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## Super-exponential bubbles in lab experiments: evidence for anchoring over-optimistic expectations on price

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# Super-exponential bubbles in lab experiments: evidence for anchoring over-optimistic expectations on price

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## Abstract

We analyze a controlled price formation experiment in the laboratory that shows evidence for bubbles. We calibrate two models that demonstrate with high statistical significance that these laboratory bubbles have a tendency to grow faster than exponential due to positive feedback. We show that the positive feedback operates by traders continuously upgrading their over-optimistic expectations of future returns based on past prices rather than on realized returns.

*Keywords:* anchoring, financial bubbles, laboratory experiments, speculation, super-exponential growth, positive feedback

**Highlights:**

- We offer an interpretation of lab experiments that exhibit financial bubbles.

- We show that bubbles in controlled experiments can grow faster than exponential.

- We find traders anchor expectations more on price than on returns in these bubbles.

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# 1. Introduction

Bubbles, defined as significant persistent deviations from fundamental value, express one of the most paradoxical behaviors of real financial markets. Here, we analyze the dynamics of bubbles in a laboratory market (Hommes et al. (2008)) and focus on the regimes of strong deviations from the fundamental values, which we refer to as the bubble regimes. Because this data is from a controlled environment, we can exclude exogenous influences such as news or private information. We show that a model with exponential growth, corresponding to a constant rate of returns, cannot account for the observed transient explosive price increases. Models that incorporate positive feedback leading to faster-than-exponential growth are found to better describe the data.

Research on financial bubbles has a rich literature (see e.g. Kaizoji and Sornette (2010) for a recent review) aiming at explaining the origin of bubbles, their persistence and other properties. The theoretical literature has classified different types of bubbles following different modeling approaches. For instance, Blanchard (1979) and Blanchard and Watson (1982) introduced rational expectation (RE) bubbles, i.e., bubbles that appear in the presence of rational investors who are willing to earn the large returns offered during the duration of the bubble as a remuneration for the risk that the bubble ends in a crash. Tirole (1982) argued that heterogeneous beliefs among traders is necessary for bubbles to develop. De-Long et al. (1990b) demonstrated that introducing noise traders in a universe of rational speculators can amplify the size and duration of bubbles. Brock and Hommes (1998) showed that endogenous switching between heterogeneous expectations rules, driven by their recent relative performance, generates bubble and crash dynamics of asset prices. Gallegati et al. (2011) presented an agent-based model of bubbles and crashes, where crashes occur after a period of financial distress. A recent review of behavioral models of bubbles and crashes with fundamentalists trading against chartists is given in Hommes (2006).

There is also a large literature on empirical tests for bubbles, see e.g. Gürkaynak (2008) for a survey. Lux and Sornette (2002) showed that the multiplicative stochastic process proposed by Blanchard and Watson (1982), together with the no-arbitrage condition, predicts a tail exponent of the distribution of returns smaller than 1, which is incompatible with empirical observations. Johansen et al. (1999) and Johansen et al. (2000) thus extended the Blanchard-Watson (1982) model of RE bubbles by proposing models in which the crash hazard rate reflects the imitation and herding behavior of the noise traders and exhibits critical bifurcation points. In these models asset bubbles are characterized by faster-than-exponential growth and this behavior has been found in many financial time series, e.g. the Chinese stock market (Jiang et al. (2010)), oil prices (Sornette et al. (2009)), the NASDAQ-index and the dot.com bubble (Johansen and Sornette (2000)) and the U.S. housing market (Zhou and Sornette (2006)). A common feature of asset bubbles with faster-than-exponential growth is that prices seem to be only loosely connected to fundamentals, leaving much room for excessive speculating and rapidly growing asset prices (see e.g. Kindleberger (1978) for qualitative examples and Sornette (2004) for a more extensive discussion). Another recent bubble example that seems to exhibit faster-than-exponential growth is the Bitcoin bubble and crash in April 2013. Bitcoins are a product of the virtual economy, intended to allow internet peer-to-peer transactions without the need for banking intermediaries. Bitcoins

have no intrinsic fundamental value. Within a couple of months, the exchange rate for a single Bitcoin quickly rose this year by a factor of 13() from 20 dollars in January 2013 to an intraday high of \$266 on April 10, before falling to \$186 later that same day and back to \$54 three days later.

Laboratory experiments with human subjects are well suited to study the emergence of bubbles in a controlled environment. Following the seminal work of Smith et al. (1988), a large literature on bubbles in laboratory experimental markets has emerged. Smith et al. (1988) showed how easily in bubbles can emerge in experimental markets, even with experienced traders. Since then, numerous papers have studied the robustness of bubbles in experimental asset markets. For example, Haruvy and Noussair (2006) impose short selling constraints, while Lei et al. (2001) preclude speculation by prohibiting buyers to resell the asset and sellers to buy, but in both cases asset bubbles are robust features. In these laboratory experiments, the fundamental value of the asset is decreasing over time (typically decreasing from 15 to 0 over 15 time periods). Prices then start below fundamental, after a few periods cross and overshoot fundamental value leading to high overvaluation of the asset and finally a crash towards the end of the experiment. Kirchler et al. (2012) have recently shown that the declining fundamental value coupled with an increasing cash-to-asset-value ratio are important drivers of the mispricing and overvaluation. They also show that, with a different context ("stocks of a depletable gold mine" instead of "stocks"), mispricing and overvaluation are reduced.

The Smith et al. laboratory experiments are too short (only 15 periods) to perform statistical testing for super-exponential bubbles. Longer asset market experiments, for 50 periods, have been performed in Hommes et al. (2005). In these learning-to-forecast experiments subjects are professional forecasters of the price of a risky asset, with the realized price depending upon average price forecasts. In their setup, one of the computerized traders is a fundamental robot trader, following a forecasting/trading strategy based upon the correct (constant) fundamental price, while the other trading decisions are derived from mean-variance maximization given subjective price forecasts. Asset prices exhibit oscillatory price fluctuations ("boom and bust cycles") around the (constant) fundamental value.

In this paper we use data from another learning-to-forecast experiment in Hommes et al. (2008) to study whether super-exponential bubbles with faster-than-exponential growth can arise in the lab. In contrast to Hommes et al. (2005), the experiment in Hommes et al. (2008) is *without* a fundamental robot trader. The markets exhibit long lasting bubbles with asset prices increasing up to more than 15 times fundamental value, before hitting an (unknown to the traders) upperbound of 1000, after which the market crashes. Figure 1 illustrates the bubble and crash dynamics. We use these experimental data, particularly focussing on the long lasting bubble phase of the experiment, to test for super-exponential bubbles.

[Figure 1 about here.]

The present paper represents the first detailed quantitative calibration of simple models with positive feedback that unambiguously demonstrates the existence of positive feedback mechanisms and super-exponential bubbles in the price formation process. It thus provides support within controlled laboratory set-ups for the empirical evidence presented by Sornette

et al. on historical financial bubbles (see Jiang et al. (2010) and Kaizoji and Sornette (2010)[1] and references therein for an overview). Understanding whether or not asset bubbles may be super-exponential is important because, if empirically relevant, behavioral models should take faster-than-exponential growth into account. Moreover, super-exponential bubbles imply much faster growth of asset prices and thus even more excessive overvaluation and more severe losses when the bubble collapses.

The paper is organized as follows. Section 2 briefly recalls the laboratory experiment of Hommes et al. (2008). Section 3 provides expressions for the rational bubbles and the models with anchoring on price and return respectively. Section 4 presents the main results on statistical testing for super-exponential bubbles in the experimental data. Discussion and Conclusions are presented in Sections 5 and 6.

## 2. The experiment

In the experiment of Hommes et al. (2008), participants ("traders") were asked to forecast the price of a single asset for 50 periods. The price of the asset evolves with the equation,

$$p_t = \frac{1}{1+r} \left[ \frac{1}{H} \sum_{h=1}^{H} p_{t+1}^h + D \right], \tag{1}$$

where the market price $p_t$ at time $t$ is given as an average of the $H = 6$ traders discounted price expectations; $r = 5\%$ is the interest rate, $p_{t+1}^h$ is the estimate of trader $h$ for the price for period $t + 1$ based on information up to time $t - 1$ and $D = 3.00$ is the dividend. Hence, today's price $p_t$ is simply the average of the current value of the traders' expectations for tomorrow $p_{t+1}^h$. Note that the traders have to make a two period forecast; for their forecast $p_{t+1}^h$, only the prices up to time $t - 1$ are available.

Traders are given the parameters above (but not the price forming Equation 1 itself) and are rewarded according to their prediction accuracy[2]. The fundamental/equilibrium price $p^f$ (which traders could calculate, but however is not relevant for the compensation) is 60[3]. In our analysis, we focus on the realized price $p_t$ and not on the traders' individual estimates $p_t^h$.

In these learning-to-forecast experiments, subjects' only task is to forecast the price of a risky asset. The demand for the risky asset is derived from mean-variance maximization, given these subjective beliefs. The price in Equation 1 is then derived from market clearing, as in a standard asset pricing model. Learning-to-forecast experiments thus provide a clear single hypothesis test of expectations, with all other model assumptions computerized in the experiment; see Hommes (2011) for an extensive discussion and survey.

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[1] An extended version is available at http://arxiv.org/abs/0812.2449

[2] The reward is proportional to the quadratic scoring rule $\max \left\{ (1300 - 1300/49(p_t - p_t^h))^2, 0 \right\}$

[3] $p^f = D/r = 3.00/5\% = 60$

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There are a number of related learning-to-forecast asset pricing experiments and it is useful to briefly discuss some differences in the experimental design. In Hommes et al. (2005), the key difference is the presence of fundamental robot traders, who always forecast the asset price to equal its fundamental value $p^f = 60$. The weight of these robot traders increases as the price moves away from its fundamental value, so that the robot traders act as a far from equilibrium stabilizing force. As a result, the amplitude of the asset price fluctuations is smaller and realized prices oscillate around the fundamental value within a bounded interval $[0, 100]$. In the experiments of Hommes et al. (2008), the fundamental robot traders were removed and only the initial forecast was restricted to lie within the interval $[0, 100]$. In later periods, realized asset prices may exceed this interval and in fact 5 of 6 markets without fundamental robot traders exhibit long lasting bubbles growing until an unknown upperbound of 1000 was hit, after which the market crashed (Figure 1).

Another related learning-to-forecast asset pricing experiment is in Heemeijer et al. (2009). Here the key difference in the experimental design is that the realized market price is derived from a price adjustment rule with subjects forecasting one-period-ahead. In contrast, Hommes et al. (2005) and Hommes et al. (2008) use a temporary equilibrium framework, where subjects have to forecast the realized market price in Equation 1 two-periods-ahead. As a result, the asset price fluctuations in Heemeijer et al. (2009) are much slower and less volatile than in the two-period ahead forecasting framework.

The asset price generating process in Equation 1 has a uniquely defined constant rational expectations price equal to its fundamental value $p^f = D/r = 3.00/5\% = 60$. This is the only constant price generating a self-fulfilling rational expectations or perfect foresight solution. Realized asset prices however are not even close to fundamental value, but exhibit rapidly growing long lasting bubbles and this is the phenomenon we seek to explain by a simple model.

## 3. Theory/calculation

### 3.1. Rational Bubble

Hommes et al. (2008) discussed the rational bubble

$$p_t = (1 + \hat{r})^t \, a_1 + b_1 \, , \tag{2}$$

where $a_1$ is a positive constant. This process fulfills the "self-confirming" nature of rational expectations if the assumed interest rate $\hat{r}$ equals the interest rate $r$ from Equation 1 and $b_1$ equals the fundamental value of $p^f = 60$[4]. In fact, Hommes et al. (2008) found that traders do use an interest rate $\hat{r}$ significantly larger than $r$ in four of the six groups and hence their expectations are no longer rational (see section 4). Furthermore, the growth rate $\hat{r}$ is not constant, but is increasing as we will see later.[5]

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[4]For a rational bubble, we have $\mathbb{E}_t[p_{t+1} + D] = c(1 + r)^{t+1} + p^f(1 + r) = (1 + r)p_t$.

[5]Evans (1991) develops a model of periodically collapsing rational bubbles, where asset prices grow at a rate larger than the risk free rate for some time, but have an exogenously given positive probability of collapsing in each period. In this model, the growth rate along the bubble is also constant.

Todd and Gigerenzer (2000) argued that "decision-making agents in the real world must arrive at their inferences using realistic amounts of time, information, and computational resources. [..] The most important aspects of an agent's environment are often created by the other agents it interacts with." Moreover, Tversky and Kahneman (1974) presented three heuristics that are employed in making judgments under uncertainty. For our purposes, the heuristic that is relevant to interpret the groups' behaviors is the "adjustment from an *anchor*, which is usually employed in numerical prediction when a relevant value is available. These heuristics are highly economical and usually effective, but they lead to systematic and predictable errors." (Emphasis is ours).

In the rest of this section, we are presenting two models that generalize the rational bubble of Equation 2, in which traders anchor their forecasts on either (1) price or (2) return. Both models have in common that they can generate price growth that is significantly faster than exponential (as observed in the data).

### 3.2. Anchoring on Price

Generalizing the constant growth generated by Equation 2, we specify a model which allows faster, slower than or just exponential growth. The growth rate $\log(\bar{p}_t/\bar{p}_{t-1})$ can be explained by the excess price $\bar{p}_{t-1}$ (which is the difference between the observed price $p_t$ and the fundamental price $p^f$) plus a constant:

$$\log\left(\frac{\bar{p}_t}{\bar{p}_{t-1}}\right) = a_2 + b_2\bar{p}_{t-1}. \tag{3}$$

$a_2 > 0$ and $b_2 > 0$ (respectively $b_2 < 0$) would imply faster (respectively slower) than exponential growth i.e. the growth rate grows (respectively decreases) itself. For $b_2 = 0$, we recover the exponential growth (equivalent to the rational bubble Equation 2 with $r = \hat{r}$). We will see below that $b_2$ is typically significantly larger than zero, indicating faster than exponential growth and positive feedback on the price.

One justification for the functional form (Equation 3) is that anchoring on price is commonly used in technical trading. One of many patterns used are support and resistance levels which is nothing else but anchoring on price. Although in violation with the efficient markets hypothesis, e.g. Brock et al. (1992) and Lo et al. (2000) studied technical trading rules and found "practical value" for such technical rules.

### 3.3. Anchoring on Return

Alternatively, we check if the growth rate can be explained by the excess log-return $\log(\bar{p}_t/\bar{p}_{t-1})$ following the following process

$$\log\left(\frac{\bar{p}_{t+1}}{\bar{p}_t}\right) = a_3 + b_3 \log\left(\frac{\bar{p}_t}{\bar{p}_{t-1}}\right). \tag{4}$$

The conditions that $a_3 > 0$ and $b_3 > 0$ implies again faster than exponential growth of the excess price $\bar{p}_t$ and positive feedback from past returns. This model can be interpreted as a second order iteration or adaptive form of the exponential growth.

## 4. Results

In this section, we estimate the parameters of the two processes and check for the statistical significance of $b_2$ and $b_3$ that express a positive feedback of price (Equation 3) or of return (Equation 4) onto future returns (for positive coefficients). In particular, we are interested in the lower 95% confidence interval for the null hypothesis that $b_2$ and $b_3$ are zero, to check for significant deviations that can confirm or not that price growth is indeed significantly faster than exponential (which, again, is the situation corresponding to $b_2$ and $b_3$ greater than zero). As the two models can be run over a multitude of different start and end points, we present the results also graphical form to provide better insight.

Hommes et al. (2008) identified bubbles in five out of the six groups. A bubble refers to a phase of growing asset prices, from the first period the price exceeds the fundamental value until a maximum is reached. Group 1 shows a somehow erratic price trajectory and no bubbles.[6] Groups 5 and 6 show large bubbles, but the time horizon is too short (9 periods in Group 5 and 7 periods in Group 6) for our statistical analysis to get significant results. Moreover, Hommes et al. (2008) found that, although the average growth of the bubble in group five is larger than for the rational bubble (1.255 versus 1.05) the null hypothesis of a rational bubble (Equation 2) could not be rejected. Hence, for statistical testing of super-exponential bubbles we focus on group number 2, 3 and 4, where the bubble lasts 20, 23 and 15 periods respectively.

[Table 1 about here.]

*4.1. Group 2*

The bubble period identified by Hommes et al. (2008) runs from $7 - 26$. Figure 2 shows that the price becomes larger than the fundamental value $p^f$ at $t = 7$. Checking the returns vs. past returns in Figure 3, we see that the bubble initially grows approximately exponentially ($r_t \approx r_{t-1}$) as confirmed by the positions of the points along the diagonal. Later, at around $t = 14$, the returns become monotonous increasing (i.e. prices become faster than exponential growth) and are plotted above the diagonal. This is also confirmed by Figure 4 where, for low starting and ending values of the analyzing time window, the parameters estimated for Equation 3 are not distinguishable from exponential growth since the parameter $b_2$ is not significantly different from zero. However, towards the middle and the end of the bubble, the growth rate accelerates ($b_2$ becomes significantly larger then zero) before the bubble finally bursts. The parameter $a_2$ is positive over the whole analysis window (lower left panel) and almost always significantly larger than zero (lower right panel). The upper panels shows that $b_2$ (for low start and ending values) is not significant different from zero, but, later in the bubble, $b_2$ becomes positive (top left panel) and even significant positive (upper right panel).

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[6]In Group 1 some sudden relatively large jumps in the asset price occur due to some subject making sudden, high price forecasts, thus preventing coordination on a long lasting bubble; see Hommes et al. (2008), p.124, footnote 9.

Checking for the existence of feedback from past returns in Figure 5, we find that Equation 4 describes less accurately the experimental results; although the parameters $a_3$ and $b_3$ are both positive (left panels), the time windows where the parameters are both significantly positive (right panels) is relatively small (only for starting values $t = 7$ and $t = 8$).

Hence, in summary, the bubble in group 2 does not only grow significantly faster then exponential in the end phase, but traders seem to anchor their expectations more on price rather than on return.

[Figure 2 about here.]

[Figure 3 about here.]

[Figure 4 about here.]

[Figure 5 about here.]

### 4.2. Group 3

Group 3 (over the time horizon from $7 - 29$) is the longest bubble among the six groups. From Figure 6 (which is plotted in log scale), the bubble seems to grow initially only exponentially (visible as a straight line in the plot), which is also confirmed by Figure 7, which shows that the growth rate is initially constant. At around $t = 20$, growth accelerates. This observation is also confirmed by the analysis of Equation 3, where $a_2$ is significant for almost all analysis windows. But, the positive feedback of the price on the growth rate embodied by $b_2$ becomes only significant in the later phase of the bubble. Analyzing this group for the possible existence of anchoring on return (Equation 4) in Figure 9, we find that the results are less clear cut: although $a_3$ and $b_3$ are positive, $a_3$ is not significantly different form zero for starting values after $t = 10$. Hence, we conclude that Equation 4 does not appropriately describe the price and traders tend to anchor their expectations on price rather than on return.

[Figure 6 about here.]

[Figure 7 about here.]

[Figure 8 about here.]

[Figure 9 about here.]

### 4.3. Group 4

As can be seen from Figure 10, the bubble formed over the time window $7 - 29$ is briefly disrupted by the intervention of trader number 6[7]. This can also be seen in Figure 11 where we plot the returns. Between $t = 7$ and $t = 13$, we have more or less a cobweb and then, starting with $t = 14$, the growth rate increases and a bubble is formed. For anchoring on

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[7]The prediction of trader number six at time point $t = 10$ seems to be off by an order of magnitude as he has misplaced the decimal.

price, we see in Figure 12 very strong evidence for faster then exponential growth; $a_2$ and $b_2$ are both significantly positive. Again, for very early and small analysis windows, only $a_2$ is positive, indicating exponential growth in the initial phase of the bubble. The analysis for Equation 4 in Figure 13 is less clear, but the signal for jointly positive $a_3$ and $b_3$ is relatively small (only for two smallest starting values), indicating that traders prefer to anchor their predictions on price and not on return.

