brief overview of minimal models and the corresponding coset models as their higher-level generalizations. We introduce the conformal
data, the central charge, the Virasoro characters and the branching functions. 

The minimal models, \( \mathcal{M}(m, m') = \mathcal{M}(m, m', 1) \), have been introduced in [49]. These are rational conformal field theories which are parametrized by the two coprime integers, \( m, m' \). The central charge is given by

\[
c = c_{m,m'} = 1 - \frac{6(m - m')^2}{mm'}, \quad 2 \leq m < m', \quad \gcd(m, m') = 1.
\] (3.3)

The minimal models are characterized by a finite number of primary fields, which are indexed by the \((r, s)\) Kac labels. The conformal weights of the primary fields and the corresponding Virasoro characters are given by

\[
\Delta_{r,s}^{m,m'} = \frac{(rm' - sm)^2 - (m - m')^2}{4mm'}, \quad 1 \leq r \leq m - 1, \quad 1 \leq s \leq m' - 1,
\] (3.4)

\[
\text{ch}_{r,s}^{m,m'} = q^{-c/24+\Delta_{r,s}^{m,m'}} \sum_{k=-\infty}^{\infty} \frac{1}{(q)_{\infty}} \left( q^{km' + rm' - sm} - q^{(km + r)(km' + s)} \right),
\] (3.5)

where the (infinite) \( q \)-factorials are defined by

\[
(q)_n = \prod_{k=1}^{n} (1 - q^k), \quad (q)_\infty = \prod_{k=1}^{\infty} (1 - q^k).
\] (3.6)

Here, \( q = e^{i\pi \tau} \) is the modular nome, and \( \tau \) is the modular parameter, determined by geometry.

Higher-level minimal models \( \mathcal{M}(M, M', n') \) with \( n' > 1 \) are constructed as GKO cosets [68, 69]:

\[
\mathcal{M}(M, M', n') \equiv \text{COSET} \left( \frac{n'M}{M' - M} - 2, n' \right),
\] (3.7)

\[
\gcd \left( \frac{n'M}{M' - M}, M' \right) = 1, \quad 2 \leq M < M', \quad n', M, M' \in \mathbb{N}
\]

where \( n' \) is the integer fusion level and \( \frac{n'M}{M' - M} - 2 \) is the fractional fusion level. The GKO cosets are Wess-Zumino-Witten (WZW) models [75, 78], built on the following affine Lie algebra

\[
\text{COSET}(k, n') : \frac{(A_{1}^{(1)})_{k+n'} + (A_{1}^{(1)})_{n'}}{(A_{1}^{(1)})_{k+n'}}, \quad k = \frac{\hat{p}}{\hat{p}'} - 2, \quad \gcd (\hat{p}, \hat{p}') = 1, \quad n', \hat{p}, \hat{p}' \in \mathbb{N}
\] (3.8)

where the level of the affine su(2) current algebra \( A_{1}^{(1)} \) is given by the subscripts \( k, n' \) and \( k+n' \). A brief overview on the coset construction is given in appendix B.1.
3.3 RSOS MODELS

The central charge of the \( \left( A_1^{(1)} \right)_k \) current algebra is given by

\[
c_k = \frac{3k}{k + 2}.
\]  

(3.9)

The central charge of the COSET\((k, n')\) model follows from the coset construction. It is given in terms of the \( \left( A_1^{(1)} \right)_k \) central charge

\[
c = c_k + c_{n'} - c_{k+n'} = \frac{3kn'(k + n' + 4)}{(k + 2)(n' + 2)(k + n' + 2)}. \]  

(3.10)

Following eqs. (3.7) and (3.10), the central charge of the fused higher level minimal models \( \mathcal{M}(M, M', n') \) is expressed by

\[
c^{M,M',n'} = \frac{3n'}{n' + 2} \left( 1 - \frac{2(n' + 2)(M' - M)^2}{n'^2 MM'} \right). \]  

(3.11)

The \( n' = 1 \) minimal models are given by \( M = m, M' = m' \). The \( n' = 2 \) minimal models coincide with superconformal minimal models.

The characters of the higher fusion level coset models are presented in appendix B.2.

3.3 RSOS models

In this section we introduce Restricted Solid-on-Solid (RSOS) lattice models \[50,51\]. We define the face weights of the model, satisfying the Yang-Baxter, the unitarity and the crossing equations. We define the \( n \times n \) fused RSOS models, and consider the \( n = 2, 3 \) cases in more details.

The RSOS\((m, m')\) models are statistical physical models, defined on a square lattice, where the degrees of freedom are height variables taking positive integer values \( a = 1, 2, \ldots, m' - 1 \) on the vertices of the lattice. Furthermore, the neighboring sites have height difference \( \pm 1 \). Hence, the connectivity graph of the heights is the Dynkin diagram of the simple Lie algebra \( A_{m'-1} \). The model is integrable, satisfying the Yang-Baxter equation with the following Boltzmann weights, associated to the plaquettes of the model in Regime III

\[
W\left( \begin{array}{cc}
a + 1 & a \\
a & a + 1 \\
a + 1 & u \\
a & a + 1 \\
a + 1 & a \\a & a + 1
\end{array} \right) = s(\lambda - u)
\]  

(3.12a)
\[ W \left( \frac{a}{a \pm 1}, \frac{a \pm 1}{a} \mid u \right) = \frac{a \pm 1}{a} = \frac{g_{a \pm 1} s((a \pm 1)\lambda) s(u)}{g_{a \mp 1} s(a\lambda)} \] (3.12b)

\[ W \left( \frac{a}{a \pm 1}, \frac{a \pm 1}{a} \mid u \right) = \frac{a \pm 1}{a} = \frac{s(a\lambda \pm u)}{s(a\lambda)} \] (3.12c)

where \( s(u) \) is a quotient of standard elliptic functions

\[ s(u) = \frac{\vartheta_1(u, t)}{\vartheta_1(\lambda, t)} \] (3.13)

\[ \vartheta_1(u, t) = 2 t^{1/4} \sin u \prod_{n=1}^{\infty} \left( 1 - 2 t^{2n} \cos(2u) + t^{4n} \right) \left( 1 - t^{2n} \right), \] (3.14)

\[ 0 < u < \lambda, \ 0 < t < 1, \]

where \( u \) is the spectral parameter, and \( g_a \) is an arbitrary gauge function. The restriction

\[ 0 < u < \lambda, \ 0 < t < 1 \] (3.15)

indicates that we consider the model in Regime III. We use \( g_a = 1 \), unless otherwise stated. The elliptic nome \( t \) is a temperature-like global variable, where \( t^2 \) measures the departure from criticality through a \( \varphi_{1,3} \) integrable perturbation \cite{79,81}. The crossing parameter \( \lambda \) is a global parameter

\[ \lambda = \frac{(m' - m)\pi}{m'}, \ 2 \leq m < m', \ m, m' \text{ coprime}. \] (3.16)

The weights satisfy the Yang-Baxter equation (in additive notion)

\[ \sum_g W \left( \frac{f}{a}, \frac{g}{b} \mid u \right) W \left( \frac{e}{f} \frac{d}{g} \mid u + v \right) W \left( \frac{d}{g} \frac{c}{b} \mid v \right) = \sum_g W \left( \frac{g}{a}, \frac{g}{b} \mid v \right) W \left( \frac{e}{g} \frac{c}{d} \mid u \right) W \left( \frac{e}{g} \frac{d}{c} \mid u + v \right) \] (3.17)
Here we use a different notation from the notation of rapidities for loop models: There the direction of the two arrows sets the direction of the \( R \)-matrix, here the arch sets it. We do not always denote the arch. In the rest of the chapter, we refer to \( R \)-matrix elements as face weights. The weights also satisfy the regularity (or initial condition) and the inversion relation (or unitarity)

\[
W^{\langle d | c \mid b, a \rangle} = \delta_{a,c} \\
\sum_g W^{\langle d | g \mid b | u \rangle} W^{\langle d | c \mid b | -u \rangle} = \delta_{a,c} \frac{s(\lambda - u)s(\lambda + u)}{s^2(\lambda)}
\]

Further properties are the symmetry under reflection about the main diagonal and height reversal symmetry

\[
W^{\langle d | c \mid b \mid u \rangle} = W^{\langle b | c \mid d \mid u \rangle} \\
W^{\langle d | c \mid b \mid u \rangle} = W^{\langle m' - d | m' - c \mid m' - a | m' - b \mid u \rangle}.
\]

### 3.3.1 Fused RSOS models and fused weights

Fused RSOS models were considered in \[67\]. The construction is based on the Yang-Baxter equation, and the following fusion property. Consider two plaquettes, forming a \( 2 \times 1 \) block, with spectral parameters \( u \) and \( u + \lambda \), respectively. Define the \( 2 \times 1 \) fused weights as

\[
W_{2,1}^{\langle d | c \mid b \mid u \rangle} = \sum_{a'} W^{\langle d | a' \mid c \mid u \rangle} W^{\langle c' | b \mid u + \lambda \rangle}.
\]
This is a proper definition of the $2 \times 1$ fused weight, as in the $g_a = 1$ gauge, after summing up to $a'$, this weight is independent of the choice of $c'$, provided that $|c' - c| = |c' - d| = 1$. In the figure, the dot indicates summation over all admissible heights, and the cross indicates independence of the allowed height. Equation (3.22) is proved by the Yang-Baxter equation, and other properties of the weights.

