Diverse methods for integrable models
Fehér, G.

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Appendix A

Currents in the dilute $O(n = 1)$ loop model

A.1 Derivation of the $n = 1$ $K$-matrix from the generic one

In this section, we derive the $n = 1$ left $K$-matrix weights from the generic weights. The generic $K$-matrix weights are the following

\[ k_1^l(u, \zeta) = \zeta \sin(2\lambda) \sin(2u) \]  
\[ k_2^l(u, \zeta) = 2 \cos \lambda \sin \left( \frac{3\lambda}{2} + u \right) \]  
\[ - n_1 \zeta^2 \sin \left( \frac{1}{2} \lambda + u \right) \sin \left( \frac{1}{2} \lambda - u \right) \sin \left( \frac{3}{2} \lambda - u \right) \]  
\[ k_3^l(u, \zeta) = -\zeta^2 \sin(2\lambda) \sin(3u) \sin \left( \frac{1}{2} \lambda - u \right) \]  
\[ k_4^l(u, \zeta) = \sin \left( \frac{3}{2} \lambda - u \right) \left( 2 \cos \lambda - n_1 \zeta^2 \sin^2 \left( \frac{1}{2} \lambda - u \right) \right) \]

We change from additive to multiplicative notation by introducing new variables (eq. (2.11))

\[ q = e^{i\lambda} \]
\[ z = e^{ix} \]
In these new variables, the $K$-matrix elements are the following

\[
\begin{align*}
k_1^I(z, \zeta) &= \frac{\zeta}{4} \frac{(q^4-1)(z^4-1)}{q^2 z^2} \tag{A.2a} \\
k_2^I(z, \zeta) &= \frac{(q^4+1)(z^4+1)}{2q^3/2 z} - \zeta^2 n_1 \frac{(q^2-1)(q-z^2)(q^2-z^2)}{8q^3/2 z^3} \tag{A.2b} \\
k_3^I(z, \zeta) &= -\zeta^2 \frac{(q^4-1)(z^4-1)(q-z^2)}{8q^3/2 z^3} \tag{A.2c} \\
k_4^I(z, \zeta) &= \frac{q^3 z^2}{2q^3/2 z} \left( \frac{q^2+1}{q} - \zeta^2 n_1 \frac{(q-z^2)^2}{4q z^2} \right) \tag{A.2d}
\end{align*}
\]

For Regime I, we take the (2.14b) solutions $q_2^{(\pm)}$, set the boundary loop weight $n_1 = 1$, and simplify the weights for the following form

\[
\begin{align*}
k_1^I(z, \zeta) &= -\frac{\zeta}{4} \frac{(1+q)(z^2-1)(1+z^2)}{q^2 z^2} \tag{A.3a} \\
k_2^I(z, \zeta) &= -\frac{(q^2+1)(1+z^2)}{2q^3/2 z} + \zeta^2 \frac{(q^2-1)(q-z^2)(1+z^2)}{8q^3/2 z^3} \tag{A.3b} \\
k_3^I(z, \zeta) &= \zeta^2 \frac{(1+q)(z^2-1)(1+z^2)(q-z^2)}{8q^3/2 z^3} \tag{A.3c} \\
k_4^I(z, \zeta) &= -\frac{(1+q^2)(1+z^2)}{2q^3/2 z} + \zeta^2 \frac{(1+z^2)(q-z^2)^2}{8q^3/2 z^3} \tag{A.3d}
\end{align*}
\]

Here, $\zeta$ is a free parameter, which we relate to the function $x(z_B)$ defined in eq. (2.21n) as follows

\[
\zeta(z_B) = -2q^{1/2} x(z_B) \tag{A.4}
\]

With this substitution, the weights take the following form

\[
\begin{align*}
k_1^I(z, \zeta) &= \frac{x(z_B)}{2} \frac{(1+q)(z^2-1)(1+z^2)}{q^3/2 z^2} \tag{A.5a} \\
k_2^I(z, \zeta) &= -\frac{(1+z^2)}{2q^3/2 z} + x^2(z_B) \frac{(q^2-1)(q-z^2)(1+z^2)}{2q^3/2 z^3} \tag{A.5b} \\
k_3^I(z, \zeta) &= x^2(z_B) \frac{(1+q)(z^2-1)(q-z^2)(1+z^2)}{2q^3/2 z^3} \tag{A.5c} \\
k_4^I(z, \zeta) &= -\frac{(1+z^2)}{2q^3/2 z} + x^2(z_B) \frac{(1+z^2)(q-z^2)^2}{2q^3/2 z^3} \tag{A.5d}
\end{align*}
\]

where we also used the identity $1 + q^2 = q$. These are the $n = 1$ weights up to a factor $\frac{1+z^2}{2q^3/2 z}$.

To make this statement explicit, here we list the $n = 1$ weights reported in eqs. (2.21a) to (2.21c). They read as follows after substituting the function $k$:
A.2. DILUTE O(1) LOOP MODEL AS PERCOLATION

(for simplicity, in the same order as in eq. (A.5))

\[ K_{\text{OpCl}}^I(z, z_B) = x(z_B) \frac{(1+q)(z^2-1)}{z} \]  \hspace{1cm} (A.6a)

\[ K_{\text{id}}^I(z, z_B) + K_{m}^I(z, z_B) = -1 + x^2(z_B) \frac{(q-z^2)(qz^2-1)}{z^2} \]  \hspace{1cm} (A.6b)

\[ K_{m}^I(z, z_B) = x^2(z_B) \frac{(1+q)(z^2-1)(q-z^2)}{z^4} \]  \hspace{1cm} (A.6c)

\[ K_{\text{id}}^I(z, z_B) = -1 + x^2(z_B) \frac{(z^2-q)^2}{z^2} \]  \hspace{1cm} (A.6d)

This shows that the used weights are gauge equivalent.

A.2 The dilute O(n=1) model and the site percolation on a triangular lattice

The dilute O(n=1) loop model on a rectangular lattice maps to the critical site percolation on a triangular lattice. The site percolation is defined as such: Consider a triangular lattice, where every site is either red or blue, both with \( p_c = \frac{1}{2} \) probability. This model maps to the dilute loop model in a certain limit. The mapping is depicted in fig. A.1. The loops became the domain walls of the percolation model. Since the domain walls live on the dual lattice of the triangular lattice, they live on a honeycomb lattice. The mapping takes place in a way that out of the nine possible plaquettes of the loop model one has to be zero, and all the others have equal weights. Since a plaquette belongs to four sites of the site percolation, if all the remaining eight plaquette configurations have equal weights, this gives back independent site probabilities. This is required, since the \( 2^4 = 16 \) possible four-site percolation configurations have equal weights, and after a factorization of the two colors, each corresponds to a loop plaquette. Since one of the weights disappears, and the others have equal weight, this corresponds to the loop model after we take the fusion limit, and the \( R \)-matrix elements factorize. The mapping can be realized in two different ways, setting either the weight of \( \) or \( \) to 0. This is realized by setting \( W_2(z, w) = 0 \) or \( W_m(z, w) = 0 \), which means \( w = zq^2 \) or \( w = zq^{-2} \), respectively. In the figure, the later realization is depicted. Either way, the remaining eight configurations have equal weights, hence the independent probabilities of the percolation model are guaranteed.
APPENDIX A. CURRENT IN DILUTE LOOP MODEL

(a) The site percolation on a triangular lattice, with the domain walls

(b) The domain walls form loops on a honeycomb lattice

(c) This maps to the dilute loops in the appropriate limit

(d) The corresponding loop configuration

Figure A.1: The mapping of a site percolation configuration on a triangular lattice to a dilute loop configuration on the square lattice. For convenience, the figures are distorted, in order to have the usual triangular and square lattice on the two sides of the mapping.
A.3 Normalization of the transfer-matrix

In order to prove the (2.40) form of the normalization, first, we map the dilute \( O(1) \) loop model to a site percolation model, and compute the normalization in the percolation model, using the all-1 left eigenvector.

There is a mapping between the dilute \( O(1) \) loop model, and an unusual site percolation model, different from the one presented in appendix A.2. The site percolation model is built up from randomly distributed spins taking the values \( s = \pm 1 \) on the vertices of the tiles. The mapping takes place, as the lines of the loop model are mapped to the domain walls of the site percolation. To implement the \( R \) and \( K \)-matrix weights, we introduce the following plaquette-interactions

- For the \( R \)-matrix: \( R = a + bs_1s_2s_3s_4 \)
- For the \( K \)-matrix: \( K = A + Bs_1s_3 \)

Here \( s_1, s_2, s_3, s_4 \) are the four spins in the corner of the \( R \)-matrix, and \( s_1, s_3 \) are the vertices in the upper and lowermost corner of the \( K \)-matrix. The aforementioned definitions coincide with the loop weights, if

\[
\begin{align*}
a &= \frac{1}{2}(W_e + W_{R_1}) \equiv \frac{1}{2}W_R , \\
b &= \frac{1}{2}(W_e - W_{R_1}) := \frac{1}{2}\tilde{W}_R , \\
A &= \frac{1}{2}(W_{K_e} + W_{K_{OpCl}}) \equiv \frac{1}{2}W_K , \\
B &= \frac{1}{2}(W_{K_e} - W_{K_{OpCl}}) := \frac{1}{2}\tilde{W}_K .
\end{align*}
\]

We define a percolation state, as a sequence of spins along the bottom edge of the \( T \)-matrix. A percolation state is equal to the sum of loop states with the same occupations, regardless the connectivity of occupied sites. There are two choices to map the occupied, and empty sites to the \( \pm 1 \) spins. If we choose to map the occupied sites to \(-1\), and the empty ones to \(+1\), e.g. \( |-1,-1,1,1\rangle_{\text{perco}} = |() \bullet \bullet \rangle + |((\bullet \bullet) + |) \bullet \bullet \rangle + |)(\bullet \bullet) \rangle \). Even the states are not in bijection, the \( T \)-matrix configurations are, consequently the normalization for both \( T \)-matrices are the same.