[Figure 10 about here.]

[Figure 11 about here.]

[Figure 12 about here.]

[Figure 13 about here.]

## 5. Discussion

We find many time windows where we can clearly reject the hypothesis of exponential growth and find evidence for faster than exponential growth. This is even more remarkable when taking into account that the data suffers some limitations which make detection of faster than exponential growth more difficult.

**Price ceiling:** Although the price is allowed to fluctuate over a relatively large range, it is capped at a maximum value of 1000. The number of observations for a bubble are thus limited.

**Stable equilibrium price:** The pricing formula Equation 1 assumes a fundamental value of 60 and thus biases the price towards this value. Even if all traders give an estimate of 1000, the realized market price from Equation 1 would be $(1000 + 3)/1.05 \approx 955$, i.e. the price is artificially deflated by almost 45 monetary units.

**Mis-trades:** There seems to be a few instances where trades' estimates are off by an order magnitude (i.e. some traders seem to fail to place the decimal point at the correct digit at some times, see also the discussion about "active participation hypothesis" in Lei et al. (2001)).

**Short data horizon:** Although the experiments run over a time horizon of 50 time-steps; the bubbles appear in much shorter time, leaving relatively few points to estimate tight confidence intervals.

Anufriev and Hommes (2012) have fitted a heuristics switching model to a positive feedback asset pricing experiment of Hommes et al. (2005) in the presence of a fundamental robot trader, whose trading drives the price back towards its fundamental value. As a consequence, long lasting bubbles do not arise in that setting, but rather asset prices oscillate around the fundamental and individuals switch between different simple forecasting heuristics such as adaptive expectations and trend following rules.

Tirole (1982) noted that "[..] speculation relies on inconsistent plans and is ruled out by rational expectations." However, in the experiments of Hommes et al. (2008) that we analyze

here, traders are rewarded, not on the basis of how well they predict the fundamental value of the assets they buy but, rather on the accuracy of their prediction of the realized price itself, similarly to real financial markets. This differs from Smith et al. (1988)'s study where subjects know that the price of the traded asset has to converge to zero at the end of the experiment when all dividends are payed out. Traders also do not need to invest their wealth into an asset, they do not worry about price fluctuations or care about supply & demand, which allows them to "ride the bubble" (see Abreu and Brunnermeier (2003), DeLong et al. (1990b) and DeLong et al. (1990a)). Haruvy and Noussair (2006) showed in their experiments that short selling constraints can considerable increase bubbles. However, in the data of the present paper, liquidity is unlimited and the market is cleared after each period, hence unlimited short selling is possible. The traders rather make a forecast as in a Keynesian beauty contest Keynes (1936), where they need to synchronize their beliefs. Such self-confirming predictions can easily lead to herding, in particular in situations where the fundamental value is not directly observable or when strong disagreement on the fundamentals between the traders occurs, such as in the dot-com bubbles, see Shiller (2005) for instance. Our results show that, in such a setting, coordination on faster than exponential growth may arise.

## 6. Conclusions

There have been many reports of super-exponential behavior in financial markets in a literature inspired by the dynamics of positive feedback leading to finite-time singularities in natural and physical systems (see for instance Johansen and Sornette (2001) and Sornette (2004) and references therein). However, the challenge has been and is still to confirm with more and more statistical evidence that the very noisy financial returns do contain a significant positive feedback component during some bubbles regimes. In the present paper, by analyzing a controlled experiments in the laboratory, we have the luxury of working with a data set with low noise (or equivalently small stochastic component). With this advantage, we have presented the first detailed quantitative calibration of simple models with positive feedback that unambiguously demonstrate the existence of positive feedback mechanisms and coordination on faster than exponential growth in the price formation process of controlled experimental financial markets.

Off course, bubbles are stochastic events, under the same conditions, a bubble may develop or may not over a given time interval. Our results show that in a simple, controlled laboratory environment in the absence of fundamental robot traders, coordination on faster-than-exponentially growing bubbles can arise. Super-exponential bubbles seem particularly relevant to markets where asset prices are only loosely connected to fundamentals. In such an environment, excessive speculation may lead to coordination on asset price bubbles with accelerating growth and a subsequent severe collapse as, for example, the recent experience with Bitcoins in April 2013 has shown once more. An interesting topic for future research is whether super-exponential bubbles occur in high-frequency tick-to-tick data."

## Appendix

Faster than exponential growth means that there is a positive feedback loop, or as Andreassen and Kraus (1990) noted that "[..] subjects were more likely [..] to buy as prices rose [..]". The table down-below illustrates the difference between constant growth and positive feedback. Note that the prices in the two bubbles can be indistinguishable in the early phase of the bubble.

[Table 2 about here.]

[Figure 14 about here.]

## Detailed numerical results

[Figure 15 about here.]

[Figure 16 about here.]

[Figure 17 about here.]

[Figure 18 about here.]

[Figure 19 about here.]

[Figure 20 about here.]

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| Group | Time window | Description | Classification |
|---|---|---|---|
| 1 | NA | erratic price trajectory | — |
| 2 | 7 − 26 | speculative bubble | anchoring on price |
| 3 | 7 − 29 | speculative bubble | anchoring on price |
| 4 | 7 − 21 | speculative bubble | anchoring on price |
| 5 | 29 − 37 | rational bubble | — |
| 6 | 23 − 29 | speculative bubble | (too short for analysis) |

Table 1: Overview of bubbles reproduced from Hommes et al. (2008) with our own classification.

| | $\log(\bar{p}_t/\bar{p}_{t-1}) = a_1$ | | $\log(\bar{p}_t/\bar{p}_{t-1}) = a_2 + b_2\bar{p}_{t-1}$ | |
| $t$ | $\bar{p}_t$ | % | $\bar{p}_t$ | % |
| --- | --- | --- | --- | --- |
| 0 | 60.00 | – | 60.00 | – |
| 1 | 66.00 | 10.00% | 65.79 | 9.66% |
| 2 | 72.60 | 10.00% | 72.19 | 9.72% |
| 3 | 79.86 | 10.00% | 79.26 | 9.79% |
| 4 | 87.85 | 10.00% | 87.08 | 9.87% |
| 5 | 96.63 | 10.00% | 95.74 | 9.95% |
| 6 | 106.29 | 10.00% | 105.36 | 10.05% |
| 7 | 116.92 | 10.00% | 116.06 | 10.15% |
| 8 | 128.62 | 10.00% | 127.99 | 10.27% |
| 9 | 141.48 | 10.00% | 141.30 | 10.40% |
| 10 | 155.62 | 10.00% | 156.21 | 10.55% |
| 11 | 171.19 | 10.00% | 172.95 | 10.72% |
| 12 | 188.31 | 10.00% | 191.80 | 10.90% |
| 13 | 207.14 | 10.00% | 213.11 | 11.11% |
| 14 | 227.85 | 10.00% | 237.30 | 11.35% |
| 15 | 250.63 | 10.00% | 264.87 | 11.62% |
| 16 | 275.70 | 10.00% | 296.45 | 11.93% |
| 17 | 303.27 | 10.00% | 332.86 | 12.28% |
| 18 | 333.60 | 10.00% | 375.09 | 12.69% |
| 19 | 366.95 | 10.00% | 424.48 | 13.17% |
| 20 | 403.65 | 10.00% | 482.74 | 13.73% |
| 21 | 444.01 | 10.00% | 552.22 | 14.39% |
| 22 | 488.42 | 10.00% | 636.09 | 15.19% |
| 23 | 537.26 | 10.00% | 738.87 | 16.16% |

Table 2: Table illustrating the difference between exponential growth ($a_1 = \log(1.1) \approx 0.095$, second column) and positive feedback by price on future returns ($a_2 = \log(1.09) \approx 0.086$, $b_2 = 0.0001$, fourth column). We let the bubbles start at $\bar{p}_t = 60 = 120 - 60 = p_t - p^f$. With the parameter above, the excess price $\bar{p}_t$ grows initially at around 10% at each time step. In the early phase, the prices grow approximately exponentially (the exponential growth is actually slightly faster). At time step $t = 10$, the bubble with positive feedback of the price on future returns overtakes the exponential growth benchmark and the growth rate start to accelerate.
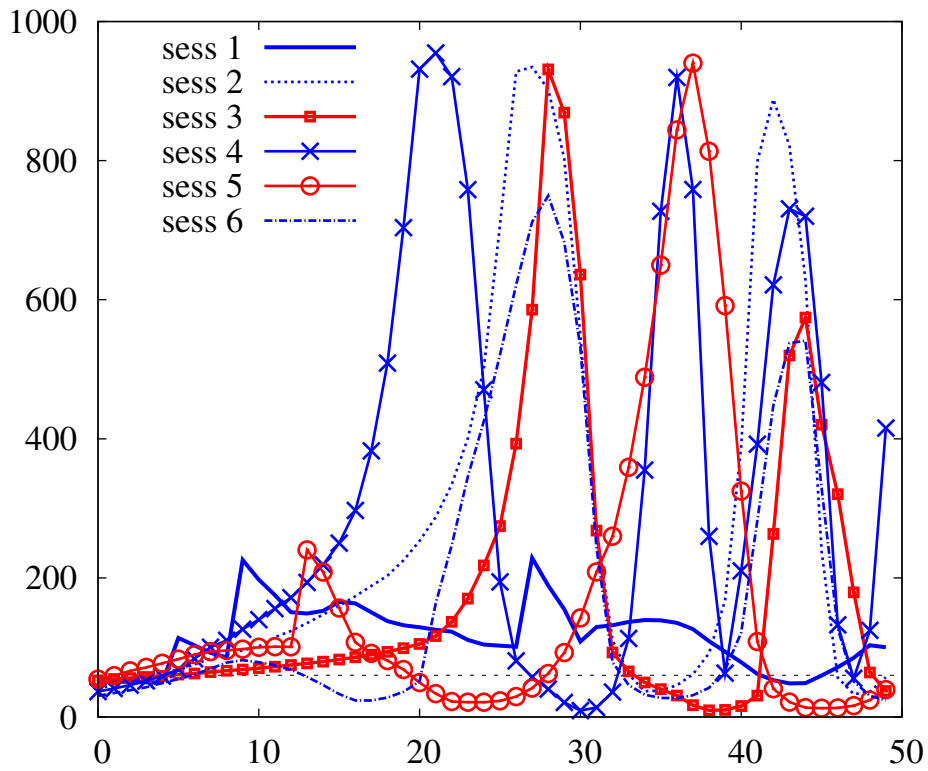
Figure 1: Prices in the learning-to-forecast market experiments Hommes et al. (2008). 5 out of 6 markets exhibit long lasting bubbles with asset prices increasing to more than 15 times fundamental value.
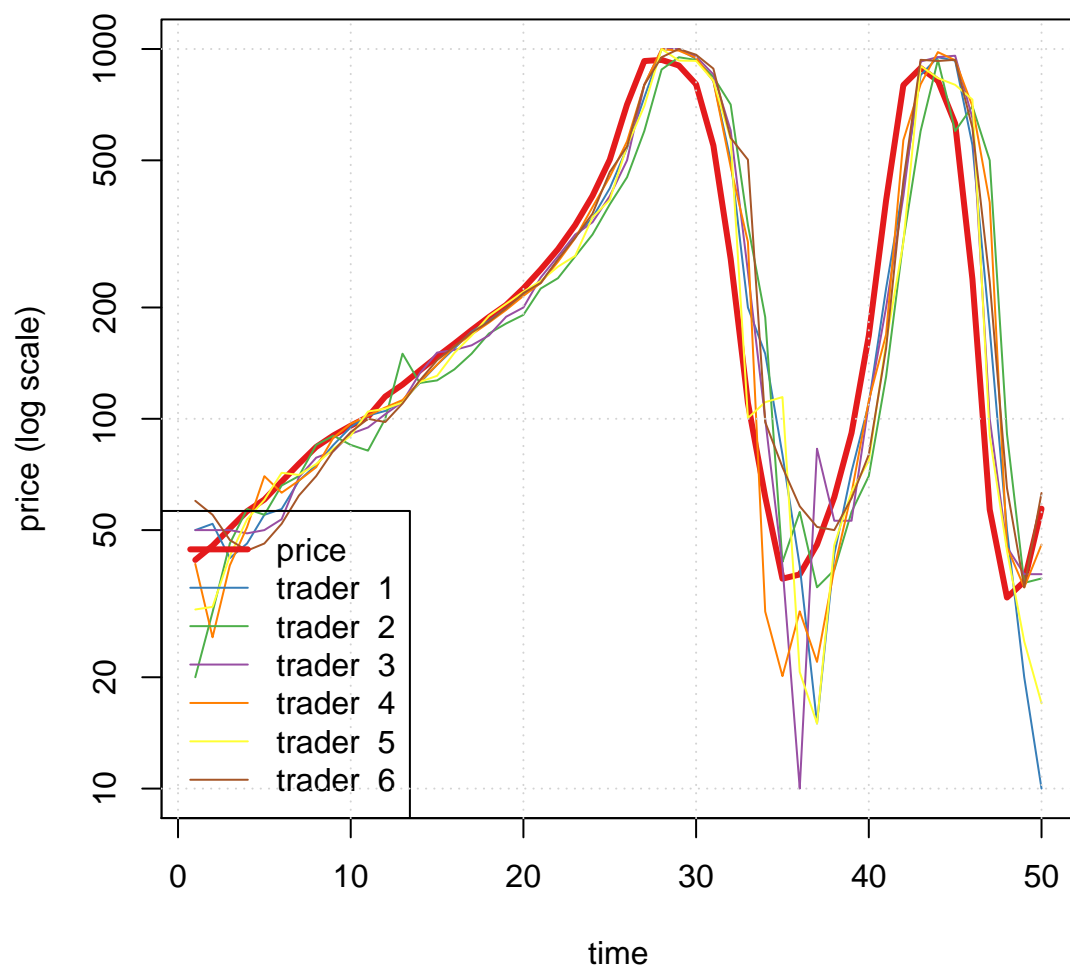
Figure 2: Price and traders' estimate over time for group 2. Note that traders' estimates $p^h$ are for time $t + 1$ and are used to form the price $p_t$ at time $t$, i.e. $p_t = 1/H(\sum_h p_{t+1}^h + D)/(1 + r)$.
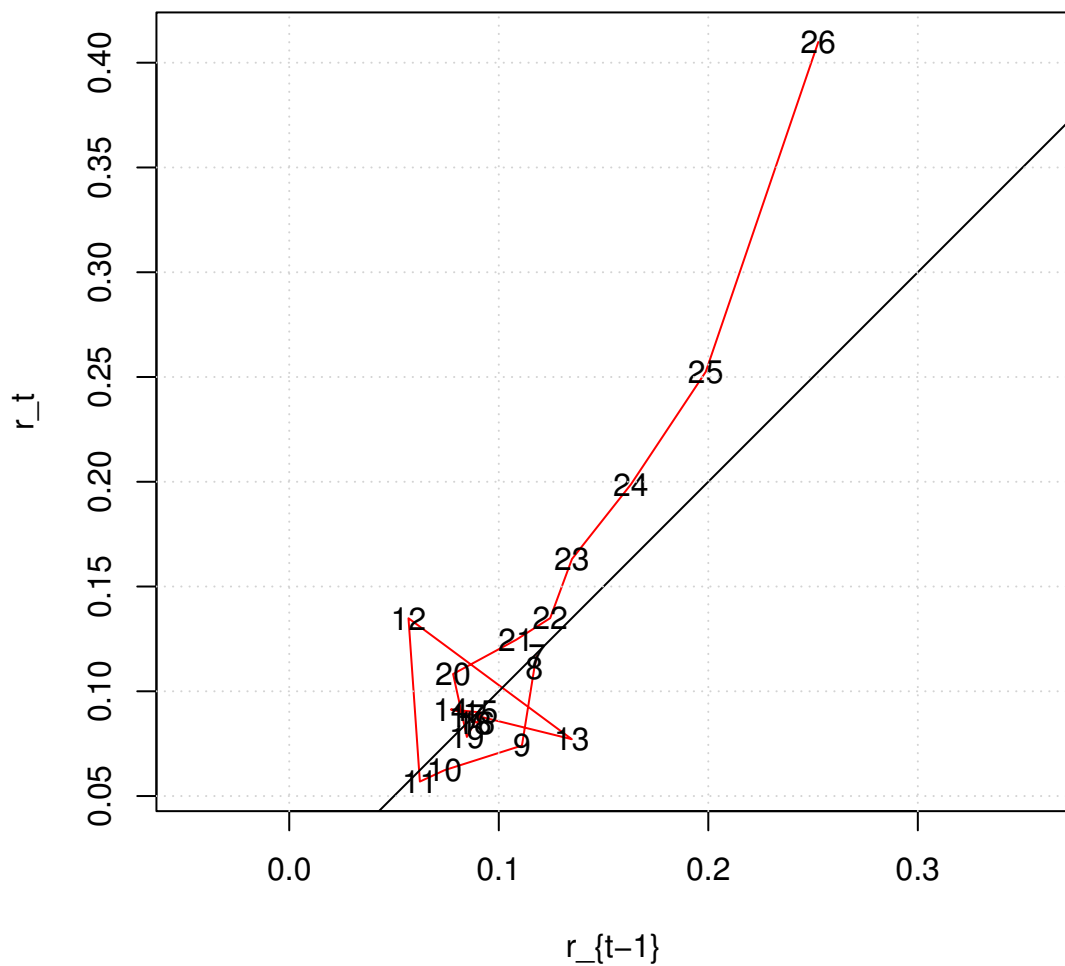
Figure 3: Next period returns $r_{t+1}$ versus current returns $r_t$ for group 2. Points on the diagonal correspond to constant growth ($r_{t+1} = r_t$), points above the diagonal ($r_{t+1} > r_t$) correspond to accelerating growth. Note that returns are defined as discrete returns, i.e. $r_{t+1} := (p_{t+1}/p_t) - 1$.
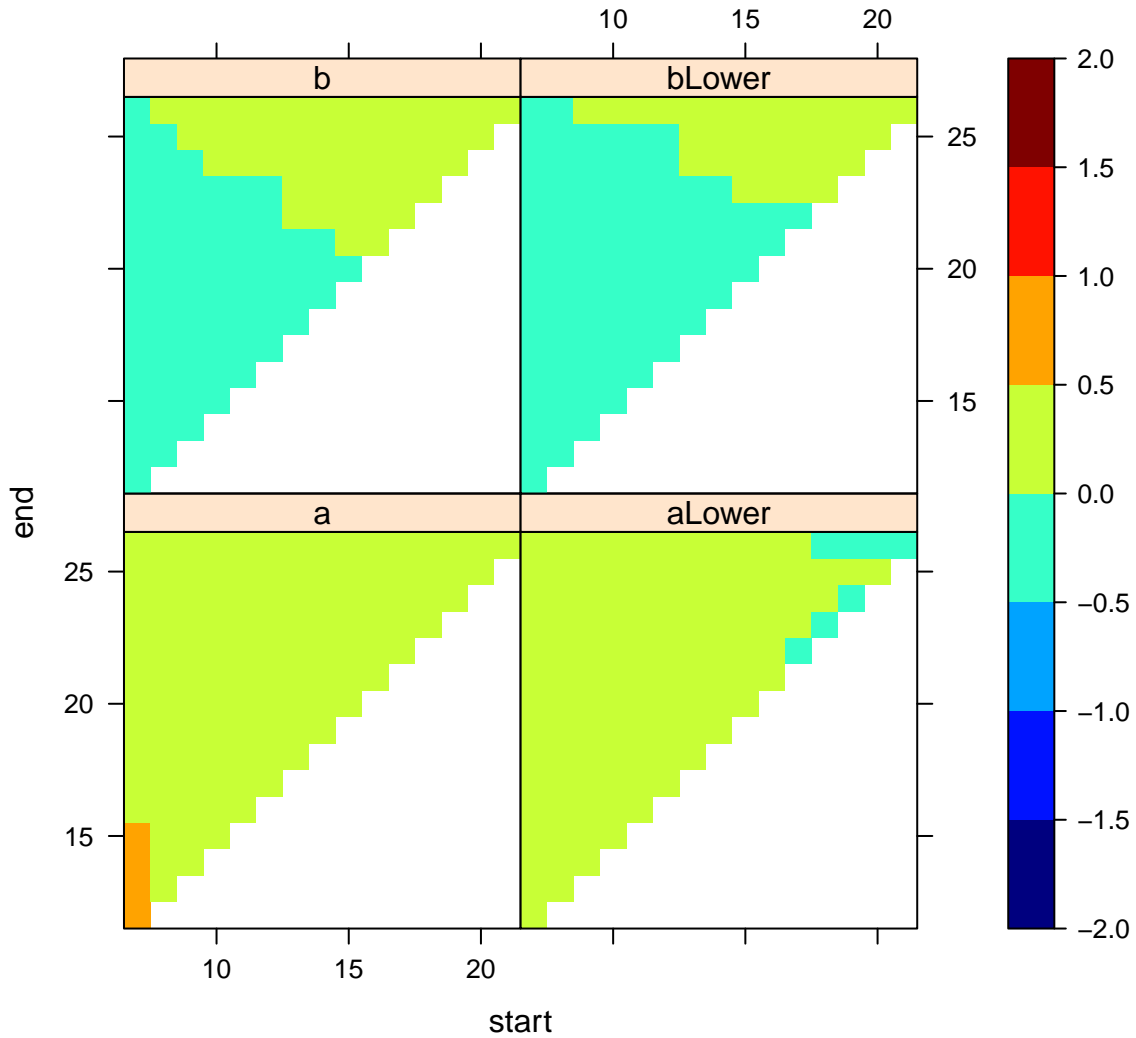
Figure 4: Parameter estimate of Equation 3 over the time interval [start, end] for group 2. The $x$-axis corresponds to the start point and the $y$-axis to the end point of the analyzed time window. The bar on the right gives the values of the parameters in color code, according to the indicated scale. aLower and bLower correspond to the lower 95% confidence level of $a_2$ and $b_2$ respectively of Equation 3. Note that $b_2$ is around 0 for small starting and end values implying exponential growth in the initial phase of the bubble. We observe a rather large domain in the parameter range describing the start time and end time of the window of calibration for which the parameter $b_2$ is positive at the 95% confidence level. The detailed numeric results can be found in the appendix in Figure 15.