The fusion generalizes to the $n \times n$ case, and gives a systematic method to define the $n \times n$ fused Boltzmann weights as

$$W_{n,n}(\begin{array}{cc} d & c \\ a & b \end{array} | u) = \frac{1}{\eta^{n,n}(u)}$$

Here again, we sum over the vertices marked with a dot, and the weight is independent of the choice of heights on vertices marked with a cross, in the $g_a = 1$ gauge. The notation of $u_k$ denotes the shifted $u$: $u_k = u + k\lambda$. The normalization factor is

$$\eta^{n,n}(u) = \prod_{k=2}^{n} s(k\lambda) \prod_{k=1}^{n-1} s^{n-k}(u - k\lambda)s^{n-k}(u + (k - 1)\lambda).$$

The adjacency restrictions on corner heights of the $n \times n$ heights are

$$\sigma - \sigma' = -n, -n + 2, \ldots, n,$$

$$\sigma + \sigma' = n + 2, n + 4, \ldots, 2m' - n - 2$$

where $\sigma, \sigma'$ are any two corner heights. The latter restriction follows since $W_{n,n}(\begin{array}{cc} a & a \\ a & a \end{array} | u)$ has a pole for $a = 1$ and $m' - 1$ which we need to exclude. Based on the properties of the RSOS weights, the $n \times n$ fused weights also satisfy the regularity, the inversion relation and the Yang-Baxter equation (in additive convention)

$$W_{n,n}(\begin{array}{cc} d & c \\ a & b \end{array} | 0) = \delta_{a,c}$$
3.3. RSOS MODELS

\[
\sum_g W_{n,n}(\begin{pmatrix} d & g \\ a & b \end{pmatrix}|u)W_{n,n}(\begin{pmatrix} d & c \\ g & b \end{pmatrix}|u - v) = \delta_{a,c} \prod_{k=1}^{n} \frac{s(k\lambda - u)s(k\lambda + u)}{s^2(k\lambda)} \tag{3.27b}
\]

\[
\sum_g W_{n,n}(\begin{pmatrix} f & g \\ a & b \end{pmatrix}|u)W_{n,n}(\begin{pmatrix} e & d \\ g & b \end{pmatrix}|u + v)W_{n,n}(\begin{pmatrix} f & c \\ a & b \end{pmatrix}|u) = \delta_{a,c} \prod_{k=1}^{n} \frac{s(k\lambda - u)s(k\lambda + u)}{s^2(k\lambda)} \tag{3.27c}
\]

The \( n \times n \) fused weights also satisfy the symmetry under reflection about the main diagonal and the height reversal symmetry

\[
W_{n,n}(\begin{pmatrix} d & c \\ a & b \end{pmatrix}|u) = W_{n,n}(\begin{pmatrix} b & c \\ a & d \end{pmatrix}|u) \tag{3.28}
\]

\[
W_{n,n}(\begin{pmatrix} d & c \\ a & b \end{pmatrix}|u) = W_{n,n}(\begin{pmatrix} m' - d & m' - c \\ m' - a & m' - b \end{pmatrix}|u) \tag{3.29}
\]

Our main interests are the \( 2 \times 2 \) and \( 3 \times 3 \) fused cases which we further examine.

2 \times 2 case

The normalized \( 2 \times 2 \) RSOS face weights are the following

\[
W_{2,2}(\begin{pmatrix} d & c \\ a & b \end{pmatrix}|u) = \frac{1}{\eta^{2,2}(u)} \tag{3.30}
\]

where \( \eta^{2,2}(u) = s(2\lambda)s(u)s(u - \lambda) \) is the common normalization factor. Here again, the dot marks summation, the cross marks independence. There are nineteen allowed face configurations, with the following explicit weights

\[
W_{2,2}(\begin{pmatrix} a & a \pm 2 \\ a & a \mp 2 \end{pmatrix}|u) = \frac{s(u - 2\lambda)s(u - \lambda)}{s(2\lambda)} \tag{3.31a}
\]

\[
W_{2,2}(\begin{pmatrix} a & a \pm 2 \\ a & a \pm 2 \end{pmatrix}|u) = W_{2,2}(\begin{pmatrix} a & a \pm 2 \\ a & a \pm 2 \end{pmatrix}|u) = - \frac{s(u - \lambda)s((a \pm 1)\lambda \mp u)}{s((a \pm 1)\lambda)} \tag{3.31b}
\]

\[
W_{2,2}(\begin{pmatrix} a & a \pm 2 \\ a & a \pm 2 \end{pmatrix}|u) = - \frac{s((a \pm 1)\lambda)s(u)s(a\lambda \pm u)}{s(2\lambda)s(a\lambda)s((a \pm 1)\lambda)} \tag{3.31c}
\]
\[ W_{2,2}\left( \begin{array}{c} a \\ a \\ a + 2 \\ a \end{array} \bigg| u \right) = -\frac{s(2\lambda)s((a \pm 2)\lambda)s(u)s(a\lambda \pm u)}{s((a - 1)\lambda)s((a + 1)\lambda)} \] (3.31d)

\[ W_{2,2}\left( \begin{array}{c} a \\ a + 2 \\ a \pm 2 \\ a \end{array} \bigg| u \right) = \frac{s((a \mp 2)\lambda)s((a \pm 1)\lambda)s(u)s(\lambda + u)}{s(2\lambda)s(a\lambda)s((a \pm 1)\lambda)} \] (3.31e)

\[ W_{2,2}\left( \begin{array}{c} a \\ a + 2 \\ a \pm 2 \\ a \end{array} \bigg| u \right) = \frac{s(a\lambda + u)s((a \pm 1)\lambda \pm u)}{s(a\lambda)s((a \pm 1)\lambda)} \] (3.31f)

\[ W_{2,2}\left( \begin{array}{c} a \\ a \\ a \pm 2 \\ a \end{array} \bigg| u \right) = W_{2,2}\left( \begin{array}{c} a \pm 2 \\ a \pm 2 \\ a \\ a \end{array} \bigg| u \right) = \frac{s((a \pm 3)\lambda)s(u)s(u - \lambda)}{s(2\lambda)s((a \pm 1)\lambda)} \] (3.31g)

\[ W_{2,2}\left( \begin{array}{c} a \pm 2 \\ a \pm 2 \\ a \\ a \end{array} \bigg| u \right) = W_{2,2}\left( \begin{array}{c} a \pm 2 \\ a \pm 2 \\ a \pm 2 \\ a \end{array} \bigg| u \right) = \frac{s((a \pm 3)\lambda)s(u)s(u - \lambda)}{s(2\lambda)s((a \pm 1)\lambda)} \] (3.31h)

Note that the two versions of the weight \( W_{2,2}\left( \begin{array}{c} a \\ a \\ a \pm 2 \\ a \end{array} \bigg| u \right) \) with the upper and lower sign are equal. The \( \pm \) signs in the other expressions correspond to the upper or lower sign on the left hand side. Here we see explicitly, why adjacent 1, 1 and \( m’ - 1 \), \( m’ - 1 \) are excluded: for these cases, the denominator is 0 in the aforementioned expressions.

Following from the general statement, the \( 2 \times 2 \) fused weights satisfy regularity, the inversion relation and the Yang-Baxter equation (in the additive convention)

\[ W_{2,2}\left( \begin{array}{c} d \\ a \\ c \\ b \end{array} \bigg| 0 \right) = \delta_{a,c} \] (3.32)

\[ \sum_{g} W_{2,2}\left( \begin{array}{c} d \\ a \\ g \\ b \end{array} \bigg| u \right) W_{2,2}\left( \begin{array}{c} d \\ c \\ b \end{array} \bigg| -u \right) = \delta_{a,c} \frac{s(2\lambda - u)s(\lambda - u)s(\lambda + u)s(2\lambda + u)}{s^2(\lambda)s^2(2\lambda)} \] (3.33)

\[ \sum_{g} W_{2,2}\left( \begin{array}{c} f \\ a \\ g \\ b \end{array} \bigg| u \right) W_{2,2}\left( \begin{array}{c} e \\ d \\ g \\ b \end{array} \bigg| u + v \right) W_{2,2}\left( \begin{array}{c} d \\ c \\ b \end{array} \begin{array}{c} v \right) = \sum_{g} W_{2,2}\left( \begin{array}{c} e \\ f \\ g \\ a \end{array} \bigg| v \right) W_{2,2}\left( \begin{array}{c} e \\ g \\ c \\ b \end{array} \bigg| u + v \right) W_{2,2}\left( \begin{array}{c} e \\ d \\ c \end{array} \begin{array}{c} u \right) \] (3.34)
The 2 × 2 case also possesses the reflection about the main diagonal and the height reversal symmetry

\[ W_{2,2} \left( \begin{array}{c} d \\ a \\ \hline b \\ \end{array} \mid u \right) = W_{2,2} \left( \begin{array}{c} b \\ a \\ \hline d \\ \end{array} \mid u \right) , \]

(3.35)

\[ W_{2,2} \left( \begin{array}{c} d \\ a \\ \hline b \\ \end{array} \mid u \right) = W_{2,2} \left( \begin{array}{c} m' - d \\ m' - a \\ \hline m' - c \\ \hline u \end{array} \right) . \]

(3.36)

3 × 3 case

The normalized 3 × 3 RSOS face weights are the following

\[ W_{3,3} \left( \begin{array}{c} d \\ a \\ \hline b \\ \end{array} \mid u \right) = \frac{1}{\eta^{3,3}(u)} \]

As previously, dots denote summation, crosses independence. The common normalization is

\[ \eta^{3,3}(u) = s(2\lambda)s(3\lambda)s(u - 2\lambda)s^2(u - \lambda)s^2(u + \lambda) . \]

(3.38)

The explicit 3 × 3 weights are in appendix B.3. As it follows from the generic case, the 3 × 3 weights satisfy the usual equations, i.e the regularity, the inversion, the Yang-Baxter equation, and the reflection and height reversal symmetry (eqs. (3.27a) to (3.27c)).

3.3.2 Corner transfer matrix

Following Baxter [82–84], we define corner transfer matrices (CTMs). There are four kinds of corner transfer matrices, as in fig. 3.1.