By definition, the \( T \)-matrix is a left stochastic matrix, so all the columns of it sum up to \( N \). Consequently, the left eigenvector is the \( (1,1,\ldots,1) \) vector. The corresponding normalization is proportional to the weight of summing over all possible spin configurations of the sites, with the exception of the bottom line,
Figure A.2: Graphical representation of a summand in the normalization of the percolation transfer-matrix. For better understanding, we denote the $K$-matrices by rectangulars, instead of triangles. The spins –we sum up to– are emphasized by dots.

where the resulting state is. The weight of a $T$-matrix configuration is $\prod R \prod K$, and the normalization is

$$N = \frac{\sum_{\text{all config.}} \prod R \prod K}{\sum_{\text{all config.}} 1} = 2^{-2(L+1)} \sum_{\text{all config.}} \prod (a + b s_1 s_2 s_3 s_4) \prod (A + B s_1 s_3).$$ (A.8)

Expanding the products, the $\prod (a + b s_1 s_2 s_3 s_4) \prod (A + B s_1 s_3)$ summands of $N$ are polynomials in $s_i$, and because $s_i$ is summed over $+1$ and $-1$, if at least one $s_i$ has odd power, the contribution of that summand cancels out as we sum over all the configurations. It is easy to see that all the summands have at least one odd-powered $s_i$, with the exception of $\prod a \prod A$ and $\prod b \prod B$. If we represent the $b s_1 s_2 s_3 s_4$ term by a cross at the given square, and $B s_1 s_3$ as a line connecting $s_1$ and $s_3$, a given summand is a partial filling of the $T$-matrix with crosses and lines (fig. A.2). The power of a spin is equal to the lines starting from that vertex. By putting somewhere a cross or a line, it is clear that the full $T$-matrix has to be filled in order to not to have vertex with odd lines (regardless the bottom edge). By this, we see that the only non-vanishing contributions are $\prod a \prod A$ (the “empty” $T$-matrix) and $\prod b \prod B$ (the “completely filled” $T$-matrix). Consequently, the normalization is $N = 2^{-2(L+1)} \sum_{\text{all config.}} \prod a \prod A + \prod b \prod B = \prod a \prod A + \prod b \prod B$. The $2^{-2(L+1)}$ prefactor is canceled by the summation over all configuration. Including
inhomogeneous weights, we get the following expression for the normalization

\[ N(w, z_0, \ldots, z_{L+1}) = \]

\[ = W_{K_i}(w, z_0) W_{K_i}(w^{-1}, -z_{L+1}) \prod_{i=1}^{L} W_R(w, z_i) W_R(z_i, w^{-1}) + \]

\[ + \tilde{W}_{K_i}(w, z_0) \tilde{W}_{K_i}(w^{-1}, -z_{L+1}) \prod_{i=1}^{L} \tilde{W}_R(w, z_i) \tilde{W}_R(z_i, w^{-1}) . \]

### A.4 L = 1 groundstate elements and X current

As an example, here we present the \( L = 1 \) groundstate elements, and the computation of the \( X \) current for this case

\[ \psi_\bullet(z_0, z_1, z_2) \equiv \psi_{EE,L=1}(z_0, z_1, z_2) = \]

\[ = z_0 + z_1 + z_2 + z_0^{-1} + z_1^{-1} + z_2^{-1} = \]

\[ = z_0 z_1 + z_0 z_2 + z_1 z_2 + z_0^2 z_1 z_2 + z_0 z_2^2 + z_0 z_1 z_2^2 \]

\[ \tilde{Z}_{f.s.,L=1}(z_0, z_1, z_2) = \]

\[ = \psi_\bullet(z_0, z_1, z_2) \psi^*_\bullet(z_0, z_1, z_2) + \left( \psi_\eta(z_0, z_1, z_2) + \psi_\gamma(z_0, z_1, z_2) \right) \times \]

\[ \times \left( \psi^*_\eta(z_0, z_1, z_2) + \psi^*_\gamma(z_0, z_1, z_2) \right) = \]

\[ = \psi_\bullet(z_0, z_1, z_2) \psi_\bullet(z_2, z_1, z_0) + \left( \psi_\eta(z_0, z_1, z_2) + \psi_\gamma(z_0, z_1, z_2) \right) \times \]

\[ \times \left( \psi_\eta(z_2, z_1, z_0) + \psi_\gamma(z_2, z_1, z_0) \right) = \]

\[ = 2(z_0 z_1 + z_0 z_2 + z_1 z_2 + z_0^2 z_1 z_2 + z_0 z_2^2 z_2 + z_0 z_1 z_2^2)^2 . \]
Here and in the followings, we utilize eq. (2.46) to relate the groundstate and the dual groundstate elements. The unnormalized current is

\[ X_{L=1}^{(1)u.n.}(z_0, z_1, z_2) = \psi_j(z_0, z_1, z_2)\psi_j^*(z_0, z_1, z_2) - \psi_j(z_0, z_1, z_2)\psi_j^*(z_0, z_1, z_2) \]

\[ = (1 - 2q) \left( \frac{z_1^2 - 1}{z_0z_1z_2} \right) \left( z_0z_1 + z_0z_2 + z_1z_2 + z_0z_1^2z_2 + z_0z_1z_2^2 \right). \tag{A.12} \]

The normalized current follows after dividing by the partition sum of the full strip

\[ X_{L=1}^{(1)}(z_0, z_1, z_2) = \frac{1 - 2q}{2} \left( \frac{z_1 - 1}{z_1} \right) \frac{1}{E_1(z_0, z_1, z_2)} = \]

\[ = \frac{1 - 2q}{2} \frac{z_1^2 - 1}{z_1} \frac{1}{z_0 + z_1 + z_2 + z_0^{-1} + z_1^{-1} + z_2^{-1}} = \]

\[ \frac{1 - 2q}{2} \frac{z_1^2 - 1}{z_1} \frac{z_0z_1z_2}{z_0z_1 + z_0z_2 + z_1z_2 + z_0z_1^2z_2 + z_0z_1z_2^2}. \tag{A.13} \]

### A.5 Proof of the fusion equation

In this section, we prove the (2.68) fusion equation

\[ R_i(zq, w)R_{i+1}(zq^{-1}, w)M_i = 2 \frac{(w - z)(w + z)}{z^2} M_i R_i(z, w). \tag{A.14} \]

If such an equation holds, every \(RR\) configuration belongs to one of the nine possible faces of the \(R\)-matrix. In other words, we should be able to group the \(RR\) configurations such a way that the sum of their weight at the fusion values \((zq, zq^{-1})\) are proportional to the corresponding \(R\)-matrix weight.

First, according to the l.h.s. of the fusion equation, we group the \(RR\) configurations: \(zq\) configuration is related to \(zq^{-1}\) configuration by the fusion equation. This implies that the weight of the \(RR\) configuration is proportional to the weight of the \(R\)-matrix element at the fusion values \((zq, zq^{-1})\).
### A.5. PROOF OF THE FUSION EQUATION

<table>
<thead>
<tr>
<th>$R$-matrix element</th>
<th>$RR$ config.</th>
<th>$RR$ config.</th>
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Table A.1: The classification of $RR$ configurations in the fusion equation. The first column contains the resulting $R$-matrix element, the second and third column contain the corresponding $RR$-configurations, grouped according to their external connectivity. The connectivity is taken on the pentagon, i.e. on the external sides of the l.h.s. of the fusion equation. The top triangle is not drawn, as it is determined by the top two sites of the $RR$ configuration.
configurations according to their external connectivity on the pentagon, i.e. on the five external sides of the l.h.s of eq. (A.14). This classification puts the possible 41 $RR$ configurations in 21 sets.

At the value $z_1 = z\omega$, $z_2 = z\omega^{-1}$, 3 of these sets have vanishing weights. The remaining 18 sets can be grouped into the expected 9 groups, according to the connectivity on the bottom, i.e. two empty sites or the two sites connected to each other turn into an empty site, one empty site and one occupied site turn into an occupied site, connected to the original connection of the occupied site.

The grouping is exactly the same, as the elements of $dLP_L$ maps to $dLP_{L-1}$ under the recursion.

The disappearing elements are exactly the ones with two not linked lines on the bottom, which cannot be mapped to a proper one site.

