Diverse methods for integrable models

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Citation for published version (APA):
Chapter 4

Supersymmetric fermion chain without particle conversation

4.1 Introduction

The study of supersymmetric lattice models dates back to the $\mathcal{N} = 1$ supersymmetry in the tricritical Ising model [52,98,99] and the fully frustrated XY-model [100]. In the introduction of [101], the historical development is further detailed.

The interest in the construction of the model originates from the model of Fendley, Schoutens, and de Boer [102], where the supersymmetric $M_1$ model was introduced. The model is an exactly solvable $\mathcal{N} = 2$ supersymmetric model. There are two supercharges, $Q_1 = Q + Q^\dagger$ and $Q_2 = i(Q - Q^\dagger)$, where $Q$ and its Hermitian conjugate $Q^\dagger$ are nilpotent: $Q^2 = (Q^\dagger)^2 = 0$. The Hamiltonian is defined as $H = \{Q, Q^\dagger\}$. For generic $\mathcal{N}$, there are $\mathcal{N}$ fermionic algebra generators with the properties $Q_k^2 = Q_n^2$, and every anticommutator $\{Q_k, Q_n\}$ is a central element of the algebra.

The consequences of the supersymmetry include the absence of negative energy states, that all positive energy states come in doublets, with the same eigenvalue, but differing in the number of fermions by one. The zero energy states (the lowest possible) might be degenerate, but form singlets with respect to the supersymmetry. This choice for the supercharge generator $Q$ in the $M_1$ model
results in a repulsive hardcore potential, i.e. fermions have to be apart at least two sites, with an empty site in between. Later, this model has been generalized to $M_k$ models, where $k$ long strings of fermions are allowed. We modify the original $M_1$ model in a different way: we change $Q$ and $Q^\dagger$ such that we restore fermion–hole symmetry. This simply means that we add extra terms to $Q^\dagger$ (and $Q$) which create (annihilate) solitary holes. This modifies the nature of the model to a large extent, e.g. fermion number is not conserved anymore. Instead, we identify domain walls with Majorana-like properties, as conserved degrees of freedom. The spectrum also changes drastically. The high degeneracy of the ground state is a common feature of supersymmetric lattice models, but here this high degeneracy is no longer limited to the ground state. While in the original model, all energy levels are two-fold degenerate, here all energy levels show extensive degeneracy: the degeneracy is a power of two, with an exponent growing linearly in the system size. Most surprisingly, despite the change in $Q$, the model preserves both supersymmetry and integrability.

The original motivation for this work comes from understanding the high degeneracy of the model. During our investigation, we identified one symmetry in terms of zero energy Cooper pair like excitations –similar to excitations in [108]– which explains the degeneracy of the groundstate. We further show that the energy gap of the system scales as $\sim 1/L^2$ with the system size $L$, which is usually associated to classical diffusive models.

This chapter is organized as follows: In section 4.2 we introduce the model. In section 4.3 we present the solution by means of the nested coordinate Bethe ansatz. First we expose the Bethe equations (section 4.3.1), and after that we derive them. Our derivation is pedagogical, we derive gradually more and more complicated cases. In section 4.4 we list the known symmetries of the model, including the most prominent one, coined as zero mode Cooper pairs (section 4.4.9), which explains the extensive degeneracy of the model. The Cooper pair description relies on the Bethe ansatz solution of the model. In section 4.5 we investigate further properties of the model and the Bethe ansatz solution. We examine the question of completeness in certain cases in sections 4.5.1 and 4.5.2. We give the exact groundstate and first excited state for $L = 4n$ system sizes in section 4.5.3. We finish with a conclusion in section 4.6. In appendix C.1 we carefully address the question of the extensiveness of the groundstate for $L = 4n$ and the highest energy level for $L = 4n + 2$ systems.
4.2 Definition of the model

Define the usual fermionic creation and annihilation operators satisfying the usual anticommutation relations

\[ \{ c_i^+, c_j \} = \delta_{ij}, \quad \{ c_i^+, c_j^+ \} = \{ c_i, c_j \} = 0. \]  

These operators create and annihilate spinless fermions on a one-dimensional chain, indexed by the site-index \( i \) and \( j \). We introduce the usual fermion- and hole-number operators

\[ n_i = c_i^+ c_i, \quad p_i = 1 - n_i. \]  

Define another number operator, \( N_i \) which counts the fermions from site 1 to \( i \), and \( N_F \), as the total fermion number operator

\[ N_i = \sum_{j=1}^{i} n_j, \quad N_F := N_L. \]  

Here and in the following \( L \) denotes the length of the system. These operators act on a Fock space built on the vacuum state \( |0\rangle \), spanned by the vectors

\[ |\tau \rangle =: \prod_{i=1}^{L} \left( c_i^+ \right)^{\tau_i} |0\rangle. \]  

The semicolons indicate the normal-ordering of the product according to their position. The vacuum state is defined by the relations

\[ c_i |0\rangle = 0 \quad i = 1, \ldots, L \]  

The \( \tau_i = 0, 1 \) indices represent a hole and fermion at site \( i \), respectively. Hence, we trivially get \( n_i |\tau \rangle = \tau_i |\tau \rangle \). We use the following notation for the vectors of the Hilbert space

\[ |\tau_1 \tau_2 \ldots \tau_L \rangle \equiv: \prod_{i=1}^{L} \left( c_i^+ \right)^{\tau_i} |0\rangle \]  

To define the modified version of the \( M_1 \) model, we introduce the operators \( d_i^\dagger \) and \( e_i \)

\[ d_i^\dagger = p_{i-1} c_i^+ p_{i+1}, \quad e_i = n_{i-1} c_i n_{i+1}. \]  

These operators create solitary fermion and hole at the \( i \)th site, respectively. The operator \( d_i^\dagger \) creates a fermion at site \( i \), given that \( i \pm 1 \) are empty. Similarly, \( e_i \)
creates a hole at site $i$, given that $i \pm 1$ are occupied. Due to the anti-commutation relations and normal ordering, the following relations hold

$$d_i^\dagger |\tau_1 \ldots \tau_{i-2} 000 \tau_{i-2} + \ldots \tau_L\rangle = (-1)^{N_{i-1}} |\tau_1 \ldots \tau_{i-2} 010 \tau_{i+2} \ldots \tau_L\rangle,$$  \hspace{1cm} (4.8)

$$e_i |\tau_1 \ldots \tau_{i-2} 111 \tau_{i+2} \ldots \tau_L\rangle = (-1)^{N_{i-1}} |\tau_1 \ldots \tau_{i-2} 101 \tau_{i+2} \ldots \tau_L\rangle,$$  \hspace{1cm} (4.9)

while acting on other states results in 0.

We define the supercharges and the Hamiltonian as follows

$$Q = \sum_{i=1}^{L} (d_i^\dagger + e_i), \hspace{1cm} Q^2 = 0 \hspace{1cm} (4.10)$$

$$Q^\dagger = \sum_{i=1}^{L} (d_i + e_i^\dagger), \hspace{1cm} (Q^\dagger)^2 = 0 \hspace{1cm} (4.11)$$

$$H = \{Q^\dagger, Q\} \hspace{1cm} (4.12)$$

Here and in the rest of the paper, unless otherwise stated, we assume periodic boundary conditions

$$c_{i+L}^\dagger = c_i^\dagger. \hspace{1cm} (4.13)$$

The nilpotent properties eqs. (4.10) and (4.11) also hold with antiperiodic boundary conditions, defined by

$$c_{i+L}^\dagger = c_i. \hspace{1cm} (4.14)$$

Our definition of $Q$ is the fermion-hole symmetric extension of the supercharge $Q_{FSdB} = \sum_i d_i^\dagger$ of Fendley, Schoutens and de Boer [102]. Unexpectedly, our modification on $Q_{FSdB}$ keeps supersymmetry, which is checked by direct calculation.

To analyze the system defined by eq. (4.12), we expand the Hamiltonian, and group the summands according to their content. $H_I$ contains terms involving $d$-type operators, $H_{II}$ contains $e$-type operators, $H_{III}$ contains mixed terms

$$H = H_I + H_{II} + H_{III}, \hspace{1cm} (4.15a)$$

$$H_I = \sum_i \left( d_i^\dagger d_i + d_i d_i^\dagger + d_i^\dagger d_{i+1} + d_{i+1}^\dagger d_i \right), \hspace{1cm} (4.15b)$$

$$H_{II} = \sum_i \left( e_i e_i^\dagger + e_i^\dagger e_i + e_i e_{i+1}^\dagger + e_{i+1} e_i^\dagger \right), \hspace{1cm} (4.15c)$$
where we assume periodic boundary conditions $c_{i+L} = c_i$. $H_I$ is the Hamiltonian of the $M_1$ model, considered in [102]. $H_{II}$ is the same expression, as $H_I$, after a $d_i \leftrightarrow e_i$ change. As the only difference that the role of fermions and holes are exchanged, physically it is equivalent to $H_I$. $H_{III}$ contains the mixed terms.

Since $d$ and $e$ are not simple fermionic operators, they do not obey the usual anticommutation relations. This means, eq. (4.15a) bilinear Hamiltonian cannot be solved by taking linear combinations of $d$- and $e$-type operators.

$H_I$ and $H_{II}$ contain terms, which are pairs of creation and annihilation operators, hence they preserve the fermion number. $H_{III}$ contains terms, which break the fermion conservation: e.g. $d_{i+1}e_i$ sends the state $|...1110...\rangle$ to $|...1000...\rangle$, annihilating two fermions. Hence, the fermions are not good quantum numbers for the Bethe ansatz. In the next section we introduce other type of excitations which are conserved: domain walls.