Each corner transfer matrix, A, B, C and D is defined in a similar fashion, so we describe only A in detail. The corner transfer matrix is an operator which maps a string of admissible heights \( \sigma = \{\sigma_0, \ldots, \sigma_N\} \) to an other string of admissible heights \( \sigma' = \{\sigma'_0, \ldots, \sigma'_N\} \). A matrix element of the CTM is realized
by summing over all internal heights in the quadrant

\[
A^{(N)}_{\sigma', \sigma} = \sum \prod W_{\left(\sigma_{i,j}+1, \sigma_{i+1,j+1}\right)}^{\left(\sigma_{i,j}, \sigma_{i+1,j}\right)} = \sum_{\sigma_0=\sigma_0'}^{\sigma_N=\sigma_N'} W_{\left(\sigma_{i,j}+1, \sigma_{i+1,j+1}\right)}^{\left(\sigma_{i,j}, \sigma_{i+1,j}\right)}
\]

(3.39)

where the product runs over all internal faces, and the sum runs over all allowed configurations with internal heights \(\sigma_{i,j}\) and with the boundary conditions \((\sigma_0 = \sigma_0' = a, \sigma_N = \sigma_N' = b, c)\). The boundary conditions fix the central height value \(a\) and the \(b, c\) height values along the outer diagonal. When it is denoted, \(N\)
3.3. RSOS MODELS

refers to the size of the corner transfer matrix. The previous definition of the CTM applies both for unfused and fused RSOS models. The height strings $\sigma$ and $\sigma'$ satisfy the adjacency restriction according to the fusion level. The other transfer matrices $B$, $C$, and $D$ are defined in a similar way. The partition function of the diamond shaped region of the lattice model (fig. 3.1) with fixed boundary conditions $(b, c)$ is given by summing over all internal configurations, hence taking the trace over the product of the CTMs

$$Z_{b,c}^{(N)} = \sum_{\sigma, \sigma', \sigma'', \sigma'''} A_{\sigma,\sigma'}^{(N)} B_{\sigma',\sigma''}^{(N)} C_{\sigma'',\sigma'''}^{(N)} D_{\sigma'''}^{(N)} = \text{Tr } A^{(N)} B^{(N)} C^{(N)} D^{(N)}.$$ 

(3.40)

The CTMs commute in the $N \to \infty$ limit [82,83]. Hence, it is sufficient to probe the spectrum of only one CTM which will be the lower right one, denoted by $A$. In the low temperature limit the Boltzmann-weights become diagonal along the counter-diagonal, i.e. $W_{d,c|a,b}(u) \sim \delta_{a,c}$. Hence, $A$ become a diagonal matrix, and taking Tr $A$ simplifies to taking the one-dimensional sums of the RSOS model.

3.3.3 RSOS paths, and shaded bands

In this section we introduce RSOS paths, and shaded bands, to understand the low temperature limit of the RSOS models. We introduce these concepts both for the unfused and fused models. The low temperature limit is taken as follows

$$t = e^{-\epsilon} \to 1, \quad u \to 0, \quad \frac{u}{\epsilon} \text{ fixed.}$$

(3.41)

In this limit the face weights become diagonal along the counter-diagonal (the SW-NE direction) of the face, and a power of $e^{-u}$

$$W_{2,2}(d,c|a,b)(u) \bigg|_{\text{low } T} \sim \frac{g_a g_c}{g_b g_d} W(d,a,b) \delta_{a,c}, \quad w \sim e^{-2\pi u/\epsilon}.$$ 

(3.42)

Here we introduced the function $H(d,a,b)$ which is the local energy function. In the low temperature limit, the two-dimensional lattice model becomes effectively one-dimensional, as the degrees of freedom freeze in the SW-NE direction. Consequently, the lower right corner transfer matrix $A$ become a diagonal matrix. The diagonal elements are labeled by the string of heights $\sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_N, \sigma_{N+1}\}$, where we included $\sigma_{N+1} = c$.

Define the $n \times n$ (fused) RSOS lattice path $\sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_N, \sigma_{N+1}\}$ as a
sequence of integers, corresponding to the height variable, hence taking values on the relevant Dynkin-diagram: \( \sigma_i \in A_{m'-1} \). The adjacency conditions are inherited from the \( n \times n \) fused RSOS weights, and hold for the fused paths too

\[
\begin{align*}
\sigma_{j+1} - \sigma_j &= -n, -n + 2, \ldots, n - 2, n, \\
\sigma_{j+1} + \sigma_j &= n + 2, n + 4, \ldots, 2m' - n - 2.
\end{align*}
\]

Note that for even fusion level \( n \) the second condition means that there are no \( \{ \ldots, 1, 1, \ldots \} \) or \( \{ \ldots, m' - 1, m' - 1, \ldots \} \) segments in a path. Unlike in the unfused case, the connectivity graph of the fused model is not \( A_{m'-1} \). Some examples are given in section 3.4.1. We define the boundary conditions of a (fused) path by fixing the first and the two last height variables

\[(\sigma_0, \sigma_N, \sigma_{N+1}) = (a, b, c), \quad c - b = -n, -n + 2, \ldots, n - 2, n, \quad (3.43)\]

which is equivalent to the boundary condition of the corner transfer matrix. The paths are in bijection with the low temperature lower right CTM configurations. The path in \( N \) steps starts at height \( a \) and finishes at heights \( b, c \), where steps can only be taken in accordance with the adjacency conditions of the model. For even fusion levels \( n \), all the heights \( \sigma_j \) take values with the same parity, while for odd \( n \)’s, the parity alternates along the path. An example of a \( 2 \times 2 \) fused path is in fig. 3.2.

We introduce the notation of shaded and unshaded bands. The shaded bands play a role in determining the groundstate boundary conditions for the one-dimensional sums. A band is a range of heights, and as in fig. 3.2 certain bands are shaded. Define the sequence

\[
\rho(r) = \left\lfloor \frac{rm'}{m} \right\rfloor, \quad r = 1, 2, \ldots, m - 1,
\]

and define the shaded bands as the bands between heights \( \rho(r) \) and \( \rho(r) + 1 \), following [85]. Define a \( k \)-band as \( k \) contiguous bands, without restricting whether these bands are shaded, unshaded or both. A shaded (unshaded) \( k \)-band is an \( k \)-band which only contains shaded (unshaded) 1-bands.

The shaded and unshaded bands interchange under the duality \( m \leftrightarrow m' - m \), and as they only depend on \( (m, m') \), they are the same for all RSOS\((m, m')_{n \times n} \) model, regardless the fusion level \( n \).

In [73], it was shown that shaded \( k \)-bands only exist for \( 0 < \lambda < \frac{\pi}{k} \), and there are no shaded \( k \)-bands for \( \frac{\pi}{k} < \lambda < \pi \). The number of shaded \( k \)-bands is given by

\[
\# \text{ of shaded } k\text{-bands} = M - 1 = \begin{cases} km - (k - 1)m' - 1, & 0 < \lambda < \frac{\pi}{k} \\
0, & \frac{\pi}{k} < \lambda < \pi \end{cases} \quad (3.45)
\]
3.3. RSOS MODELS

Figure 3.2: A typical path $\sigma = \{3, 5, 3, 5, 7, 5, 3, 3\}$ in model $(m, m') = (5, 13)$ with boundary conditions $(s, \rho, \rho') = (3, 3, 3)$ and $N = 7$ steps from $s = 3$ to $\rho = 3$. The shaded 1-bands are labelled by $r = 1, 2, 3, \ldots$ from the bottom.

This expression depends on $m, m'$ through $\lambda = \frac{(m-m')\pi}{m}$. In the unitary cases, with $m = m' - 1$ all of the 1-bands are shaded. For non-unitary cases, with $2 \leq m \leq m' - 1$, there are both shaded and unshaded bands. Despite the ease with which the position of shaded $k$-bands can be computed diagrammatically for any given case and $k$, there is no known general expression for it.

For the interval $\frac{\pi}{2} < \lambda < \pi$ which is our primary interest here, there are only shaded 1-bands, and no shaded $k$-bands for $k > 1$. The shaded bands are separated by contiguous unshaded $k$-bands with $k \geq 1$.

3.3.4 Low temperature limit and local energy functions

In [82, 83], Baxter argued that the corner transfer matrices commute in the $N \to \infty$ massive, low temperature limit.

The low temperature limit is realized by the limit

$$t = e^{-\epsilon} \to 1, \quad u \to 0, \quad \frac{u}{\epsilon} \text{ is fixed.} \quad (3.46)$$
In this limit, the Boltzmann weights become diagonal along the counter-diagonal

\[ W_{2,2}(d | c , u) \sim \frac{g_d g_c}{g_b g_a} w^{H(d,a,b)} \delta_{a,c}, \]

\[ w = e^{-2\pi u/\epsilon}, \quad t = e^{-\epsilon}, \quad \epsilon \to 0, \quad u \to 0, \quad \frac{u}{\epsilon} \text{ fixed.} \]

This defines the local energy functions \( H(d,a,b) \) which inherits reflection and height reversal symmetry from the Boltzmann weights

\[ H(d,a,b) = H(b,a,d) = H(m - d, m' - a, m' - b) \] (3.48)

There is a gauge freedom from the arbitrary gauge functions \( g_a \) which allows us to put the local energy functions in a convenient form.

1 \times 1 local energy function

The 1 \times 1 RSOS local energy functions were computed by Forrester and Baxter [51], in the \( g_a = 1 \) gauge

\[ H^{FB}(a \mp 1, a) = \pm \left\lfloor \frac{a \lambda}{\pi} \right\rfloor \] (3.49)

\[ H^{FB}(a \pm 1, a) = \frac{1}{2} \] (3.50)

By changing the gauge, a more suitable set of local energy functions is constructed [73]

\[ H(a + 1, a, a + 1) = \frac{1}{2}(h_{a+1} - h_a) \] (3.51)

\[ H(a - 1, a, a - 1) = \frac{1}{2}(h_a - h_{a-1}) \] (3.52)

\[ H(a \pm 1, a, a \mp 1) = \frac{1}{2} - \frac{1}{4}(h_{a+1} - h_{a-1}) \] (3.53)

(3.54)

where we introduce the auxiliary function

\[ h_a = \left\lfloor \frac{a(m - m')}{m'} \right\rfloor = \left\lfloor \frac{a \lambda}{\pi} \right\rfloor \] (3.55)

This function counts the number of unshaded 1-bands below the height \( a \). With
3.3. RSOS MODELS

Figure 3.3: The gauged local energies of the $1 \times 1$ RSOS models in the interval $0 < \lambda < \pi$. In this gauge, the local energies take the values $0, \frac{1}{4}, \frac{1}{2}$.