The classification is depicted on Tab. A.1. The first column is the corresponding $R$-matrix (the r.h.s. of the equation), the second and third are the corresponding $RR$ configurations, where we keep the original classification to 18 sets according to the external connectivity on the pentagon. The top triangle operator is not shown, as it is uniquely defined by the top two sites. Also, as the triangles have equal weights, we do not have to take them into account at computing the weight. The proportionality factor $\frac{2(z-w)(z+w)}{z^2}$ is exposed in eq. (A.14). It is even true that the 18 sets independently proportional to their corresponding $R$-matrix weight, with factor $\frac{(z-w)(z+w)}{z^2}$.

All of the aforementioned statements are computed directly.

A.6 Construction of the open boundary $K$-matrix from the closed boundary $K$-matrix via insertion of a line

In this section, we show the construction of the open boundary $K$-matrix weights from the closed boundary case, by the well known method of insertion of a line. This description of the open boundary $K$-matrix allows us to extend certain symmetry arguments to the boundary rapidity. Unfortunately, we cannot use this construction to prove the symmetry of the bulk and boundary rapidity in the currents. Using the fusion equation, we get the boundary fusion equation as a corollary.

The closed boundary $K$-matrix consist two elements, with identical weights, and
A.6. OBC K-MATRIX CONSTRUCTION

Figure A.3: The construction of a new $K$-matrix via insertion of a line. In this figure— for convenience— we use different style, then in the previous equations. Straight lines represent the rapidity-lines, a crossing of two rapidity lines is an $R$-matrix, a cusp in a line is a $K$-matrix.

satisfies the reflection equation

$$K_{\text{closed b.c.}} = \begin{array}{c}
\end{array}$$

$K_{\text{cbc}} R(v^{-1}, u) K_{\text{cbc}} R(u, v) = R(v^{-1}, u^{-1}) K_{\text{cbc}} R(u^{-1}, v) K_{\text{cbc}}.$ \hspace{1cm} (A.16)

The idea of the insertion of a line is as in fig. A.3. We multiply the closed boundary reflection equation from the right with a column of four $R$-matrices, and by the means of the Yang-Baxter equation, we move the $R$-matrices inside (fig. A.3 (b)). In this configuration we regard the $KRR$ blocks, as the elements of the new $K$-matrix, and the weight of the new $K$-matrix is equal to the sum of the weights of the corresponding $KRR$ blocks (fig. A.4). Our aim is to follow this procedure to create the open boundary (left) $K$-matrix. The procedure is the same for the right boundary. Since we want to create independent weights, and the possible $KRR$ configurations depend on if a line or an empty site enters on the top of the top $R$-matrix, we can elaborate our procedure. For every open boundary $K$-matrix element, we want to have two groups of $KRR$ configurations, one with an entering line on the top, one without, and we expect the sum of these weights to be equal, in order to produce independent open $K$-matrix weights. Since we expect the right sides of the two $R$-matrices to be the top and bottom half of the open boundary $K$-matrix, the occupancy on the left and on the top
already defines the six groups associated with the empty, the top, and the bottom type $K$-matrix. (First three row of table [A.2])

Distinguishing the two remaining elements (the "line": $\bigtriangleup$ and the "monoid": $\bigtriangledown$) is a bit more tricky, and done in the following way: We look at configurations with a line entering, and we group them according to their connectivity on the left: If the two left side are connected, they belong to the "line", if not, they belong to the "monoid". Now we have to choose the other two groups according to the criteria that with and without the line, the weights should be the same. Based on this criteria, we can uniquely make the choice, however, there is one $KRR$ configuration which has to "divided" between the line and the monoid. Regardless these divided cases, the following statement holds for the weight of the open BC $K$-matrix and the weight of the $KRR$ configuration

$$\frac{(1 + z_B^2)^2}{z B^2} z_1^2 W_{K_{OBC}}(z_1, z_B) = \sum_{i \in G_{K_{OBC}}} R_{i, \text{top}}(z_B, z_1) R_{i, \text{bottom}}(z_1^{-1}, z_B).$$

Here $W_{K_{OBC}}$ denotes the weigh of a specific open B.C. $K$-matrix element, and the sum on the other side runs over the $KRR$ configurations which contribute to the given open BC $K$-matrix element (As given on Table [A.2]). The divided cases have the prefactors $\frac{3}{4}$ and $\frac{1}{4}$ in the homogeneous case, in the inhomogeneous
Table A.2: Open B.C. $K$-matrix elements expressed by closed B.C. $KRR$ configurations. Note the $\frac{3}{4}$ and $\frac{1}{4}$ factors: In order to reproduce the open B.C. weights, we have to split the configuration into two part, with certain probabilities. These probabilities goes into $\frac{3}{4}$ and $\frac{1}{4}$ in the homogenous limit. For the general expression, consult the text.
case, the following relations hold

\[
\left(1 + \frac{z_B^2}{2}\right) W_m(z_1, z_B) = W_t(z_B, z_1) W_t(z_1^{-1}, z_B) + \\
\left(\frac{1}{f(z_1^{-1}, z_B)} + 1\right) \times \left(\frac{1}{f(z_B, z_1) + 1}\right) + \frac{1}{f(z_B, z_1) + 1} \times W_1(z_B, z_1) W_1(z_1^{-1}, z_B),
\]

\[
\left(1 + \frac{z_B^2}{2}\right) W_i d(z_1, z_B) = \frac{f(z_B, z_1^{-1}) + 1}{f(z_1, z_B) + 1} \times W_1(z_B, z_1) W_1(z_1^{-1}, z_B).
\]

Here \(f\) is defined as

\[
f(z_1, z_2) = \frac{W_2(z_1, z_2)}{W_m(z_1, z_2)},
\]

with the property: \(f^{-1}(z_1, z_2) = f(z_2, z_1)\). The prefactors, involving the \(f\)'s turn into \(\frac{3}{4}\) and \(\frac{1}{4}\), if the rapidities are equal to 1.

An intuitive understanding of the division of this \(KRR\) configuration is missing, however, the aforementioned relations have been thoroughly checked analytically.

Since the vertical rapidity becomes the boundary rapidity, the previous argument about the symmetry in the rapidities in certain cases –e.g. the partition sum– extends to the boundary rapidities too.

It is easy to prove the boundary fusion relation, based on this construction and the fusion relation. If we extend the \(KRR\) configuration into a \(KRRRR\) configuration, and we apply the recursion relation on the four \(R\)-matrix, we get the boundary fusion relation, as a corollary.
Appendix B

One-dimensional sums and finitized characters of $2 \times 2$ fused RSOS models

B.1 Coset construction

In this appendix, we give a brief overview of the GKO (Goddard-Kent-Olive) coset construction [68]. For a more complete introduction, we suggest standard textbooks, as [115][116]. The GKO cosets are constructed based on quotients of affine Lie algebras. Affine Lie algebra arises from the central extension of the loop algebra, where a $\tilde{g}$ loop algebra is generated as a tensor product of a simple Lie algebra $g$ and Laurent polynomials in a variable $t$

$$\tilde{g} = g \otimes \mathbb{C}[t, t^{-1}]$$ (B.1)

The generators of the loop algebra are expressed in terms of the $\{J^a\}$ generators of the Lie algebra as

$$J^a_n \equiv J^a \otimes t^n, \quad n \in \mathbb{Z},$$ (B.2)

with the following commutation relations

$$[J^a_n, J^b_m] = \sum_c i f^{a b c} J^c_{n+m}$$ (B.3)
Consider the central extension of the loop algebra, which leads to the following commutation relation

\[
[J^a_n, J^b_m] = \sum_c i f^{ab}_c J^c_{n+m} + \hat{k} n \delta_{ab} \delta_{n+m,0} \quad (B.4)
\]

\[
[J^a_n, \hat{k}] = 0 \quad (B.5)
\]

where \(\hat{k}\) is the central element. The \(\hat{g}\) affine Lie algebra is defined as the central extension of the \(\tilde{g}\) loop algebra, supplemented with an element \(L_0\) which measures the grade, the polynomial degree of \(t\)

\[
L_0 := -t \frac{d}{dt} \quad (B.6)
\]

\[
[L_0, J^a_n] = -n J^a_n \quad (B.7)
\]

\[
[L_0, \hat{k}] = 0 \quad (B.8)
\]

With this definitions, the untwisted affine Lie algebra is defined as follows

\[
\hat{g} = \tilde{g} \oplus \mathbb{C} \hat{k} \oplus \mathbb{C} L_0 \quad (B.9)
\]

Considering the representation theory of affine Lie algebras, there are major differences compared to the simple Lie algebras. Every representation of an affine algebra is characterized by its \(\hat{k}\) eigenvalue, denoted by \(k\), which is called the level of the representation. The physically interesting highest-weight representations are infinite dimensional, and are such that any \(su(2)\) subalgebra associated with real roots are finite. These are called integrable highest-weight representations. For any integrable highest-weight representation, \(k\) must be a positive integer. The number of integral highest-weight representation is bounded by \(k\).