### 4.2.1 Domain walls

Introduce the 01-domain wall, as an interface between a string of 0’s followed by a string of 1’s, and similarly, the 10-domain wall, as an interface between a string of 1’s followed by a string of 0’s. We index the domain wall by the index of the site to the left of it. For example, the next state contains 8 domain walls, four-four of both type

$$00|11|0\rangle_1 |10|11|0\rangle_1 |111\rangle_1$$

(4.16)

To describe a state $|\tau_1 \tau_2 \ldots \tau_L\rangle$, we list the positions of the domain walls, and the type of the first domain wall. This is sufficient to make the description bijective. Let us revisit the terms in the Hamiltonian with respect to the domain walls. $H_{III}$ corresponds to the hopping of a domain wall, creating or annihilating two fermions

$$|...1000...\rangle \xrightarrow{e_{i+1}d_i} |...1110...\rangle,$$

(4.17)

$$|...0111...\rangle \xrightarrow{d_{i+1}e_i} |...0001...\rangle,$$

(4.18)

where the minus sign arises because of the fermionic nature of the model. In $H_I$ and $H_{II}$, there are diagonal and nondiagonal terms. The diagonal terms are $d_i^\dagger d_i + d_i d_i^\dagger$ and $e_i e_i^\dagger + e_i^\dagger e_i$. These terms count the number of 010, 000, 101 and
111 strings. The nondiagonal terms describe the hopping of a fermion or a hole

\[
\left(\ldots 00\right|10\ldots) \xrightarrow[d_{i+1}^\dagger d_i^\dagger]{} \left(\ldots 0|00\ldots\right) \quad (4.19)
\]

\[
\left(\ldots 11\right|01\ldots) \xrightarrow[e_i^\dagger e_{i+1}^\dagger]{} \left(\ldots 1|11\ldots\right) \quad (4.20)
\]

The hopping of a fermion or hole is equivalent to the interaction between two domain walls. As we see, all the processes, included in \(H\), preserve the number of domain walls. The coordinate of any domain wall changes by two during these processes, hence on an even sized system \(L = 2n, n \in \mathbb{N}\), with periodic boundary conditions, the number of domain walls at odd and even positions are separately conserved. We call these two types of domain walls simply odd and even domain walls. The total number of domain walls is even, due to periodic boundary conditions. In the following, unless otherwise stated, we consider even sized systems. Hence the Hilbert space is naturally divided into the direct sum of sectors. An \((m, k)\) sector contains \(m\) domain walls, \(m - k\) domain walls with even positions, \(k\) with odd. The Hamiltonian is consequently block-diagonal in these sectors.

### 4.3 Solution by Bethe ansatz

In this section we derive the solution to the supersymmetric chain by means of the nested coordinate Bethe ansatz \([109][110]\). First, we expose the most general Bethe equations, and give some notes on it. Then we derive these equations gradually: We start from the simplest case, i.e. the Bethe solution for the \((2, 0)\) sector, and case by case derive solutions to more complicated cases, ending up at the most general \((m, k)\) sector.

#### 4.3.1 Bethe equations

Consider a generic \((m, k)\) sector for an even \(L\) sized system, with \(m\) domain walls (\(m\) must be even) and \(k\) odd domain walls, with periodic boundary conditions \((c_i = c_{L+i})\). The Bethe ansatz is performed in the domain walls, as particles. There are two kinds of particles, even and odd domain walls. Each domain wall is associated with a Bethe root \(z_j\), where \(\log z_j\) is proportional to the momentum of the particle, and each odd domain wall carries an additional nested Bethe root,
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$u_l$. The eigenenergy of $H$ for a Bethe solution is given in terms of $z_j$’s

$$\Lambda = L + \sum_{i=1}^{m} (z_i^2 + z_i^{-2} - 2) \quad (4.21)$$

The set of $z_1, z_2, \ldots, z_m$ and $u_1, u_2, \ldots u_k$ satisfies the equations

$$z_j^L = \pm i^{-L/2} \prod_{l=1}^{k} \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2}, \quad j = 1, \ldots, m \quad (4.22)$$

$$1 = \prod_{j=1}^{m} \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2}, \quad l = 1, \ldots, k \quad (4.23)$$

where the choice of $\pm$ is the same for all $j$ in the first equation.

Certain solutions of the Bethe equations belong to the same Bethe vector: If $z_1, \ldots, z_m; u_1, \ldots, u_k$ and $z'_1, \ldots, z'_m; u'_1, \ldots, u'_k$ are two solutions, they correspond to the same Bethe vector, if there is a set of signs $\delta_i = \pm$, and permutations $\pi \in S_m, \sigma \in S_k$, such that $z_j = \delta_j z'_{\pi(j)}, \quad j = 1, \ldots, m$ and $u_l = u'_{\sigma(l)}$. In other words, the solutions are invariant under permutations of the Bethe roots, and the sign of $z$’s. In order to have nonzero Bethe vector, all the $u$’s have to be different, and all the $z$’s differ up to a sign, $u_r \neq u_s$ and $z_i \neq \pm z_j$.

In the absence of odd domain walls, the second set of equations is missing, and the first set simplifies to the free-fermionic equations

$$z_j^L = \pm i^{-L/2}, \quad j = 1, \ldots, m \quad (4.24)$$

This is physically clear, if we take into account that only with even domain walls, the domain walls satisfy simple hardcore repulsion. The solution to the free-fermion case is

$$z_j = i^{-1/2} e^{2i\pi I_j / L}, \quad j = 1, \ldots, m \quad (4.25)$$

where $I_j$’s are (half-)integers.

Having the odd domain walls at the nested level is an arbitrary choice: we can chose the even domain walls to be nested. Hence, the sectors $(m, k)$ and $(m, m - k)$ display the same physics after an odd–even change. This is not apparent in the Bethe equations, however, this will be clear during the discussion of the symmetries of the model.

As a preparation for the derivation, introduce the domain wall notation for the $|\tau_1 \tau_2 \ldots \tau_L \rangle$ basis vectors of the Fock space. Let $1 \leq x_1 < x_2 < \ldots < x_m \leq L$
denote the position of the \( m \) domain walls, of which \( p_1, \ldots, p_k \) are the indices of odd domain walls
\[
|x_1, x_2, \ldots, x_m; p_1, \ldots, p_k\rangle_\epsilon
\] (4.26)
To fully specify the vector, the type of the first domain wall is encoded in \( \epsilon = 0, 1 \), 0 denoting 01-type domain wall, and 1 denoting 10-type domain wall. The information contained in \( p_1, \ldots, p_k \) is redundant, but it is useful during the derivation of the Bethe equations. The following equation is an example for this notation
\[
|000110010011\rangle = |3, 5, 7, 8, 10, 12; 1, 2, 3\rangle_0 .
\] (4.27)
To conveniently describe how the Hamiltonian acts on a state, introduce two operators, \( S^\pm_i \) which by definition shifts the \( i \)th domain wall
\[
S^\pm_i |x_1, \ldots, x_i, \ldots, x_m; p_1, \ldots, p_k\rangle = |x_1, \ldots, x_i \pm 1, \ldots, x_m; p_1, \ldots, p_k\rangle .
\] (4.28)
The \( S^\pm_i \) operators change the \( p \) content of the vector, which we did not denote here.
To diagonalize the Hamiltonian in the \((m, k)\) sector, introduce eigenvectors of \( H \) expanded in the domain wall notation. Define \(|\Psi(m, k)\rangle\) as an eigenvector in the sector \((m, k)\)
\[
H |\Psi(m, k)\rangle = \Lambda |\Psi(m, k)\rangle,
\] (4.29)
where we give the most general form to the vector by
\[
|\Psi(m, k)\rangle = \sum_{\{x_i\}} \sum_{\{p_j\}} \sum_{\epsilon=0,1} \psi_\epsilon(x_1, \ldots, x_m; p_1, \ldots, p_k) |x_1, \ldots, x_m; p_1, \ldots, p_k\rangle_\epsilon .
\] (4.30)
Here we sum up for all possible domain wall configurations with \( m \) domain walls and \( k \) odd domain walls. The Bethe ansatz takes place in the wavefunction amplitudes \( \psi_\epsilon(x_1, \ldots, x_m; p_1, \ldots, p_k) \). We start with the simplest case, the two even, no odd domain wall case.

### 4.3.2 Two domain walls, no odd domain wall \((m, k) = (2, 0)\)
We start by examining, what is the effect of \( H \) on a state with two even domain walls. If the two domain walls are far apart (\( |x_1 - x_2| > 2 \)), both domain walls can move in both direction. If they are close, they cannot move towards each other
\[
H |x_1, x_2\rangle_\epsilon = \left( L - 4 + \sum_{i=1,2} (-1)^{i+\epsilon} (S^+_{i+2} + S^-_{i-2}) \right) |x_1, x_2\rangle_\epsilon,
\] (4.31)
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\[ H |x, x + 2\rangle_\epsilon = (L - 4 + (-1)^\epsilon (-S_1^{-2} + S_2^{+2}) |x, x + 2\rangle_\epsilon. \] (4.32)

The diagonal terms \((L - 4) |x_1, x_2\rangle_\epsilon\) and \( |x, x + 2\rangle_\epsilon\) come from the diagonal terms in the Hamiltonian. The only terms without contribution are the 001, 011, 110 and 100 strings around the domain walls. Taking the scalar product with an eigenstate \(\langle \Psi(2, 0) |\), it follows from the first equation

\[ \Lambda \psi_\epsilon(x_1, x_2) = (L - 4) \psi_\epsilon(x_1, x_2) + (-1)^\epsilon (-\psi_\epsilon(x_1 + 2, x_2) - \psi_\epsilon(x_1 - 2, x_2) + \psi_\epsilon(x_1, x_2 + 2) + \psi_\epsilon(x_1, x_2 - 2)) \] (4.33)