This choice of gauge, the groundstate paths indeed have minimal, $0$ energy, and all the local energy functions are fractional values: $0, \frac{1}{4}, \frac{1}{2}$. The local energy functions for RSOS$(m, m')_{1 \times 1}$ are depicted in fig. 3.3.

There is a duality, described by

$$m \leftrightarrow m' - m,$$
$$\lambda \leftrightarrow \pi - \lambda,$$

shaded 1-bands $\leftrightarrow$ unshaded 1-bands,

$$h_a \leftrightarrow a - 1 - h_a,$$

which leads to the identity

$$H^{m, m', 1}(a, b, c) = \frac{1}{2} H^{m' - m, m', 1}(a, b, c).$$

2 × 2 local energy function

The $2 \times 2$ local energy functions are computed after a conjugate modulus transformation is performed on the Boltzmann weights. The conjugate modulus transformation was originally used by Forrester and Baxter to make the computation feasible \cite{51}. The conjugate modulus transformation involves a change from elliptic nome $t = e^{-\epsilon}$ to conjugate nome $p = e^{-\pi^2/\epsilon}$ and the replacement of $\lambda$ and $u$ with $x$ and $w$, respectively

$$x = e^{-2\pi \lambda/\epsilon} = p^{\lambda/\pi},$$

$$w = e^{-2\pi u/\epsilon} = p^{u/\pi}. $$

$= 0$

$= \frac{1}{2}$

$= \frac{1}{4}$
The conjugate modulus transformation is the following
\[
\theta_1(u, t = e^{-\epsilon}) = ie^{-\epsilon/4}e^{-iu}E(e^{2iu}, e^{-2\epsilon}) = \sqrt{\frac{\pi}{\epsilon}}e^{-(u-\pi/2)^2/\epsilon}E(e^{-2\pi u/\epsilon}, e^{-2\pi^2/\epsilon}) .
\]
where we introduce the function
\[
E(w) = E(w, p) = \sum_{k=-\infty}^{\infty} (-1)^n p^{n(n-1)/2} w^n = \prod_{n=1}^{\infty} (1-p^{n-1}w)(1-p^nw)(1-p^n) .
\]

The diagonal fused weights are expressed as such
\[
W_{2,2} \left( \begin{array}{cc} a + 2 & a \\ a & a + 2 \end{array} \right) = \frac{g_a^2}{g_a - g_{a+2}} \frac{wE(x^2w^{-1})E(w^{-1}x)}{E(x^2)E(x)} ,
\]
\[
W_{2,2} \left( \begin{array}{cc} a & a \\ a & a + 2 \end{array} \right) = W_{2,2} \left( \begin{array}{cc} a + 2 & a \\ a & a \end{array} \right) = \frac{g_a}{g_{a+2}} \frac{E(xw^{-1})E(x^{a+1}w^{1})}{E(x)E(x^{a+1})} ,
\]
\[
W_{2,2} \left( \begin{array}{cc} a & a + 2 \\ a & a \end{array} \right) = \frac{E(x^a w^{-1})E(x^{a+1})E(x^{a-1})E(x^{a+2})}{E(x)E(x^2)E(x^a)E(x^{a+1})} + \frac{E(x^{a-1})E(x^a w^{-1})}{E(x^{a-1})E(x^a)} ,
\]
where the gauge factors \(g_a\) are arbitrary. The low temperature limit is taken as \(x \to 0\), or equivalently \(p \to 0\) with \(w\) kept constant. The function \(E\) satisfies the following relations which are used for taking the limit
\[
E(w, p) = E(pw^{-1}, p) = -wE(w^{-1}, p) ,
\]
\[
E(p^n w, p) = p^{-n(n-1)/2}(-w)^{-n}E(w, p) ,
\]
\[
\lim_{p \to 0} E(p^a w, p^b) = \begin{cases} 1, & 0 < a < b \\ 1 - w, & a = 0 \end{cases}
\]
\[
\lim_{x \to 0} \frac{E(x^a w^{-1})}{x^a} = w^{[a \lambda/\pi]} ,
\]
where \(n\) is an integer. These identities are used to derive the limit. By choosing the gauge \(g_a = w^{a(\lambda-\pi)/(4\pi)}\), after removing an overall scale factor \(e^{-2u(\lambda-u)/\epsilon}\) the local energy functions are
\[
0 < \lambda < \frac{\pi}{2}: \quad H(a \pm 2, a, a \mp 2) = 1
\]
3.3. RSOS MODELS

\[ H(a \pm 2, a, a) = H(a, a, a \pm 2) = \frac{1}{2} \pm h_{a \pm 1} \]  (3.64b)

\[ H(a, a \pm 2, a) = \mp (h_a + h_{a \pm 1}) \]  (3.64c)

\[ H(a, a, a) = \begin{cases} 
0, & h_{a-1} = h_a = h_{a+1} \\
1, & \text{otherwise}
\end{cases} \]  (3.64d)

\[ \frac{\pi}{2} < \lambda < \pi : \quad H(a \pm 2, a, a \mp 2) = 2 \]  (3.65a)

\[ H(a \pm 2, a, a) = H(a, a, a \pm 2) = \frac{1}{2} \pm h_{a \pm 1} \]  (3.65b)

\[ H(a, a \pm 2, a) = \mp (h_a + h_{a \pm 1}) \]  (3.65c)

\[ H(a, a, a) = \begin{cases} 
0, & h_{a-1} = h_a \text{ or } h_a = h_{a+1} \\
1, & \text{otherwise}
\end{cases} \]  (3.65d)

Note that the local energy functions take different values for the intervals \( 0 < \lambda < \frac{\pi}{2} \) and \( \frac{\pi}{2} < \lambda < \pi \). These local energies can be put in a more desirable form, with only non-negative, half-integer values by a gauge transformation. In present form, they also take negative and positive values. We introduce a new gauge \( g'_a = w^G_a \) which defines the new local energies

\[ H'(a, b, c) = H(a, b, c) + 2G_b - G_a - G_c. \]  (3.66)

The requirement \( H'(a, b, c) \geq 0 \) is satisfied by the following choice for \( G_a \)

\[ G_a = \begin{cases} 
h_1 + h_3 + \ldots h_{a-1}, & a \text{ even} \\
h_2 + h_4 + \ldots h_{a-1}, & a \text{ odd}
\end{cases} \quad G_{a+1} - G_{a-1} = h_a \]  (3.67)

This gauge-transformation can be defined through defining \( G(a, b) = G_b - G_a \) and applying

\[ H'(a, b, c) = H(a, b, c) + G(a, b) - G(b, c), \quad G(a, b) = \frac{1}{2} (b - a) h_{(a+b)/2}. \]  (3.68)

These local energies are depicted in figs. 3.4 and 3.5. They take non-negative half-integer values, namely \( 0, \frac{1}{2}, 1 \). They take the following values for the two intervals (leaving the prime)

\[ 0 < \lambda < \frac{\pi}{2} : \quad H(a \pm 2, a, a \mp 2) = 1 - (h_{a+1} - h_{a-1}) \]  (3.69a)

\[ H(a \pm 2, a, a) = H(a, a, a \pm 2) = \frac{1}{2} \]  (3.69b)

\[ H(a, a \pm 2, a) = \pm (h_{a \pm 1} - h_a) \]  (3.69c)
CHAPTER 3. FUSED RSOS MODELS AND FINITIZED CHARACTERS

\[
H(a, a, a) = \begin{cases} 
0, & h_{a-1} = h_a = h_{a+1} \\
1, & \text{otherwise}
\end{cases}
\quad (3.69d)
\]

\[
\frac{\pi}{2} < \lambda < \pi : \quad H(a \pm 2, a, a \mp 2) = 2 - (h_{a+1} - h_{a-1}) 
\]

\[
H(a \pm 2, a, a) = H(a, a, a \pm 2) = \frac{1}{2} 
\]

\[
H(a, a \pm 2, a) = \pm(h_{a\pm1} - h_a) 
\]

\[
H(a, a, a) = \begin{cases} 
0, & h_{a-1} = h_a \text{ or } h_a = h_{a+1} \\
1, & \text{otherwise}
\end{cases}
\quad (3.70d)
\]

The shaded band structure, hence the \( h_a \) functions differ for the two intervals, \( 0 < \lambda < \frac{\pi}{2} \) and \( \frac{\pi}{2} < \lambda < \pi \). Hence, the \( h_a \) dependent choice of \( G_a \) leads to the two sets of equations \((3.69)\) and \((3.70)\). The \( H(a, b, c) \) functions depend on differences of the \( h_a \) function which is in fact a dependence on the shaded band structure, as

\[
\delta_a = h_{a+1} - h_a = \begin{cases} 
0, & \text{the 1-band } (a, a + 1) \text{ is shaded} \\
1, & \text{the 1-band } (a, a + 1) \text{ is unshaded}
\end{cases}
\quad (3.71)
\]

This means the local energies can be associated to height independent local configurations, as in figs. 3.4 and 3.5. The duality relation, described for the \( 1 \times 1 \) case (eq. 3.56), interchanges the two intervals, \( 0 < \lambda < \frac{\pi}{2} \) and \( \frac{\pi}{2} < \lambda < \pi \). The shaded and unshaded bands are also interchanged, hence \( \delta_a \leftrightarrow 1 - \delta_a \). The duality leads to the following equation for the local energy functions

\[
H^{m, m', 2}(a, b, c) = 1 - H^{m' - m, m', 2}(a, b, c) .
\quad (3.72)
\]

**3 \times 3 local energy function**

The \( 3 \times 3 \) local energy functions are computed for the interval \( 0 < \lambda < \frac{\pi}{3} \) in [73]. By the duality eq. (3.56), the local energies of the interval \( \frac{2\pi}{3} < \lambda < \frac{4\pi}{3} \) has not yet been found.