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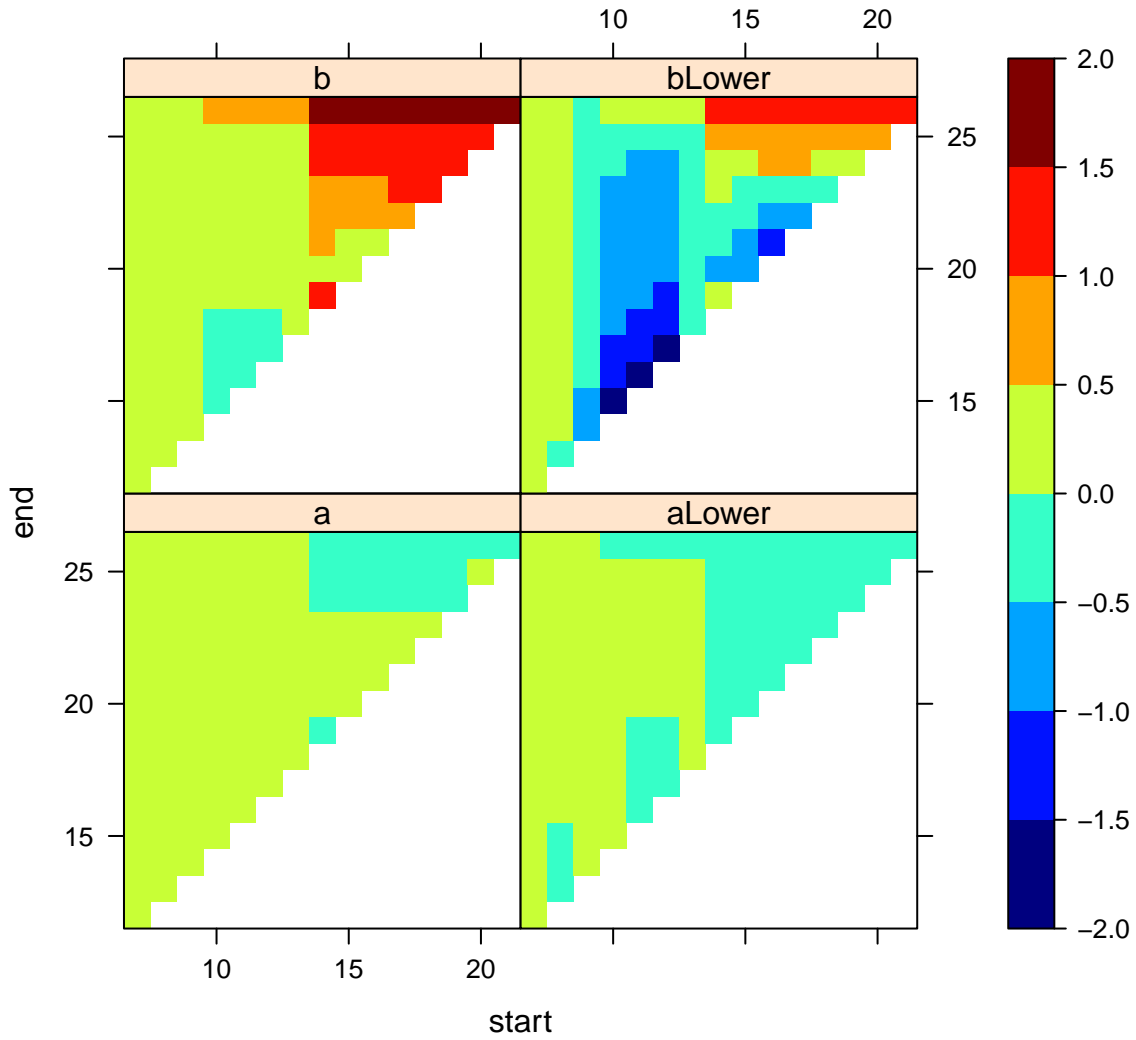
Figure 5: Parameter estimate of Equation 4 over the time interval [time, start] for group 2. The $x$-axis corresponds to the start point and the $y$-axis to the end point of the analyzed time window. aLower and bLower correspond to the lower 95% confidence level for $a_3$ and $b_3$ respectively of Equation 4. Note that the domain where $a_3$ and $b_3$ are both significantly larger than zero is restricted to the earliest two starting points. The detailed numeric results can be found in the appendix in Figure 16.
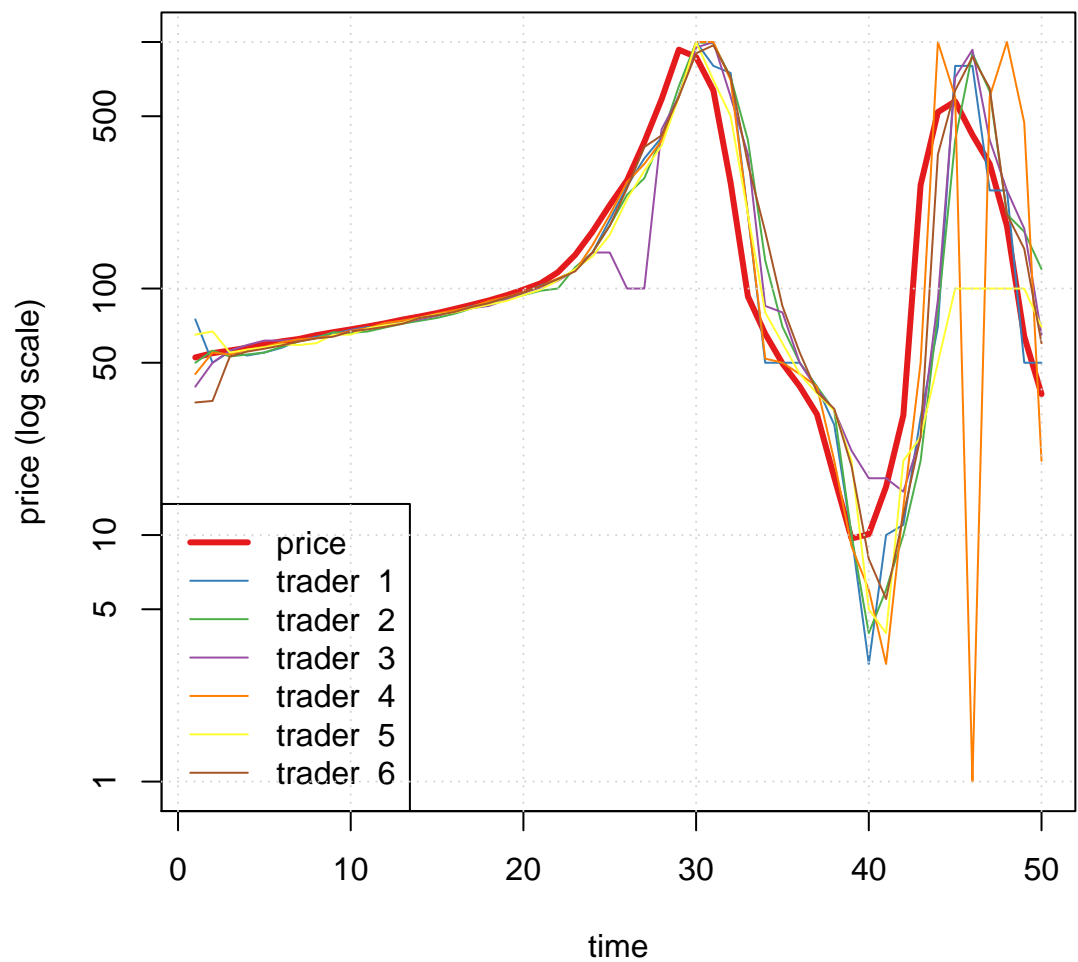
Figure 6: Price and traders' estimate over time for group 3. Same representation as Figure 2.
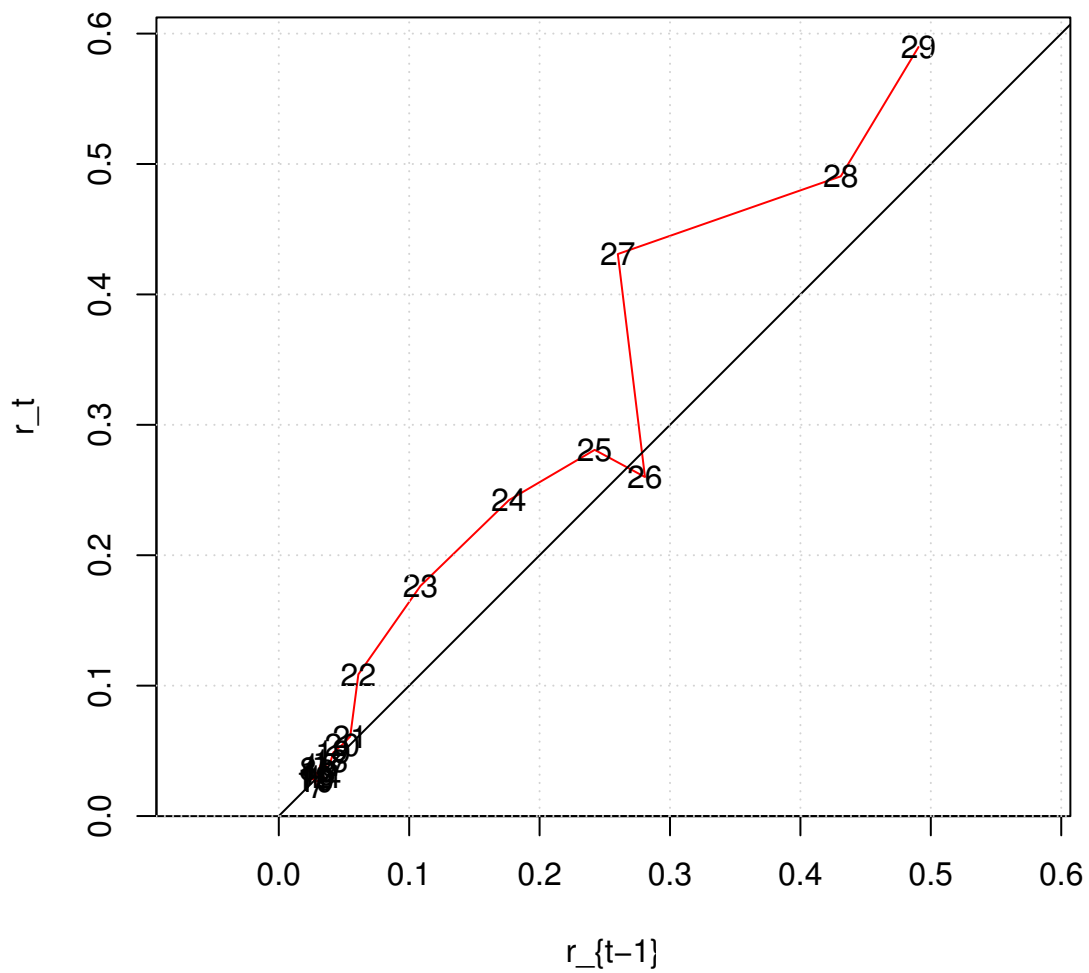
Figure 7: Next period returns $r_{t+1}$ versus current returns $r_t$ for group 3. Same representation as Figure 3.
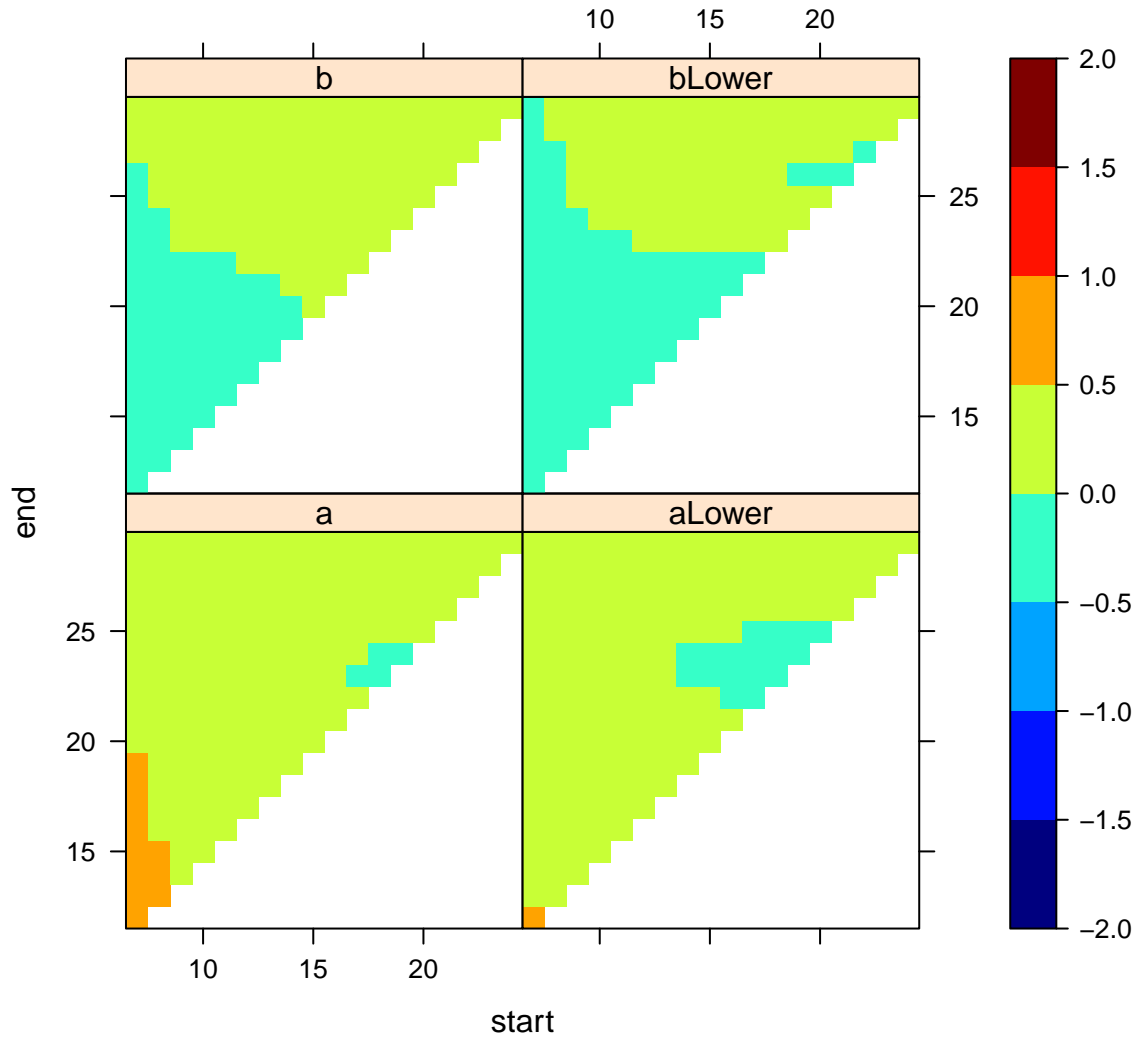
Figure 8: Parameter estimate of Equation 3 over the time interval [start, end] for group 3. Same representation as Figure 4. The detailed numeric results can be found in the appendix in Figure 17.
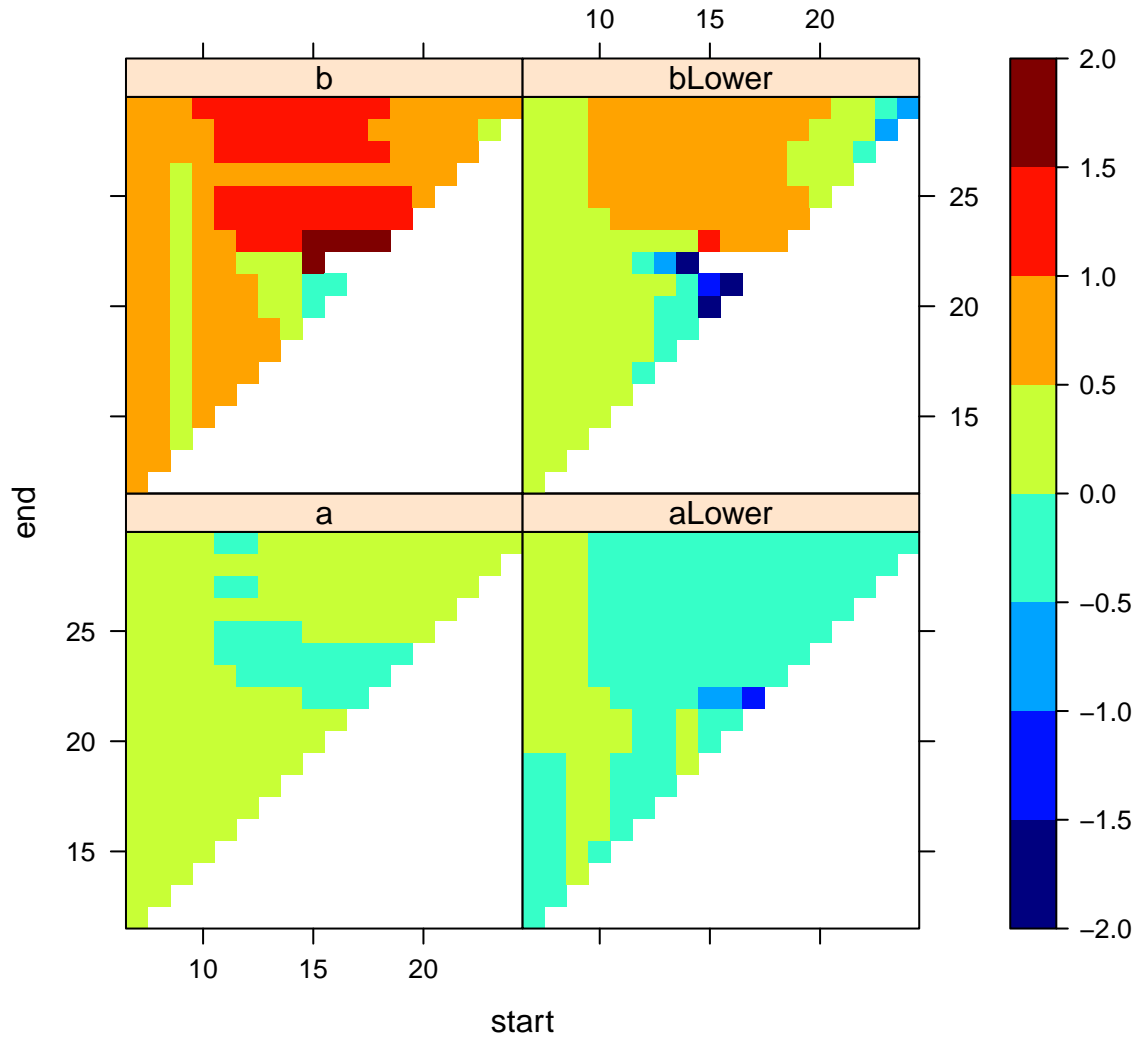
Figure 9: Parameter estimate of Equation 4 over the time interval [time, start] for group 3. Same representation as Figure 5. The detailed numeric results can be found in the appendix in Figure 18.
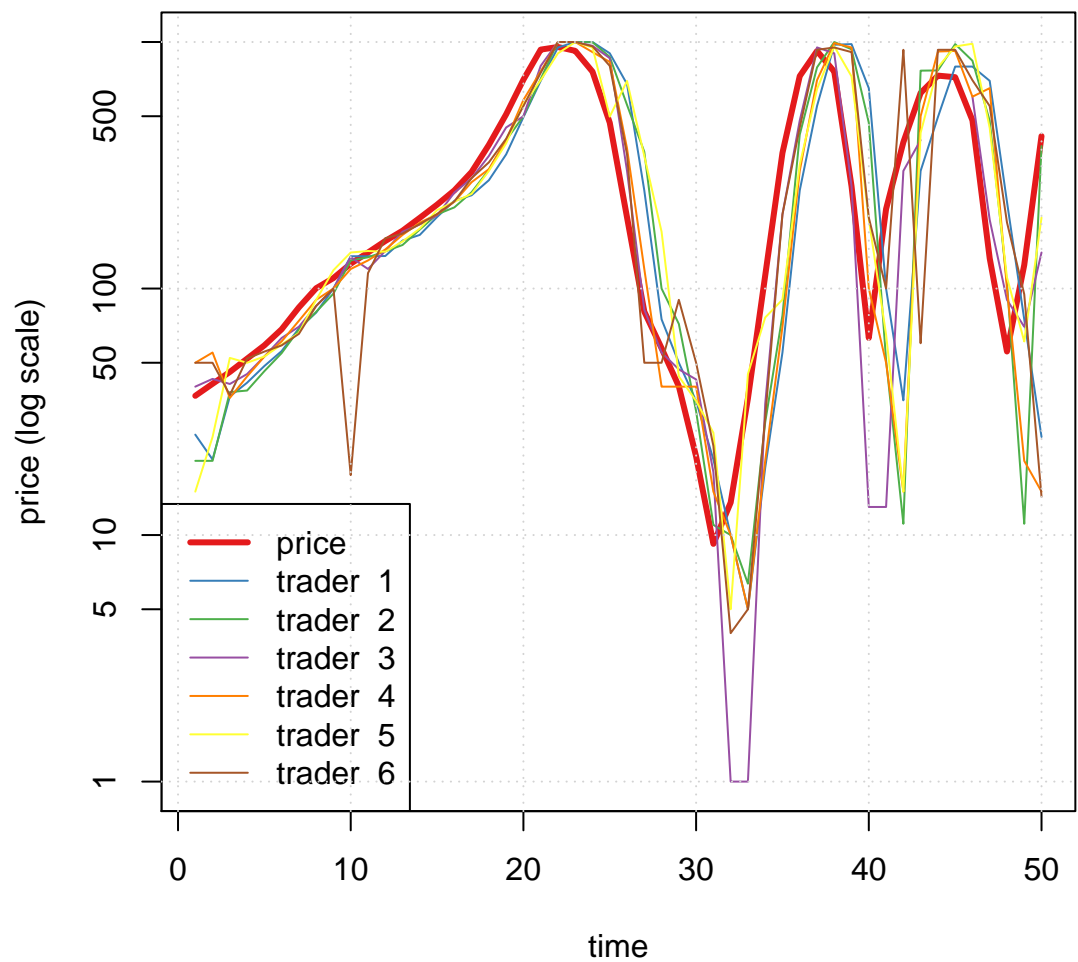
Figure 10: Price and traders' estimate over time for group 4. Same representation as Figure 2.
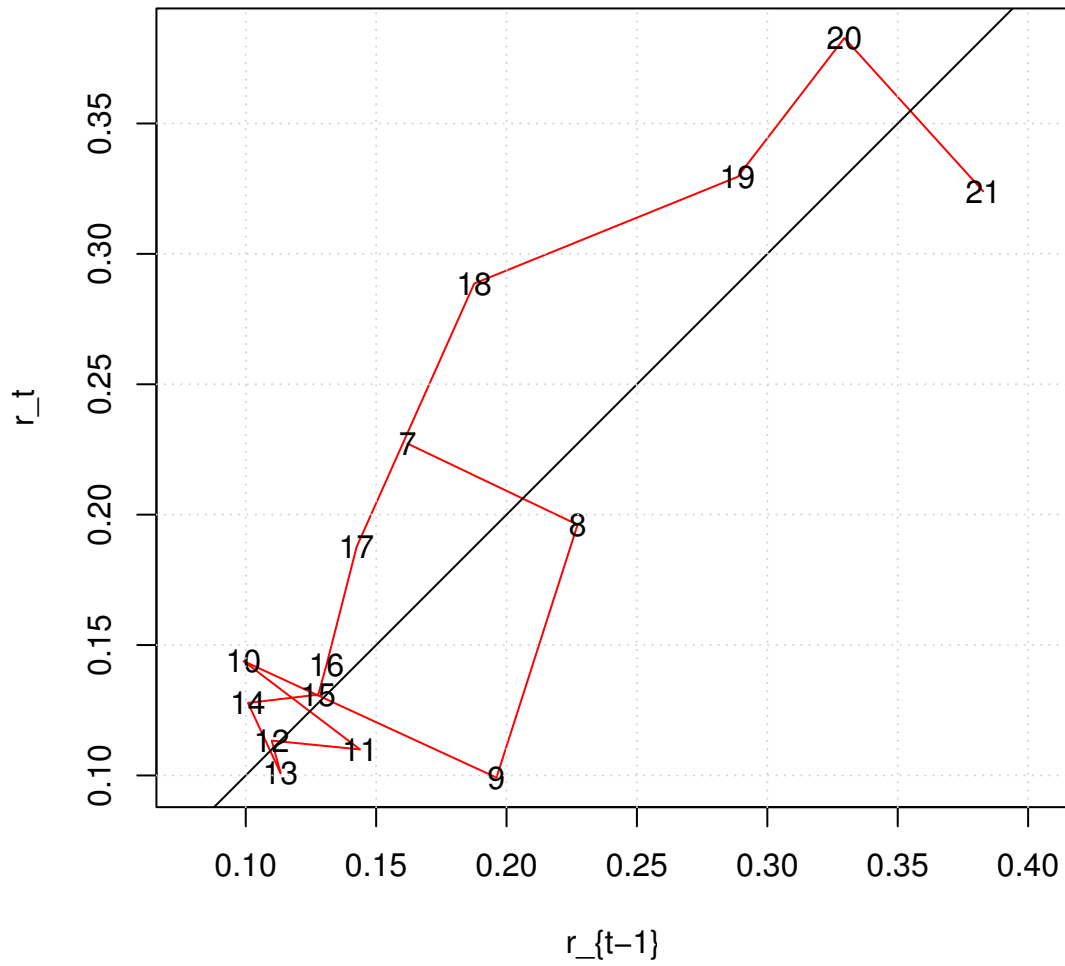
Figure 11: Next period returns $r_{t+1}$ versus current returns $r_t$ for group 4. Same representation as Figure 3.
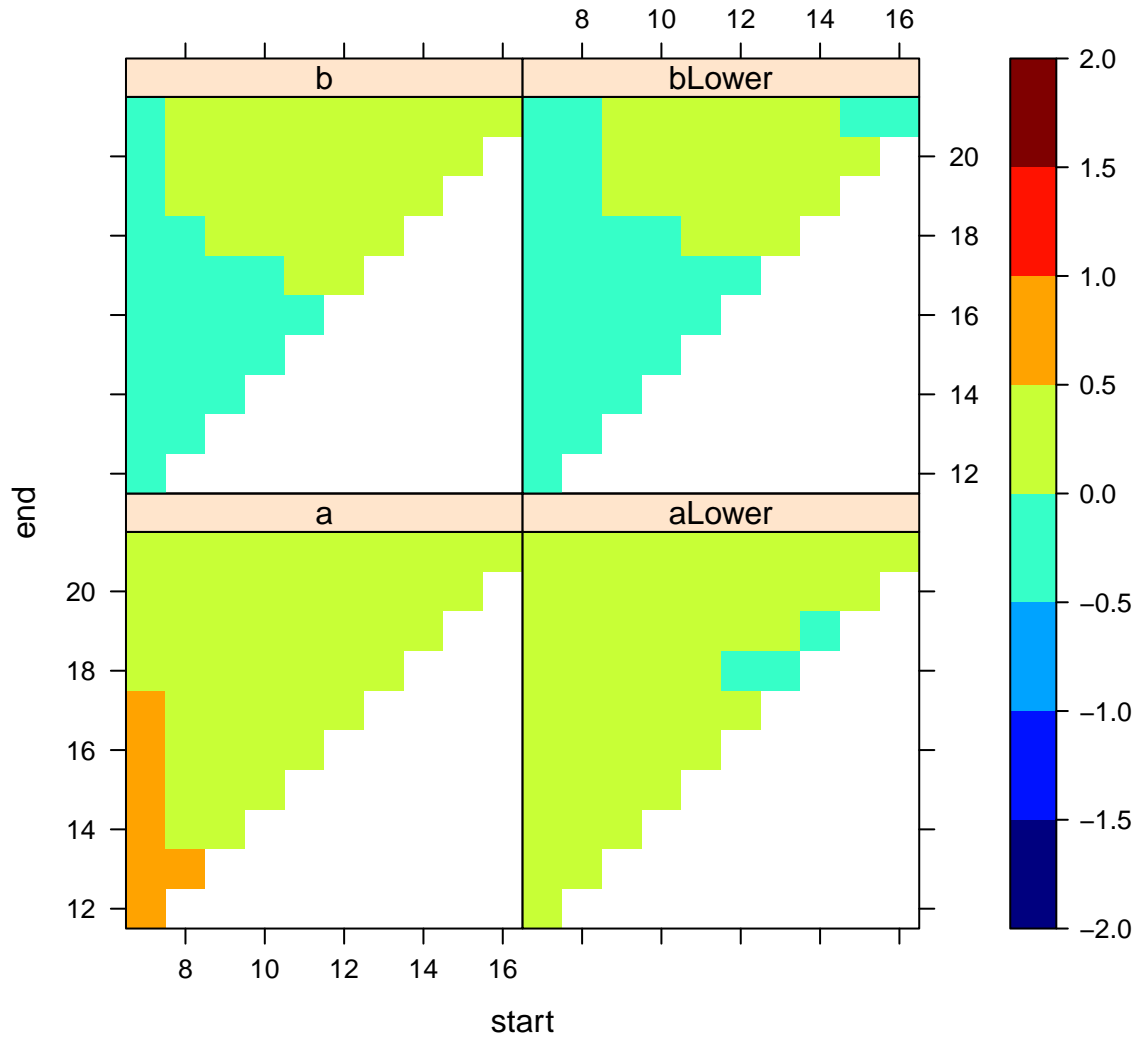
Figure 12: Parameter estimate of Equation 3 over the time interval [start, end] for group 4. Same representation as Figure 4. The detailed numeric results can be found in the appendix in Figure 19.
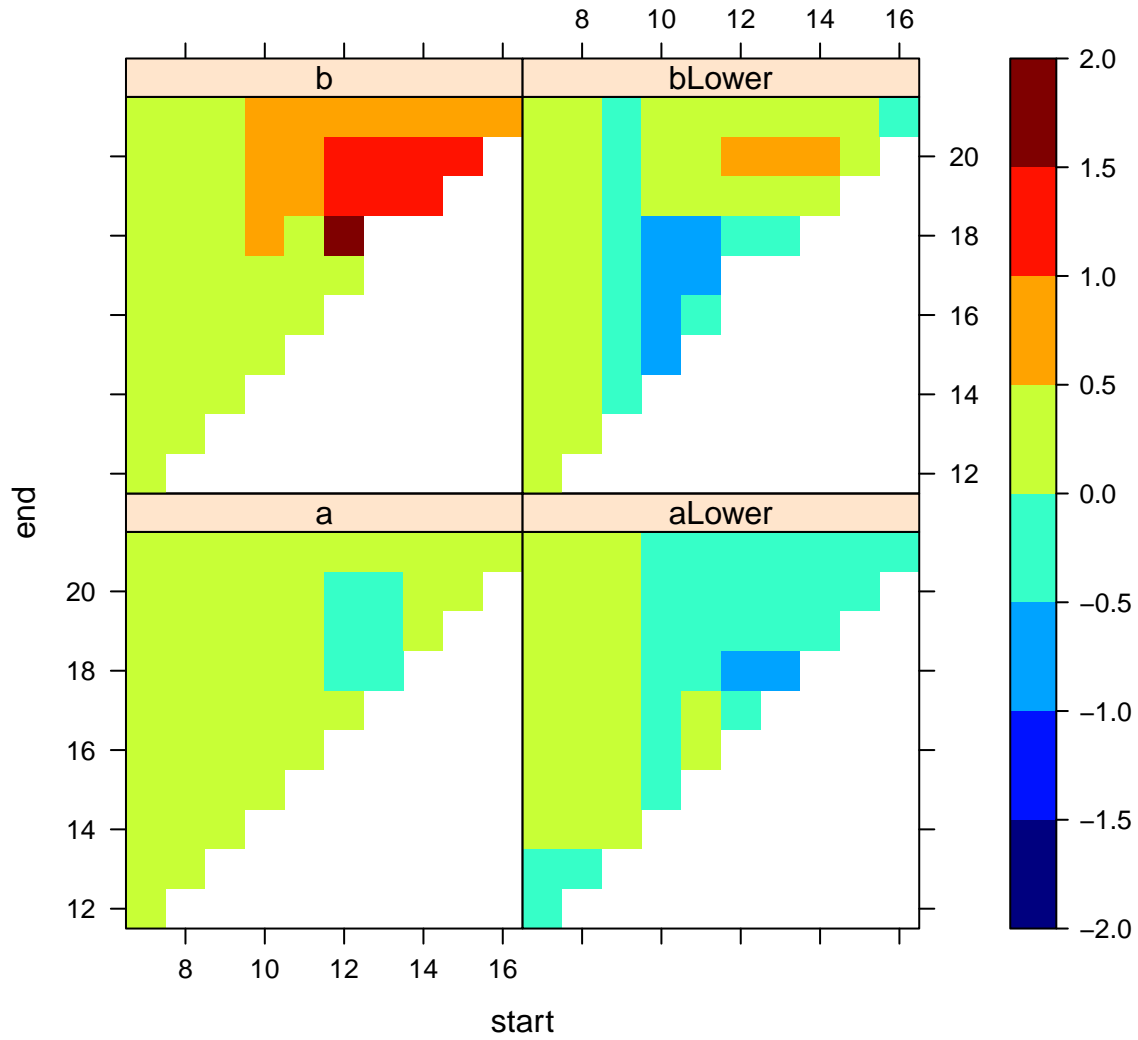
Figure 13: Parameter estimate of Equation 4 over the time interval [time, start] for group 4. Same representation as Figure 5. The detailed numeric results can be found in the appendix in Figure 20.
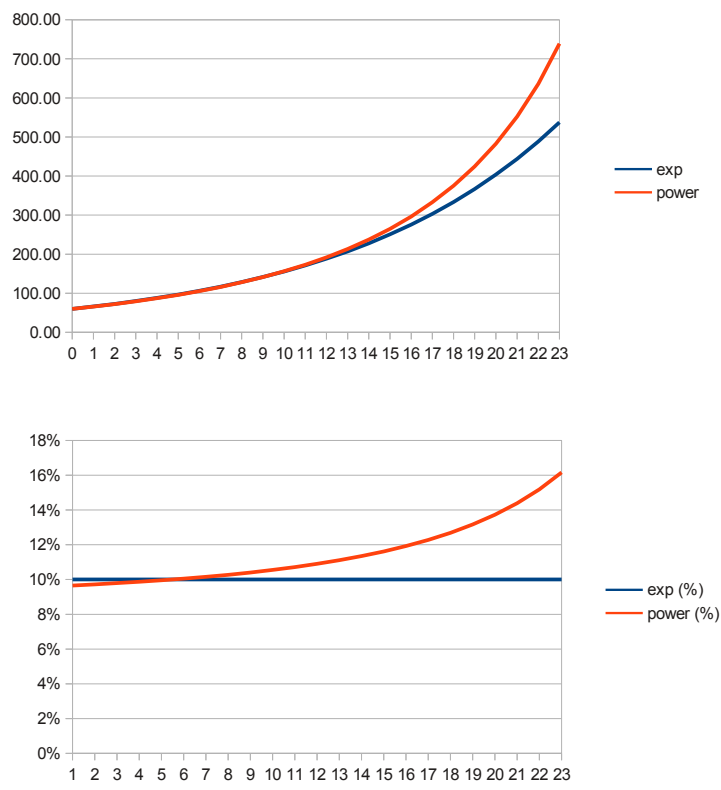
Figure 14: Graphical representation of Table 2. Top panel: prices. Bottom panel: returns.

```
26 ┤ -3.7e-05   2.2e-04   3.6e-04   4.1e-04   4.3e-04   4.4e-04   5.6e-04   6.0e-04   6.5e-04   6.9e-04   7.3e-04   7.6e-04   7.9e-04   7.7e-04   8.1e-04
     (±6.1e-04)(±3.7e-04)(±2.5e-04)(±2.5e-04)(±2.6e-04)(±2.8e-04)(±1.7e-04)(±1.7e-04)(±1.5e-04)(±1.3e-04)(±1.3e-04)(±1.2e-04)(±1.4e-04)(±1.7e-04)(±2.1e-04)
      2.3e-01   1.7e-01   1.3e-01   1.2e-01   1.1e-01   1.1e-01   7.9e-02   7.0e-02   5.6e-02   4.3e-02   3.2e-02   2.1e-02   1.4e-02   1.8e-02   6.7e-03
     (±1.0e-01)(±6.5e-02)(±4.5e-02)(±4.6e-02)(±5.0e-02)(±5.6e-02)(±3.4e-02)(±3.6e-02)(±3.3e-02)(±3.1e-02)(±3.1e-02)(±3.1e-02)(±3.6e-02)(±4.7e-02)(±6.1e-02)

     -4.0e-04  -5.4e-05   1.4e-04   2.0e-04   2.3e-04   2.2e-04   3.9e-04   4.3e-04   4.9e-04   5.5e-04   5.9e-04   6.3e-04   6.5e-04   5.9e-04
     (±7.6e-04)(±4.5e-04)(±2.9e-04)(±2.8e-04)(±3.1e-04)(±3.4e-04)(±1.7e-04)(±1.8e-04)(±1.6e-04)(±1.3e-04)(±1.2e-04)(±1.0e-04)(±1.3e-04)(±7.8e-05)
      2.6e-01   1.9e-01   1.5e-01   1.4e-01   1.4e-01   1.4e-01   1.0e-01   9.3e-02   7.9e-02   6.5e-02   5.5e-02   4.4e-02   4.0e-02   5.5e-02
     (±1.1e-01)(±6.6e-02)(±4.4e-02)(±4.5e-02)(±5.0e-02)(±5.7e-02)(±3.0e-02)(±3.2e-02)(±2.9e-02)(±2.5e-02)(±2.4e-02)(±2.2e-02)(±2.8e-02)(±1.8e-02)

24 ┤ -7.3e-04  -2.8e-04  -2.0e-05   5.7e-05   7.6e-05   5.6e-05   2.9e-04   3.3e-04   4.2e-04   5.0e-04   5.6e-04   6.3e-04   6.6e-04
     (±9.3e-04)(±5.5e-04)(±3.5e-04)(±3.5e-04)(±3.9e-04)(±4.4e-04)(±2.2e-04)(±2.4e-04)(±2.1e-04)(±1.9e-04)(±1.9e-04)(±1.7e-04)(±2.3e-04)