From the second equation, a similar equation follows

\[ \Lambda \psi_\epsilon(x, x + 2) = (L - 4) \psi_\epsilon(x, x + 2) + (-1)^\epsilon \left( \psi_\epsilon(x - 2, x + 2) + \psi_\epsilon(x, x + 4) \right) \] (4.34)

Analytically continue eq. (4.33) to \(x_2 = x_1 + 2\) and by subtracting eq. (4.34), we get

\[ -\psi_\epsilon(x - 2, x - 2) + \psi_\epsilon(x, x) = 0 \] (4.35)

We use the following ansatz to satisfy eqs. (4.33) and (4.35)

\[ \psi_\epsilon(x_1, x_2) = c_\epsilon \left( A_{12}^{12} (1^{1-\epsilon} z_1^{x_1} (i^\epsilon z_2)^{x_2}) + A_{21}^{21} (1^{1-\epsilon} z_2^{x_2} (i^\epsilon z_1)^{x_1}) \right) \] (4.36)

The \(z_j\) variables are the Bethe roots, complex parameters to be determined by further conditions. By substituting the ansatz, we find that the eigenvalue \(\Lambda\) and the amplitudes \(A\) satisfy the conditions

\[ \Lambda = L + \sum_{i=1}^{2} (z_i^2 + z_i^{-2} - 2), \quad A_{12}^{12} + A_{21}^{21} = 0 \] (4.37)

The periodic boundary condition on a chain of even length \(L\) imposes the following further condition

\[ \psi_\epsilon(x, L + 2) = \psi_{1-\epsilon}(2, x) \] (4.38)

Substituting the ansatz in this equation results in

\[ c_0 A_{12}^{12} z_2^L = c_1 A_{12}^{21} \quad c_1 A_{12}^{12} (iz_2)^L = c_0 A_{21}^{21} \] (4.39)

Hence we find that \((c_0/c_1)^2 = i^L\) and

\[ z_1^{2L} = z_2^{2L} = i^{-L}. \] (4.40)
Taking the square root of the equation, we find two different sets of solutions

\[
\frac{c_0}{c_1} = i^{L/2} \quad \frac{c_0}{c_1} = -i^{L/2} \quad (4.41a)
\]

\[
z_1^L = -\frac{c_1}{c_0} = -i^{-L/2} \quad z_1^L = -\frac{c_1}{c_0} = i^{-L/2} \quad (4.41b)
\]

\[
z_2^L = -\frac{c_1}{c_0} = -i^{-L/2} \quad z_2^L = -\frac{c_1}{c_0} = i^{-L/2} \quad (4.41c)
\]

For certain solutions, the wavefunction amplitude is identically zero, which is a manifestation of the Pauli principle: \(z_1 \neq \pm z_2\). Two solutions, \((z_1, z_2)\) and \((z_1', z_2')\) are independent and the corresponding Bethe vectors are orthogonal, if taking the squares of the Bethe roots are not equal up to interchange. If \(z\) is a solution, \(-z\) is also a solution, but does not give new Bethe vector. Hence it is enough to deal with half of the solutions of eq. (4.41). This gives \(2\left(\frac{L}{2}\right)\) different solutions (where the prefactor 2 is coming from the two different set of solutions). The dimension of the \((2, 0)\) sector is \(2\left(\frac{L}{2}\right)\), consequently the Bethe ansatz gives the full solution in the free-fermionic \((2, 0)\) sector.

### 4.3.3 Two domain walls, one odd domain wall \((m, k) = (2, 1)\)

Probe the effect of \(H\) on a state with two domain walls, the first being an odd domain wall. Similarly as in the previous case, we distinguish two cases. The walls are either far apart \(|x_1 - x_2| > 1\), and they diffuse without interaction, or next to each other and they interact

\[
H |x_1, x_2; 1\rangle_\epsilon = \left( L - 4 + \sum_{i=1,2} (-1)^{i+\epsilon} \left( S_i^{+2} + S_i^{-2} \right) \right) |x_1, x_2; 1\rangle_\epsilon, \quad (4.42a)
\]

\[
H |x, x+1; 1\rangle_\epsilon = \left( L - 2 + (-1)^\epsilon (-S_1^{-2} + S_2^{+2}) \right) |x, x+1; 1\rangle_\epsilon \quad + (-1)^\epsilon \left( |x-1, x; 2\rangle_\epsilon + |x+1, x+2; 2\rangle_\epsilon \right) \quad (4.42b)
\]

In the second case, there is no hardcore exclusion as for two even domain walls close to each other, but the walls can pass each other. Similar equations hold for the case when the second domain wall is the odd one.

From eq. (4.42a), the same way, as previously, by taking the scalar product with an eigenstate \(\langle \Psi(2, 1) |\) we derive the following equation

\[
\Lambda \psi_\epsilon(x_1, x_2; 1) = (L - 4) \psi_\epsilon(x_1, x_2; 1) + (-1)^\epsilon \left( - \psi_\epsilon(x_1 + 2, x_2; 1) - \psi_\epsilon(x_1 - 2, x_2; 1) \right. \]

\[
\left. + \psi_\epsilon(x_1, x_2 + 2; 1) + \psi_\epsilon(x_1, x_2 - 2; 1) \right). \quad (4.43)
\]
For eq. (4.42b), when the domain walls are distance one apart, we derive the following equation

\[ \Lambda \psi_\epsilon(x, x+1; 1) = (L-2)\psi_\epsilon(x, x+1; 1) + (-1)^\epsilon \left( -\psi_\epsilon(x-2, x+1; 1) + \psi_\epsilon(x-1, x; 2) + \psi_\epsilon(x+1, x+2; 2) \right). \]  (4.44)

Analytically continue eq. (4.43) to \( x_2 = x_1 + 1 \), and by subtracting eq. (4.44), it follows

\[ 2\psi_\epsilon(x, x+1; 1) + (-1)^\epsilon \left( \psi_\epsilon(x+2, x+1; 1) - \psi_\epsilon(x, x-1; 1) \right) + \psi_\epsilon(x-1, x; 2) + \psi_\epsilon(x+1, x+2; 2) = 0. \]  (4.45)

The previous derivation is for the case, when out of the two domain walls, the first one is at the odd position. The same equation is derived for the case, when the odd domain wall is the second one

\[ 2\psi_\epsilon(x, x+1; 2) + (-1)^\epsilon \left( \psi_\epsilon(x+2, x+1; 2) - \psi_\epsilon(x, x-1; 2) \right) + \psi_\epsilon(x-1, x; 2) + \psi_\epsilon(x+1, x+2; 2) = 0. \]  (4.46)

To solve these equations (eqs. (4.45) and (4.46)), we use a nested wavefunction ansatz. With a view to later generalization, we make the following ansatz

\[ \psi_\epsilon(x_1, x_2; p) = \sum_{\pi \in S_2} B_{\pi_1, \pi_2}^\epsilon(p) (1^\epsilon z_{\pi_1})^{x_1} (i^\epsilon z_{\pi_2})^{x_2}, \]  (4.47)

where \( B_{\pi_1, \pi_2}^\epsilon(p) \) is the nested amplitude in the wavefunction

\[ B_{\pi_1, \pi_2}^\epsilon(p) = c_\epsilon (-1)^{(p+\epsilon-1)/2} A_{\pi_1, \pi_2}^\epsilon(u, z_{\pi_1}) \prod_{j=1}^{p-1} f(u, z_{\pi_j}). \]  (4.48)

Here the extra phase factors have the purpose of canceling the \((-1)\) factors of the Hamiltonian.

We assume the same form of the eigenenergy and the wavefunction amplitudes as previously

\[ \Lambda = L + \sum_{i=1}^{2} (z_i^2 + z_i^{-2} - 2), \quad A_{12}^\epsilon + A_{21}^\epsilon = 0. \]  (4.49)

Substituting these assumptions leads to the following equations for the nested ansatz functions \( f \) and \( g \) of eqs. (4.45) and (4.46)

\[ \sum_{\pi \in S_2} A_{\pi_1, \pi_2}^\epsilon (z_{\pi_2} g(u, z_{\pi_1}) (2 - z_{\pi_1}^2 - z_{\pi_2}^{-2}) - iz_{\pi_1} f(u, z_{\pi_1}) g(u, z_{\pi_2}) (z_{\pi_1}^{-2} - z_{\pi_2}^2)) = 0, \]  (4.50)
\[ \sum_{\pi \in S_2} A_{\pi \pi_1 \pi_2} (z_{\pi_2} f(u, z_{\pi_1}) g(u, z_{\pi_2}) (2 - z_{\pi_1}^2 - z_{\pi_2}^{-2}) - iz_{\pi_1} g(u, z_{\pi_1}) (z_{\pi_1}^{-2} - z_{\pi_2}^2)) = 0. \] (4.51)

These equations are solved by the following expressions, which is checked by direct calculation
\[ f(u, z) = \frac{u - (z - 1/z)^2}{u + (z - 1/z)^2}, \] (4.52)
\[ g(u, z) = \frac{z - 1/z}{u + (z - 1/z)^2}. \] (4.53)

Here \( u \) is a additional complex parameter, the nested Bethe root, associated only with odd domain walls.