The computation for the interval \( 0 < \lambda < \frac{\pi}{3} \) is the same as for the \( 2 \times 2 \) case. We use initially the \( g_a = u^{a(a\lambda - \pi)/(4\pi)} \) gauge, use the conjugate modulus
Figure 3.4: Local energies for $2 \times 2$ fused RSOS models in the interval $0 < \lambda < \frac{\pi}{2}$.
Figure 3.5: Local energies for $2 \times 2$ fused RSOS models in the interval $\frac{\pi}{2} < \lambda < \pi$. 
transformation, and normalize by a factor $e^{3u(\lambda - u)/\epsilon}$. In the low temperature limit $x \to 0$, initially the local energies take the following values

\[
H(a \pm 3, a, a \pm 3) = \pm (h_{a\pm1} + h_{a\pm2} + h_{a\pm3})
\]

\[
H(a \pm 3, a, a \pm 1) = H(a \pm 1, a, a \pm 3) = \frac{1}{2} \pm (h_{a\pm1} + h_{a\pm2})
\]

\[
H(a \pm 3, a, a \mp 1) = H(a \mp 1, a, a \pm 3) = 1 \pm h_{a\pm1}
\]

\[
H(a \pm 3, a, a \mp 3) = \frac{3}{2}
\]

\[
H(a \pm 1, a, a \pm 1) = \left\{
\begin{array}{ll}
1 \pm h_{a\pm1}, & h_{a+1} = h_{a+1} = h_{a-1} + 1 = h_{a-2} + 1 \\
\pm h_{a\pm2}, & \text{otherwise}
\end{array}
\right.
\]

\[
H(a \pm 1, a, a \mp 1) = \frac{1}{2} + h_{a+1} - h_{a-1}
\]

To introduce non-negative, fractional weights, we perform a similar transformation, as before, to introduce $H'(a, b, c)$

\[
H'(a, b, c) = H(a, b, c) + 2G_b - G_a - G_c,
\]

where we use the gauge function

\[
G_a = \frac{1}{4} \sum_{k=1}^{a-1} h_{a+1} + h_a, \quad G_{a+1} - G_a = \frac{1}{4}(h_{a+1} + h_a).
\]

The change of gauge leads to the following local energies (again, we omit the prime)

\[
H(a \pm 3, a, a \pm 3) = \pm \frac{1}{2}(h_{a\pm3} - h_a)
\]

\[
H(a \pm 3, a, a \pm 1) = H(a \pm 1, a, a \pm 3) = \frac{1}{2} \pm \frac{1}{4}(-2h_a + h_{a\pm1} + 2h_{a\pm2} - h_{a\pm3})
\]

\[
H(a \pm 3, a, a \mp 1) = H(a \mp 1, a, a \pm 3) = 1 \pm \frac{1}{4}(h_{a\mp1} + 2h_{a\pm1} - 2h_{a\pm2} - h_{a\pm3})
\]

\[
H(a \pm 3, a, a \mp 3) = \frac{3}{2} + \frac{1}{4}(h_{a-3} + 2h_{a-2} + 2h_{a-1} - 2h_{a+1} - 2h_{a+2} - h_{a+3})
\]

\[
H(a \pm 1, a, a \pm 1) = \left\{
\begin{array}{ll}
\pm(h_{a\pm2} - h_{a\pm1}), & h_{a+1} = h_a = h_{a-1} \\
1 \pm \frac{1}{2}(h_{a\pm1} - h_a), & h_{a+1} = h_{a-1} + 1
\end{array}
\right.
\]

\[
H(a \pm 1, a, a \mp 1) = \frac{1}{2} + \frac{3}{4}(h_{a+1} - h_{a-1})
\]
Figure 3.6: The gauged local energies of the $n = 3$ RSOS models in the interval $0 < \lambda < \frac{\pi}{3}$. In this gauge, the local energies take the values $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}$. 

\begin{align*}
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\end{align*}

\begin{align*}
= & \frac{1}{4} = \frac{1}{2} = \frac{3}{4} = 1 = \frac{5}{4} = \frac{3}{2} = 0
\end{align*}
3.3. RSOS MODELS

The local energy functions, expressed in the last gauge, are listed in fig. 3.6.

In the dual interval, $\frac{2\pi}{3} < \lambda < \pi$ the local energy functions follow from the duality

$$H^{m,m',3}(a,b,c) = \frac{3}{2} - H^{m'-m,m',3}(a,b,c).$$  \hspace{1cm} (3.77)

In general, the duality is conjectured [2] to be

$$H^{m,m',n}(a,b,c) = \frac{n}{2} - H^{m'-m,m',n}(a,b,c).$$  \hspace{1cm} (3.78)

3.3.5 One-dimensional sums and energy statistics

Utilizing Baxter’s corner transfer matrix (CTM) method [82, 84] we define the energy statistics of an RSOS path

$$E(\sigma) = \sum_{j=1}^{N} jH(\sigma_{j-1},\sigma_{j},\sigma_{j+1}).$$ \hspace{1cm} (3.79)

The energy statistics is the statistical weight of the corresponding CTM configuration in the lower-right (SE) quadrant in the low temperature limit. As in the low temperature limit, the configuration is diagonal in the SW-NE direction, the configuration of the CTM is described by a path. The boundary conditions $(a, b, c)$ of the path correspond to the boundary conditions of the CTM, i.e. to the value of the central height, and the boundary values of the CTM, respectively. The summation runs from the center to the edge of the CTM. Hence the factor $j$ in the summands is due to the increasing number of faces.

With the help of the energy statistics, we define the one-dimensional sum, by summing over every path with boundary conditions $(a, b, c)$

$$X^{(N)}_{a,b,c}(q) = \sum_{\sigma} q^{E(\sigma)} = \sum_{\sigma} q^{\sum_{j=1}^{N} jH(\sigma_{j-1},\sigma_{j},\sigma_{j+1})},$$ \hspace{1cm} (3.80)

$$\sigma_{0} = a, \sigma_{N} = b, \sigma_{N+1} = c.$$

The one-dimensional sums satisfy the recursion relation

$$X^{(N)}_{a,b,c}(q) = \sum_{d \sim b} q^{NH(d,b,c)} X^{(N-1)}_{a,d,c}(q),$$ \hspace{1cm} (3.81)

where $d \sim b$ means that we sum over all values of $d$ which can form an allowed path with $b$. The recursion relations satisfy the following boundary and initial conditions

$$X^{(N)}_{a,0,c}(q) = X^{(N)}_{a,m',c}(q) = 0, \quad X^{(0)}_{a,b,c}(q) = \delta_{a,b}. \hspace{1cm} (3.82)$$
In this section, based on the shaded band structure, local energy functions and the one-dimensional sums, we define groundstate boundary conditions and groundstate paths. As local energies are known explicitly only for \( n = 1, 2 \) and for \( n = 3 \) in the interval \( 0 < \frac{\lambda}{\pi} < \frac{1}{3} \), we restrict our statements to these cases. As a conjecture, we extend our statements to generic \( n \) in the interval \( 0 < \frac{\lambda}{\pi} < \frac{1}{n} \).

Loosely speaking, groundstate boundary conditions are boundary conditions where the lowest energy path can be extended in length without increment in its
3.3. **RSOS MODELS**

energy statistics. We define a boundary condition \((a, b, c)\) to be groundstate boundary for RSOS fusion level \(n\) and \(0 < \lambda < \frac{1}{n}\), if \(b\) and \(c\) are in the same shaded \(n\)-band, and they are symmetrically placed about its center. The further necessary condition for \((a, b, c)\) that the fused paths can be actually formed (e.g. for fusion level \(n = 2\), \(a, b, c\) have to share the same parity).

If \(n\) is even, then the central value \(d\) is an integer height, and \((b, c)\) can take the following values: \((d, d)\), \((d \pm 1, d \mp 1)\), \((d \pm 2, d \mp 2)\),..., \((d \pm \frac{n}{2}, d \mp \frac{n}{2})\). If \(n\) is odd, the central value \(d\) of the shaded \(n\)-band is a half integer, and the allowed \((b, c)\) values are \((d \pm \frac{1}{2}, d \mp \frac{1}{2})\), \((d \pm \frac{3}{2}, d \mp \frac{3}{2})\),..., \((d \pm \frac{n}{2}, d \mp \frac{n}{2})\). Shaded \(n\)-bands occur for \(0 < \lambda < \frac{1}{n}\), and not for larger \(\lambda\). This means that the \((d \pm \frac{n}{2}, d \mp \frac{n}{2})\) values of \((b, c)\) are not supported here, and we cannot extend our definition here. However, we expect that the other elements of the previous lists are groundstate boundary condition, with some \(d\) which is not necessary central.

Out primary interest is fusion level \(n = 2\) which means that the fused path, \(a\) and \(b\) share the same parity.

For the interval \(0 < \lambda < \frac{1}{2}\) there are shaded 2-bands, and at fusion level \(n = 2\), the associated groundstates are related to the superconformal minimal models in the Neveu-Schwarz and Ramond sectors.

In the interval \(\frac{1}{2} < \lambda < 1\), there are no shaded 2-bands, only shaded 1-bands. Consider the union of two sequences \(\rho(r)\) and \(\rho(r) + 1\). Separate this union into the union of two new sequences \(\rho_0(r)\) and \(\rho_1(r)\) consisting of the even and odd members respectively (fig. 3.7). As there are only shaded 1-bands, if we wish to end a \(2 \times 2\) fused path in the same, we must have

\[
b = c = \rho_\mu(r), \quad \mu = 0, 1
\]  

(3.83)

for some \(r = 1, 2, \ldots, m - 1\). The supported zero energy groundstate paths are flat paths \(\sigma = \{\rho, \rho, \ldots, \rho\}\) with \(\rho = \rho_0(r)\) or \(\rho = \rho_1(r)\) with some \(r\).