      2.8e-01   2.1e-01   1.7e-01   1.6e-01   1.5e-01   1.6e-01   1.1e-01   1.0e-01   8.8e-02   7.2e-02   5.9e-02   4.5e-02   3.9e-02
     (±1.2e-01)(±7.0e-02)(±4.6e-02)(±4.8e-02)(±5.5e-02)(±6.3e-02)(±3.3e-02)(±3.7e-02)(±3.5e-02)(±3.2e-02)(±3.3e-02)(±3.3e-02)(±4.5e-02)

     -1.1e-03  -5.2e-04  -1.9e-04  -1.1e-04  -9.5e-05  -1.4e-04   1.7e-04   2.2e-04   3.3e-04   4.4e-04   5.3e-04   6.3e-04
     (±1.1e-03)(±6.6e-04)(±4.1e-04)(±4.3e-04)(±4.8e-04)(±5.5e-04)(±2.7e-04)(±3.1e-04)(±2.9e-04)(±2.8e-04)(±3.0e-04)(±3.1e-04)
      3.1e-01   2.3e-01   1.8e-01   1.7e-01   1.7e-01   1.7e-01   1.3e-01   1.2e-01   9.9e-02   7.9e-02   6.4e-02   4.4e-02
     (±1.2e-01)(±7.3e-02)(±4.8e-02)(±5.1e-02)(±6.0e-02)(±7.0e-02)(±3.6e-02)(±4.3e-02)(±4.3e-02)(±4.2e-02)(±4.8e-02)(±5.2e-02)

22 ┤ -1.5e-03  -7.8e-04  -3.8e-04  -2.8e-04  -2.9e-04  -3.8e-04   2.6e-05   6.5e-05   2.0e-04   3.6e-04   4.7e-04
     (±1.3e-03)(±7.8e-04)(±4.9e-04)(±5.2e-04)(±5.9e-04)(±6.7e-04)(±3.2e-04)(±3.9e-04)(±4.0e-04)(±4.2e-04)(±5.2e-04)
      3.3e-01   2.5e-01   2.0e-01   1.8e-01   1.8e-01   2.0e-01   1.4e-01   1.3e-01   1.1e-01   9.0e-02   7.1e-02
     (±1.3e-01)(±7.8e-02)(±5.0e-02)(±5.5e-02)(±6.6e-02)(±7.8e-02)(±3.9e-02)(±4.9e-02)(±5.3e-02)(±5.7e-02)(±7.5e-02)

     -1.9e-03  -1.1e-03  -5.7e-04  -4.6e-04  -4.9e-04  -6.5e-04  -1.2e-04  -9.8e-05   7.4e-05   2.8e-04
     (±1.6e-03)(±9.2e-04)(±5.8e-04)(±6.4e-04)(±7.4e-04)(±8.4e-04)(±4.0e-04)(±5.1e-04)(±5.9e-04)(±7.0e-04)
      3.6e-01   2.6e-01   2.1e-01   1.9e-01   2.0e-01   2.2e-01   1.5e-01   1.5e-01   1.3e-01   9.8e-02
     (±1.4e-01)(±8.3e-02)(±5.4e-02)(±6.1e-02)(±7.4e-02)(±8.7e-02)(±4.4e-02)(±5.9e-02)(±7.1e-02)(±8.8e-02)

20 ┤ -2.4e-03  -1.4e-03  -7.9e-04  -6.8e-04  -7.5e-04  -1.0e-03  -3.7e-04  -4.0e-04  -2.4e-04
     (±1.8e-03)(±1.1e-03)(±6.8e-04)(±7.7e-04)(±9.0e-04)(±9.9e-04)(±4.1e-04)(±5.6e-04)(±7.3e-04)
      3.8e-01   2.8e-01   2.2e-01   2.1e-01   2.2e-01   2.5e-01   1.7e-01   1.8e-01   1.6e-01
     (±1.4e-01)(±8.8e-02)(±5.7e-02)(±6.7e-02)(±8.3e-02)(±9.5e-02)(±4.1e-02)(±5.8e-02)(±8.1e-02)

     -2.9e-03  -1.7e-03  -1.0e-03  -8.9e-04  -1.0e-03  -1.4e-03  -6.6e-04  -7.9e-04
     (±2.2e-03)(±1.3e-03)(±8.1e-04)(±9.4e-04)(±1.1e-03)(±1.2e-03)(±2.7e-04)(±2.2e-04)
      4.0e-01   2.9e-01   2.3e-01   2.2e-01   2.3e-01   2.8e-01   2.0e-01   2.1e-01
     (±1.5e-01)(±9.4e-02)(±6.2e-02)(±7.5e-02)(±9.4e-02)(±1.0e-01)(±2.5e-02)(±2.1e-02)

18 ┤ -3.4e-03  -1.9e-03  -1.1e-03  -9.1e-04  -1.1e-03  -1.7e-03  -6.1e-04
     (±2.6e-03)(±1.6e-03)(±1.1e-03)(±1.3e-03)(±1.6e-03)(±1.6e-03)(±4.2e-04)
      4.2e-01   3.0e-01   2.3e-01   2.2e-01   2.4e-01   2.9e-01   1.9e-01
     (±1.7e-01)(±1.1e-01)(±7.3e-02)(±9.1e-02)(±1.2e-01)(±1.3e-01)(±3.5e-02)

     -4.1e-03  -2.2e-03  -1.1e-03  -9.5e-04  -1.2e-03  -2.1e-03
     (±3.3e-03)(±2.1e-03)(±1.4e-03)(±1.8e-03)(±2.3e-03)(±2.4e-03)
      4.5e-01   3.2e-01   2.4e-01   2.2e-01   2.4e-01   3.2e-01
     (±1.8e-01)(±1.2e-01)(±8.9e-02)(±1.2e-01)(±1.6e-01)(±1.8e-01)

16 ┤ -5.0e-03  -2.6e-03  -1.2e-03  -1.0e-03  -1.4e-03
     (±4.1e-03)(±2.7e-03)(±2.0e-03)(±2.6e-03)(±3.5e-03)
      4.7e-01   3.3e-01   2.4e-01   2.3e-01   2.5e-01
     (±2.0e-01)(±1.4e-01)(±1.1e-01)(±1.5e-01)(±2.2e-01)

     -6.1e-03  -3.1e-03  -1.2e-03  -8.8e-04
     (±5.2e-03)(±3.8e-03)(±3.0e-03)(±4.2e-03)
      5.1e-01   3.5e-01   2.4e-01   2.2e-01
     (±2.3e-01)(±1.8e-01)(±1.5e-01)(±2.2e-01)

14 ┤ -7.8e-03  -3.9e-03  -1.2e-03
     (±6.9e-03)(±5.5e-03)(±4.9e-03)
      5.5e-01   3.7e-01   2.4e-01
     (±2.6e-01)(±2.2e-01)(±2.1e-01)

     -1.0e-02  -5.2e-03
     (±9.2e-03)(±8.5e-03)
      6.1e-01   4.1e-01
     (±3.1e-01)(±3.1e-01)

12 ┤ -1.4e-02
     (±1.3e-02)
      6.9e-01
     (±3.8e-01)
     └────────┬─────────┬─────────┬─────────┬─────────┬─────────┬─────────
              8        10        12        14        16        18        20
                                       start
 end
```