Consider the current algebra, with the following OPE

\[
J^a(z)J^b(w) \sim \frac{k \delta_{ab}}{(z-w)^2} + \sum_c i f^{abc} J^c(w) \frac{J^a(z)}{(z-w)^3} \quad (B.10)
\]

Here \(J^a(z)\) is expanded in the Laurent modes of the generators of the underlying affine Lie algebra

\[
J^a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J^a_n \quad (B.11)
\]

Similarly to the aforementioned holomorphic module, depending on \(z = x^0 + ix^1\), there is an antiholomorphic section, depending on \(\bar{z} = x^0 - ix^1\), which commutes with the holomorphic section. We define the Wess-Zumino-Witten
B.1. **COSET CONSTRUCTION**

(WZW) models in the Sugawara construction, based on the energy-momentum tensor, defined as

\[ T(z) = \frac{1}{2(k+g)} \sum_a (J^a J^a)(z) \]  

(B.12)

where \( k \) is the level of the algebra, and \( g \) is the dual Coxeter number. The normal ordering \((\ldots)\) is defined by contour integral. The central charge of this model is given by

\[ c = \frac{k \dim g}{k + g} \]  

(B.13)

where \( \dim g \) is the dimension of the Lie algebra corresponding to the \( \hat{g} \) affine algebra.

In terms of the affine generators, we define the following Virasoro generators

\[ L_n = \frac{1}{2(k+g)} \sum_a \sum_m : J^a_m J^a_{n-m} : \]  

(B.14)

where the normal order defined by the semicolons as the term with larger subindex is placed on the rightmost position. These Virasoro generators satisfy the following commutation relations

\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0} \]  

(B.15)

\[ [L_n, J^a_m] = -m J^a_{n+m} \]  

(B.16)

Consider the affine Lie algebra \( \hat{g} \), and \( \hat{p} < \hat{g} \) subalgebra of \( \hat{g} \). The \( \{ \tilde{J}^a_n \} \) generators of \( \hat{p} \) are linear combinations of the \( \{ J^a_n \} \) generators of \( \hat{g} \)

\[ \tilde{J}^a_n = \sum_a m_{a'a} J^a_n \]  

(B.17)

Using the definition of \( L^g_m \) and \( L^p_m \), the Virasoro generators of the affine algebras \( \hat{g} \) and \( \hat{p} \) respectively

\[ [L^g_m, \tilde{J}^a_n] = -n \tilde{J}^a_n \]  

(B.18)

\[ [L^p_m, \tilde{J}^a_n] = -n \tilde{J}^a_n \]  

(B.19)

Define the Virasoro generators of the \( \hat{g}/\hat{p} \) coset model, as

\[ L^{(g/p)}_m := L^g_m - L^p_m \]  

(B.20)

which leads to the following commutation relation

\[ [L^{(g/p)}_m, L^{(g/p)}_n] = (m - n) L^{(g/p)}_{m+n} + (c(\hat{g}_k) - c(\hat{p}_{x,k})) \frac{m(m^2-1)}{12} \delta_{m+n,0} \]  

(B.21)
Here \( c(\hat{g}_k) \) and \( c(\hat{p}_{x_e k}) \) are the central charges of the two affine algebras, and \( k \) and \( x_e k \) are the levels, respectively. The level of the subalgebra \( \hat{p} \) is necessarily an integer multiply of \( k \), which is encoded in the \( x_e \) embedding index. Hence, \( L_{\hat{n}}^{(g/p)} \) satisfies the Virasoro algebra, and the central charge of it is the difference of the constituent pieces

\[
c(\hat{g}_k/\hat{p}_{x_e k}) = \frac{k \dim g}{k + g} - \frac{x_e k \dim p}{x_e k + p}
\]  

(B.22)

where \( g \) and \( p \) are the dual Coxeter numbers of the two models, respectively. This construction is the GKO coset construction. Coset construction based on the form \( \hat{g} \oplus \hat{g} \) is called diagonal coset construction. The generators of the direct sum algebra are given by simply the sum of the two copies of \( \hat{g} \)

\[
J^a_{\text{diag}} = J^a_{(1)} + J^a_{(2)}
\]

(B.23)

where (1) and (2) refers to the first and second copy of \( \hat{g} \). As they commute

\[
[J^a_{(1)}, J^b_{(2)}] = 0
\]

(B.24)

the embedding index is always 1. Consequently, the levels of the three copies are \( k, n' \) and \( k + n' \), and such a diagonal coset construction is denoted as \( \hat{g}_k \oplus \hat{g}_{n'} g_{k+n'} \). The central charge of this coset is given as

\[
c = c(\hat{g}_k) + c(\hat{g}_{n'}) - c(\hat{g}_{k+n'}) = \\
= \dim g \left( \frac{k}{k + g} + \frac{n'}{n' + g} - \frac{k + n'}{k + n' + g} \right)
\]

(B.25)

(B.26)

Current algebras with fractional level \( k \) do not have integrable highest-weight representations. Rather, they are represented in admissible representations which are highest-weight representations with certain modular transformation properties.

### B.2 Characters of higher fusion level coset models

The Virasoro characters of the higher fusion level minimal models \( \mathcal{M}(M, M', n') \) are given by the branching functions \( b^{M,M',n'}_r(q) \) of the coset (3.7). The branching functions are expressed in terms of the string functions of \( \mathbb{Z}_{n'} \) parafermionic models with central charges \( c = \frac{2n'}{n' + 2} \). The \( \mathbb{Z}_{n'} \) parafermionic
model is constructed as the $\hat{su}(2)_{n'}/\hat{u}(1)$ coset model. The string function is given by the expression

$$c_\ell^\ell(q) = \frac{q^{-\frac{1}{22}(2n'-2)+\frac{\ell(\ell+2)-n'^2}{4n'}}}{(q^3)^{\infty}} \sum_{i,j=0}^{\infty} (-1)^{i+j} q^{ij(n'+1)+\frac{1}{2}(i+1)+\frac{1}{2}(j+1)} \times$$

$$\times \left( q^{\frac{1}{2}(\ell+\bar{m})} + \frac{1}{2}(\ell-\bar{m}) - q^{n'-\ell+1+\frac{1}{2}(2n'+2-\ell-\bar{m})} \right) , \quad (B.27)$$

for the fundamental domain

$$0 \leq \bar{m} \leq \ell \leq n', \quad \ell - \bar{m} \in 2\mathbb{Z}, \quad \bar{m}, \ell = 0, 1, \ldots, n', \quad n' \in \mathbb{N}. \quad (B.28)$$

Note that the parameter $\bar{m}$ of the parafermionic model should not be confused with the $n'=1$ minimal model parameter $m$. The previous definition is extended to general values $\ell = 0, 1, \ldots, n', m \in \mathbb{Z}, \bar{m} \in \mathbb{Z}, n' \in \mathbb{N}$ by setting

$$c_\ell^\ell(q) = 0 \text{ if } \ell - \bar{m} \notin 2\mathbb{Z} \quad (B.29)$$

$$c_\ell^\ell(q) = c_{\ell-\bar{m}}^{\ell}(q) = c_{n'-\bar{m}}^{n'-\ell}(q) = c_{\bar{m}+2n'}^{\ell}(q). \quad (B.30)$$

By this extension, $c_\ell^\ell$ is an even and periodic function in $\bar{m}$ with period $2n'$. The branching functions are defined as follows \cite{119,123}, for the values $r + s = \ell \mod 2$

$$b_{r,s,\ell}^{M,M',n'}(q) = q^{\frac{\Delta_{r,s}^{M,M',n'} - \frac{\ell M' - s M}{24} + \frac{n'-1}{12(n'+2)}}} \sum_{0 \leq \bar{m} \leq n'/2} c_\ell^{\ell}(q) \times$$

$$\times \left( \sum_{j \in \mathbb{Z}} q^{\frac{1}{n'}(jMM'+rM'-sM)} - \sum_{j \in \mathbb{Z}} q^{\frac{1}{n'}(jMM'+s)(jM+r)} \right) \quad (B.31)$$

$$1 \leq r \leq M - 1, \quad 1 \leq s \leq M' - 1, \quad \ell = 0, 1, \ldots, n \quad (B.32)$$

where the mod 1 notation in the first sum over $\bar{m}$ indicates summing over either integer or half-integer values, and in the second and third sum, $m_{r,\pm s}(j)$ is defined as follows

$$m_{a}(j) = \frac{a}{2} + jM'. \quad (B.33)$$
Appendix B. Fused RSOS Models and Finitized Characters

The branching functions satisfy the branching rules [117–119] of the characters of the \(A_1^{(1)}\) affine current algebras

\[
\hat{c}_{r,s}^{n',l}(q,z) = \sum_{\sigma=1}^{n'_r+n'_s+n'_l} \hat{p}_{r,s}^{n'_r+n'_s+n'_l}(q) \hat{c}_{\sigma,s}^{n'_r+n'_s+n'_l}(q,z), \quad (B.34)
\]

\[
\hat{p} = M, \quad \hat{p}' = \frac{M' - M}{n'}. \quad (B.35)
\]

The branching rules relate admissible characters of affine current algebras \((A_1^{(1)})_k, (A_1^{(1)})_{n'}, (A_1^{(1)})_{k+n'}\), where

\[
r' = \begin{cases} 
n' + 1 - \ell, & s \text{ odd} \\
\ell + 1, & s \text{ even} 
\end{cases} \quad \ell = 0, 1, \ldots, n. \quad (B.36)
\]

For \(n' = 1, 2\), the branching functions give back the Virasoro minimal and superconformal characters, respectively.