Based on *Physical Combinatorics* [74][85][92] and the correspondence principle of the Kyoto school [67][93][95], one-dimensional sums with groundstate boundary conditions are interpreted as finitized conformal characters

\[
\chi_{\Delta}^{(N)}(q) = q^{-c/24 + \Delta} X_{abc}^N(q) = q^{-c/24 + \Delta} \sum_{\sigma} q^{E(\sigma)},
\]  

(3.84)

where on the CFT side, \(E(\sigma)\) are identified as conformal energies of the infinite system. These are spectrum generating functions for a finite truncated set of conformal energy levels in a given sector labeled by \((a, b, c)\). The connection between the one-dimensional sum with boundary conditions \((a, b, c)\) and conformal character with Kac indices \((r, s)\) is the following: \(s = a\), and
(b, c) is in the rth continuous shaded band, in other words, in the shaded band associated to \( \rho(r) \). The precise value of \( b \) and \( c \) depend on the type of groundstate and the fusion level and the length \( N \).

For fusion level \( n = 2 \) and \( \frac{1}{2} \frac{\lambda}{\pi} < 1 \), the correspondence is given by

\[
  a = s, \quad b = c = \rho_\mu(r), \quad \mu = s \mod 2 .
\]

(3.85)

Here, the previous definition of \( \rho_0 \) and \( \rho_1 \) ensures the required parity property.

### 3.4 RSOS\((m, 2m + 1)_{2 \times 2}\) sequence

In [74], Jacob and Mathieu (JM) defined one-dimensional sums associated with a particular choice of local energy functions for RSOS lattice paths with half-integer steps. These local energy functions were not related to integrable lattice models. However quite remarkably, in the thermodynamic limit, these one-dimensional sums reproduce the Virasoro characters of the sequence of non-unitary minimal models \( \mathcal{M}(m, 2m + 1) \) with \( m = 2, 3, 4, \ldots \). Here, we present an energy preserving bijection between the JM paths and RSOS\((m, 2m + 1)_{2 \times 2}\) paths. Hence we show that the underlying lattice models of the JM paths are the RSOS\((m, 2m + 1)_{2 \times 2}\) fused models.

#### 3.4.1 Adjacency graph for RSOS\((m, 2m + 1)_{2 \times 2}\) models

The heights of the unfused model RSOS\((m, m')\) live on the Dynkin diagram \( A_{m'-1} \), which possesses a \( \mathbb{Z}_2 \) symmetry. Specifically, the Boltzmann weights are invariant under the height reversal \( \sigma \leftrightarrow m' - \sigma \). As a consequence, when nodes related by the symmetry are identified, the connectivity graph of the \( 1 \times 1 \) unfused models with odd \( m' \) folds into a \( T_{m'-1}^{(n=2)} \) tadpole diagram. Examples are shown in fig. 3.8.

At fusion level \( n = 2 \), the allowed adjacent heights fulfill the following conditions

\[
  \sigma - \sigma' = 0, \pm 2 \quad (3.86)
\]

\[
  \sigma + \sigma' = 4, 6, \ldots, 2m' - 4 \quad (3.87)
\]

For \( m' = 2m + 1 \) models, this means that their adjacency diagram split into the product of two tadpole diagrams \( T_m^{(n=2)} \). For the RSOS\((m, 2m + 1)\) series, the unfused tadpole diagrams \( T_m \) have only one self-connected node (figs. 3.8a and 3.8c), while the \( 2 \times 2 \) fused tadpole diagrams \( T_m^{(n=2)} \) have only one not self-connected node (figs. 3.8d and 3.8d). Hence, the fused and unfused tadpole diagrams do not coincide, with the exception of RSOS\((2, 5)\) (fig. 3.8d), where the fused and unfused models are equivalent, as one can easily prove.
3.4. **RSOS**($M, 2M + 1$)$_{2 \times 2}$ **SEQUENCE**

![Diagram](image)

(a) RSOS(2, 5)$_{1 \times 1}$: folding of the $A_4 = A_4^{(n=1)}$ and the corresponding $T_2$ tadpole.

(b) RSOS(2, 5)$_{2 \times 2}$: decomposition of $A_4^{(n=2)}$ into the tensor product of two $T_2$ tadpoles.

(c) RSOS(3, 7)$_{1 \times 1}$: folding of the $A_6 = A_6^{(n=1)}$ and the corresponding $T_3^{(n=1)}$ tadpole.

(d) RSOS(3, 7)$_{2 \times 2}$: decomposition of $A_6^{(n=2)}$ into the tensor product of two $T_3^{(n=2)}$ tadpoles.

Figure 3.8: The RSOS(2, 5) and RSOS(3, 7) lattice models are identified with tadpole diagrams which encode the adjacency rules between heights. The RSOS(2, 5)$_{1 \times 1}$ (a) and RSOS(2, 5)$_{2 \times 2}$ (b) models share the same tadpole diagram. This results from the fact that the $2 \times 2$ lattice fusion gives back the original lattice model. Instead, the RSOS(3, 7)$_{1 \times 1}$ (c) and RSOS(3, 7)$_{2 \times 2}$ (d) models show different tadpole diagrams because the $2 \times 2$ lattice fusion produces a new lattice model. In all figures, we denoted the $\sigma^{(RSOS)}$ heights.
3.4.2 Bijection between the JM and $2 \times 2$ RSOS paths

Jacob and Mathieu defined their path as follows: It is a sequence of half-integer heights $\sigma_j^{(JM)}$, with half integer steps. The heights are restricted to $1 \leq \sigma_j^{(JM)} \leq m$, $\sigma_j^{(JM)} \in \frac{1}{2}\mathbb{Z}$, and the steps goes from 0 to $L$: $0 \leq j \leq L$, $j \in \frac{1}{2}\mathbb{Z}$. The steps are either up or down by half unit: $|\sigma_{j+\frac{1}{2}}^{(JM)} - \sigma_j^{(JM)}| = \frac{1}{2}$. The path must start at integer value, $\sigma_0^{(JM)} \in \mathbb{N}$, and the local maxima occur at integer $j$’s with integer height $\sigma_j^{(JM)}$. Every path of length $N + \frac{1}{2}$ ends at a half-integer value $\sigma_{N+\frac{1}{2}}^{(JM)} = \frac{1}{2}, \frac{3}{2}, \ldots, m - \frac{3}{2}$ with a down step.

The energy of a JM path is given by

$$E = \sum_{j=\frac{1}{2}, j \in \frac{1}{2}\mathbb{N}}^N jw(j), \quad w(j) = \frac{1}{2} \left| \sigma_{j+\frac{1}{2}}^{(JM)} - \sigma_{j-\frac{1}{2}}^{(JM)} \right|, \quad \text{(3.88)}$$

$$1 \leq \sigma_j^{(JM)} \leq m, \quad \sigma_j^{(JM)} \in \frac{1}{2}\mathbb{N}.$$

To understand the bijection between the RSOS and JM path, consider the RSOS path as a walk on the fused tadpole diagram $T_m^{(n=2)}$, where we relabel the nodes corresponding to the heights as $\sigma^{(TP)} = 1, 2, \ldots, m$. Every $\sigma^{(TP)}$ value corresponds to two original RSOS heights, $\sigma_1^{(RSOS)} = 2\sigma^{(TP)} - 1$, $\sigma_2^{(RSOS)} = 2m + 2 - 2\sigma^{(TP)}$. As $2\sigma^{(TP)} - 1$ is odd and $2m + 2 - 2\sigma^{(TP)}$ is even, a path expressed in the $\sigma^{(TP)}$ variables corresponds to two –an odd and an even– paths in the $\sigma^{(RSOS)}$ variables. These two paths are equivalent: the corresponding energy statistics are the same, what follows from the height reversal symmetry (eq. 3.29). The two paths are also equivalent walks on the two copies of the $T_m^{(n=2)}$ tadpole diagrams.

Consider the RSOS path, as a path in the $\sigma^{(TP)}$ variables. The bijection between the RSOS and the JM path is given as follows. The RSOS path on the tadpole diagram –in the $\sigma_j^{(TP)}$ variables– takes values of the JM path at integer positions, so that two JM steps become one RSOS step

$$\sigma_j^{(TP)} = \sigma_j^{(JM)} \quad j \in \mathbb{N} \quad \text{(3.89)}$$

Where the JM path has a local minimum at a half integer position and value, the RSOS path has a horizontal step. As a result, the RSOS path follows almost completely the JM path, with the exception of local minima, as shown in fig. 3.9.

To prove the bijection, we prove that the energy defined by JM is gauge
3.4. RSOS($M, 2M + 1)_{2 \times 2}$ SEQUENCE

(a) JM($2, 5$) and RSOS($2, 5$)$_{2 \times 2}$ paths with $N = 17$ integer steps. Due to the $2 \times 2$ fusion, RSOS paths have definite parity, with only even or odd heights. Due to the $\mathbb{Z}_2$ symmetry of the $A_4$ Dynkin diagram we made the identification: $\sigma^{(TP)} = \sigma^{(JM)} = 1, 2$ correspond to $\sigma^{(RSOS)} = 1, 4$ and $\sigma^{(RSOS)} = 3, 2$, respectively.

(b) JM($3, 7$) and RSOS($3, 7$)$_{2 \times 2}$ paths with $N = 17$ integer steps: $\sigma^{(TP)} = \sigma^{(JM)} = 1, 2, 3$ correspond to $\sigma^{(RSOS)} = 1, 6$, $\sigma^{(RSOS)} = 3, 4$ and $\sigma^{(RSOS)} = 5, 2$, respectively.