Figure 15: Parameter estimate of Equation 3 over the time interval [start, end] for group 2. The $x$-axis corresponds to the start point and the $y$-axis to the end point of the analyzed time window. The four values grouped at each grid point correspond to $b_2$ (top line), the 95% confidence interval of $b_2$, $a_2$ and the 95% confidence interval of $a_2$ (bottom line), respectively.

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2.2e–01 4.5e–01 3.9e–01 7.0e–01 7.9e–01 7.9e–01 8.9e–01 1.7e+00 1.7e+00 1.7e+00 1.7e+00 1.7e+00 1.7e+00 1.6e+00 1.9e+00
(±5.7e–02)(±2.2e–01)(±4.2e–01)(±6.8e–01)(±7.2e–01)(±7.4e–01)(±6.6e–01)(±3.7e–01)(±4.0e–01)(±3.8e–01)(±3.7e–01)(±4.2e–01)(±4.7e–01)(±5.6e–01)(±4.9e–01)
1.5e–01 1.0e–01 1.1e–01 6.1e–02 4.8e–02 5.1e–02 2.5e–02 –9.6e–02 –9.6e–02 –9.4e–02 –8.8e–02 –8.6e–02 –7.8e–02 –6.7e–02 –1.3e–01
(±3.8e–02)(±5.6e–02)(±8.5e–02)(±1.2e–01)(±1.3e–01)(±1.3e–01)(±1.2e–01)(±6.2e–02)(±6.6e–02)(±6.4e–02)(±6.3e–02)(±7.2e–02)(±8.2e–02)(±1.0e–01)(±9.5e–02)

2.2e–01 4.3e–01 2.5e–01 2.7e–01 3.1e–01 3.2e–01 4.2e–01 1.3e+00 1.3e+00 1.4e+00 1.3e+00 1.3e+00 1.3e+00 1.1e+00
(±4.6e–02)(±1.7e–01)(±3.2e–01)(±5.9e–01)(±6.7e–01)(±6.9e–01)(±5.1e–01)(±4.4e–01)(±4.7e–01)(±4.6e–01)(±4.3e–01)(±4.8e–01)(±5.0e–01)(±4.7e–01)
1.4e–01 9.6e–02 1.3e–01 1.2e–01 1.2e–01 1.2e–01 9.2e–02 –3.9e–02 –4.0e–02 –4.3e–02 –3.9e–02 –3.5e–02 –2.3e–02 3.5e–03
(±3.2e–02)(±4.3e–02)(±6.3e–02)(±1.0e–01)(±1.1e–01)(±1.2e–01)(±8.7e–02)(±6.8e–02)(±7.3e–02)(±7.1e–02)(±6.7e–02)(±7.5e–02)(±8.0e–02)(±7.8e–02)

2.2e–01 4.2e–01 2.2e–01 1.2e–01 1.2e–01 1.3e–01 2.2e–01 1.1e+00 1.1e+00 1.2e+00 1.2e+00 1.2e+00 1.2e+00
(±4.5e–02)(±1.6e–01)(±3.0e–01)(±5.8e–01)(±6.7e–01)(±7.0e–01)(±4.2e–01)(±6.3e–01)(±6.7e–01)(±7.0e–01)(±6.8e–01)(±7.7e–01)(±8.3e–01)
1.3e–01 9.2e–02 1.3e–01 1.4e–01 1.4e–01 1.4e–01 1.2e–01 –1.2e–02 –1.2e–02 –2.3e–02 –2.5e–02 –2.0e–02 –6.5e–03
(±3.2e–02)(±4.2e–02)(±5.9e–02)(±9.7e–02)(±1.1e–01)(±1.1e–01)(±7.0e–02)(±9.3e–02)(±9.9e–02)(±1.0e–01)(±9.9e–02)(±1.1e–01)(±1.2e–01)

2.2e–01 4.3e–01 2.1e–01 7.1e–02 4.3e–02 5.5e–02 1.4e–01 8.4e–01 8.1e–01 9.3e–01 1.1e+00 1.0e+00
(±4.6e–02)(±1.6e–01)(±3.0e–01)(±5.8e–01)(±6.7e–01)(±7.2e–01)(±3.4e–01)(±7.9e–01)(±8.5e–01)(±1.0e+00)(±1.1e+00)(±1.3e+00)
1.3e–01 8.9e–02 1.2e–01 1.5e–01 1.5e–01 1.5e–01 1.3e–01 2.6e–02 3.0e–02 1.4e–02 2.9e–04 3.9e–03
(±3.3e–02)(±4.3e–02)(±5.8e–02)(±9.6e–02)(±1.1e–01)(±1.1e–01)(±5.5e–02)(±1.1e–01)(±1.2e–01)(±1.4e–01)(±1.5e–01)(±1.8e–01)

2.2e–01 4.3e–01 2.2e–01 6.9e–02 3.2e–02 4.3e–02 1.2e–01 6.1e–01 5.4e–01 5.7e–01 7.2e–01
(±4.8e–02)(±1.7e–01)(±3.1e–01)(±5.9e–01)(±7.0e–01)(±7.5e–01)(±2.8e–01)(±8.0e–01)(±8.4e–01)(±1.1e+00)(±1.4e+00)
1.3e–01 8.6e–02 1.2e–01 1.4e–01 1.5e–01 1.5e–01 1.2e–01 5.6e–02 6.4e–02 6.0e–02 4.3e–02
(±3.6e–02)(±4.5e–02)(±6.1e–02)(±9.9e–02)(±1.1e–01)(±1.2e–01)(±4.6e–02)(±1.1e–01)(±1.2e–01)(±1.6e–01)(±1.9e–01)

2.2e–01 4.3e–01 2.2e–01 7.0e–02 3.1e–02 4.1e–02 1.2e–01 5.1e–01 3.9e–01 2.6e–01
(±5.1e–02)(±1.8e–01)(±3.3e–01)(±6.3e–01)(±7.5e–01)(±8.1e–01)(±2.8e–01)(±9.1e–01)(±9.6e–01)(±1.5e+00)
1.3e–01 8.5e–02 1.2e–01 1.4e–01 1.5e–01 1.5e–01 1.2e–01 6.8e–02 8.3e–02 9.8e–02
(±3.9e–02)(±4.9e–02)(±6.5e–02)(±1.1e–01)(±1.2e–01)(±1.3e–01)(±4.5e–02)(±1.3e–01)(±1.3e–01)(±2.0e–01)

2.2e–01 4.4e–01 2.2e–01 7.3e–02 3.4e–02 4.3e–02 1.3e–01 4.7e–01 3.1e–01
(±5.3e–02)(±1.9e–01)(±3.5e–01)(±6.8e–01)(±8.1e–01)(±8.9e–01)(±2.9e–01)(±9.7e–01)(±9.6e–01)
1.3e–01 8.3e–02 1.2e–01 1.4e–01 1.5e–01 1.5e–01 1.2e–01 7.2e–02 9.0e–02
(±4.2e–02)(±5.3e–02)(±7.1e–02)(±1.2e–01)(±1.3e–01)(±1.4e–01)(±4.7e–02)(±1.3e–01)(±1.3e–01)

2.2e–01 4.4e–01 2.2e–01 5.3e–02 1.1e–02 1.8e–02 1.6e–01 1.1e+00
(±5.7e–02)(±2.1e–01)(±3.9e–01)(±7.8e–01)(±9.4e–01)(±1.1e+00)(±3.3e–01)(±1.0e+00)
1.3e–01 8.1e–02 1.2e–01 1.5e–01 1.5e–01 1.5e–01 1.1e–01 –1.5e–02
(±4.6e–02)(±5.9e–02)(±8.2e–02)(±1.4e–01)(±1.6e–01)(±1.8e–01)(±5.7e–02)(±1.5e–01)

2.2e–01 4.3e–01 1.9e–01 –5.4e–02 –1.0e–01 –9.6e–02 9.9e–02
(±5.9e–02)(±2.2e–01)(±4.1e–01)(±8.5e–01)(±1.0e+00)(±1.2e+00)(±3.3e–01)
1.4e–01 8.6e–02 1.3e–01 1.7e–01 1.8e–01 1.8e–01 1.3e–01
(±5.0e–02)(±6.6e–02)(±9.1e–02)(±1.5e–01)(±1.8e–01)(±2.0e–01)(±5.9e–02)

2.2e–01 4.2e–01 1.5e–01 –1.5e–01 –2.0e–01 –2.0e–01
(±6.2e–02)(±2.4e–01)(±4.5e–01)(±9.4e–01)(±1.2e+00)(±1.4e+00)
1.4e–01 9.1e–02 1.4e–01 1.9e–01 2.0e–01 2.0e–01
(±5.5e–02)(±7.5e–02)(±1.0e–01)(±1.7e–01)(±2.1e–01)(±2.5e–01)

2.1e–01 4.1e–01 1.2e–01 –2.7e–01 –3.2e–01
(±6.7e–02)(±2.7e–01)(±5.1e–01)(±1.1e+00)(±1.4e+00)
1.5e–01 9.4e–02 1.5e–01 2.2e–01 2.3e–01
(±6.2e–02)(±8.8e–02)(±1.2e–01)(±2.1e–01)(±2.6e–01)

2.1e–01 4.0e–01 7.6e–02 –3.9e–01
(±7.2e–02)(±3.1e–01)(±5.8e–01)(±1.2e+00)
1.5e–01 1.0e–01 1.7e–01 2.5e–01
(±7.0e–02)(±1.1e–01)(±1.4e–01)(±2.4e–01)

2.1e–01 3.9e–01 4.1e–02
(±8.0e–02)(±3.7e–01)(±7.1e–01)
1.6e–01 1.1e–01 1.8e–01
(±8.2e–02)(±1.3e–01)(±1.8e–01)

2.0e–01 3.9e–01
(±9.3e–02)(±4.7e–01)
1.7e–01 1.1e–01
(±1.0e–01)(±1.8e–01)

2.0e–01
(±1.0e–01)
1.8e–01
(±1.2e–01)

end

start

26  24  22  20  18  16  14  12

8  10  12  14  16  18  20

Figure 16: Parameter estimate of Equation 4 over the time interval [start, end] for group 2. The $x$-axis corresponds to the start point and the $y$-axis to the end point of the analyzed time window. The four values grouped at each grid point correspond to $b_3$ (top line), the 95% confidence interval of $b_3$, $a_3$ and the 95% confidence interval of $a_3$ (bottom line), respectively.