Imposing periodic boundary conditions leads to further conditions, where again we have to carefully take minus signs into account
\[ \psi_\epsilon(x, L + 2; 1) = \psi_{1-\epsilon}(2, x; 2), \quad \psi_\epsilon(x, L + 1; 2) = (-1)^{N_F-1} \psi_{1-\epsilon}(1, x; 1). \] (4.54)

As for this case, we have odd number of fermions, this leads to
\[ B_{\epsilon}^{\pi_1 \pi_2}(1)(i^\epsilon z_{\pi_2})^L = B_{1-\epsilon}^{\pi_2 \pi_1}(2), \quad B_{\epsilon}^{\pi_1 \pi_2}(2)(i^\epsilon z_{\pi_2})^L = B_{1-\epsilon}^{\pi_2 \pi_1}(1). \] (4.55)

Substituting this into eq. (4.48) we arrive to the following equation
\[ c_0/c_1 = \pm i^{L/2+1}. \] (4.56)

The following equations are derived from eq. (4.55), after substituting the ansatz
\[ z_{\pi_2}^L = -\frac{c_1}{c_0} \frac{A_{\pi_2 \pi_1}}{A_{\pi_1 \pi_2}} f(u, z_{\pi_2}) = \frac{c_1}{c_0} \frac{A_{\pi_2 \pi_1}}{A_{\pi_1 \pi_2}} f(u, z_{\pi_1})^{-1} \] (4.57)

Using the previous results, we derive the following Bethe equations
\[ z_1^L = \pm i^{-L/2} \frac{u - (z_1 - 1/z_1)^2}{u + (z_1 - 1/z_1)^2}, \] (4.58)
\[ z_2^L = \pm i^{-L/2} \frac{u - (z_2 - 1/z_2)^2}{u + (z_2 - 1/z_2)^2}, \] (4.59)

with the further nested Bethe equation
\[ 1 = -f(u, z_1) f(u, z_2) = \frac{u - (z_1 - 1/z_1)^2}{u + (z_1 - 1/z_1)^2} \frac{u - (z_2 - 1/z_2)^2}{u + (z_2 - 1/z_2)^2}. \] (4.60)
4.3. SOLUTION BY BETHE ANSATZ

The simultaneous solution of eqs. (4.58) to (4.60) gives the Bethe vectors for the $(2, 1)$ sector. The same exclusion rules apply for these solutions, as for the $(2, 0)$ sector: $z_1 \neq \pm z_2$. For the $(2, 0)$ sector, the Bethe equations are free-fermionic. In the $(2, 1)$ sector, the free-fermionic spectrum is produced by the $u = 0$ or $u = \infty$ solutions.

4.3.4 Arbitrary number of domain walls, no odd domain wall $(m, k = 0)$

Taking a solution with an arbitrary number of domain walls, without odd domain wall is the straightforward generalization of the $(2, 0)$ case. If all the walls are far apart ($|x_i - x_j| > 2$, $\forall i, j \in 1, \ldots, m$), the following equation holds for the wavefunction amplitudes

$$
\Lambda \psi_\epsilon(x_1, \ldots, x_m) = (L - 2m)\psi_\epsilon(x_1, \ldots, x_m) + 
+ (-1)^\epsilon \sum_{j=1}^{m} (-1)^j \psi_\epsilon(\ldots, x_j - 2, \ldots) + (-1)^j \psi_\epsilon(\ldots, x_j + 2, \ldots) 
$$

(4.61)

If the distance between two walls is 2, $x_{i+1} = x_i + 2$, then $\psi_\epsilon(\ldots, x_i, x_{i+1} - 2)$ and $\psi_\epsilon(x_i + 2, x_{i+1}, \ldots)$ are not present in the sum. This, and taking the $x_{i+2} = x_i + 2$ limit in eq. (4.61) leads to

$$
0 = (-1)^\epsilon(-1)^i \psi_\epsilon(\ldots, x_i + 2, x_i + 2, \ldots) + (-1)^\epsilon(-1)^{i+1} \psi_\epsilon(\ldots, x_i, x_i, \ldots) 
$$

(4.62)

In the absence of odd domain walls, we use the generalization of the simple version of the ansatz

$$
\psi_\epsilon(x_1, \ldots, x_m) = c_\epsilon \sum_{\pi \in S_m} A^\pi \prod_{j=1}^{m/2} (1^\epsilon z_{2j-1})^{x_{2j-1}} (i^\epsilon z_{2j})^{x_{2j}}. 
$$

(4.63)

This generalized ansatz solves the conditions with the generalizations on the eigenvalue and amplitudes

$$
\Lambda = L + \sum_{j=1}^{m} (z_j^2 + z_j^{-2} - 2), \quad A^\pi = \text{sign}(\pi). 
$$

(4.64)

Periodic boundary conditions fully specify the solution. Prescribing PBC leads to

$$
\psi_\epsilon(x_2, \ldots, x_m, x_1 + L) = \psi_{1-\epsilon}(x_1, \ldots, x_m), 
$$

(4.65)
which results in one of the following equations

\[ z_j^L = -i^{-L/2}, \quad \forall j = 1, \ldots, m, \quad (4.66) \]

\[ z_j^L = i^{-L/2}, \quad \forall j = 1, \ldots, m. \quad (4.67) \]

These are the Bethe equations for \( m \) free-fermionic particle, where all the Bethe roots have to satisfy the equations either with + or − sign. As for the \((2, 0)\) sector, certain solutions for the Bethe equations lead to identical or zero Bethe vectors. If two roots are equal up to a sign, the corresponding vector is identically zero, which is the same manifestation of the Pauli principle as before. If \( z_1, \ldots, z_m \) and \( z'_1, \ldots, z'_m \) are two solutions of eq. (4.66) they give the same vector, if for a set of signs \( \delta_i = \pm 1 \) and permutation \( \pi \in S_m \), \{\( z_1, \ldots, z_m \)\} = \{\( \delta_1 z'_{\pi(1)}, \ldots, \delta_m z'_{\pi(m)} \)\}. In other words, if the two sets only differ by signs up to permutation, the two Bethe vectors coincide. There are \( 2^{(L/2)} \) states in the \((m, 0)\) sector. Since \( z \) and \( -z \) give the same solution, out of the \( L \) roots of the Bethe equation, only \( L/2 \) should be taken into account. This leads to \( 2^{(L/2)} \) different solutions, fully determining the spectrum of these sectors.

### 4.3.5 Arbitrary number of domain walls, one odd domain wall \((m, k = 1)\)

Consider the case where one odd wall is present at position \( x_p \), the \( p \)th wall is odd. Analogously, as before, we can derive an equation for the case, when all the walls are well separated \( (x_{i+1} - x_i > 2, \) and particularly \( |x_p - x_{p\pm1}| > 3) \). This equation is essentially the same as eq. (4.61), with the only difference that the \( p \)th wall is odd. Taking the \( x_{p+1} = x_p + 1 \) case, when otherwise everything else is well separated, and subtracting the two equations leads to

\[
2\psi_{\epsilon}(\ldots, x_p, x_p + 1, \ldots; p) + \underbrace{(-1)^{\epsilon+p-1}[\psi_{\epsilon}(\ldots, x_p + 2, x_p + 1, \ldots; p) - \psi_{\epsilon}(\ldots, x_p, x_p - 1, \ldots; p)]}_{(4.68)} + \psi_{\epsilon}(\ldots, x_{p-1}, x_p, \ldots; p + 1) + \psi_{\epsilon}(\ldots, x_{p+1}, x_p + 2, \ldots; p + 1)\] = 0.

In this case, the odd wall is the start of the 1-long string of fermion or hole. The other possibility that the odd wall is the end of the 1-long string, which leads to (denoting the original position of the odd wall by \( x_{p+1} \))

\[
2\psi_{\epsilon}(\ldots, x_p, x_p + 1, \ldots; p + 1) + (-1)^{\epsilon+p-1}[\psi_{\epsilon}(\ldots, x_p + 2, x_p + 1, \ldots; p + 1) - \psi_{\epsilon}(\ldots, x_p, x_p - 1, \ldots; p + 1)] + \psi_{\epsilon}(\ldots, x_{p+1}, x_p, \ldots; p + 1) + \psi_{\epsilon}(\ldots, x_{p+2}, x_p + 2, \ldots; p + 1)\] = 0.
where the one odd domain wall wave function is given by
\[ \psi^\epsilon(\ldots, x_p + 1, x_p + 2, \ldots; p) = 0. \]

There are three additional cases to consider, i.e. three wall the closest possible to each other, the odd one being in the middle, being the last one, or being the first one.

The first case is, when \( x_{p+2} = x_{p+1} + 1 = x_p + 2 \), the domain wall at position \( x_{p+1} \) is the odd wall. This leads to the following equation
\[
4\psi(\ldots, x_p, x_p + 1, x_p + 2, \ldots; p + 1) + \\
+ (1)^{\epsilon+p-1}[\psi(\ldots, x_p, x_p + 2, x_p + 1, x_p + 2, \ldots; p + 1) + \\
- \psi(\ldots, x_p, x_p + 3, x_p + 2; p + 1) + \psi(\ldots, x_p - 1, x_p, x_p + 2; p) - \\
- \psi(\ldots, x_p, x_p + 2, x_p + 3, \ldots; p + 2)] = 0.
\]

In the case when \( x_{p+2} = x_{p+1} + 1 = x_p + 3 \), with \( x_{p+2} = x_p + 3 \) being the odd domain wall, we derive the following equation
\[
2\psi(\ldots, x_p, x_p + 2, x_p + 3, \ldots; p + 2) + \\
+ (1)^{\epsilon+p-1}[\psi(\ldots, x_p, x_p + 2, x_p + 1, \ldots; p + 2) - \\
- \psi(\ldots, x_p, x_p + 4, x_p + 3; p + 2) + \psi(\ldots, x_p + 2, x_p + 2, x_p + 3; p + 2) - \\
- \psi(\ldots, x_p, x_p, x_p + 3, \ldots; p + 2) - \psi(\ldots, x_p, x_p + 1, x_p + 2; p + 1) + \\
+ \psi(\ldots, x_p, x_p + 3, x_p + 4; p + 1)] = 0.
\]

