Chapter 5

Conclusion and outlook

During my doctorate, my research focused on three main topics. The common theme in all three projects is integrability. Being involved in diverse researches during the four years of my doctorate allowed me to learn many techniques to probe integrable systems.

I studied integrable quantum field theories during my MSc, and my interest turned towards integrable lattice models and integrable quantum chains after starting my doctorate. Lattice models and quantum chains are closely related. This is most apparent in the transfer matrix formalism, as typically the logarithmic derivative of the transfer matrix of the lattice model corresponds to the Hamiltonian of the related quantum chain.

In the first and second main chapters I studied two different lattice models in statistical physics. The first model, I studied, is the dilute $O(n = 1)$ loop model, a model which is based on the dilute version of the Temperley-Lieb algebra, and related to the Izergin-Korepin 19-vertex model. I considered the model on a strip with finite width and infinite height, with open boundary conditions. The model describes the statistical ensemble of closed loops, possibly connected to the boundaries.

We considered the boundary to boundary spin-1 current. On the lattice, the current between two arbitrary points is expressed in terms of two "elementary" currents, $X$ and $Y$ which describe the current through a horizontal or a vertical edge, respectively. We found explicit expressions for these two types of currents on the inhomogeneous lattice. The $Y$ current turned out to be translation invariant. We expressed these currents in terms of the partition sum of the model, and symmetric polynomials.

Our solution is based on the quantum Knizhnik-Zamolodchikov equations. The
$qKZ$ equations give relations between the eigenvector of the transfer matrix with different permutations of the inhomogeneities. Based on the $qKZ$ equations and some plausible minimal assumption on certain elements of the groundstate vector, we recovered the full groundstate vector of transfer matrix – a program similar to form factor bootstrap in integrable field theories. This method was found to be powerful to compute exact results in a number of cases, including our result for the spin-1 current. There is no Bethe ansatz approach directly for loop models, for the Temperley-Lieb algebra, Bethe ansatz was considered in the vertex (or spin) basis. Lacking of Bethe ansatz makes $qKZ$ equations particularly important for loop models. However, this technique has some limitations: The loop weight is limited to $n = 0$ or $1$, otherwise the eigenvector does not constitute polynomial entries, making the computation practically impossible. Achieving the analytic result also includes considerable amount of guesswork.

There are some further possibilities to consider: We can investigate further to compute different quantities with the same technique. We made some sizable efforts to compute different one point functions and correlators, however, these attempts did not lead to success. As the current was computed on the lattice, the continuum limit of the current is naturally interesting.

In the second chapter, we considered a whole family of lattice models, the $n \times n$ fused RSOS models, and studied the $n = 2$ case in more detail. These fused RSOS models are characterized by the $\lambda$ crossing parameter, and the fusion level $n$. The $n$th level fused RSOS model is constructed based on the $n = 1$ unfused RSOS model, by considering the $n \times n$ configurations of the unfused $R$-matrix. We investigated the continuum limit of fused models via the Corner Transfer Matrix method, and finitized characters. The continuum limit is a higher level minimal model, where the exact correspondence depends on $n$ and $\lambda$. The identification is based on the comparison of one-dimensional sums computed by Corner Transfer Matrix method and finitized characters of the conformal field theory. According to the Correspondence Principle, the low-temperature one-dimensional sums approximate the characters of the field theory. We found that the $n = 2$ fused RSOS models for the interval $\frac{\pi}{2} < \lambda < \pi$ correspond to minimal models $\mathcal{M}(m, m')$, where $m$ and $m'$ are coprime integers, and $\lambda = \frac{(m' - m)\pi}{m}$.

By the same method, Tartaglia and Pearce conjectured that for $n \times n$ RSOS models, for $0 < \lambda < \pi/n$, the corresponding models are $\mathcal{M}(M, M', n)$ higher level minimal models, where $\lambda = \frac{(m' - m)\pi}{m}$ and $(M, M') = (nm - (n-1)m', m')$. The conjecture is based on calculations performed on the $n = 2, 3$ cases.

Considering the project on the fused RSOS models, a clear further direction is the full exploration of the tower of models: for $n \geq 3$, only partial results are known even as a conjecture. As the $3 \times 3$ fused $R$-matrix is known for all values
of $\lambda$, there is no obvious obstacle to extend our investigation there. However, taking the low temperature limit does not lead to the expected form for the $R$-matrix. This problem has to be resolved in order to make further steps in finding the corresponding conformal field theory.

The third main chapter is focuses on the exact solution of a spinless supersymmetric fermionic chain with periodic boundary conditions. The model is a supersymmetric extension of the $M_1$ model of Fendley, Schoutens and de Boer. One striking feature of the model that fermion number is not conserved. However domain walls – separating particles from empty sites– are conserved degrees of freedom. The parity of domain walls (regarding that they occupy an even or odd site on the dual lattice) is also conserved which is a result of the parity conservation of the number of fermions. We solved the model by nested coordinate Bethe ansatz, and found the Bethe equations for the model. We checked back our result by comparison to direct diagonalization results. We checked the completeness of the Bethe ansatz, for small systems. We found that with supplementary use of the symmetries of the model, the whole spectrum is recovered. The model exhibit large number of symmetry, which we studied in details. The energy levels are extensively degenerate, i.e. the degeneracy of the levels exponentially increases with the system size. We explained the extensive degeneracy in terms of Bethe ansatz, in zero energy Cooper pair like excitations. The model is solved in terms of Bethe roots, however, there are many open questions: Mainly, the algebraic structure and the $R$-matrix are still to be discovered. There is a possible mapping of the model to a different chain model, which could help answering these questions.