5.1e-04  6.4e-04  7.5e-04  7.8e-04  8.0e-04  8.1e-04  8.1e-04  8.1e-04  8.0e-04  7.8e-04  7.6e-04  7.3e-04  6.9e-04  6.4e-04  5.8e-04  4.8e-04  4.1e-04  3.8e-04
(±6.2e-04)(±4.4e-04)(±2.6e-04)(±2.4e-04)(±2.5e-04)(±2.6e-04)(±2.7e-04)(±2.9e-04)(±3.0e-04)(±3.1e-04)(±3.3e-04)(±3.5e-04)(±3.6e-04)(±3.7e-04)(±3.6e-04)(±2.8e-04)(±2.4e-04)(±3.1e-04)
2.5e-01  2.2e-01  1.9e-01  1.8e-01  1.7e-01  1.7e-01  1.7e-01  1.7e-01  1.7e-01  1.8e-01  1.8e-01  1.9e-01  2.1e-01  2.2e-01  2.5e-01  2.8e-01  3.1e-01  3.1e-01
(±9.1e-02)(±6.5e-02)(±4.0e-02)(±3.8e-02)(±4.0e-02)(±4.2e-02)(±4.6e-02)(±5.0e-02)(±5.4e-02)(±5.8e-02)(±6.4e-02)(±7.0e-02)(±7.5e-02)(±8.1e-02)(±8.3e-02)(±6.8e-02)(±6.3e-02)(±8.7e-02)

5.6e-04  8.1e-04  1.0e-03  1.1e-03  1.1e-03  1.1e-03  1.1e-03  1.2e-03  1.1e-03  1.1e-03  1.1e-03  1.1e-03  1.0e-03  9.2e-04  8.1e-04  6.4e-04  5.0e-04
(±1.0e-03)(±7.1e-04)(±4.0e-04)(±3.5e-04)(±3.5e-04)(±3.6e-04)(±3.8e-04)(±4.1e-04)(±4.3e-04)(±4.6e-04)(±5.0e-04)(±5.3e-04)(±5.7e-04)(±6.0e-04)(±6.1e-04)(±5.1e-04)(±5.0e-04)
2.5e-01  2.1e-01  1.7e-01  1.6e-01  1.6e-01  1.5e-01  1.5e-01  1.5e-01  1.5e-01  1.5e-01  1.5e-01  1.6e-01  1.7e-01  1.8e-01  2.0e-01  2.2e-01  2.9e-01
(±9.9e-02)(±7.1e-02)(±4.1e-02)(±3.7e-02)(±3.8e-02)(±4.1e-02)(±4.4e-02)(±4.8e-02)(±5.3e-02)(±5.8e-02)(±6.6e-02)(±7.4e-02)(±8.2e-02)(±9.1e-02)(±9.9e-02)(±8.7e-02)(±9.1e-02)

4.7e-04  9.0e-04  1.3e-03  1.4e-03  1.5e-03  1.5e-03  1.5e-03  1.6e-03  1.6e-03  1.6e-03  1.5e-03  1.5e-03  1.4e-03  1.3e-03  1.1e-03  8.0e-04
(±1.6e-03)(±1.1e-03)(±6.0e-04)(±5.2e-04)(±5.0e-04)(±5.0e-04)(±5.2e-04)(±5.4e-04)(±5.7e-04)(±6.2e-04)(±6.7e-04)(±7.4e-04)(±8.2e-04)(±9.1e-04)(±1.0e-03)(±1.1e-03)(±1.0e-03)
2.5e-01  2.0e-01  1.6e-01  1.5e-01  1.4e-01  1.3e-01  1.3e-01  1.2e-01  1.3e-01  1.3e-01  1.3e-01  1.4e-01  1.5e-01  1.7e-01  1.9e-01  2.5e-01
(±1.1e-01)(±7.9e-02)(±4.4e-02)(±3.8e-02)(±3.9e-02)(±4.1e-02)(±4.4e-02)(±4.8e-02)(±5.4e-02)(±6.1e-02)(±7.0e-02)(±8.1e-02)(±9.5e-02)(±1.1e-01)(±1.3e-01)(±1.3e-01)

−4.3e-05  6.5e-04  1.2e-03  1.4e-03  1.6e-03  1.6e-03  1.7e-03  1.8e-03  1.8e-03  1.7e-03  1.7e-03  1.7e-03  1.5e-03  1.4e-03  1.1e-03
(±2.3e-03)(±1.6e-03)(±9.0e-04)(±7.7e-04)(±7.6e-04)(±7.8e-04)(±8.2e-04)(±8.7e-04)(±9.5e-04)(±1.1e-03)(±1.2e-03)(±1.3e-03)(±1.5e-03)(±1.8e-03)(±2.1e-03)
2.7e-01  2.1e-01  1.7e-01  1.5e-01  1.4e-01  1.3e-01  1.2e-01  1.2e-01  1.2e-01  1.2e-01  1.2e-01  1.3e-01  1.4e-01  1.6e-01  2.0e-01
(±1.2e-01)(±8.7e-02)(±4.9e-02)(±4.4e-02)(±4.4e-02)(±4.7e-02)(±5.2e-02)(±5.6e-02)(±6.4e-02)(±7.3e-02)(±8.6e-02)(±1.0e-01)(±1.2e-01)(±1.5e-01)(±1.9e-01)

−4.7e-04  7.3e-04  1.7e-03  2.1e-03  2.4e-03  2.6e-03  2.8e-03  3.0e-03  3.1e-03  3.1e-03  3.2e-03  3.2e-03  3.2e-03  3.0e-03
(±3.4e-03)(±2.5e-03)(±1.3e-03)(±1.1e-03)(±9.9e-04)(±9.6e-04)(±9.4e-04)(±9.2e-04)(±1.0e-03)(±1.1e-03)(±1.3e-03)(±1.5e-03)(±1.9e-03)(±2.4e-03)
2.8e-01  2.1e-01  1.3e-01  1.1e-01  1.1e-01  1.0e-01  8.8e-02  7.5e-02  7.0e-02  6.6e-02  5.9e-02  5.8e-02  6.4e-02  7.5e-02
(±1.4e-01)(±1.0e-01)(±5.5e-02)(±4.6e-02)(±4.4e-02)(±4.4e-02)(±4.5e-02)(±4.5e-02)(±5.1e-02)(±6.0e-02)(±7.2e-02)(±9.0e-02)(±1.2e-01)(±1.6e-01)

−2.1e-03  −1.9e-04  1.5e-03  2.2e-03  2.7e-03  3.1e-03  3.5e-03  3.9e-03  4.2e-03  4.4e-03  4.8e-03  5.1e-03  5.3e-03
(±4.9e-03)(±3.6e-03)(±2.0e-03)(±1.6e-03)(±1.5e-03)(±1.5e-03)(±1.4e-03)(±1.2e-03)(±1.3e-03)(±1.4e-03)(±1.5e-03)(±1.6e-03)(±2.1e-03)
3.1e-01  2.3e-01  1.6e-01  1.3e-01  1.1e-01  8.8e-02  6.9e-02  4.7e-02  3.4e-02  2.2e-02  1.3e-03  −1.5e-02  −2.6e-02
(±1.5e-01)(±1.2e-01)(±6.6e-02)(±5.5e-02)(±5.3e-02)(±5.3e-02)(±5.3e-02)(±4.8e-02)(±5.3e-02)(±6.1e-02)(±6.5e-02)(±7.7e-02)(±1.1e-01)

−5.1e-03  −2.4e-03  1.2e-04  1.1e-03  1.7e-03  2.3e-03  2.9e-03  3.6e-03  4.1e-03  4.5e-03  5.3e-03  6.1e-03
(±6.2e-03)(±4.7e-03)(±2.6e-03)(±2.1e-03)(±2.1e-03)(±2.1e-03)(±2.1e-03)(±2.0e-03)(±2.2e-03)(±2.5e-03)(±2.7e-03)(±3.0e-03)
3.6e-01  2.7e-01  1.8e-01  1.5e-01  1.3e-01  1.1e-01  8.3e-02  5.5e-02  3.7e-02  1.9e-02  −1.6e-02  −5.2e-02
(±1.6e-01)(±1.2e-01)(±7.0e-02)(±6.1e-02)(±6.1e-02)(±6.4e-02)(±6.5e-02)(±6.6e-02)(±7.6e-02)(±9.1e-02)(±1.0e-01)(±1.2e-01)

−8.6e-03  −5.1e-03  −2.0e-03  −8.1e-04  −1.3e-04  4.5e-04  1.0e-03  1.9e-03  2.2e-03  2.5e-03  3.5e-03
(±7.2e-03)(±5.4e-03)(±2.6e-03)(±2.0e-03)(±2.0e-03)(±2.0e-03)(±2.1e-03)(±1.9e-03)(±2.3e-03)(±3.0e-03)(±3.6e-03)
4.1e-01  3.1e-01  2.2e-01  1.8e-01  1.6e-01  1.4e-01  1.3e-01  9.7e-02  8.5e-02  7.4e-02  4.0e-02
(±1.6e-01)(±1.2e-01)(±6.2e-02)(±5.0e-02)(±5.0e-02)(±5.4e-02)(±5.8e-02)(±5.6e-02)(±7.1e-02)(±9.6e-02)(±1.2e-01)

−1.2e-02  −7.6e-03  −3.8e-03  −2.5e-03  −1.8e-03  −1.2e-03  −7.2e-04  1.1e-04  2.1e-04  1.3e-04
(±8.2e-03)(±6.1e-03)(±2.5e-03)(±1.7e-03)(±1.5e-03)(±1.5e-03)(±1.4e-03)(±6.6e-04)(±8.5e-04)(±1.2e-03)
4.4e-01  3.4e-01  2.4e-01  2.1e-01  1.9e-01  1.8e-01  1.6e-01  1.4e-01  1.3e-01  1.4e-01
(±1.6e-01)(±1.3e-01)(±5.4e-02)(±3.7e-02)(±3.4e-02)(±3.5e-02)(±3.7e-02)(±1.7e-02)(±2.3e-02)(±3.5e-02)

−1.4e-02  −9.5e-03  −4.8e-03  −3.2e-03  −2.4e-03  −1.8e-03  −1.2e-03  −8.4e-05  1.0e-05
(±9.7e-03)(±7.4e-03)(±2.9e-03)(±1.9e-03)(±1.9e-03)(±1.8e-03)(±1.8e-03)(±1.9e-03)(±9.4e-04)(±1.3e-03)
4.7e-01  3.7e-01  2.6e-01  2.2e-01  2.0e-01  1.9e-01  1.7e-01  1.4e-01  1.4e-01
(±1.7e-01)(±1.4e-01)(±5.6e-02)(±3.9e-02)(±3.7e-02)(±4.0e-02)(±4.5e-02)(±2.3e-02)(±3.4e-02)

−1.8e-02  −1.2e-02  −6.1e-03  −4.2e-03  −3.3e-03  −2.6e-03  −1.9e-03  −4.1e-04
(±1.1e-02)(±8.8e-03)(±3.4e-03)(±2.2e-03)(±2.1e-03)(±2.3e-03)(±2.7e-03)(±1.4e-03)
5.0e-01  3.9e-01  2.7e-01  2.3e-01  2.1e-01  2.0e-01  1.8e-01  1.5e-01
(±1.8e-01)(±1.5e-01)(±5.9e-02)(±3.9e-02)(±3.9e-02)(±4.5e-02)(±5.6e-02)(±3.1e-02)

−2.1e-02  −1.5e-02  −7.5e-03  −5.3e-03  −4.3e-03  −3.6e-03  −2.9e-03
(±1.3e-02)(±1.1e-02)(±3.9e-03)(±2.4e-03)(±2.4e-03)(±2.8e-03)(±3.8e-03)
5.4e-01  4.2e-01  2.9e-01  2.5e-01  2.3e-01  2.1e-01  2.0e-01
(±1.9e-01)(±1.6e-01)(±6.2e-02)(±4.0e-02)(±4.1e-02)(±5.2e-02)(±7.3e-02)

−2.6e-02  −1.8e-02  −9.0e-03  −6.3e-03  −5.1e-03  −4.4e-03
(±1.6e-02)(±1.3e-02)(±4.8e-03)(±2.9e-03)(±3.2e-03)(±4.3e-03)
5.7e-01  4.5e-01  3.1e-01  2.6e-01  2.4e-01  2.3e-01
(±2.0e-01)(±1.7e-01)(±6.7e-02)(±4.4e-02)(±5.0e-02)(±7.1e-02)

−3.1e-02  −2.2e-02  −1.1e-02  −7.6e-03  −6.3e-03
(±1.9e-02)(±1.6e-02)(±5.8e-03)(±3.6e-03)(±4.4e-03)
6.1e-01  4.8e-01  3.2e-01  2.7e-01  2.5e-01
(±2.1e-01)(±1.9e-01)(±7.4e-02)(±4.8e-02)(±6.2e-02)

−3.8e-02  −2.7e-02  −1.4e-02  −9.8e-03
(±2.2e-02)(±2.0e-02)(±6.7e-03)(±3.1e-03)
6.5e-01  5.2e-01  3.5e-01  3.0e-01
(±2.2e-01)(±2.1e-01)(±7.5e-02)(±3.7e-02)

−4.6e-02  −3.3e-02  −1.6e-02
(±2.7e-02)(±2.7e-02)(±9.4e-03)
6.9e-01  5.6e-01  3.7e-01
(±2.4e-01)(±2.5e-01)(±9.4e-02)

−5.6e-02  −4.2e-02
(±3.4e-02)(±3.8e-02)
7.4e-01  6.1e-01
(±2.6e-01)(±3.1e-01)

−7.2e-02
(±4.1e-02)
8.0e-01
(±2.6e-01)

Figure 17: Parameter estimate of Equation 3 over the time interval [start, end] for group 3. The $x$-axis corresponds to the start point and the $y$-axis to the end point of the analyzed time window. The four values grouped at each grid point correspond to $b_2$ (top line), the 95% confidence interval of $b_2$, $a_2$ and the 95% confidence interval of $a_2$ (bottom line), respectively.

```
end
25 ─

   6.9e-01  6.9e-01  6.7e-01  1.0e+00  1.1e+00  1.1e+00  1.1e+00  1.1e+00  1.1e+00  1.0e+00  1.0e+00  1.0e+00  9.8e-01  9.4e-01  8.8e-01  7.4e-01  6.8e-01  7.1e-01
 (±1.9e-01)(±1.9e-01)(±2.6e-01)(±2.3e-01)(±2.2e-01)(±2.2e-01)(±2.2e-01)(±2.3e-01)(±2.3e-01)(±2.4e-01)(±2.7e-01)(±2.8e-01)(±3.1e-01)(±3.6e-01)(±4.2e-01)(±4.8e-01)(±7.8e-01)(±1.3e+00)
   7.4e-02  7.4e-02  7.6e-02  2.0e-03 −3.5e-03 −1.3e-03  1.5e-03  3.8e-03  1.2e-02  1.7e-02  1.9e-02  2.8e-02  4.0e-02  5.3e-02  7.8e-02  1.3e-01  1.5e-01  1.4e-01
 (±6.3e-02)(±6.3e-02)(±7.5e-02)(±6.0e-02)(±5.5e-02)(±5.5e-02)(±5.7e-02)(±6.0e-02)(±6.1e-02)(±6.6e-02)(±7.4e-02)(±8.2e-02)(±9.3e-02)(±1.1e-01)(±1.3e-01)(±1.6e-01)(±2.8e-01)(±4.9e-01)

   6.5e-01  6.5e-01  6.0e-01  9.8e-01  1.0e+00  1.1e+00  1.1e+00  1.0e+00  1.0e+00  1.0e+00  1.0e+00  9.9e-01  9.5e-01  9.0e-01  8.2e-01  6.4e-01  4.7e-01
 (±1.8e-01)(±1.8e-01)(±2.6e-01)(±2.7e-01)(±2.6e-01)(±2.6e-01)(±2.7e-01)(±2.8e-01)(±2.8e-01)(±3.0e-01)(±3.2e-01)(±3.5e-01)(±3.8e-01)(±4.5e-01)(±5.2e-01)(±6.1e-01)(±1.0e+00)
   7.7e-02  7.7e-02  8.8e-02  1.0e-02  1.3e-03  2.5e-03  5.0e-03  7.2e-03  1.5e-02  2.0e-02  2.3e-02  3.3e-02  4.6e-02  6.1e-02  9.0e-02  1.6e-01  2.1e-01
 (±6.0e-02)(±6.0e-02)(±7.3e-02)(±6.6e-02)(±6.2e-02)(±6.2e-02)(±6.4e-02)(±6.8e-02)(±6.8e-02)(±7.5e-02)(±8.4e-02)(±9.3e-02)(±1.1e-01)(±1.3e-01)(±1.6e-01)(±1.9e-01)(±3.6e-01)

   6.3e-01  6.3e-01  5.3e-01  9.6e-01  1.1e+00  1.1e+00  1.1e+00  1.1e+00  1.1e+00  1.0e+00  1.0e+00  1.0e+00  9.6e-01  9.1e-01  8.0e-01  5.5e-01
 (±1.8e-01)(±1.8e-01)(±2.7e-01)(±3.3e-01)(±3.3e-01)(±3.3e-01)(±3.4e-01)(±3.6e-01)(±3.5e-01)(±3.8e-01)(±4.2e-01)(±4.6e-01)(±5.1e-01)(±6.1e-01)(±7.4e-01)(±9.0e-01)
   7.9e-02  7.9e-02  9.9e-02  1.5e-02 −8.3e-04 −1.2e-03  7.2e-04  2.8e-03  9.9e-03  1.6e-02  1.8e-02  2.9e-02  4.3e-02  6.0e-02  9.5e-02  1.8e-01
 (±5.9e-02)(±5.9e-02)(±7.2e-02)(±7.6e-02)(±7.2e-02)(±7.3e-02)(±7.5e-02)(±7.9e-02)(±7.9e-02)(±8.7e-02)(±9.9e-02)(±1.1e-01)(±1.3e-01)(±1.6e-01)(±2.1e-01)(±2.6e-01)

   6.2e-01  6.2e-01  4.9e-01  8.4e-01  9.3e-01  9.5e-01  9.5e-01  9.5e-01  9.4e-01  9.3e-01  9.2e-01  8.9e-01  8.5e-01  8.0e-01  7.1e-01
 (±1.6e-01)(±1.6e-01)(±2.2e-01)(±2.7e-01)(±2.8e-01)(±2.8e-01)(±2.9e-01)(±3.1e-01)(±3.0e-01)(±3.2e-01)(±3.6e-01)(±3.8e-01)(±4.3e-01)(±5.1e-01)(±6.2e-01)
   7.3e-02  7.3e-02  1.0e-01  3.2e-02  1.8e-02  1.7e-02  1.8e-02  1.9e-02  2.5e-02  3.0e-02  3.3e-02  4.2e-02  5.6e-02  7.2e-02  1.0e-01
 (±5.0e-02)(±5.0e-02)(±5.7e-02)(±6.1e-02)(±5.9e-02)(±6.0e-02)(±6.2e-02)(±6.6e-02)(±6.5e-02)(±7.1e-02)(±8.1e-02)(±9.1e-02)(±1.1e-01)(±1.3e-01)(±1.7e-01)

   6.2e-01  6.2e-01  4.7e-01  8.9e-01  1.1e+00  1.1e+00  1.1e+00  1.1e+00  1.1e+00  1.1e+00  1.1e+00  1.1e+00  1.0e+00  9.7e-01
 (±1.6e-01)(±1.6e-01)(±2.3e-01)(±3.2e-01)(±3.1e-01)(±3.2e-01)(±3.3e-01)(±3.4e-01)(±3.2e-01)(±3.4e-01)(±3.8e-01)(±4.1e-01)(±4.6e-01)(±5.7e-01)
   7.3e-02  7.3e-02  1.0e-01  2.3e-02 −1.6e-03 −5.3e-03 −5.0e-03 −3.8e-03  1.2e-03  6.3e-03  7.9e-03  1.7e-02  3.0e-02  4.4e-02
 (±5.2e-02)(±5.2e-02)(±5.9e-02)(±6.7e-02)(±6.3e-02)(±6.3e-02)(±6.4e-02)(±6.7e-02)(±6.3e-02)(±6.9e-02)(±8.0e-02)(±8.9e-02)(±1.0e-01)(±1.4e-01)

   6.0e-01  6.0e-01  4.2e-01  8.4e-01  1.1e+00  1.2e+00  1.3e+00  1.3e+00  1.3e+00  1.3e+00  1.3e+00  1.3e+00  1.2e+00
 (±1.6e-01)(±1.6e-01)(±2.2e-01)(±4.0e-01)(±4.4e-01)(±4.6e-01)(±4.7e-01)(±4.9e-01)(±4.3e-01)(±4.6e-01)(±5.3e-01)(±5.7e-01)(±6.7e-01)
   7.2e-02  7.2e-02  1.1e-01  3.2e-02 −1.3e-02 −2.7e-02 −3.2e-02 −3.3e-02 −3.3e-02 −2.8e-02 −2.7e-02 −1.7e-02 −4.7e-03
 (±5.2e-02)(±5.2e-02)(±5.6e-02)(±7.9e-02)(±8.1e-02)(±8.1e-02)(±8.2e-02)(±8.6e-02)(±7.5e-02)(±8.2e-02)(±9.6e-02)(±1.1e-01)(±1.3e-01)

   5.9e-01  5.9e-01  3.7e-01  6.4e-01  9.4e-01  1.2e+00  1.4e+00  1.5e+00  1.9e+00  1.8e+00  1.8e+00  1.8e+00
 (±1.5e-01)(±1.5e-01)(±1.8e-01)(±4.3e-01)(±6.5e-01)(±8.1e-01)(±9.1e-01)(±1.0e+00)(±8.4e-01)(±9.1e-01)(±1.1e+00)(±1.1e+00)
   6.8e-02  6.8e-02  1.1e-01  6.3e-02  1.6e-02 −1.8e-02 −4.5e-02 −6.2e-02 −1.1e-01 −1.0e-01 −1.0e-01 −9.5e-02
 (±4.9e-02)(±4.9e-02)(±4.5e-02)(±8.0e-02)(±1.1e-01)(±1.3e-01)(±1.4e-01)(±1.6e-01)(±1.4e-01)(±1.4e-01)(±1.6e-01)(±1.8e-01)

   6.0e-01  6.0e-01  3.7e-01  5.0e-01  5.5e-01  5.0e-01  3.9e-01  3.4e-02  1.9e+00  2.1e+00  2.2e+00
 (±1.4e-01)(±1.4e-01)(±1.2e-01)(±2.9e-01)(±5.2e-01)(±7.8e-01)(±1.1e+00)(±1.7e+00)(±5.2e+00)(±5.9e+00)(±7.4e+00)
   6.0e-02  6.0e-02  1.0e-01  7.9e-02  7.2e-02  7.9e-02  9.4e-02  1.4e-01 −1.1e-01 −1.4e-01 −1.5e-01
 (±4.4e-02)(±4.4e-02)(±2.9e-02)(±5.3e-02)(±8.6e-02)(±1.2e-01)(±1.7e-01)(±2.4e-01)(±7.3e-01)(±8.2e-01)(±1.0e+00)

   6.1e-01  6.1e-01  3.9e-01  5.6e-01  6.5e-01  6.1e-01  4.9e-01  3.1e-01 −1.2e-01 −6.2e-02
 (±1.3e-01)(±1.3e-01)(±7.1e-02)(±1.3e-01)(±2.3e-01)(±3.5e-01)(±4.8e-01)(±3.3e-01)(±1.2e+00)(±1.4e+00)
   5.0e-02  5.0e-02  9.2e-02  6.2e-02  4.8e-02  5.4e-02  7.1e-02  1.3e-01  1.6e-01  1.5e-01
 (±4.2e-02)(±4.2e-02)(±1.8e-02)(±2.5e-02)(±3.9e-02)(±5.5e-02)(±7.3e-02)(±4.8e-02)(±1.7e-01)(±2.0e-01)

20 ─
   6.2e-01  6.2e-01  3.9e-01  5.7e-01  6.6e-01  6.2e-01  5.0e-01  3.8e-01 −2.4e-01
 (±1.4e-01)(±1.4e-01)(±7.7e-02)(±1.5e-01)(±2.5e-01)(±3.9e-01)(±5.4e-01)(±3.5e-01)(±1.4e+00)
   4.9e-02  4.9e-02  9.2e-02  6.1e-02  4.7e-02  5.2e-02  7.0e-02  1.3e-01  1.7e-01
 (±4.6e-02)(±4.6e-02)(±2.0e-02)(±2.8e-02)(±4.3e-02)(±6.1e-02)(±8.2e-02)(±5.1e-02)(±1.9e-01)

   6.2e-01  6.2e-01  3.8e-01  5.7e-01  6.7e-01  6.3e-01  5.1e-01  5.1e-02
 (±1.5e-01)(±1.5e-01)(±8.3e-02)(±1.6e-01)(±2.8e-01)(±4.4e-01)(±6.3e-01)(±4.1e-01)
   4.8e-02  4.8e-02  9.3e-02  6.0e-02  4.4e-02  4.9e-02  6.7e-02  1.3e-01
 (±5.2e-02)(±5.2e-02)(±2.2e-02)(±3.1e-02)(±4.9e-02)(±7.1e-02)(±9.7e-02)(±6.1e-02)

   6.2e-01  6.2e-01  3.8e-01  5.8e-01  7.0e-01  6.7e-01  5.5e-01
 (±1.6e-01)(±1.6e-01)(±9.1e-02)(±1.9e-01)(±3.3e-01)(±5.4e-01)(±8.2e-01)
   4.6e-02  4.6e-02  9.4e-02  5.8e-02  3.9e-02  4.2e-02  6.0e-02
 (±5.8e-02)(±5.8e-02)(±2.6e-02)(±3.7e-02)(±5.9e-02)(±8.9e-02)(±1.3e-01)

   6.2e-01  6.2e-01  3.7e-01  5.6e-01  6.8e-01  6.5e-01
 (±1.8e-01)(±1.8e-01)(±9.8e-02)(±2.1e-01)(±4.1e-01)(±7.0e-01)
   4.7e-02  4.7e-02  9.7e-02  6.2e-02  4.2e-02  4.7e-02
 (±6.8e-02)(±6.8e-02)(±2.8e-02)(±4.4e-02)(±7.5e-02)(±1.2e-01)

   6.2e-01  6.2e-01  3.6e-01  5.4e-01  6.5e-01
 (±2.0e-01)(±2.0e-01)(±1.0e-01)(±2.5e-01)(±5.2e-01)
   4.7e-02  4.7e-02  1.0e-01  6.7e-02  4.8e-02
 (±8.0e-02)(±8.0e-02)(±3.2e-02)(±5.3e-02)(±9.6e-02)

15 ─
   6.3e-01  6.3e-01  3.6e-01  5.7e-01
 (±2.3e-01)(±2.3e-01)(±1.3e-01)(±3.3e-01)
   4.2e-02  4.2e-02  1.0e-01  6.0e-02
 (±9.7e-02)(±9.7e-02)(±4.0e-02)(±7.3e-02)

   6.3e-01  6.3e-01  3.4e-01
 (±2.8e-01)(±2.8e-01)(±1.4e-01)
   4.1e-02  4.1e-02  1.1e-01
 (±1.3e-01)(±1.3e-01)(±4.9e-02)

   6.2e-01  6.2e-01
 (±3.6e-01)(±3.6e-01)
   4.7e-02  4.7e-02
 (±1.7e-01)(±1.7e-01)

   6.4e-01
 (±5.1e-01)
   3.5e-02
 (±2.6e-01)

          10            15            20
                    start
```