Figure 3.9: Two examples of the bijection between JM and RSOS paths. The shared path is highlighted in blue. The edges of JM paths are in black while their equivalent in the RSOS description in purple. The bijection between the JM and RSOS paths preserves the contour of the path except at local minima where RSOS path replaces two consecutive steps with a single straight step.

Equivalent with the corresponding RSOS energy

\[
\sum_{j=\frac{1}{2}, j \in \mathbb{Z}}^{N+\frac{1}{2}} jw(j) =
\]

\[
= \frac{1}{2} \sum_{j=1}^{N} j \left( w(j - \frac{1}{2}) + 2w(j) + w(j + \frac{1}{2}) \right) + \frac{1}{2}(N + 1)w(N + \frac{1}{2})
\]

\[
= \sum_{j=1}^{N} j \left( H(\sigma_{j-1}, \sigma_j, \sigma_{j+1}) - 2G_j + G_{j-1} + G_{j+1} \right) + E_0 ,
\]

where the constant energy shift is

\[
E_0 = \frac{1}{2}(N + 1)w(N + \frac{1}{2}) = \frac{1}{4}(N + 1)|c - b|
\]

with boundary conditions $(\sigma_N, \sigma_{N+1}) = (b, c)$.

To prove the bijection between the JM and the fused RSOS description, we need to find a gauge $G_j = g(\sigma_j)$ such that eq. (3.91) holds

\[
H(\sigma_{j-1}, \sigma_j, \sigma_{j+1}) = \frac{1}{2} \left( w(j - \frac{1}{2}) + 2w(j) + w(j + \frac{1}{2}) \right) + 2G_j - G_{j-1} - G_{j+1}.
\]
To find the appropriate gauge transformation, consider \(\sigma_j = a, \sigma_{j+1} = a \pm 2, \sigma_{j+1/2} = a \pm 1\), where \(a\) is the height in the equivalent RSOS description (in the original height variables), write out eq. (3.93) for each possible local energy function, and substitute the correct value of \(w(j)\) from eq. (3.88)

\[
a \in 2N - 1 : \\
H(a \pm 2, a, a \mp 2) = 1 + 2g(a) - g(a + 2) - g(a - 2) \quad (3.94a) \\
H(a + 2, a, a) = H(a, a, a + 2) = \frac{3}{4} + g(a) - g(a + 2) \quad (3.94b) \\
H(a - 2, a, a) = H(a, a, a - 2) = \frac{1}{4} + g(a) - g(a - 2) \quad (3.94c) \\
H(a, a + 2, a) = \frac{1}{2} + 2g(a + 2) - 2g(a) \quad (3.94d) \\
H(a, a - 2, a) = \frac{1}{2} + 2g(a - 2) - 2g(a) \quad (3.94e) \\
H(a, a, a) = 0 \quad (3.94f)
\]

Now, we solve all these equations together. This computation was performed for \(a\) odd; the case where \(a\) is even follows from the \(\mathbb{Z}_2\) symmetry \(\sigma \leftrightarrow m' - \sigma\). The last equation (3.94f) is an identity, whenever \(h_a = h_{a+1}\) or \(h_{a-1} = h_a\) which is the case for the RSOS\((m, 2m + 1)_{2 \times 2}\) series described in JM theory. The sequence of integers, describing the shaded bands is explicitly computable for this case

\[
\rho = \left\lfloor \frac{r(2m+1)}{m} \right\rfloor = \left\lfloor 2r + \frac{r}{m} \right\rfloor = 2r \quad r = 1, 2, \ldots, m - 1. \quad (3.95)
\]

This shows that every other band is shaded starting from the lowest one \(a = 1\), what implies that eq. (3.94f) is satisfied. Excluding this identity, the remaining equations simplify to

\[
a \in 2N - 1 : \\
h_{a+1} - h_{a-1} = 1 \quad (3.96a) \\
g(a + 2) - g(a) = \frac{1}{4} \quad (3.96b) \\
g(a - 2) - g(a) = -\frac{1}{4} \quad (3.96c) \\
h_{a+1} - h_a = 1 \quad (3.96d) \\
h_{a-1} - h_a = 0 \quad (3.96e)
\]

Here eqs. (3.96a), (3.96d) and (3.96e) are again satisfied because of the shaded band structure of the JM series. Finally, the desired solution to the gauge is the solution to eqs. (3.96b) and (3.96c), where \(C\) is an arbitrary constant

\[
a \in 2N - 1 : \quad g(a) = \frac{1}{8} a + C. \quad (3.97)
\]
3.5. **FINITIZED CHARACTERS**

This solutions completes the bijection between the RSOS\((m, 2m + 1)_{2 \times 2}\) models and the JM models, proving that these fused models are indeed the underlying lattice models of the JM one-dimensional sums.

### 3.5 Finitized characters

In this section, we present the known results on the finitized characters. In [73], a formula was conjectured for RSOS\(_{n \times n}(m, m')\) for \(0 < \lambda < \frac{\pi}{n}\) which based on explicit calculations carried out for \(n = 2, 3\). In [2], we conjectured the finitized characters for \(n = 2, \frac{\pi}{2} < \lambda < \pi\). Neither of these conjectures is proven using the recursion relation: The groundstate energies and certain one-dimensional sums which do not correspond to characters have not been calculated.

#### 3.5.1 Conjectured finitized branching functions corresponding to \(n \times n\) fused RSOS models at \(0 < \lambda < \frac{\pi}{n}\)

Finitized characters of RSOS\((m, m')_{n \times n}\) models with \(0 < \lambda < \frac{\pi}{n}\) were considered in [73]. These give the finitized branching functions for non-unitary minimal cosets \(\mathcal{M}(M, M', n')\) with \(n' = n\), establishing the correspondence between them. The unitary case was considered in [72].

For simplicity, we assume that the system size \(N\) is even and consequently \(b - a\) also even. Following Schilling [72] we introduce \(q\)-multinomials

\[
T_{\ell}^{(n)}(N, \mu) = \sum_{\tilde{v} \in \mathbb{Z}_{\geq 0}^{n+1}} q^{vC^{-1}v - e_{\ell}C^{-1}v} (q)_{N} \prod_{i=0}^{n} \frac{1}{(q)_{v_i}}
\]

where

\[
\tilde{v} = (v_0, v_1, \ldots, v_n),
\]

\[
v = (v_1, v_2, \ldots, v_{n-1}), \quad v_i \in \mathbb{Z}, \quad i = 0, \ldots, n
\]

and the sum means summing over all \(v_i \in \mathbb{Z}_{\geq 0}, \ i = 0, \ldots, n\) with the constrains

\[
v_0 = \frac{N}{2} - \frac{\mu}{n} - e_1C^{-1}v,
\]

\[
v_n = \frac{N}{2} + \frac{\mu}{n} - e_{n-1}C^{-1}v,
\]
\[ \mu = -\frac{nN}{2} \quad , -\frac{nN}{2} + 1, \ldots, \frac{nN}{2} \quad (3.101c) \]

The matrix \( C^{-1} \) is the inverse of the Cartan matrix \( C \) with entries \( C_{i,j} = 2 - \delta_{i,j+1} - \delta_{i,j-1} \). The vectors \( e_{\ell} \) are standard \((n - 1)\) dimensional basis vectors

\[ C_{i,j}^{-1} = \begin{cases} \frac{(n-i)j}{n}, & j \leq i \\ \frac{(n-j)i}{n}, & j > i \end{cases} \] \quad \quad (\delta_{i,j}, \quad i,j = 1,\ldots,n-1 \quad 0, \quad \text{otherwise} \quad (3.102) \]

The \( q \)-factorials are defined in eq. (3.6) as

\[ (q)_n = \prod_{k=1}^{n}(1-q^k), \quad (q)_\infty = \prod_{k=1}^{\infty}(1-q^k). \]

The \( q \)-multinomials satisfy the following recursion relation and initial condition which is an equivalent definition of them

\[ T_{\ell}^{(n)}(N, \mu) = \sum_{k=0}^{\ell-1} q^{(\ell-k)(\frac{n}{2}+\frac{\mu}{2})} T_{n-k}^{(n)}(N-1, \frac{n}{2} + \mu - k) + \]

\[ + \sum_{k=\ell}^{n} q^{(k-\ell)(\frac{n}{2}-\frac{\mu}{2})} T_{n-k}^{(n)}(N-1, \frac{n}{2} + \mu - k) \]

\[ T_{\ell}^{(n)}(0, \mu) = \delta_{\mu,0} \quad (3.104) \]

The finitized branching functions of the \( \mathcal{M}(M, M', n) \) models corresponding to one-dimensional sums of RSOS\((m, m')_{n \times n} \) models are

\[ b_{r,s,\ell}^{M,M',n,(N)}(q) \cong \sum_{k=-\infty}^{\infty} \left( q^{\frac{k}{2}(knMM'+M'r-Ms)} T_{(n+b-c)/2}^{(n)}(N, \frac{1}{2}(b-a) + kM') \right. \]

\[ -q^{\frac{k}{2}(knM+r)(knM'+s)} T_{(n+b-c)/2}^{(n)}(N, \frac{1}{2}(b+a) + kM') \) \quad (3.105) \]

where \( \cong \) means that the equality holds up to the leading power of \( q \). The relations between the parameters of the equation

\[ M = nm - (n-1)m' \quad , \quad (3.106) \]

\[ M' = m' \quad , \quad (3.107) \]

\[ s = a \quad , \quad (3.108) \]

\[ r = r(\rho) = r^{m,m',n}(\rho) \quad , \quad (3.109) \]
Note that in the case $h_b$ is odd, $b$ and $c$ are interchanged. Whenever $\gcd(m, m') = 1$, $\gcd\left(\frac{M'-M}{n}, M'\right) = \gcd(m'-m, m') = 1$ also. In case, $nm-(n-1)m' = M = 1$, the fused model is not defined. For the unitary cases $m = m' - 1$, the formula reduces to the one in Schilling [72]. This formula has been checked for $n = 1, 2, 3$, for many different models, for different branching functions, up to $N = 14$.