A similar equation holds for the \( x_{p+2} = x_{p+1} + 2 = x_p + 3 \) case, with \( x_p \) being the odd wall. These equations are satisfied by a similar ansatz as in the (2, 1) case: The \( f \) and \( g \) functions take the same form, and the full wavefunction ansatz is in the following form
\[
\psi^\epsilon(x_1, \ldots, x_{2n}; p) = c_\epsilon \sum_{\pi \in S_n} A^{\pi_1 \ldots \pi_{2n}} \phi^\epsilon_p(u; \pi) \prod_{j=1}^{n} [(1 - z_{\pi_{2j-1}}) x_{2j-1} (i z_{\pi_{2j}}) x_{2j}],
\]

where the one odd domain wall wave function is given by
\[
\phi^\epsilon_p(u; \pi) = g(z_{\pi_p})(-1)^{(p+\epsilon-1)/2} \prod_{j=1}^{p-1} f(u, z_{\pi_j}).
\]

The solution for the eigenvalue and amplitudes are
\[
\Lambda = L + \sum_{i=1}^{2n} (z_i^2 + z_i^{-2} - 2), \quad A^{\pi_1 \ldots \pi_{2n}} = \text{sign}(\pi_1 \ldots \pi_{2n}).
\]
Periodic boundary conditions lead to the conditions
\[ \psi_{\epsilon}(x_1, \ldots, x_{2n-1}, L + 2; p) = \psi_{1-\epsilon}(2, x_1, \ldots, x_{2n-1}; p + 1), \]  
(4.75)
\[ \psi_{\epsilon}(x_1, \ldots, x_{2n-1}, L + 1; 2n) = (-1)^{N_F-1} \psi_{1-\epsilon}(1, x_1, \ldots, x_{2n-1}; 1). \]  
(4.76)

The parity of the number of fermions \( N_F \) is equal to the number of odd domain walls, hence we conclude the following conditions
\[ c_\epsilon A^{\pi_1 \ldots \pi_{2n}} (i^\epsilon z_{\pi_{2n}}) L (-1)^{(p+\epsilon-1)/2} = c_{1-\epsilon} A^{\pi_{2n} \pi_1 \ldots \pi_{2n-1}} (-1)^{(p+1-\epsilon)/2} f(u, z_{\pi_{2n}}), \]  
(4.77)
\[ c_{1-\epsilon} (-1)^{(1-\epsilon)/2} A^{\pi_{2n+1} \ldots \pi_{2n-1}} = c_\epsilon A^{\pi_1 \ldots \pi_{2n}} (i^\epsilon z_{\pi_{2n}}) L (-1)^{(2n+\epsilon-1)/2} \prod_{j=1}^{2n-1} f(u, z_j). \]  
(4.78)

Using the \((-1)^{(p+\epsilon-1)/2} = (-1)\epsilon^1(-1)^{(p+1-\epsilon)/2}\) identity, we obtain the same condition as before
\[ c_0/c_1 = \pm i^{L/2+1}, \]  
(4.79)
and similar consistency conditions, which are the Bethe equations for this case
\[ z_j^L = \pm i^{-L/2-1} f(u, z_j) = \pm i^{-L/2} \frac{u - (z_j - 1/z_j)^2}{u + (z_j - 1/z_j)^2} \quad (j = 1, \ldots, 2n) \]  
(4.80)
\[ \prod_{j=1}^{2n} f(u, z_j) = (-1)^n \Leftrightarrow \prod_{j=1}^{2n} \frac{u - (z_j - 1/z_j)^2}{u + (z_j - 1/z_j)^2} = 1, \]  
(4.81)

Here again, we have free-fermionic solutions, corresponding to the \( u = 0 \) and \( u = \infty \) solution.

### 4.3.6 Arbitrary number of domain walls, two odd domain walls \( (m, k = 2) \)

With similar technique, the equivalent equation of eq. (4.70) describes three wall being together, the two outermost being odd. The indices of the odd walls are \( p_1 \equiv p \) and \( p_2 \), hence \( p_2 = p_1 + 2 \equiv p + 2 \). There are two extra terms due to two more prohibited processes (the two odd walls cannot hop towards each other due to exclusion)
\[ 4\psi_{\epsilon}(..., x_p, x_p + 1, x_p + 2, \ldots; p, p + 2) + \]  
(4.82)
+ (-1)^{\epsilon + p - 1} \left[ \psi_\epsilon(\ldots, x_p + 2, x_p + 1, x_p + 2, \ldots; p, p + 2) + \\
+ \psi_\epsilon(\ldots, x_p, x_p + 1, x_p, \ldots; p, p + 2) - \psi_\epsilon(\ldots, x_p, x_p - 1, x_p + 2, \ldots; p, p + 2) + \\
- \psi_\epsilon(\ldots, x_p, x_p + 3, x_p + 2, \ldots; p, p + 2) + \\
+ \psi_\epsilon(\ldots, x_p - 1, x_p, x_p + 2, \ldots; p + 1, p + 2) - \\
- \psi_\epsilon(\ldots, x_p + 2, x_p + 3, \ldots; p, p + 1) \right] = 0.

The equivalent equation of eq. (4.71) is similar. These equations are satisfied by the two particle generalization of the nested ansatz

$$\psi_\epsilon(x_1, \ldots, x_{2n}; p_1, p_2) = c_\epsilon \sum_{\pi \in S_{2n}} A^{\pi_1 \ldots \pi_{2n}} \sum_{\sigma \in S_2} B^{\pi_1 \sigma_2} \phi^{(\epsilon)}_{p_1}(u_{\sigma_1}; \pi) \phi^{(\epsilon)}_{p_2}(u_{\sigma_2}; \pi) \prod_{j=1}^n \left[ (i^{-\epsilon} z_{\pi_{2j-1}})^{x_{2j-1}} (i^{\epsilon} z_{\pi_{2j}})^{x_{2j}} \right],$$

(4.83)

Here the eigenvalue and the wavefunction amplitudes are

$$\Lambda = L + \sum_{i=1}^m (z_i^2 + z_i^{-2} - 2), \quad A^\pi = \text{sign}(\pi), \quad B^\sigma = \text{sign}(\sigma).$$

(4.84)

Periodic boundary conditions lead to

$$\psi_\epsilon(x_1, \ldots, x_{2n-1}, L + 2; p_1, p_2) = \psi_{1-\epsilon}(2, x_1, \ldots, x_{2n-1}; p_1 + 1, p_2 + 1),$$

(4.85)

$$\psi_\epsilon(x_1, \ldots, x_{2n-1}, L + 1; p_1, 2n) = (-1)^{\epsilon - 1} \psi_{1-\epsilon}(1, x_1, \ldots, x_{2n-1}; 1, p_1 + 1).$$

(4.86)

These result in $c_0/c_1 = \pm i^{L/2+2}$, and the Bethe equations for two odd particles follow

$$z_j^L = \pm i^{-L/2} \prod_{k=1,2} \frac{u_k - (z_j - 1/z_j)^2}{u_k + (z_j - 1/z_j)^2}, \quad j = 1, \ldots, 2n$$

(4.87)

$$1 = \prod_{j=1}^{2n} \frac{u_k - (z_j - 1/z_j)^2}{u_k + (z_j - 1/z_j)^2}, \quad k = 1, 2.$$  

(4.88)

### 4.3.7 Arbitrary number of domain walls, arbitrary number of odd domain walls ($m = 2n, k$)

This is a straightforward generalization of the previous cases. The similar, but more general equations are solved by the general form of the ansatz
\[ \psi_\epsilon(x_1, \ldots, x_{2n}; p_1, \ldots, p_m) = c_\epsilon \sum_{\pi \in S_{2n}} A^{\pi_1 \ldots \pi_{2n}} \sum_{\sigma \in S_m} B^\sigma \prod_{j=1}^m \phi_\epsilon^{(c)}(u_{\sigma_j}; \pi) \prod_{j=1}^n \left[ (i^{1-\epsilon} z_{\pi_{2j-1}})^{x_{2j-1}} - (i^\epsilon z_{\pi_{2j}})^{x_{2j}} \right], \] (4.89)

where the nested wavefunction of the odd domain walls is

\[ \phi_\epsilon^{(c)}(u; \pi) = g(z_{\pi_p})(-1)^{(p+\epsilon-1)/2} \prod_{j=1}^{p-1} f(u, z_{\pi_j}). \] (4.90)

The amplitudes and the eigenvalue take the usual form

\[ \Lambda = L + \sum_{i=1}^{2n} (z_i^2 + z_i^{-2} - 2), \quad A^\pi = \text{sign}(\pi), \quad B^\sigma = \text{sign}(\sigma). \] (4.91)

and the Bethe equations take their most general form

\[
z_j^L = \pm i^{-L/2} \prod_{k=1}^m \frac{u_k - (z_j - 1/z_j)^2}{u_k + (z_j - 1/z_j)^2}, \quad j = 1, \ldots, 2n \] (4.92)

\[ 1 = \prod_{j=1}^{2n} \frac{u_k - (z_j - 1/z_j)^2}{u_k + (z_j - 1/z_j)^2}, \quad k = 1, \ldots, m. \] (4.93)

By this, we derived the Bethe equations for the most general case.

### 4.4 Symmetries

The original motivation of this work is to understand the high degeneracy at all energy levels. Table 4.1 shows the degeneracy of the groundstate, the least degenerate energy level and the number of energy levels both for periodic and antiperiodic boundary conditions. The degeneracies seem to be extensive: the degeneracy of all energy levels scales exponentially with the system size.