Figure 18: Parameter estimate of Equation 4 over the time interval [start, end] for group 3. The $x$-axis corresponds to the start point and the $y$-axis to the end point of the analyzed time window. The four values grouped at each grid point correspond to $b_3$ (top line), the 95% confidence interval of $b_3$, $a_3$ and the 95% confidence interval of $a_3$ (bottom line), respectively.
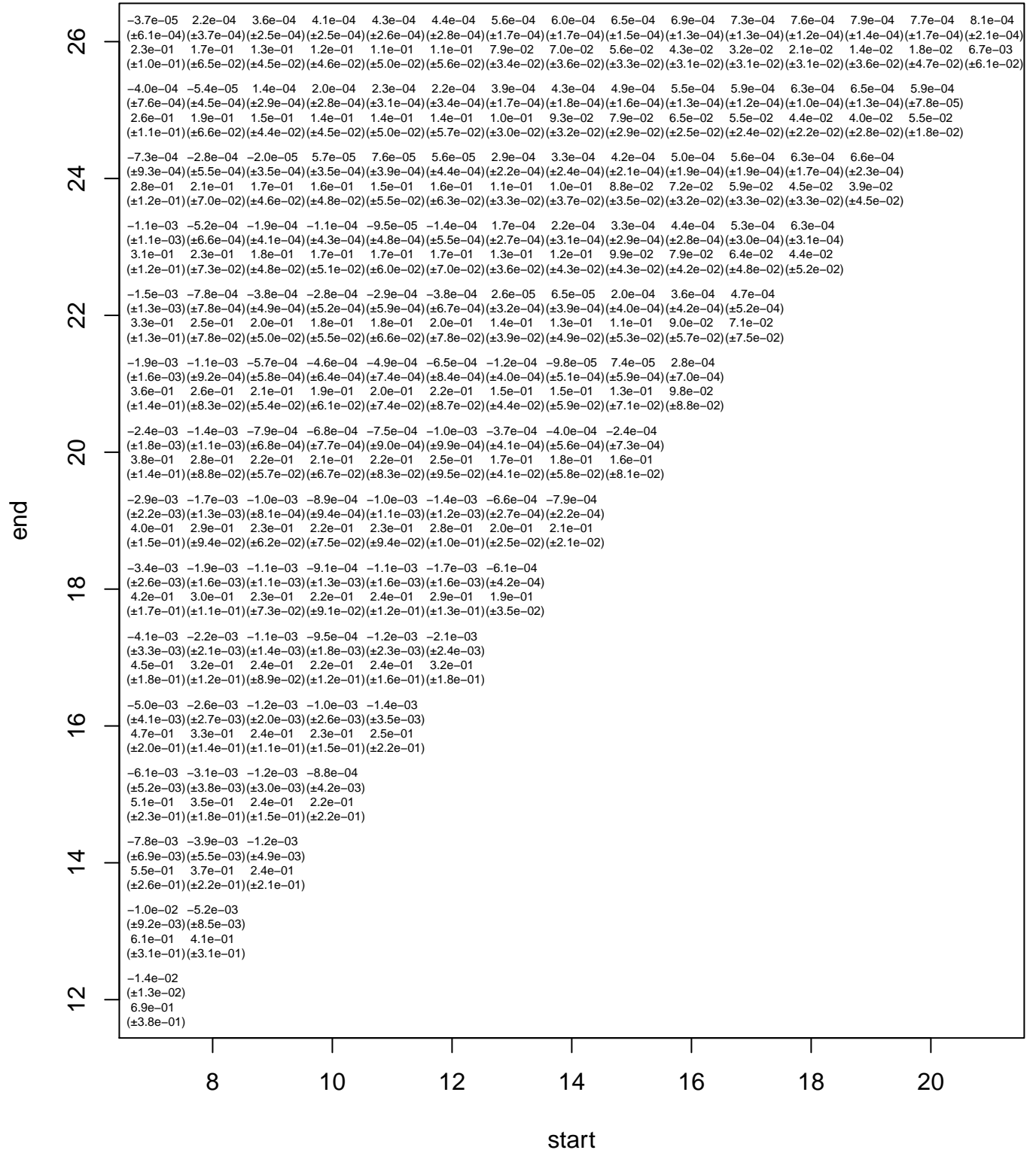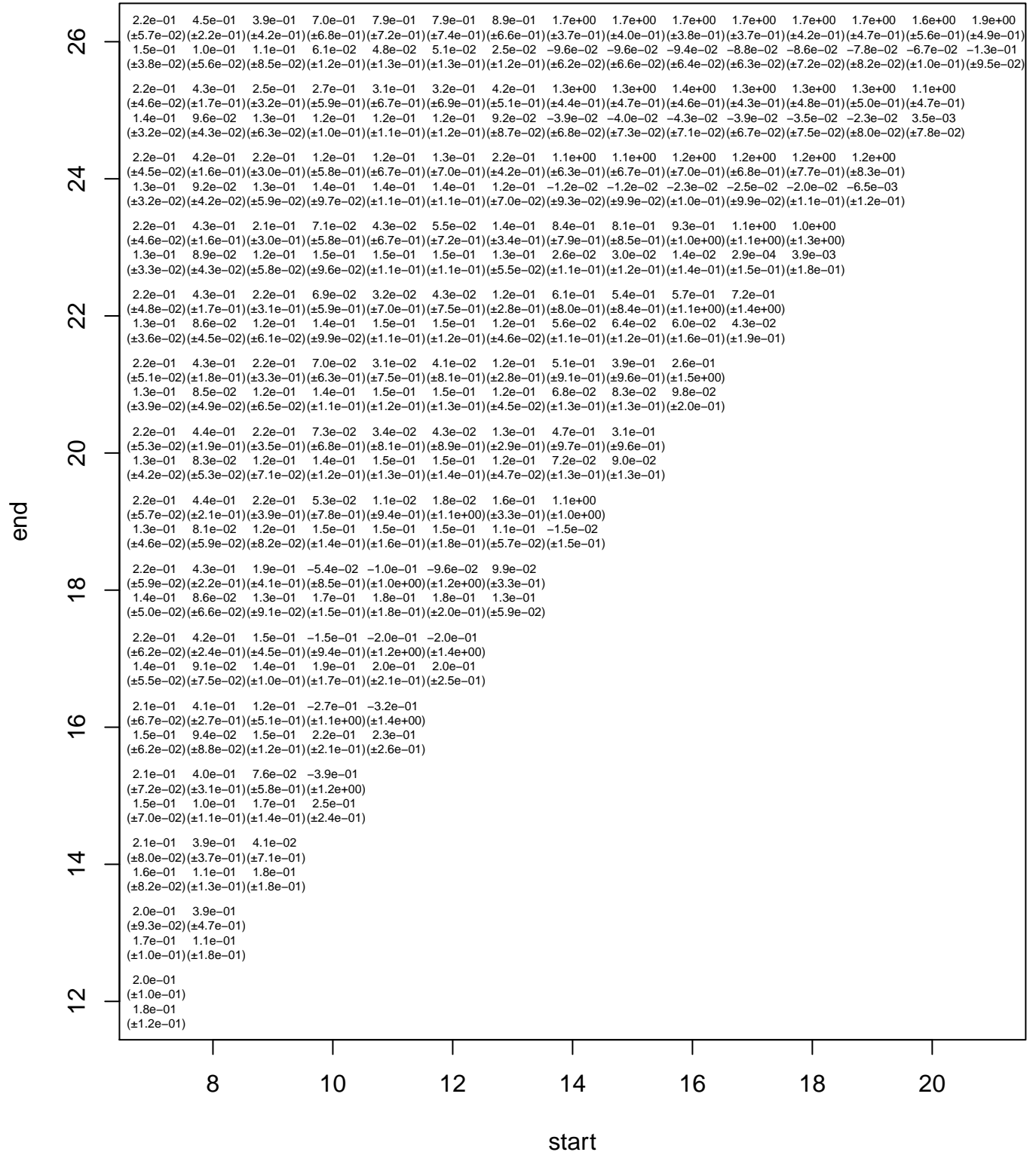
end

start



```
-1.7e-04    1.4e-04    2.8e-04    2.9e-04    3.3e-04    3.4e-04    3.4e-04    3.1e-04    2.8e-04    2.3e-04
(±7.6e-04) (±3.5e-04) (±1.9e-04) (±2.0e-04) (±1.9e-04) (±2.1e-04) (±2.5e-04) (±2.8e-04) (±3.4e-04) (±4.1e-04
 3.4e-01    2.3e-01    1.8e-01    1.8e-01    1.6e-01    1.6e-01    1.6e-01    1.7e-01    1.8e-01    2.1e-01
(±1.9e-01) (±8.9e-02) (±4.9e-02) (±5.5e-02) (±5.4e-02) (±6.4e-02) (±7.7e-02) (±9.2e-02) (±1.2e-01) (±1.5e-01

-3.5e-04    1.9e-04    4.4e-04    4.8e-04    5.8e-04    6.1e-04    6.4e-04    6.2e-04    6.3e-04
(±1.2e-03) (±5.5e-04) (±2.6e-04) (±2.7e-04) (±1.9e-04) (±2.0e-04) (±2.3e-04) (±2.8e-04) (±3.7e-04)
 3.6e-01    2.2e-01    1.6e-01    1.5e-01    1.2e-01    1.1e-01    1.0e-01    1.1e-01    1.1e-01
(±2.2e-01) (±1.1e-01) (±5.1e-02) (±5.7e-02) (±4.1e-02) (±4.6e-02) (±5.5e-02) (±7.1e-02) (±1.0e-01)

-8.7e-04    9.7e-06    4.2e-04    4.9e-04    6.8e-04    7.7e-04    8.5e-04    8.5e-04
(±1.7e-03) (±8.3e-04) (±4.1e-04) (±4.5e-04) (±3.0e-04) (±3.0e-04) (±3.0e-04) (±4.1e-04)
 4.1e-01    2.4e-01    1.6e-01    1.5e-01    1.1e-01    8.8e-02    6.9e-02    6.8e-02
(±2.5e-01) (±1.3e-01) (±6.4e-02) (±7.4e-02) (±5.1e-02) (±5.4e-02) (±5.9e-02) (±8.5e-02)

-1.7e-03   -4.0e-04    2.5e-04    3.4e-04    6.6e-04    8.2e-04    1.0e-03
(±2.3e-03) (±1.2e-03) (±6.0e-04) (±7.0e-04) (±5.0e-04) (±5.3e-04) (±5.5e-04)
 4.8e-01    2.8e-01    1.8e-01    1.6e-01    1.1e-01    8.1e-02    4.7e-02
(±2.7e-01) (±1.5e-01) (±7.8e-02) (±9.6e-02) (±7.3e-02) (±8.1e-02) (±8.9e-02)

-2.8e-03   -1.1e-03   -2.2e-04   -1.7e-04    2.4e-04    3.9e-04
(±2.8e-03) (±1.4e-03) (±6.1e-04) (±7.7e-04) (±4.6e-04) (±5.2e-04)
 5.5e-01    3.3e-01    2.2e-01    2.1e-01    1.5e-01    1.3e-01
(±2.9e-01) (±1.5e-01) (±7.0e-02) (±9.2e-02) (±5.8e-02) (±6.9e-02)

-3.9e-03   -1.7e-03   -5.7e-04   -5.9e-04   -5.5e-05
(±3.5e-03) (±1.8e-03) (±7.2e-04) (±9.6e-04) (±5.2e-04)
 6.1e-01    3.7e-01    2.5e-01    2.5e-01    1.8e-01
(±3.1e-01) (±1.7e-01) (±7.2e-02) (±1.0e-01) (±5.8e-02)

-5.2e-03   -2.3e-03   -8.4e-04   -9.5e-04
(±4.4e-03) (±2.3e-03) (±9.9e-04) (±1.4e-03)
 6.8e-01    4.1e-01    2.7e-01    2.8e-01
(±3.4e-01) (±1.9e-01) (±8.7e-02) (±1.3e-01)

-6.7e-03   -3.1e-03   -1.2e-03
(±5.5e-03) (±3.1e-03) (±1.5e-03)
 7.5e-01    4.6e-01    2.9e-01
(±3.7e-01) (±2.3e-01) (±1.1e-01)

-8.6e-03   -4.2e-03
(±6.9e-03) (±4.2e-03)
 8.2e-01    5.1e-01
(±4.1e-01) (±2.7e-01)

-1.1e-02
(±9.3e-03)
 9.0e-01
(±4.8e-01)
```