Setting $q \to 1$ gives the correct counting of states. The $N \to \infty$ limit correctly recovers the full branching function, as it was proven in [73].

### 3.5.2 Conjectured finitized minimal characters corresponding to $2 \times 2$ fused RSOS models at $\frac{\pi}{2} < \lambda < \pi$

For the RSOS($m, m'$)$_{2 \times 2}$ models, two intervals are distinguished: $0 < \lambda < \frac{\pi}{2}$ and $\frac{\pi}{2} < \lambda < \pi$, which are covered in [73] and [2] respectively. Here, we consider the latter one.

Define the $q$-trinomial coefficients in terms of $q$-factorials

$$
\left[ \begin{array}{c} n \\ \ell, m \end{array} \right]_q = \begin{cases} \frac{(q)_n}{(q)_\ell (q)_m} & \ell, m, n-\ell-m \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise} \end{cases}
$$

The $\left[ \begin{array}{c} n \\ \ell, m \end{array} \right]_q$ trinomial is a $q$-deformed counting of $2 \times 2$ fused paths with $\ell$ up steps, $m$ down steps, and $n-\ell-m$ flat steps. The counting does not respect the height restrictions eqs. (3.25) and (3.26).

In the $q \to 1$ limit, the $q$-deformed trinomials simplify to trinomial coefficients

$$
\lim_{q \to 1} \left[ \begin{array}{c} n \\ \ell, m \end{array} \right]_q = \left( \begin{array}{c} n \\ \ell, m \end{array} \right) = \frac{n!}{\ell! m! (n-\ell-m)!}
$$

Introduce an auxiliary function, a special case of a trinomial, originally introduced in [96]

$$
T_k^{(N)}(q) = \sum_{j=0}^{N} q^{j+k} \left[ \begin{array}{c} N \\ j, j+k \end{array} \right]_q.
$$

Note that it does not have direct connection to the $T_{\ell}^{(n)}(N, \mu)$ Schilling multinomials. The $T_k^{(N)}(q)$ function is a $q$-deformed counting of an other set of $2 \times 2$ fused paths. It counts the $N$ step long $2 \times 2$ fused paths on $A_\infty$ Dynkin-diagram, only respecting $\sigma_{i+1} - \sigma_i = 0, \pm 2$, with $2k$ height difference between
the initial and final height: \(\sigma_N - \sigma_0 = 2k\). This ensures correct counting, as \(j + k\) is the number of up-steps, \(j\) is the number of down-steps, and \(N - 2j - k\) is the number of flat steps.

The finitized characters corresponding to \(\mathcal{M}(m, m')\) minimal models, realized by \(\text{RSOS}(m, m')_{2 \times 2}\) model, with \(\pi < \lambda = \frac{(m' - m)\pi}{m'} < \pi\) are the following

\[
\text{ch}^{m, m', (N)}_{r, s}(q) \equiv X^{m, m', (N)}_{a, b, c}(q) = \\
= \sum_{k = -\infty}^{\infty} \left( q^{k(km'+m'-r-m)}T^{(N)}_{km'+\frac{\mu}{2}}(q) - q^{(km+r)(km'+s)}T^{(N)}_{-km'-\frac{\mu}{2}}(q) \right)
\]

where

\[
a = s, \quad b = c = \rho_\mu(r), \quad \mu \equiv 0 \mod 2
\]

as it is defined in section 3.3.5.

This result was extensively checked for numerous \(\text{RSOS}(m, m')_{2 \times 2}\) models, for all possible \(1 \leq r \leq m, 1 \leq s \leq m' - 1\) (\(r, s\) Kac indices, up to size \(N = 12\), for \((m, m') = (2, 5), (2, 7), (3, 7), (3, 8)\), up to size \(N = 11\) for \((m, m') = (2, 9), (4, 9)\), up to size \(N = 10\) for \((m, m') = (3, 10)\) and up to size \(N = 9\) for \((m, m') = (2, 11), (3, 11), (4, 11), (5, 11)\).

The result correctly recover the counting of the paths in the \(q \to 1\) limit.

In the \(N \to \infty\) limit the finitized characters are expected to recover the full CFT character, up to an overall factor. The \(q\)-deformed trinomial \(T^{(N)}_k(q)\) takes the following value in the \(N \to \infty\) limit

\[
\lim_{N \to \infty} T^{(N)}_k(q) = \lim_{N \to \infty} \sum_{j=0}^{\infty} \left[ \begin{array}{c} N \\ j, j + k \end{array} \right]_q = \frac{1}{(q)_\infty}
\]

To prove this, first we take the limit inside the summation. Then we take the limit of the trinomial inside

\[
\lim_{N \to \infty} \left[ \begin{array}{c} N \\ j, j + k \end{array} \right]_q = \lim_{N \to \infty} \frac{(q)_N}{(q)_j(q)_{j+k}(q)_{N-2j-k}} = \frac{1}{(q)_j(q)_{j+k}}
\]

This shows that \(\lim_{N \to \infty} T^{(N)}_k(q) = \sum_{j=0}^{\infty} \frac{q^{j(j+k)}}{(q)_j(q)_{j+k}}\). Use the \(q\)-deformed version of Kummer’s theorem (see (2.2.8) of [97]), which is

\[
\sum_{j=0}^{\infty} \frac{q^{j(j-1)z^j}}{(1-q) \ldots (1-q^j)(1-z)(1-zq) \ldots (1-zq^{j-1})} = \prod_{j=0}^{\infty} \frac{1}{1-zq^j}, \quad (3.118)
\]
3.5. FINITIZED CHARACTERS

with \( z = q^{k+1} \), to obtain the identity

\[
\lim_{N \to \infty} T_k^{(N)}(q) = \sum_{j=0}^{\infty} \frac{q^{j(k+1)}}{(q)_j(q)_{j+k}} = \frac{1}{(q)_\infty}.
\] (3.119)

From here, it is straightforward to prove that the finitized characters reconstruct the \( n' = 1 \) Virasoro characters in the \( N \to \infty \) limit

\[
\lim_{N \to \infty} q^{-c/24+\Delta'_{r,s}^{m',m'}} \mathcal{ch}^{m',m'}_{r,s}(N)(q) = \mathcal{ch}^{m,m'}_{r,s}(q).
\] (3.120)

The \( n' = 1 \) Virasoro characters are symmetric under the change \((r, s) \leftrightarrow (m - r, m' - s)\), which means

\[
\mathcal{ch}^{m,m'}_{r,s}(q) = \mathcal{ch}^{m,m'}_{m-r,m'-s}(q)
\] (3.121)

This symmetry also holds for the finitized characters

\[
\mathcal{X}_{a,b,c}^{m,m',}(N)(q) = \mathcal{X}_{m'-a,m'-b,m'-c}^{m,m',}(N)(q)
\] (3.122)

The shaded band structure is symmetric, hence the Kac-indices corresponding to the right hand side of the equation are \((m - r, m' - s)\). To prove eq. (3.122), we use the property

\[
T_k^{(N)}(q) = T_{-k}^{(N)}(q)
\] (3.123)

which follows from direct calculation.

By direct calculation, using eq. (3.123), we prove eq. (3.122)

\[
\mathcal{X}_{m'-a,m'-b,m'-c}^{m,m',}(N) = \sum_{k=-\infty}^{\infty} \left( q^{k(kmm'+m'(m-r)-m(m'-s))} T_k^{(N)}_{km'+\frac{m'-b+m'-s}{2}}(q) - q^{(k+1)(m-r)(m'-s)} T_{-km'-\frac{m'+b+m'-s}{2}}^{(N)}(q) \right)
\] (3.124a)

\[
= \sum_{k=-\infty}^{\infty} \left( q^{k(kmm'+m'r+ms)} T_k^{(N)}_{km'-\frac{b-a}{2}}(q) - q^{(k+1)(m-r)((k+1)m'-s)} T_{-(k+1)m'+\frac{b+a}{2}}^{(N)}(q) \right)
\] (3.124b)

\[
= \sum_{k=-\infty}^{\infty} \left( q^{k(kmm'+m'r-ms)} T_k^{(N)}_{km'+\frac{b-a}{2}}(q) \right)
\] (3.124c)
\[- q^{(km+r)((km'+s)m'-s)} T_{-km'-\frac{km'+s}{2}}^{(N)}(q) = X_{a,b,c}^{m,m',(N)}(q) \]

where in eq. (3.124d), we modified the summation indices, \( k \rightarrow -k \) in the first summand and \( k \rightarrow -(k+1) \) in the second summand. This proves that the finitized characters \( X_{a,b,c}^{m,m',(N)}(q) \) are symmetric under \((r, s) \leftrightarrow (m-r, m'-s)\).

### 3.6 Conclusion and outlook

In this chapter, we considered fused RSOS\((m, m')_{n \times n}\) models. These models are lattice realizations of higher level non-unitary minimal models \( \mathcal{M}(M, M', n') \). We considered fusion level \( n = 2 \) in the regime \( \frac{\pi}{2} < \lambda < \pi \), for crossing parameter \( \lambda = \frac{m'-m}{m} \), and found that in this interval the lattice models are new realizations of \( n' = 1 \) minimal models.

Our calculation is based on Baxter’s corner transfer matrix method, making correspondence between the low temperature one-dimensional sums of the statistical physical model and finitized versions of the conformal characters. We found explicit expressions for the one-dimensional sums, expressed in terms of \( q \)-deformed trinomials. These one-dimensional sums correspond to the finitized characters up to overall factors. We checked our result for numerous models up to system size \( N = 12 \), to support our conjecture. Our result is a conjecture. In order to prove it we need to make use of recursion relations, which require certain other one-dimensional sums that do not correspond to CFT characters. The project has some clear directions to continue: To finalize the \( n = 2 \) case, we have to find the groundstate energies, and some more one-dimensional sums. This has started, and some partial results on the required one-dimensional sums are achieved. Higher fusion levels \( n \) are also in the focus of further research.