For a better understanding we also consider the model with antiperiodic boundary conditions (APBC), defined by \( c_{i+L} = c_i^1 \). Table 4.2 shows the groundstate energies for both boundary conditions. As table 4.2 suggests, with periodic boundary conditions (PBC), for \( L = 4n \), the groundstate energies are 0, while for \( L = 4n - 2 \), they are positive. However, the antiperiodic groundstate energies are 0 for \( L = 4n - 2 \). The groundstate degeneracy \( G \) shows the following extensiveness with PBC and APBC for \( L = 4n \) and \( L = 4n - 2 \), respectively

\[ \# \text{ of GS: } 2^{L/2 + 1}. \] (4.94)
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Table 4.1: The degeneracy $G$ of the groundstate, the degeneracy $\ell$ of the least degenerate energy level and the number of energy levels $e$ both for periodic and antiperiodic boundary conditions.

Later we give a natural explanation why we consider periodic and antiperiodic BC for $L = 4n$ and $L = 4n - 2$, respectively. The abundance of degeneracies suggests a high number of symmetries. There are many symmetries in the model, most obviously the supersymmetry and the fermion–hole symmetry. The extensive degeneracy is explained by a process similar to Cooper pairs in the BCS theory of superconductivity [108,111]. From an existing solution a new one is created by turning two even domain walls into odd ones, with $u$ and $-u$ as nested Bethe roots. Such pairs of odd domain walls are much like Cooper pairs.

4.4.1 Supersymmetry

By construction, the model possesses supersymmetry, the Hamiltonian exchanges with the supersymmetry generators

$$[H, Q] = 0, \quad [H, Q^\dagger] = 0.$$  \hfill (4.95)

Examine the effect of these generators on a sector $(m, k)$. As they are sums of $d_i^\dagger + e_i$ and $d_i + e_i^\dagger$ respectively, they add or remove a fermion or hole, introducing or removing two domain walls, one even and one odd

$$Q : (m, k) \mapsto (m + 2, k + 1), \quad Q^\dagger : (m, k) \mapsto (m - 2, k - 1).$$  \hfill (4.96)


\[ \begin{array}{c|cc|cc}
L & \text{periodic} & \Lambda_{GS} & \text{antiperiodic} & \Lambda_{GS} \\
\hline
4 & 8 & 0.000 & 4 & 0.000 \\
5 & 8 & 0.000 & 8 & 0.000 \\
6 & 16 & 0.268 & 16 & 0.000 \\
7 & 16 & 0.000 & 16 & 0.000 \\
8 & 32 & 0.000 & 32 & 0.000 \\
9 & 32 & 0.000 & 64 & 0.000 \\
10 & 32 & 0.087 & 64 & 0.000 \\
11 & 64 & 0.000 & 64 & 0.000 \\
12 & 128 & 0.000 & \\
\end{array} \]

Table 4.2: The degeneracy \( G \) of the groundstate, and the \( \Lambda_{GS} \) groundstate energy both for periodic and antiperiodic boundary conditions.

### 4.4.2 Domain wall number conversation and translation symmetry

The Bethe ansatz solution based on the property of the model that for even system sizes \( L \), the number of domain walls (\( D \)), and separately the number of even and odd domain walls (\( D_{e} \) and \( D_{o} \)) are conserved

\[
[H, D] = 0, \quad [H, D_{e}] = 0, \quad [H, D_{o}] = 0. \tag{4.97}
\]

By the periodic boundary conditions, the model is also invariant under the translation operator \( T \). By applying \( T \), the role of even and odd domain walls is exchanged, hence

\[
T: (m, k) \mapsto (m, m - k). \tag{4.98}
\]

### 4.4.3 Fermion parity symmetry

Despite the number of fermions is not conserved, the number of fermions always changes by two, hence the parity of the fermions is conserved. Define the fermion parity operator

\[
P |\tau_{1}, \ldots, \tau_{L}\rangle = (-1)^{N_{L}} |\tau_{1}, \ldots, \tau_{L}\rangle. \tag{4.99}
\]

The supersymmetry generators change the parity, hence they anti-commute

\[
\{P, Q\} = 0, \quad \{P, Q^{\dagger}\} = 0. \tag{4.100}
\]
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From this, the symmetry is a trivial consequence

\[ [P, H] = 0. \tag{4.101} \]

4.4.4 Fermion–hole symmetry

The model is introduced in a fermion–hole symmetric way. Define the Majorana operators

\[ \Gamma =: \prod_{i=1}^{L} \gamma_i :, \quad \gamma_i = c_i + c_i^\dagger. \tag{4.102} \]

The operator \( \Gamma \) acts on a state \( |\tau_1 \tau_2 \ldots\rangle \) as exchanging fermions and holes. It is easy to prove that this is symmetry of the model,

\[ [H, \Gamma] = 0. \tag{4.103} \]

Depending on system size, \( \Gamma \) either commutes or anticommutes with the generators

\[ Q \Gamma + (-1)^L \Gamma Q = 0, \quad Q^\dagger \Gamma + (-1)^L \Gamma Q^\dagger = 0. \tag{4.104} \]

4.4.5 Domain wall–non-domain wall symmetry

Define non-domain walls, as particles on the dual lattice of domain walls, occupying sites without domain walls. Non-domain walls have similar properties to domain walls: Their number and parity are conserved.

It is physically motivated that a domain wall–non-domain wall symmetry is present in the model: The displacement of domain walls (eqs. (4.17) and (4.18)), e.g. \( |\ldots 1000\ldots\rangle \leftrightarrow |\ldots 1110\ldots\rangle \) and the interaction of even and odd domain walls (eqs. (4.19) and (4.20), e.g. \( |\ldots 0100\ldots\rangle \leftrightarrow |\ldots 0010\ldots\rangle \) are identical of interaction of even and odd non-domain walls, and displacement of non-domain walls, respectively. Define the domain wall–non-domain wall symmetry operator

\[ E =: \prod_{i=1}^{L/2} (c_{2i} - c_{2i}^\dagger) :, \tag{4.105} \]

which satisfies the following commutation relations

\[ EQ = Q^\dagger E, \quad EQ^\dagger = QE, \quad [E, H] = 0. \tag{4.106} \]

The domain wall–non-domain wall symmetry maps the sectors \( (m, k) \) and \( (L - m, L/2 - m + k) \) into each other. This solves the so called "over the equator" problem \cite{112,114}. Finding Bethe solutions over half-filling of the pseudo-vacuum raises difficulties, but utilizing the domain wall–non-domain wall symmetry, it is enough to probe the states up to half-filling.
4.4.6 Shift symmetry

A further symmetry is found, by examination of small systems. Define the shift operator

\[ S = \sum_{i=1}^{L} n_{i-1} \gamma_i p_{i+1} + p_{i-1} \gamma_i n_{i+1}, \quad \gamma_i = c_i + c_i^\dagger. \] (4.107)

This operator shifts a domain wall either to the left or to the right by one. \( S \) is self-adjoint by construction, and easy to see that it is a symmetry

\[ \{Q, S\} = 0, \quad \{Q^\dagger, S\} = 0, \quad [H, S] = 0. \] (4.108)

The shift operator relates the sector \((m, k)\) with the sectors \((m, k \pm 1)\).

4.4.7 Reflection symmetry of the spectrum for \( L = 4n \)

For \( L = 4n, \quad n \in \mathbb{N} \) the groundstate energy is exactly zero: \( \Lambda_0 = 0 \), which we compute in section 4.5.3. Similarly, the largest energy level is given by \( \Lambda_{\text{max}} = L \). We observed that the spectrum has a reflection symmetry, i.e. for every energy level \( \Lambda = L - \Delta \) there is a corresponding reflected one \( \tilde{\Lambda} = \Delta \) with the same degeneracy. The states on these two energy levels are related by a bijective operator, defined in the following way. Define an other type of Majorana operator

\[ \delta_j = i(c_j - c_j^\dagger), \quad \delta_j^\dagger = \delta_j. \] (4.109)

The reflection operator then defined as

\[ M := \prod_{i=0}^{n-1} \delta_{4i+1} \delta_{4i+2} := (-1)^n \prod_{i=0}^{n-1} (c_{4i+1} - c_{4i+1}^\dagger)(c_{4i+2} - c_{4i+2}^\dagger). \] (4.110)

The operator \( M \) is (anti)hermitian depending on the parity of \( n \), and squares to a multiple of the identity,

\[ M^\dagger = (-1)^n M, \quad M^2 = (-1)^n \mathbb{I}. \] (4.111)

Obviously, this operator is not expected to commute with \( H \), rather the reflecting property is encoded in the following equation

\[ M(L \mathbb{I} - H) = HM, \] (4.112)

which means exactly that for every eigenstate, there is a corresponding reflected one

\[ H |\Psi\rangle = \Lambda |\Psi\rangle \Leftrightarrow HM |\Psi\rangle = (L - \Lambda)M |\Psi\rangle. \] (4.113)
A good example of the pairing is the pseudo-vacuum state $|00\ldots0\rangle$, which maps into the half-filled true groundstate, $M|00\ldots0\rangle = \pm |110011001\ldots100\rangle$, where the sign depends on the system size. It is easy to see that the pseudo-vacuum has energy $\Lambda = L$, while the other state is indeed a groundstate, $\Lambda = 0$.

4.4.8 Antiperiodic boundary conditions and reflection symmetry of the spectrum for $L = 4n - 2$

The reflection symmetry can be extended to $L = 4n - 2$ system sizes by taking into account the model with antiperiodic boundary conditions. The intuition is the following: Compare the same mapping for $L = 4n$ and $L = 4n - 2$

$$M|00\ldots0\rangle_{L=4n} = \pm |110011001\ldots100\rangle_{L=4n} \quad (4.114)$$
$$M|00\ldots0\rangle_{L=4n-2} = \pm |110011001\ldots011\rangle_{L=4n-2} \quad (4.115)$$

For $L = 4n$ systems, the 1100 pattern is repeated, while for the $L = 4n - 2$ systems, there is a “defect”, an “accumulated” 1111 or 0000 string. This intuitively shows that this state is not a $\Lambda = 0$ state (as it is indeed not). Considering the model with antiperiodic boundary conditions removes this “defect”, as under antiperiodic boundary conditions, the role of fermions and holes exchanges under a full turn.