### B.3 3 × 3 fused RSOS face weights

\[
W_{3,3} \left( \begin{array}{c| c} a \pm 1 & a \\ a & a \pm 3 \end{array} \right) = \frac{s((a \pm 1)\lambda \mp u)s((a \pm 2)\lambda \mp u)s((a \pm 3)\lambda \mp u)}{s((a \pm 1)\lambda \mp u)s((a \pm 2)\lambda \mp u)s((a \pm 3)\lambda \mp u)} \quad (B.37a)
\]

\[
W_{3,3} \left( \begin{array}{c| c} a \mp 1 & a \\ a & a \pm 3 \end{array} \right) = \frac{s(\lambda - u)s(2\lambda - u)s(3\lambda - u)}{s(2\lambda)s(3\lambda)} \quad (B.37b)
\]

\[
W_{3,3} \left( \begin{array}{c| c} a \pm 1 & a \\ a & a \pm 3 \end{array} \right) = W_{3,3} \left( \begin{array}{c| c} a \pm 3 & a \\ a & a \pm 1 \end{array} \right) = \quad (B.37c)
\]

\[
\quad = \frac{s(\lambda - u)s((a \pm 1)\lambda \mp u)s((a \pm 2)\lambda \mp u)}{s((a \pm 1)\lambda \mp u)s((a \pm 2)\lambda \mp u)}
\]

\[
W_{3,3} \left( \begin{array}{c| c} a \mp 1 & a \\ a & a \pm 3 \end{array} \right) = W_{3,3} \left( \begin{array}{c| c} a \pm 3 & a \\ a & a \mp 1 \end{array} \right) = \quad (B.37d)
\]

\[
\quad = \frac{s(\lambda - u)s(2\lambda - u)s((a \pm 1)\lambda \mp u)}{s(2\lambda)s((a \pm 1)\lambda \mp u)}
\]

\[
W_{3,3} \left( \begin{array}{c| c} a \pm 1 & a \\ a & a \pm 1 \end{array} \right) = \frac{s((a \pm 1)\lambda \mp u)s((a \pm 1)\lambda \mp u)s((a \pm 2)\lambda \mp u)}{s((a \pm 1)\lambda \mp u)s((a \pm 2)\lambda \mp u)} - (B.37e)
\]
\[ W_{3,3} \left( \frac{a \mp 1}{a} \ a \mp 1 \mid u \right) = \frac{s(2\lambda)s((a-2)\lambda)s((a+2)\lambda)s(\lambda-u)s(u)s((a\pm 1)\lambda \mp u)}{s(3\lambda)s((a\mp 1)\lambda)s((a\pm 1)\lambda)^2} - (B.37f) \]

\[ W_{3,3} \left( \frac{a \mp 3}{a} \ a \mp 6 \mid u \right) = \frac{s((a\pm 4)\lambda)s((a\pm 5)\lambda)s((a\pm 6)\lambda)s(u)s((\lambda+u))s((2\lambda+u))}{s(2\lambda)s(3\lambda)s((a\pm 1)\lambda)s((a\pm 2)\lambda)s((a\pm 3)\lambda)} = (B.37g) \]

\[ W_{3,3} \left( \frac{a \mp 3}{a} \ a \mp 4 \mid u \right) = \frac{s((a\pm 4)\lambda)s((a\pm 5)\lambda)s(u)s((\lambda+u))s((2\lambda+u))}{s(2\lambda)s(3\lambda)s((a\pm 1)\lambda)s((a\pm 2)\lambda)s((a\pm 3)\lambda)} = (B.37h) \]

\[ W_{3,3} \left( \frac{a \pm 1}{a} \ a \pm 4 \mid u \right) = W_{3,3} \left( \frac{a \pm 3}{a} \ a \pm 1 \mid u \right) = \] \[ W_{3,3} \left( \frac{a \pm 1}{a} \ a \pm 2 \mid u \right) = \frac{s((a\pm 4)\lambda)s(u)s((a\pm 2)\lambda \pm u)s((a\pm 3)\lambda \pm u)}{s(3\lambda)s((a\pm 1)\lambda)s((a\pm 2)\lambda)} = (B.37j) \]

\[ W_{3,3} \left( \frac{a \pm 1}{a} \ a \pm 2 \mid u \right) = W_{3,3} \left( \frac{a \pm 3}{a} \ a \pm 1 \mid u \right) = \] \[ W_{3,3} \left( \frac{a \mp 1}{a} \ a \mp 2 \mid u \right) = W_{3,3} \left( \frac{a \mp 3}{a} \ a \mp 1 \mid u \right) = \] \[ W_{3,3} \left( \frac{a \pm 1}{a} \ a \pm 3 \mid u \right) = \frac{s((a\pm 4)\lambda)s(2\lambda-u)s(\lambda - u)s(u)}{s(2\lambda)s(3\lambda)s((a\mp 1)\lambda)} = (B.37l) \]

\[ W_{3,3} \left( \frac{a \pm 1}{a} \ a \pm 4 \mid u \right) = \frac{s(3\lambda)s((a\pm 3)\lambda)s((a\pm 4)\lambda)s(u)s((u+\lambda)s((a\pm 1)\lambda \pm u))}{s(2\lambda)s((a\pm 1)\lambda)s((a\pm 1)\lambda)} = (B.37m) \]
\begin{align*}
W_{3,3}(\begin{pmatrix} a & a+2 \\ a+1 & a+2 \end{pmatrix} | u) &= -\frac{s((a+1)\lambda) s((a+1)\lambda) s((a+2)\lambda) s(u) s(u-\lambda)}{s(2\lambda) s(3\lambda) s((a+1)\lambda) s((a+2)\lambda)} - \frac{s((a+1)\lambda) s((a+1)\lambda) s((a+2)\lambda) s(u)}{s((a+1)\lambda) s((a+2)\lambda)} \\
W_{3,3}(\begin{pmatrix} a & a+2 \\ a+1 & a+2 \end{pmatrix} | u) &= W_{3,3}(\begin{pmatrix} a+1 & a \\ a+1 & a \end{pmatrix} | u) = \\
&= \frac{s((a+1)\lambda) s((a+2)\lambda) s((a+1)\lambda) s(u)}{s((a+1)\lambda) s((a+2)\lambda)} \\
W_{3,3}(\begin{pmatrix} a & a+2 \\ a+1 & a+1 \end{pmatrix} | u) &= \frac{s(3\lambda) s((a+1)\lambda) s((a+1)\lambda) s((a+2)\lambda)}{s((a-1)\lambda) s((a+2)\lambda)} (B.37n)
\end{align*}
Appendix C

Supersymmetric fermion chain without particle conservation

C.1 Groundstate degeneracy for $L = 4n$

In this appendix we discuss the degeneracy of the $L = 4n$ groundstates in details. We give comments on the $L = 4n + 2$ case, and give a conjecture on the exact groundstate degeneracy based on numerical studies.

For $L = 4n$, the groundstate energy $\Lambda_G S^{(L=4n)} = 0$. Our aim is to explain the extensiveness of the groundstate degeneracy, based on the counting of zero mode solutions built on the groundstates in sector $(2n, 0)$, described in section 4.5.3.

A solution in sector $(2n, k)$ satisfies the following equations

\[
\begin{align*}
    z_j^L &= \pm i^{-L/2} \prod_{l=1}^{k} \frac{u_l}{u_l + (z_j - 1/z_j)^2}, \quad j = 1, \ldots, 2n \tag{C.1a} \\
    1 &= \prod_{j=1}^{k} \frac{u_l}{u_l + (z_j - 1/z_j)^2}, \quad l = 1, \ldots, k. \tag{C.1b}
\end{align*}
\]
Plugging in the groundstate solution (eq. (4.128), as \((z_j^{(\pm)})^L = \pm 1\), from eq. (C.1a) we get the consistency conditions

\[
\prod_{l=1}^{k} \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2} = \pm 1.
\]  

(C.2)

This can be satisfied by \(u = 0\), \(u = \infty\) and by purely imaginary complex conjugate \(u\) pairs, as for these

\[
\frac{u - (z_j - 1/z_j)^2}{u + (z_j - 1/z_j)^2} = \pm 1.
\]  

(C.3)

The possible values of \(u_l\)'s are fixed by (C.1b) which is a rational function in variable \(u\) after substituting the \(z_j^{(\pm)}\)'s of the groundstate solution. Whether the consistency condition eq. (C.2) gives \(z_L^j = 1\) or \(z_L^j = -1\) depends on the number of domain walls, and if \(u = \infty\) is part of the solution or not.

- \(+1\) equation:
  - even \(k\), all the \(u_l\)'s form complex conjugate pairs
  - odd \(k\), \(u_1 = \infty\), the rest of the \(u_l\)'s form c.c. pairs

- \(-1\) equation:
  - even \(k\), \(u_1 = 0\), \(u_2 = \infty\), and the rest of the \(u_l\)'s are c.c. pairs
  - odd \(k\), \(u_1 = 0\) and the other \(u_l\)'s are c.c. pairs

We have to take into account that the self consistency condition (C.2) has different number of solutions depending on \(z_j\)'s. If \(z_j^L = -i^{-L/2}\), it has \(2n\) solutions for even \(n\)'s, and \(2n-1\) for odd ones. Out of these solutions, \(2n-2\) are nonzero c.c. pairs, one is the \(u = \infty\) solution, and for even number of solutions, \(u = 0\) is also a solution.