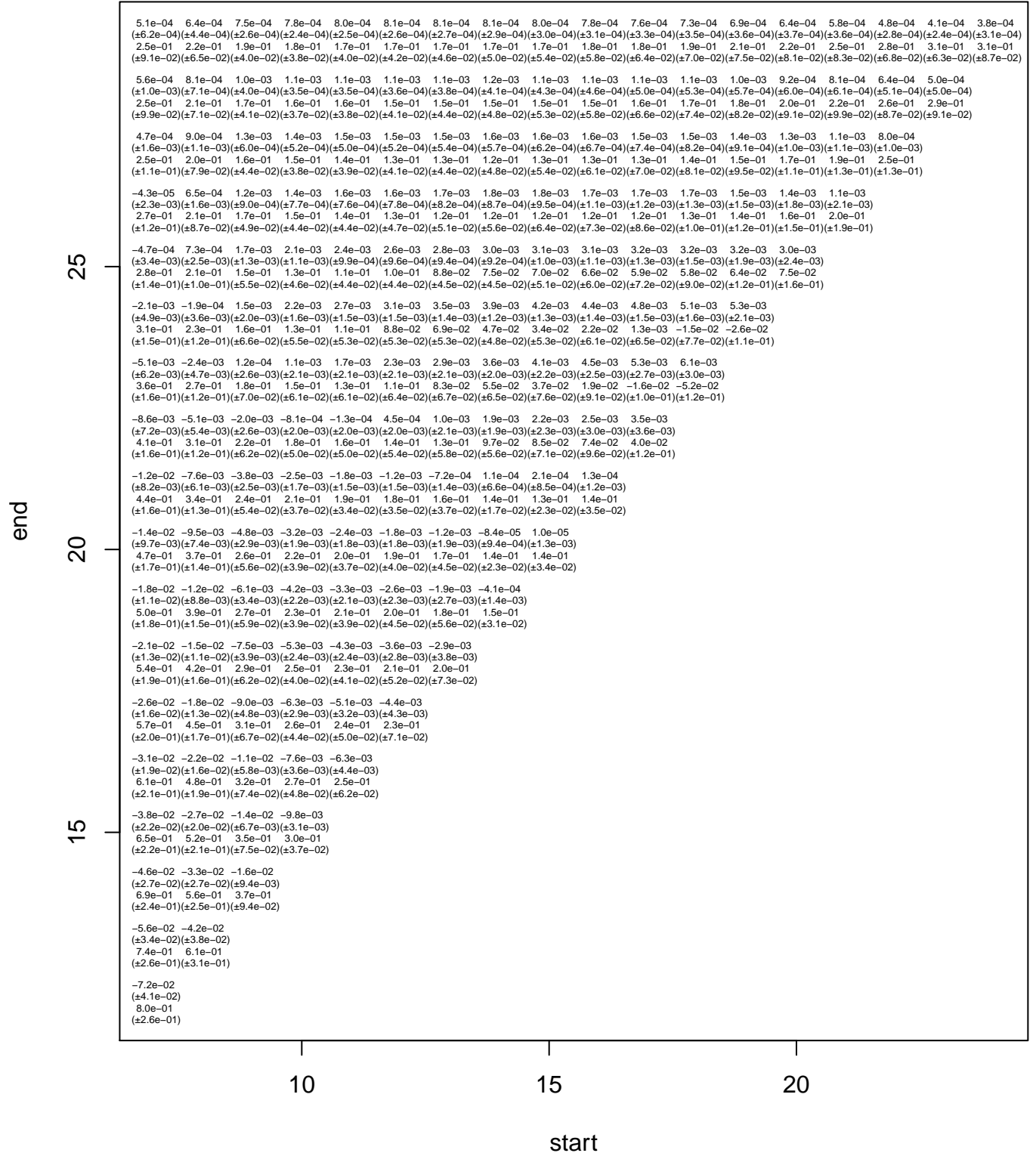y-axis (end): 20, 18, 16, 14, 12

x-axis (start): 8, 10, 12, 14, 16

Figure 19: Parameter estimate of Equation 3 over the time interval [start, end] for group 4. The $x$-axis corresponds to the start point and the $y$-axis to the end point of the analyzed time window. The four values grouped at each grid point correspond to $b_2$ (top line), the 95% confidence interval of $b_2$, $a_2$ and the 95% confidence interval of $a_2$ (bottom line), respectively.
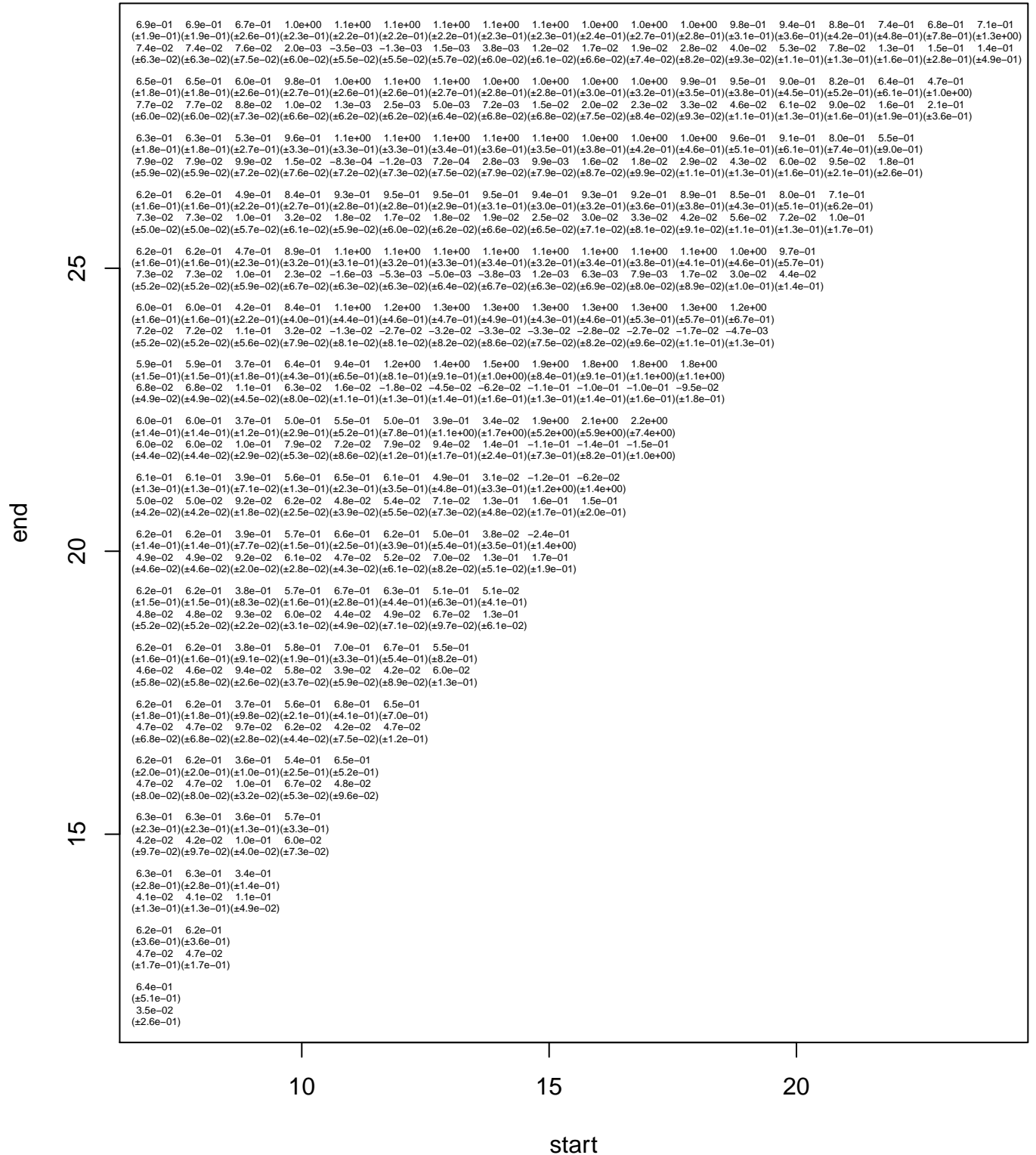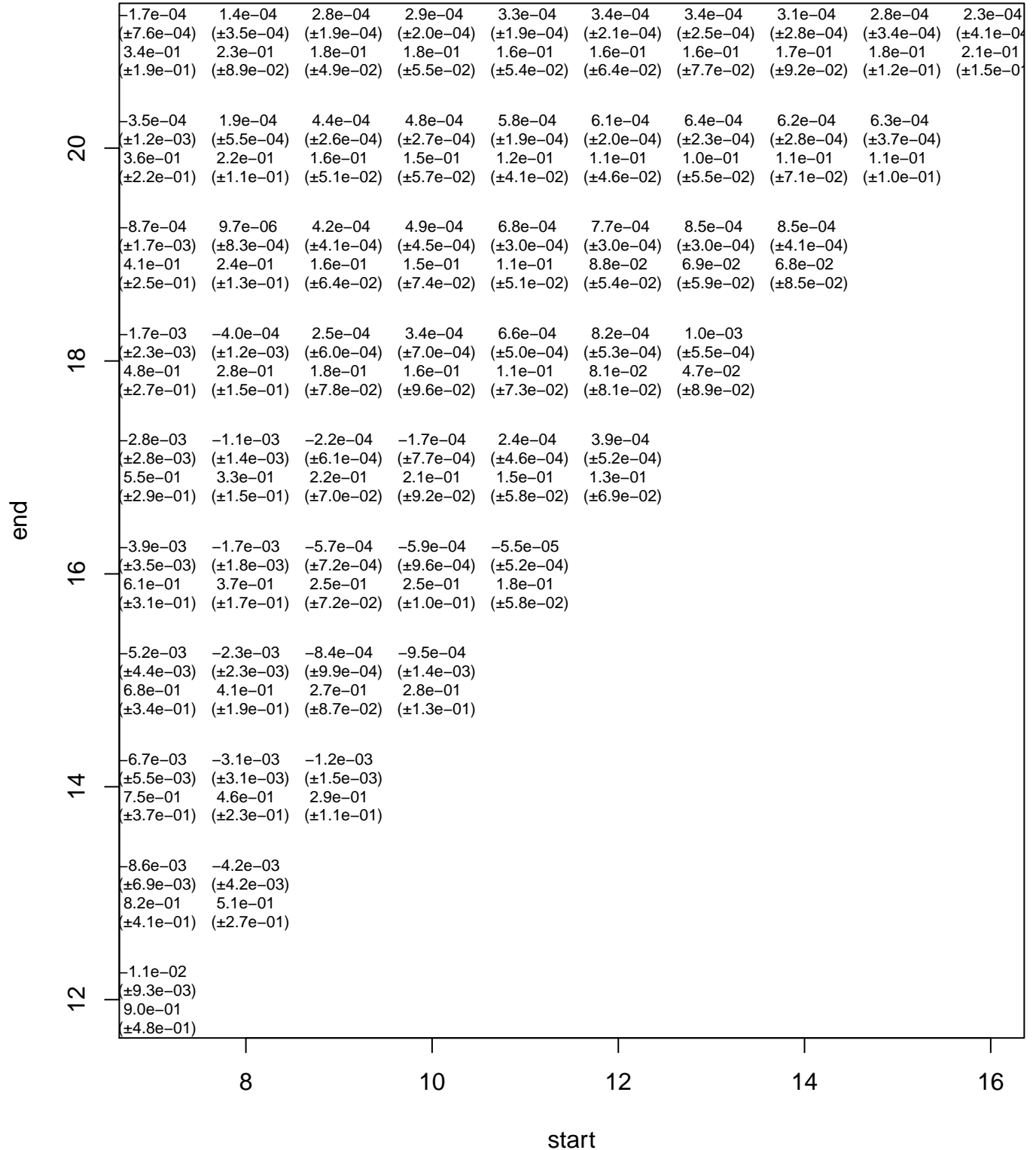
3.5e−01 3.5e−01 2.9e−01 8.1e−01 8.2e−01 8.9e−01 8.7e−01 8.1e−01 7.9e−01 6.9e−01
(±1.6e−01) (±1.6e−01) (±4.0e−01) (±4.6e−01) (±4.7e−01) (±4.2e−01) (±4.6e−01) (±4.7e−01) (±5.9e−01) (±7.0e−0
1.5e−01 1.5e−01 1.6e−01 5.1e−02 4.5e−02 3.7e−02 4.4e−02 6.5e−02 7.1e−02 1.0e−01
(±6.2e−02) (±6.2e−02) (±1.1e−01) (±1.1e−01) (±1.1e−01) (±1.0e−01) (±1.1e−01) (±1.2e−01) (±1.5e−01) (±1.9e−0

**end = 20**
3.5e−01 3.5e−01 2.6e−01 9.6e−01 9.6e−01 1.1e+00 1.1e+00 1.1e+00 1.1e+00
(±1.7e−01) (±1.7e−01) (±4.3e−01) (±5.8e−01) (±6.0e−01) (±4.7e−01) (±5.1e−01) (±4.9e−01) (±6.5e−01)
1.5e−01 1.5e−01 1.7e−01 2.1e−02 1.7e−02 −1.0e−02 −2.5e−03 1.8e−02 1.7e−02
(±6.5e−02) (±6.5e−02) (±1.1e−01) (±1.3e−01) (±1.4e−01) (±1.0e−01) (±1.1e−01) (±1.1e−01) (±1.6e−01)

3.4e−01 3.4e−01 1.7e−01 8.4e−01 8.1e−01 1.2e+00 1.2e+00 1.1e+00
(±1.6e−01) (±1.6e−01) (±4.0e−01) (±7.8e−01) (±8.1e−01) (±7.3e−01) (±8.0e−01) (±8.0e−01)
1.4e−01 1.4e−01 1.8e−01 4.4e−02 4.5e−02 −1.6e−02 −9.9e−03 1.1e−02
(±6.2e−02) (±6.2e−02) (±1.0e−01) (±1.7e−01) (±1.7e−01) (±1.5e−01) (±1.6e−01) (±1.7e−01)

**end = 18**
3.4e−01 3.4e−01 9.4e−02 5.7e−01 4.2e−01 1.9e+00 2.0e+00
(±1.5e−01) (±1.5e−01) (±3.6e−01) (±1.2e+00) (±1.2e+00) (±1.9e+00) (±2.1e+00)
1.3e−01 1.3e−01 1.9e−01 9.5e−02 1.2e−01 −1.5e−01 −1.7e−01
(±6.0e−02) (±6.0e−02) (±9.1e−02) (±2.3e−01) (±2.4e−01) (±3.5e−01) (±3.8e−01)

3.5e−01 3.5e−01 1.0e−01 3.3e−01 8.7e−02 1.1e−03
(±1.3e−01) (±1.3e−01) (±2.7e−01) (±9.4e−01) (±6.0e−01) (±2.4e+00)
1.2e−01 1.2e−01 1.7e−01 1.3e−01 1.7e−01 1.8e−01
(±5.2e−02) (±5.2e−02) (±6.8e−02) (±1.9e−01) (±1.2e−01) (±4.2e−01)

**end = 16**
3.6e−01 3.6e−01 1.2e−01 3.8e−01 1.4e−01
(±1.3e−01) (±1.3e−01) (±2.8e−01) (±1.0e+00) (±3.8e−01)
1.1e−01 1.1e−01 1.7e−01 1.2e−01 1.5e−01
(±5.8e−02) (±5.8e−02) (±7.5e−02) (±2.0e−01) (±7.4e−02)

3.6e−01 3.6e−01 1.1e−01 3.7e−01
(±1.5e−01) (±1.5e−01) (±3.3e−01) (±1.3e+00)
1.1e−01 1.1e−01 1.7e−01 1.2e−01
(±6.9e−02) (±6.9e−02) (±9.1e−02) (±2.6e−01)

**end = 14**
3.6e−01 3.6e−01 1.0e−01
(±1.8e−01) (±1.8e−01) (±4.1e−01)
1.1e−01 1.1e−01 1.7e−01
(±8.6e−02) (±8.6e−02) (±1.2e−01)

3.7e−01 3.7e−01
(±2.3e−01) (±2.3e−01)
1.1e−01 1.1e−01
(±1.2e−01) (±1.2e−01)

**end = 12**
3.6e−01
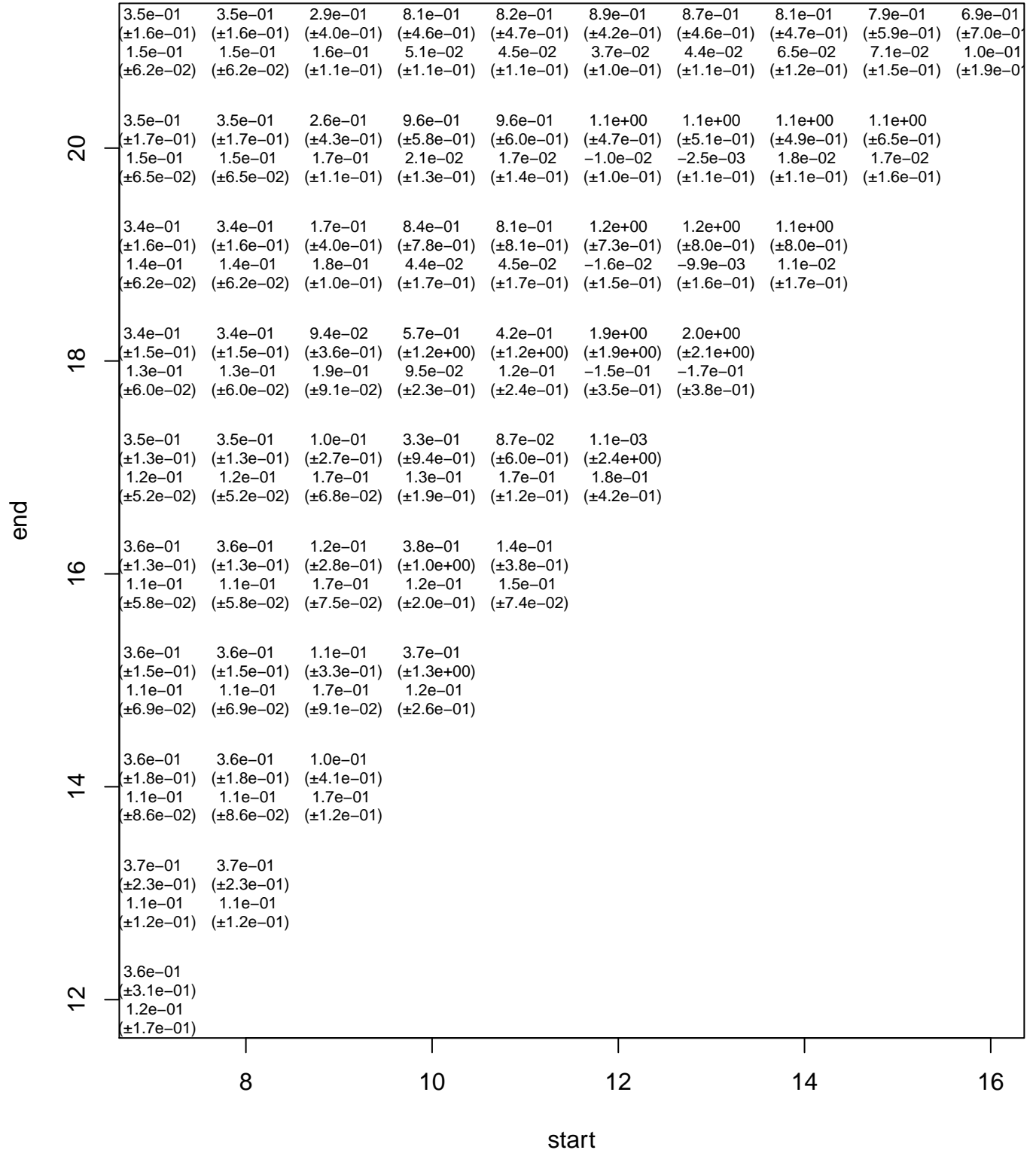(±3.1e−01)
1.2e−01
(±1.7e−01)

start: 8    10    12    14    16

Figure 20: Parameter estimate of Equation 4 over the time interval [start, end] for group 4. The $x$-axis corresponds to the start point and the $y$-axis to the end point of the analyzed time window. The four values grouped at each grid point correspond to $b_3$ (top line), the 95% confidence interval of $b_3$, $a_3$ and the 95% confidence interval of $a_3$ (bottom line), respectively.