Introduce antiperiodic boundary conditions by

$$c^\dagger_{i+L} = c_i. \quad (4.116)$$

This modifies the Hamiltonian which we denote by $H_{AP}$. The spectrum of the antiperiodic Hamiltonian has the same reflection symmetry as the periodic for $L = 4n$. $M$ is boundary condition independent. The following equation describes the antiperiodic case for $L = 4n$ system size

$$M(L\mathbb{1} - H_{AP}^{(L=4n)}) = H_{AP}^{(L=4n)} M, \quad (4.117)$$

where for clarity we emphasized the system size $L = 4n$.

More interestingly, for $L = 4n - 2$, the period and antiperiodic cases are related: For an eigenstate of $H_{AP}$ with energy $\Lambda_{AP}$, there is a corresponding state of $H$ with energy $L - \Lambda_{AP}$. The largest energy for $H$ is $\Lambda_{P,max} = L$, which corresponds to the antiperiodic groundstate with zero energy, $\Lambda_{AP,GS} = 0$. This reflection is realized by the following operator equation

$$M(L\mathbb{1} - H_{AP}^{(L=4n-2)}) = H_{P}^{(L=4n-2)} M, \quad (4.118)$$
4.4.9 Zero mode Cooper pairs

Consider a free-fermionic solution of the Bethe equations, satisfying the equation

\[ z_j^L = \pm i^{-L/2}, \quad j = 1, \ldots, m. \]  

(4.119)

The same equation is satisfied, if we turn two even domain walls into odd ones (and keep the total number of domain walls the same) with nested rapidities.

where we stressed the system size \( L = 4n - 2 \) and the periodic Hamiltonian by \( H_P \).

A good example again if we take e.g. the state \( |000\ldots00⟩ \) with \( L = 4n - 2 \), which is an eigenvector of \( H_P \) with the largest eigenvalue \( \Lambda_P = L \). This state is mapped to \( |1100110\ldots0011⟩ \), where the first two and last two sites are occupied. This is a groundstate of the antiperiodic Hamiltonian, as the role of fermions and holes exchanges during a full round.
which are each other’s negative, \( u_2 = -u_1 \). The new Bethe equations are

\[
\begin{align*}
    z_j^L &= \pm i^{-L/2} \frac{u_1 - (z_j - 1/z_j)^2}{u_1 + (z_j - 1/z_j)^2} u_2 - (z_j - 1/z_j)^2, \quad j = 1, \ldots, m \\
    1 &= \prod_{j=1}^m \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2}, \quad l = 1, 2.
\end{align*}
\]  

(4.120)  

(4.121)

The first equations are automatically satisfied, if \( u_2 = -u_1 \), since the product of the two fractions is one

\[
\frac{u_1-(z_j-1/z_j)^2}{u_1+(z_j-1/z_j)^2} \times \frac{u_2-(z_j-1/z_j)^2}{u_2+(z_j-1/z_j)^2} = \frac{u_1-(z_j-1/z_j)^2}{u_1+(z_j-1/z_j)^2} \times \frac{u_1+(z_j-1/z_j)^2}{u_1-(z_j-1/z_j)^2} = 1. \tag{4.122}
\]

Since the set of \( z \)'s is already given, we can regard the second equations as rational equations for the unknown \( u \)'s, and hence determine the possible \( u_1, u_2 \) pairs. Starting from a free-fermionic \( z \) solution, the solutions always contain such pairs which are each other’s negative.

We can continue like this, introducing new pairs of \( u \)'s, as long as \( m \) is large enough to generate new solutions from the \( 1 = \prod_{j=1}^m \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2} \) equation. We address this question in more details in appendix C.1. Similarly, we can start from a non free-fermionic sector, with \( k \neq 0 \). Some typical solutions to the Bethe equations are shown in figs. 4.1 to 4.3.

We have not found an explicit operator that creates the Cooper pairs, acting on appropriate states. If such an operator can be constructed, it must either select one of the solution pairs \((u, -u)\) of eq. (4.121), or more likely create a linear combination of all such solution pairs. Since the pairs do not affect the energy, such linear combination is an eigenstate of the Hamiltonian, but not a pure Bethe state.

### 4.5 Completeness of the Bethe solution and exact solutions

One of the main question regarding the Bethe ansatz solution is its completeness: Whether all the eigenvectors are Bethe vectors with associated Bethe roots. We have addressed this question in certain finite size cases. We found that the Bethe solutions only form a subset of all the eigenvectors, and utilizing symmetry operators, we recover the full Hilbert space. Our examination is based on comparing the direct diagonalization results with numerical solutions to the Bethe equations.
There are two symmetries which are systematically used: the translation symmetry $T$ and the domain wall–non-domain wall symmetry $E$. $T$ maps the $(m, k)$ sector to $(m, m - k)$, while $E$ maps it to $(L - m, L/2 - k)$. Applying both, the image of $(m k)$ is $(L - m, L/2 - m + k)$

\[
T : \quad (m, k) \rightarrow (m, m - k) \quad \text{(4.123a)}
\]

\[
E : \quad (m, k) \rightarrow (L - m, L/2 - k) \quad \text{(4.123b)}
\]

\[
ET = TE \quad (m, k) \rightarrow (L - m, L/2 - m + k) \quad \text{(4.123c)}
\]

These mappings allow us to reduce our examination to sectors with $m \leq L/2$ domain walls, and $k \leq m/2$ odd domain walls.

We recovered all the states in the probed cases, but other than $T$ and $E$, we
4.5. COMPLETENESS OF BETHE EQUATIONS, EXACT SOLUTIONS

<table>
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</tr>
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</tr>
<tr>
<td>2.000</td>
<td>16</td>
</tr>
<tr>
<td>3.732</td>
<td>16</td>
</tr>
<tr>
<td>6.000</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
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<th>$k$</th>
<th>dim.</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0, 2</td>
<td>6, 6</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>1, 3</td>
<td>6, 6</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 4.3: $L = 6$ sector energy levels and degeneracies.

Table 4.4: $L = 6$ sectors. Certain sectors have the same dimensions and are listed in the same line.

did not find a general scheme to do it: the application of the various symmetry operators was otherwise ad hoc without any apparent pattern. To address this question more carefully, much more extensive studies of small systems is needed, which might be a subject of further research.

The studied examples were the full spectrum of the $L = 6$ system and the most degenerate eigenvalue of $L = 10$, the $\Lambda = 6$ eigenvalue.

### 4.5.1 $L = 6$, full spectrum

The $L = 6$ Hamiltonian is easily solved by direct diagonalization. There are four energy levels, all of them are 16-folded degenerate (table 4.3).

There are eight sectors in the Hilbert space, and three of them should be considered: $(0, 0)$, $(2, 0)$, and $(2, 1)$, the rest follows via $T$ and $E$. The list of the sectors is in table 4.4.

#### $\Lambda = 6$ eigenvalue

The $(0, 0)$ sector contains two trivial Bethe vectors: $|b_1\rangle = |000000\rangle$ and $|b_2\rangle = |111111\rangle$, both are eigenvectors with $\Lambda = 6$. By applying $Q$, two more states are created in $(2, 1)$, $Q|b_1\rangle$ and $Q|b_2\rangle$. By applying $E$, we create four more states, two in $(4, 2)$ and two in $(6, 3)$: $EQ|b_1\rangle = Q^i E|b_1\rangle$, $EQ|b_2\rangle = Q^i E|b_2\rangle$, $E|b_1\rangle$, $E|b_2\rangle$.

In the $(2, 1)$ sector, the Bethe equations are

$$z_j^6 = \pm i^{-3} \frac{u - (z_j - 1/z_j)^2}{u + (z_j - 1/z_j)^2}, \quad j = 1, 2,$$

(4.124)
\[ 1 = \prod_{j=1}^{2} \frac{u - (z_j - 1/z_j)^2}{u + (z_j - 1/z_j)^2}. \] \hspace{1cm} (4.125)

This has two independent non-free-fermionic solutions, with \( \Lambda = 6 \) eigenvalues
\[ (z_1^2, z_2^2, u_{\pm}) = \left( \frac{\sqrt{3}}{2} \left( \sqrt{3} + i \right), \frac{1}{2\sqrt{3}} \left( \sqrt{3} + i \right), \frac{1}{39} \left( -9 \pm 14\sqrt{3} \right) \right). \] \hspace{1cm} (4.126)

Denote the corresponding two Bethe vectors by \( |b_3\rangle \) and \( |b_4\rangle \). Starting from these two, we generate all the remaining eight vectors. There are four vectors in \((2, 1)\), namely: \( |b_3\rangle, |b_4\rangle, Q|b_3\rangle = EQ|b_3\rangle, Q|b_4\rangle = EQ|b_4\rangle \), and these vectors mapped to \((4, 2)\) by \( E: E|b_3\rangle, E|b_4\rangle, Q|b_3\rangle, Q|b_4\rangle \), respectively.

With using all these symmetries, we recovered all the sixteen eigenvectors corresponding to \( \Lambda = 6 \), for \( L = 6 \).