The self consistency condition induced by \(z_j^L = +i^{-L/2}\) has \(2n - 2\) solutions for even \(n\)'s, and \(2n\) for odd ones. Out of these solutions, \(2n - 2\) are nonzero c.c. pairs, one is the \(u = \infty\) solution, and for odd number of solutions, \(u = 0\) is also a solution. We have to count the number of \(u\) solutions in c.c. pairs, as the degeneracy is from the possible choices among them, while we use \(u = 0\) and \(u = \infty\) to “tune” (C.1a) to \(\pm 1\).

In order to construct new groundstate solutions in the \((2n, k)\) sectors, we have to find self consistent solutions: we have to find a set of \(u_l\)’s which give the expected \(+1\) or \(-1\) for (C.1a), and compute the degeneracy case by case. We have to distinguish eight cases: even or odd \(n\), even or odd \(k\), \(+1\) or \(-1\) equation and discuss these case by case:
C.1. GS DEGENERACY FOR $L = 4N$

- $n$ even, $k$ even, $-1$ equation: In order to get the $-1$ equation with even number of $u_i$’s, $u_1 = 0$, $u_2 = \infty$, and the rest form c.c. pairs. The degeneracy is $\binom{n-1}{(k-2)/2}$.

- $n$ even, $k$ odd, $-1$ equation: To get the $-1$ equation, $u_1 = 0$, and the other $k-1$ $u_i$’s form c.c. pairs. The degeneracy is $\binom{n-1}{(k-1)/2}$.

- $n$ odd, $k$ even, $-1$ equation: To get the $-1$ equation with even number of $u$’s, we should have $u_1 = 0$, $u_2 = \infty$, but in this sector $u = 0$ is not a solution. Consequently, there is no consistent solution.

- $n$ odd, $k$ odd, $-1$ equation: To get the $-1$ equation with odd number of $u$’s, we should have $u_1 = 0$, but in this sector $u = 0$ is not a solution. Consequently, there is no consistent solution.

- $n$ even, $k$ even, $+1$ equation: The $k$ $u$’s have to form $k/2$ c.c. pairs. No 0 or $\infty$ is involved. The degeneracy is $\binom{n-1}{k/2}$.

- $n$ even, $k$ odd, $+1$ equation: To satisfy the $+1$ equation, $u_1 = \infty$ and the rest form c.c. pairs. The degeneracy is $\binom{n-1}{(k-1)/2}$.

- $n$ odd, $k$ even, $+1$ equation: The $k$ $u$’s have to form $k/2$ c.c. pairs. No 0 or $\infty$ is involved. The degeneracy is $\binom{n-1}{k/2}$.

- $n$ odd, $k$ odd, $+1$ equation: To satisfy the $+1$ equation, $u_1 = \infty$ and the rest form c.c. pairs. The degeneracy is $\binom{n-1}{(k-1)/2}$.

Based on this, we can count the states in a certain sector. Instead of counting the explicit results, we would like to point out that summing over $k$ results in a degeneracy proportional to $2^n$. The exact number is not so interesting because this is only a partial degeneracy, with other symmetries, we can construct more states, however the exact number seems to be complicated to find.

Based on the numerical studies, we give the following conjecture for the degeneracy of the groundstate with $L = 4n$ system size

$$\# \text{ of GS: } 2^{\frac{L}{2}+1} \quad (C.4)$$

For $L = 4n - 2$ systems, the previous degeneracy gives the degeneracy of the highest level state, which –according to eq. (4.118)– equals with the degeneracy of the groundstate of the antiperiodic system.
Summary

This thesis is centered around three topics, sharing integrability as a common theme. This thesis explores different methods in the field of integrable models. The second and third chapters are about integrable lattice models in statistical physics. The fourth chapter describes an integrable quantum chain. Integrable lattice models and quantum chains are closely related, both by the motivation to study them and by method to probe them. Typically, the Hamiltonian of the quantum chain is the logarithmic derivative of the transfer matrix of the corresponding lattice model: \( H = \frac{d}{d\lambda} \log T(\lambda)|_{\lambda=0} \). One reason which makes these models interesting is exact solvability. They are typically strongly correlated systems which are impossible to approach with perturbation theory in interaction strength. In the context of statistical physics, integrable models can be used to compute universal quantities. They are also used as toy-models to learn about new phenomena, e.g. spin-charge separation is observed in the one-dimensional Hubbard-model. Low-dimensional systems with tunable parameters are realized in laboratories recently, giving direct experimental relevance to integrable systems.

Spin-1 current in the dilute \( O(n = 1) \) loop model

In the second chapter, we considered the dilute \( O(n = 1) \) loop model on a strip, with open boundary conditions. It is a lattice model, built up on the following face configurations

\[
\begin{array}{ccccccc}
\square & \square & \square & \square & \square & \square & \square \\
\end{array}
\]
The interaction with the left boundary is described by the following plaquettes, by the elements of the left $K$-matrix

![Plaquettes](image)

Interaction with the right boundary is described by the five vertically mirrored plaquettes. This model describes the statistical ensemble of dilute closed loops, possibly connected to the boundaries, with $n = 1$ loop weight. The model is treated in the framework of Sklyanin’s double row transfer matrix, with inhomogeneous weights depending on rapidities (spectral parameters). At this special point, the largest eigenvalue of the $T$-matrix is simple, and the elements of the groundstate vector are polynomials in the rapidities. The quantum Knizhnik-Zamolodchikov equations (more precisely, an equivalent set of equations, which are loosely also called $q$KZ equations) are used to derive the groundstate elements.

We compute the spin-1 boundary to boundary current on a strip with finite width and infinite height. The spin-1 property means that each path between the two boundaries carries a unit current which –depending on the configuration– contributes to the mean current with a relative $\pm$ sign. The current passing between two arbitrary points is additive, consequently, it is sufficient to compute the current through a horizontal and vertical edge. We denote these quantities by $X$ and $Y$, respectively. We find closed expressions for $X$ and $Y$, in terms of rational functions in the rapidities. The expression for $Y$ involves the partition sum, and symmetric in the rapidities, meaning that the current is translation invariant. The expression for $X$ is substantially simpler, however it is not symmetric.

We explicitly calculated these expressions for small systems. Based on recursion relations, and some technical assumption we proved the form of the current for general system size. The technical assumptions are slightly different for $X$ and $Y$. For $X$, we did not prove some of the symmetry in the rapidities, and used it as an assumption. We proved recursion relations in the width of the strip. The interaction with a boundary is characterized by a parameter, the so called boundary rapidity. It plays a similar role to the rapidities, and we observed for small systems that the symmetry in the rapidities includes the boundary rapidities also. For both $X$ and $Y$, we assumed the symmetry between the normal and boundary rapidities. We constructed the $K$-matrix weights by the insertion of a line, which explains for certain other quantities the symmetry, however the proof does not apply for the current. We give numerical evidence
of the symmetry by computing small but highly nontrivial cases.

**Fused RSOS models and finitized characters**

In the third chapter, we considered the $n \times n$ fused RSOS($m, m'$) models, further parametrized by the crossing parameter $\lambda = (m' - m)\pi / m'$. RSOS models are lattice models, where height variables on the vertices take values in a restricted range. Fusion is a construction by which one forms new solvable models from existing ones by effectively keeping only the variables of a sublattice of the original lattice. By $n \times n$ fusion, we consider the $n \times n$ blocks of the original lattice.

At criticality, the unfused $n = 1$ RSOS models are described by the minimal models, the simplest rational conformal field theories. At higher RSOS fusion level $n$, the continuum limit is expected to be described by higher $n'$ fusion level coset models. Note that $n$ and $n'$ are different parameters. The parameter $n$ characterizes the fused RSOS model which is constructed by the fusion of $n \times n$ faces of the unfused model. The parameter $n'$ characterizes the level of the diagonal coset construction. Previous research by Tartaglia and Pearce conjectured that for $0 < \lambda < \pi$, $n \times n$ RSOS models are related to $n = n'$ coset models. The conjecture is based on computations performed for the $n = 2, 3$ cases.

Based on the Corner Transfer Matrix method, one-dimensional sums are computed in the low-temperature limit. With increasing size, the one-dimensional sums approximate the Virasoro characters of the corresponding conformal field theory. We use the same method to compute the $n = 2$ case for $\pi / 2 < \lambda < \pi$, and we find the finitized characters. Based on our computations, the corresponding conformal theories for the RSOS($m, m'$)$_{2 \times 2}$ models are the $\mathcal{M}(m, m')$ minimal models. For both cases, closed expressions for the one-dimensional sums are found which we call finitized characters. The correspondence between the RSOS model and the CFT is proven, if based on the recursion relation and initial conditions, the one-dimensional sums are proven to be in the conjectured form. The proof is not complete for either case, as some further one-dimensional sums (not corresponding to finitized characters) are needed.

Jacob and Matthieu studied certain one-dimensional sums corresponding to paths with half-integer steps. These one-dimensional sums reproduce the Virasoro characters of the $\mathcal{M}(m, 2m + 1)$ models. By a bijection, we showed that the underlying lattice models of their one-dimensional sums are the $2 \times 2$
fused RSOS$(m, 2m + 1)$ models.

**Supersymmetric fermion chain**

In Chapter 4 we introduce a new supersymmetric chain model, based on the $M_1$ model of Fendley, Schoutens and de Boer. The $M_1$ model describes spinless lattice fermions on a chain which interact via repulsion, i.e. the fermions are separated by at least one lattice site. It is an exactly solvable supersymmetric model with $N = 2$ supercharges.

We consider a fermion–hole symmetric modification of the model. The characteristics of the model change to a large extent, however some properties remain. All the energy levels become highly degenerate, and the fermion number is not a conserved quantity anymore. The degeneracy of all energy levels seems to be extensive, i.e. the degeneracy is a power of two with an exponent growing linearly in the system size. We give counting evidence that the degeneracy of the groundstate is extensive, and give the conjecture that it is exactly $2^{\frac{L}{2}+1}$.

Supersymmetry and integrability are kept, providing powerful tools to solve the model.

We solved the model by means of the nested coordinate Bethe ansatz. We give the Bethe equations. This reduces the problem of diagonalizing a $2^L \times 2^L$ matrix to solving $O(L)$ coupled nonlinear algebraic equations. The Bethe ansatz gives the solution in terms of the Bethe roots. As for the supersymmetric chain the fermion number is not conserved, it is not a good quantum number for the Bethe ansatz. We notice that the number of domain walls –separating strings of fermions from strings of empty sites– is conserved. There are two types of domain walls, domain walls on even or odd site of the dual lattice, which are separately conserved. Hence, the Bethe ansatz is nested, and there are two sets of Bethe roots, $\{z_j\}_{j=1}^m$ and $\{u_l\}_{l=1}^k$. Both types of domain walls carry variable $z$, while odd domain walls also have variable $u$ associated with them. We study the completeness of the Bethe solution by probing two explicit cases, the full spectrum of $L = 6$ and the most degenerate $\Lambda = 6$ eigenvalue at $L = 10$. We find that the Bethe solution does not give the full spectrum, rather utilizing various symmetry transformations gives the full degeneracy of all the energies. The used symmetry transformations seems to be *ad hoc*, we do not know the complete symmetry group.

We study the symmetries characterizing the model, and give an exhaustive list of them. We explain the extensive degeneracy in terms of Cooper pair like excitations which are pairs of odd domain walls with $u$ and $-u$ roots. Such new solutions do not change the eigenenergy, as the energy only depends on the $z$’s.
We find the exact groundstate and first excited state solutions for the Bethe equations, for $L = 4n$, $n \in \mathbb{N}$. The energy follows for these cases, as $\Lambda_0 = 0$ and $\Lambda_1 = 4(1 - \cos(\pi/2n))$. This also shows that in the large size limit, the gap disappears as $\sim 1/L^2$, which is a characteristic of the classical diffusive systems.
Samenvatting

Dit proefschrift behandelt drie onderwerpen, die integreerbaarheid als een gemeenschappelijke thema gemeen hebben. Verschillende methodes uit de theorie van integreerbare modellen worden verkend. De hoofdstukken 2 en 3 gaan over integreerbaar roostermodellen in klassieke statistische fysica. Het laatste hoofdstuk behandelt een integreerbare quantumketen. Klassieke roostermodellen in twee dimensies en quantumketens zijn zeer verwant, zowel in de motivatie om ze te bestuderen als in de manier om ze te onderzoeken. Voor integreerbare roostermodellen geldt dat de Hamiltoniaan van de quantumketen de logaritmische afgeleide is van de transfermatrix van het overeenkomstige roostermodel naar de zogenaamde spectrale parameter $\lambda$: $H = \frac{d}{d\lambda} \log T|_{\lambda=0}$. De spectrale parameter kenmerkt een familie van onderling commuterende transfermatrices. De eigenvectoren hangen dus niet af van $\lambda$, maar de eigenwaarden (het spectrum) wel, vanwaar de naam spectrale parameter.

De belangrijkste reden waarom deze modellen interessant zijn, is de oplosbaarheid of ook wel integreerbaarheid, het feit dat eigenschappen berekend kunnen worden zonder benadering. Het zijn typisch sterk gecorreleerde systemen, die niet benaderd kunnen worden met een storingsreeks in wisselwerkingsterkte, zodat exacte oplossing overblijft als methode voor analyse. In de context van kritieke verschijnselen worden integreerbare modellen gebruikt om universele grootheid te berekenen. Voor universele grootheden is het geen bezwaar als de realistische beschrijving van microscopische details van de wisselwerkingen wordt opgeofferd aan integreerbaarheid. Ook worden integreerbare modellen gebruikt als toy-models om over nieuwe verschijnselen te leren, zoals de scheiding van spin en lading die wordt waargenomen in het één-dimensionale Hubbard-model. Laag-dimensionale systemen met instelbare parameters worden steeds vaker gerealiseerd in laboratoria, een nieuwe link tussen integreerbare systemen en het experiment. Zo is ook de vraag in hoeverre integreerbare systemen uitzonderlijk zijn in hun fysische eigenschappen experimenteel toegankelijk.
Spin-1 stroom in het verdunde \( O(n=1) \) lussenmodel

Het onderwerp van het tweede hoofdstuk is het verdunde \( O(n=1) \) lussenmodel op een strip, met open randvoorwaarden. De naam \( O(n) \) verwijst naar de orthogonale groep, de symmetriegroep van spinmodellen waarvan dit lussenmodel een herformulering is. De lussen zijn gesloten paden op het rooster, en het model wordt verdund genoemd, omdat de paden maar een fractie van de roosterplaatsen bezoeken. De configuraties van het roostermodel zijn opgebouwd met één van de volgende figuren op elk veld van het vierkante rooster

![Diagram](image)

De interactie met de linker rand wordt beschreven door de volgende figuren

![Diagram](image)

Deze figuren laten toe dat de paden eindigen op de rand. De interactie met de rechter rand wordt beschreven door de vijf analoge figuren, gespiegeld ten opzichte van die aan de linker rand.

Dit model beschrijft een statistische verdeling van paden, die gesloten lussen vormen, of eindigen op de randen. Het gewicht van elk pad is \( n \). Het model wordt behandeld met behulp van Sklyanin’s twee-rij-transfermatrix. De transfermatrix hangt af van de spectrale parameter, en daarnaast van een vrije parameter op elke roosterplaats, ook wel rapiditeit genoemd, omdat het analoog is aan een snelheid. Deze plaatsafhankelijkheid maakt het model inhomogeen. In het hier bestudeerde geval \( n = 1 \), is de grootste eigenwaarde van de \( T \)-matrix voor elke systeemgrootte eenvoudig uit te drukken, en zijn de elementen van de grondtoestandsvector polynomen in de rapiditeiten. De quantum Knizhnik-Zamolodchikov vergelijkingen (een reeks vergelijkingen, die ook qKZ vergelijkingen genoemd worden) worden gebruikt om de elementen van de grondtoestandsvector van de transfermatrix af te leiden.

We berekenen de verwachtingswaarde van de locale stroom tussen de twee randen op een strip met eindige breedte en oneindige hoogte, waarbij elk pad dat de twee randen verbindt, een eenheidsstroom voert van de linker naar de rechter rand. Voor elke configuratie van paden, zal dit, door een gegeven georiënteerde link, resulteren in een stroom van grootte 0 of ±1. Deze grootheid wordt ook wel
spin-1 stroom genoemd, om te benadrukken dat de stroom een vector grootheid is en de stroom door een gegeven link van teken omkeert als het pad dat de stroom voert, de link in omgekeerde richting passeert. De stroom die tussen twee willekeurige punten passeert is additief, d.w.z. de stroom tussen punten A en C is de som van de stroom tussen de punten A en B, en die tussen de punten B en C. Daarom is het voldoende om de stroom door een willekeurige horizontale en een willekeurige verticale link te berekenen. Deze twee grootheden duiden we aan met respectievelijk $X$ en $Y$. Ons belangrijkste resultaat is een gesloten uitdrukking voor $X$ en $Y$, in de vorm van van rationale functies in de rapiditeiten. De uitdrukking voor $Y$ is symmetrisch onder verwisseling van de rapiditeiten en hangt ook af van de spectrale parameter. De symmetrie impliceert translatie invariancie, dit ondanks het feit dat het model inhomogene is. De uitdrukking voor $X$ is aanmerkelijk simpeler, en is symmetrisch in de rapiditeiten uitgezonderd de rapiditeit van de link waardoor de stroom wordt gemeten.