### Other eigenvalues

The \((2, 0)\) sector is completely determined by the free-fermionic solution. The Bethe equations here are \( z_j^0 = \pm i^{-3}, j = 1, 2 \). The (+) equation has 3 distinct solution, 1-1-1 for the three lowermost eigenvalues \((\Lambda_1 = 0.268, \Lambda_2 = 2, \Lambda_3 = 3.732)\). The same hold for the (−) solution. These solutions are mapped to \((2, 2), (4, 1)\) and \((4, 3)\) by \( T, E \) and \( TE \) respectively. So far, we have found an 8-fold degeneracy for the lower three eigenvalues, we need to find a further 8-fold degeneracy. There are indeed four-four states in sectors \((2, 1)\) and \((4, 2)\) belonging to each of the lower three eigenvalues. These four-four solutions are achieved as follows: \( u = 0 \) and \( u = \infty \) leads to one-one free-fermionic solution for each eigenvalues, call them \( |f_1^{(i)}\rangle \) and \( |f_2^{(i)}\rangle \), where \( i = 1, 2, 3 \) denotes the corresponding eigenvalue. In this case, \( \{ |f_1^{(i)}\rangle, |f_2^{(i)}\rangle, EQ|f_1^{(i)}\rangle, EQ|f_2^{(i)}\rangle \} \) and \( \{ E|f_1^{(i)}\rangle, E|f_2^{(i)}\rangle, Q|f_1^{(i)}\rangle, Q|f_2^{(i)}\rangle \}, i = 1, 2, 3 \) are the sets of remaining states in \((2, 1)\) and \((4, 2)\) respectively.

These concludes that we indeed recovered the full Hilbert space from the Bethe solutions via symmetries.

#### 4.5.2 \( L = 10, \Lambda = 6 \) eigenvalue

The other example, we probed, is the \( \Lambda = 6 \) eigenvalue of \( L = 10 \), which is the most, 64-fold degenerate eigenvalue of the system. The domain wall–non-domain wall symmetry \( E \) allows us, to consider only the sectors \((m, k)\) with \( m < L/2 = 4 \). The sector by sector degeneracies are listed in table 4.5. The four states in \((2, 0)\) are Bethe states, this is always the case for \((m, 0)\) sectors. Denote the subspace
4.5. **Completeness of Bethe Equations, Exact Solutions**

<table>
<thead>
<tr>
<th>$m$</th>
<th>$k$</th>
<th>deg. of $\Lambda = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0,2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1,3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 4.5: $L = 10, \Lambda = 6$ degeneracies sector by sector. The unlisted sectors follow by DW–nonDW symmetry.

of these four solutions by $\mathcal{B}^{(2,0)}$. By translation, we create the corresponding subspace in $(2,2)$, $\mathcal{B}^{(2,2)} = TB^{(2,0)}$. There are further four free-fermionic solutions in $(2, 1)$, spanning $\mathcal{B}^{(2,1)}$. Acting with $Q$ on these three subspaces, we generate 4-4-4 further solutions: Action of $Q$ is $(m, k) \mapsto (m + 2, k + 1)$, hence $QB^{(2,0)}$, $QB^{(2,1)}$, $QB^{(2,2)}$ are subspaces in $(4, 1)$, $(4, 2)$, $(4, 3)$, respectively.

$S$ is a symmetry operator which shifts the walls by one position, hence it maps $(m, k) \mapsto (m, k + 1) \oplus (m, k - 1)$. Acting with $S$ on $QB^{(2,0)}$ creates two linearly independent vector in $(4, 2)$ (and two linearly dependent one). Similarly, $SQB^{(2,2)}$ creates further two linearly independent vector in $(4, 2)$ (and again, two linearly dependent one). This gives the expected 8-fold degeneracy of the sector $(4, 2)$. From these four vectors, $Q$ creates four further independent vectors in $(2, 1)$. $QSQB^{(2,0)}$ and $QSQ \dagger B^{(2,2)}$ contain 2-2 linearly independent vectors in $(2, 1)$. This gives the expected degeneracy in the sector $(2, 1)$, and the remaining degeneracy is constructed by the domain wall–non-domain wall symmetry, $E$.

The scheme of the full process is in fig. 4.4.

It would be interesting to find the full underlying symmetry algebra, i.e. the general algorithm to create all the eigenvectors. This is however seems fairly nontrivial as there is no obvious similarity in the discussed examples.

### 4.5.3 The groundstate and first excited state for $L = 4n$

In the half-filled sector $(2n, 0)$ with $L = 4n$, the Bethe equations are

$$z_j^{4n} = \pm i^{-4n/2} = \pm (-1)^n = \pm 1, \quad j = 1, \ldots, 2n,$$

(4.127)

which are satisfied by the free-fermionic solutions:

$$z_j^{(+)} = e^{\frac{imj}{2n}} \quad (j = 1, \ldots, 2n), \quad z_j^{(-)} = e^{\frac{im(2j+1)}{4n}} \quad (j = 1, \ldots, 2n).$$

(4.128)
Both solutions produce a groundstate, as it is easy to check

$$\Lambda^{(\pm)} = 4n + \sum_{j=1}^{2n} (z_j^2 + z_j^{-2} - 2) = 0.$$  \hspace{1cm} (4.129)

These two solutions span the subspace $(2n, 0)$, which is 2 dimensional, where an other basis is formed by $|0011\ldots0011\rangle$ and $|1100\ldots1100\rangle$.

There are further groundstates in the $(2n, k)$ sectors which are generated by the described Cooper-pair process. In the presence of $k$ odd domain walls, the $u$'s satisfy the constrain

$$1 = \prod_{j=1}^{2n} \frac{u - (z_j - 1/z_j)^2}{u + (z_j - 1/z_j)^2} \quad (l = 1, \ldots, k). \hspace{1cm} (4.130)$$

Setting $z_j = z_j^{(+)}$ or $z_j = z_j^{(-)}$ in eq. (4.130) gives the rational function for $u$'s to satisfy. This has purely imaginary roots in complex conjugate pairs. Any combination of these pairs forms a new, valid solution, due to zero mode Cooper pairs.

The Cooper pairs give an overall degeneracy to the groundstate which grows exponentially with the system size $L$. This is intuitively clear, since $L = 4n$, and in a sector $(2n, k)$, (with even $k$) there are roughly $\binom{n}{k/2}$ states, generated by zero modes. Summing over $k$ gives the full degeneracy which is consequently exponential. A detailed examination is in appendix C.1.
4.6 CONCLUSION AND OUTLOOK

Based on the explicit solutions of the $L = 4, 8, 12$ systems, we observed that the first excited states occur in the sectors $(2n \pm 2, k)$, with arbitrary $k$, and $(2n, k)$ with $k \neq 0, 2n$. As for the $(2n - 2, 0)$ sector, the Bethe equations are free-fermionic, we can determine the first excited state energy.

The Bethe equations for $L = 4n$, in the $(2n - 2, 0)$ sector are

$$z_j^L = \pm i^{-L/2} = \pm (-1)^n = \pm 1, \quad j = 1, \ldots, 2n - 2$$  \hspace{1cm} \text{(4.131)}

which have the solutions as described in eq. \(4.128\). The only difference is that out of the $2n$ solutions, we have to choose $2n - 2$ which minimizes the energy $\Lambda = 4 + \sum_{i=1}^{2n-2} z_i^2 + z_i^{-2}$. To do so, we have to leave out the two Bethe roots contributing the most. The two largest contribution is coming from the Bethe roots $z_{2n}^{(+)} = 1$, $z_1^{(+)} = e^{i\pi/(2n)}$ for the $(+)$ case and $z_1^{(-)} = e^{i\pi/(4n)}$, $z_{2n-1}^{(-)} = e^{-i\pi/(4n)}$ for the $(-)$ case. The two associated energy levels are

$$\Lambda^{(+)}(L = 4n) = 4 - 2 - 2 \cos(\pi/n) = 2(1 - \cos(\pi/n))$$  \hspace{1cm} \text{(4.132)}

$$\Lambda^{(-)}(L = 4n) = 4 - 4 \cos(\pi/2n) = 4(1 - \cos(\pi/2n))$$  \hspace{1cm} \text{(4.133)}

As $\Lambda^{(-)} < \Lambda^{(+)}$ gives the lower energy, $\Lambda^{(-)}$ is the energy of the first excited state.

This result correctly reproduces the $L = 4, 8, 12$ first excited state energies. Note that this investigation is based on the assumption that the $(2n - 2, 0)$ sector contains first excited states.

The ground state energy is $\Lambda_0 = 0$, hence the energy gap of the system is

$$\Delta \Lambda_n = \Lambda_n^{(-)} - \Lambda_0 = 4(1 - \cos(\pi/2n)) \approx \frac{\pi^2}{2n^2}.$$  \hspace{1cm} \text{(4.134)}

The gap vanishes as $\sim 1/L^2$, which is a sign of classical diffusive mode. As this is strictly speaking a conjecture, this result is strictly speaking an upper bound on the energy gap.

4.6 Conclusion and outlook

We have introduced a new lattice supersymmetric chain in which fermion number conservation is violated. The model turns out to be integrable and we give a detailed derivation of the equations governing the spectrum using nested coordinate Bethe ansatz. The energy spectrum is highly degenerate, all states with a finite density have an extensive degeneracy. This degeneracy is explained by the identification of several symmetry operators, but most significantly by the possibility at each level to create modes that do not cost any energy. These modes
are analogous to Cooper pairs in BCS theory, and our model contains a direct realisation of these which can be explicitly identified in the Bethe equations. The class of finite solutions to the Bethe ansatz does not provide all eigenvectors. We give circumstantial evidence that all eigenvectors are obtained by the application of the symmetry operators on Bethe vectors. We furthermore find that the energy gap to the first excited state scales as $\sim 1/L^2$ where $L$ is the system size which is a signature of classical diffusion.