We hebben deze uitdrukkingen expliciet berekend voor kleine systemen (i.e. smalle strips). We hebben recursierelaties aangetoond in de breedte van de strip. Hiermee en met enkele technische veronderstellingen, vinden we daaruit de stroom voor willekeurige systeemgrootte. Over de technische veronderstellingen het volgende. Voor $X$ hebben we de symmetrie in de rapiditeiten niet volledig bewezen, maar hebben volledige symmetrie verondersteld op basis van observatie. De interactie met een grens wordt gekenmerkt door een parameter, de zogenaamde grensrapiditeit. Deze speelt een soortgelijke rol als de rapiditeiten, en we zagen in kleine systemen, dat de symmetrie in de rapiditeiten ook de grensrapiditeit bevat. Daarom is voor zowel $X$ als $Y$ hebben we de symmetrie tussen de normale en grens rapiditeiten verondersteld. We hebben de randgewichten geconstrueerd door een lijn in te voegen, die voor bepaalde andere grootheden de symmetrie verklaart, maar net niet voor de stroom. Tenslotte hebben we de veronderstelling van symmetrie door een numerieke berekening ondersteund.

**Gefuseerde RSOS modellen en finitized characters**

In het derde hoofdstuk, beschouwen we $n \times n$ gefuseerd $RSOS(m, m')$ modellen, met een belangrijke rol voor de *crossing* parameter $\lambda = (m' - m)\pi/m'$. RSOS modellen zijn roostermodellen, waar hoogtevariabelen op de vertices van een rooster waarden in een beperkt bereik aannemen. Fusie is een constructie waarmee nieuwe oplosbare modellen uit bestaande worden gemaakt, door alleen de variabelen van een subrooster van het oorspronkelijke rooster te behouden. De parameters in het oorspronkelijke model worden zo gekozen dat de overige
variabelen kunnen worden uitgesommeerd. Door een $n \times n$ fusie beschouwen we
de $n \times n$ blokken van het oorspronkelijke rooster.
De oorspronkelijke $n = 1$ modellen hebben een kritiek punt waarin de
schaallimiet beschreven wordt door de zogenaamde minimale modellen, de
eenvoudigste rationele conformale veldtheorieën. Na fusie van het RSOS model
in $n \times n$ blokken, wordt verwacht dat de schaallimiet wordt beschreven door
hogere $n'$ level coset modellen. Merk op dat $n$ en $n'$ differente parameters zijn.
De parameter $n$ karakteriseert de fusie-constructie van het RSOS-model. De
parameter $n'$ karakteriseert het niveau van de diagonale coset constructie die de
basis vormt voor de conforme veldentheorie. Eerder onderzoek door Tartaglia
en Pearce heeft geleid tot het voorstel dat voor $0 < \lambda < \frac{\pi}{n}$, $n \times n$ RSOS
modellen via de schaallimiet gerelateerd zijn aan $n' = n$ coset modellen, zodat
beide parameters dezelfde waarde aanmenen. In dit proefschrift wordt dat werk
uitgebreid naar $\frac{\pi}{n} < \lambda < \pi$.
Op basis van de corner transfermatrix methode werden toestandssommen
berekend in een limiet waarin lage temperatuur wordt gecombineerd met
hoge anisotropie. In deze limiet wordt het model vrijwel ééndimensionaal.
Met de toenemende grootte van het systeem benaderen de ééndimensionale
toestandssommen de Virasoro-karakters van de bijbehorende conforme
veldentheorie. We hebben deze methode gebruikt om het $n = 2$ geval voor $\frac{\pi}{2} < 
\lambda < \pi$ te berekenen en de karakters te vinden. Onze berekeningen laten zien dat
voor de RSOS$(m, m')_{2 \times 2}$ de corresponderende conforme theoriën de $M(m, m')$
minimale modellen zijn. Voor beide zijden van de correspondentie worden
gesloten uitdrukkingen gevonden voor de eindige ééndimensionale sommen,
finitized characters genoemd. Dat de uitdrukkingen gelijk zijn is niet in alle
gevallen volledig bewezen. Voor het bewijs zijn nog enkele ééndimensionale
toestandssommen nodig die niet overeenkomen met finitized characters.
Voor de gevallen $(m, m') = (m, 2m + 1)$, is het bewijs wel volledig gevonden
op basis van werk van Jacob en Matthieu. Zij bestudeerden bepaalde
ééndimensionale sommen die overeenkomen met paden met halve integerstappen.
Deze ééndimensionale sommen reproduceren de Virasoro karakters van de
$M(m, 2m + 1)$ modellen. Door een bijectie hebben we aangetoond dat de
onderliggende roostermodellen van hun ééndimensionale sommen zijn de $2 \times 2$
gefuseerd RSOS$(m, 2m + 1)$ modellen.

**Supersymmetrisch fermionketen**

In het vierde hoofdstuk wordt een nieuwe supersymmetrische quantumketen
ingevoerd, gebaseerd op het $M_1$-model van Fendley, Schoutens en de Boer. Het
$M_1$-model beschrijft spinvrije fermionen op een keten die inwerken op elkaar via afstoting, d.w.z. de fermionen worden tenminste gescheiden door één lege roosterplaats. Het is een exact oplosbaar supersymmetrisch model met $\mathcal{N} = 2$ supercharges.

Hier wordt het model $M_1$ gemodificeerd om de gat-deeltje symmetrie te herstellen. De kenmerken van het model veranderen sterk, maar sommige eigenschappen blijven overeind. Alle energie niveaus zijn sterk gedegenereerd, en het fermion nummer is niet meer een behouden grootheid. Terwijl in het $M_1$ model de generieke energie-niveaus tweevoudig ontaard zijn, is hier de ontaardingsgraad van elk niveau steeds een macht van twee, met een exponent die lineair lijkt te groeien met de systeengrootte. Voor de ontaardingsgraad van de grondtoestand hebben we argumenten dat deze gelijk is aan $2^L + 1$ is. Supersymmetrie en integreerbaarheid zijn beide van toepassing, zodat krachtige instrumenten beschikbaar zijn om het model te analyseren.

We bestuderen het model door middel van de geneste coördinaten Bethe Ansatz. We hebben de Bethe vergelijkingen uitgewerkt. Deze vergelijkingen alleen al reduceren het probleem van het diagonaliseren van een $2^L \times 2^L$ matrix tot het oplossen van $O(L)$ gekoppelde niet-lineaire algebraïsche vergelijkingen. De Bethe Ansatz geeft de eigenwaarden van de Hamiltoniaan in termen van de oplossingen van die vergelijkingen, de Bethe-wortels. Omdat voor de supersymmetrische keten het fermiongetal niet behouden is, is het ook geen goed quantumgetal voor de Bethe Ansatz. Wat wel behouden blijkt is het aantal domeingrenzen tussen bezette en onbezette posities. Als we de domeingrenzen op even en oneven posities onderscheiden, blijken beide aantallen afzonderlijk behouden te zijn. Als gevolg hiervan is de Bethe Ansatz genest, en er zijn twee sets van Bethe-wortels, $\{z_j\}_{j=1}^{n}$ en $\{u_l\}_{l=1}^{k}$. Beide soorten domeingrenzen worden gekarakteriseerd door de parameter $z$, terwijl oneven domeingrenzen tevens een tweede parameter $u$ voeren.

We bestuderen de volledigheid van de Bethe-oplossing door twee gevallen te onderzoeken, het volledige spectrum van $L = 6$ en de meest ontaard $\Lambda = 6$ eigenwaarde bij $L = 10$. We hebben gevonden dat de Bethe-oplossingen wel het volledige spectrum, maar niet de volledige ontaardingsgraad geeft. Door gebruik te maken van verschillende symmetrieën wordt toch de volledige ontaardingsgraad gevonden. De precieze structuur van de symmetriegroep blijkt erg gecompliceerd te zijn.

Een groot aandeel in de ontaardingsgraad is het gevolg van tweetallen van domeingrenzen op oneven posities met tegengestelde waarden van de variabele $u$. In analogie met supergeleiding hebben we deze tweetallen Cooperparen genoemd. We hebben een operator geconstrueerd die twee even domeingrenzen omzet in
oneven domeingrenzen, en aldus een Cooperpaar creëert. Dergelijke nieuwe oplossingen veranderen de eigenwaarde voor de energie niet, omdat de energie bepaald wordt door de \( z \)-variabelen. We hebben de Bethe-oplossing voor de grondtoestand en de eerste aangeslagen toestand gevonden voor \( L = 4n, \ n \in \mathbb{N} \). De energie volgt voor deze gevallen, als \( \Lambda_0 = 0 \) en \( \Lambda_1 = 4(1 - \cos(\pi/2n)) \). Dit impliceert dat de eerste aangeslagen toestand een energie heeft \( \mathcal{O}(L^{-2}) \) boven die van de grondtoestand. Het is opmerkelijk dat deze SUSY keten deze eigenschap gemeen heeft met een klassiek diffusief systeem.
Contribution to publications

Here, I give a detailed overview on my contribution to the publications, this basis is based on.


2. György Z. Feher, Paul A. Pearce, and Alessandra Vittorini-Orgeas. One-dimensional sums and finitized characters of $2 \times 2$ fused RSOS models. 2016, unpublished
   Construction of fused face weights. Local energy functions without gauge modification. Exact expressions for the finitized characters. Numerical checks for the finitized characters.

   Little modification of Bethe equations. Completeness of Bethe equations. Certain symmetries, including the domain wall–non-domain wall symmetry, and the reflection of the spectrum. Exact Bethe solution for the groundstate, and the first excited state. Large system asymptotics of the system. Systematic counting of groundstates, conjecture on groundstate degeneracy.
Bibliography


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