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COLLECTIVE MODES AND GENERALIZED TRANSPORT COEFFICIENTS FOR A DENSE ONE-COMPONENT PLASMA IN A MAGNETIC FIELD

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The collective modes for a one-component Coulomb plasma in a magnetic field are derived from the microscopic balance equations. The mode frequencies contain nine independent generalized transport coefficients for which Green–Kubo expressions are determined. The discontinuity in the mode spectrum for wave vectors orthogonal to the magnetic field is discussed in detail. The connexion with the results obtained by means of kinetic theory is established.

1. Introduction

The decay of microscopic disturbances in a fluid or a plasma is determined by the collective modes of the system. These modes are essential for the study of the large-scale structure of the dynamical structure factor. Furthermore, expressions for the mode frequencies and the mode amplitudes are needed if the long-time behaviour of time-correlation functions is studied by means of mode-coupling theory.

In a recent paper[1]) the spectrum of the collective mode frequencies for a one-component plasma in a magnetic field has been determined on the basis of a formal kinetic equation for the one-particle time correlation function in momentum space. The mode frequencies could be written in terms of matrix elements of the memory kernel that occurs in the kinetic equation.

An alternative method to obtain the mode spectrum starts by deriving the microscopic balance equations of particle number, momentum and energy, and establishing fluctuation formulae for the densities and flows. Upon using projection operator techniques expressions for both the mode frequencies and the mode amplitudes may be derived then by solving a low-dimensional eigenvalue problem. This method has been used to determine the collective modes for a neutral fluid[2,3]) and for an unmagnetized plasma[4]).

The collective modes of a plasma in a magnetic field depend on the angle between the wave vector and the field. Transverse and longitudinal modes can no longer be distinguished. As a consequence both the mode amplitudes and the mode frequencies are given by expressions that are rather more complicated than those for an unmagnetized plasma. Moreover, the spectrum is expected to show a discontinuity if the wave vector is perpendicular to the magnetic field[1]). The purpose of this paper is to derive the complete set of the mode amplitudes and frequencies for arbitrary strength of the magnetic field, and to study the discontinuity in the spectrum. Moreover we wish to establish the connexion with the results obtained by using kinetic theory.

The model adopted is the classical one-component plasma, consisting of charged particles which are immersed in a neutralizing inert background and which interact through a Coulomb potential. The external magnetic field is assumed to be static and uniform in space.

2. Balance equations for a magnetized plasma

To derive the collective modes we shall need the microscopic balance equations of particle number, momentum and energy. For an unmagnetized plasma these balance equations have been discussed before[5,6]). As the equations are only slightly modified if a magnetic field is turned on we shall omit the derivations and present the results only, merely to fix the notation.

The balance equation for the particle density in Fourier space

$$n(\boldsymbol{k}) = \sum_\alpha \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} \,, \tag{2.1}$$

with wave vector $\boldsymbol{k}$, reads

$$\mathrm{i}Ln(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}\cdot\frac{\boldsymbol{g}(\boldsymbol{k})}{m} \,. \tag{2.2}$$

Here L is the Liouville operator in phase space, which for an arbitrary function F determines its time derivative as $\dot{F} = \mathrm{i}LF$. Furthermore $\boldsymbol{r}_\alpha$ is the position of particle α, with mass m, and $\boldsymbol{g}(\boldsymbol{k})$ is the momentum density

$$\boldsymbol{g}(\boldsymbol{k}) = \sum_\alpha \boldsymbol{p}_\alpha \,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} \,, \tag{2.3}$$

with $\boldsymbol{p}_\alpha$ the momentum of particle α.

The momentum balance is given by:

$$\mathrm{i}L\boldsymbol{g}(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\tau}(\boldsymbol{k}) - \mathrm{i}ne^2\,\frac{\boldsymbol{k}}{k^2}\,n(\boldsymbol{k}) + \omega_B\,\boldsymbol{g}(\boldsymbol{k})\wedge\hat{\boldsymbol{B}}\,. \tag{2.4}$$

It contains a pressure tensor $\boldsymbol{\tau}$ defined through:

$$\boldsymbol{k}\cdot\boldsymbol{\tau}(\boldsymbol{k}) = \boldsymbol{k}\cdot\boldsymbol{\tau}^{\mathrm{kin}}(\boldsymbol{k}) + \boldsymbol{k}\cdot\boldsymbol{\tau}^{\mathrm{pot}}(\boldsymbol{k})\,, \tag{2.5}$$

with kinetic and potential parts:

$$\boldsymbol{k}\cdot\boldsymbol{\tau}^{\mathrm{kin}}(\boldsymbol{k}) = \boldsymbol{k}\cdot\sum_{\alpha}\frac{\boldsymbol{p}_\alpha\boldsymbol{p}_\alpha}{m}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}\,, \tag{2.6}$$

$$\boldsymbol{k}\cdot\boldsymbol{\tau}^{\mathrm{pot}}(\boldsymbol{k}) = \frac{1}{2V}\sum_{\boldsymbol{q}(\neq\boldsymbol{0},\neq\boldsymbol{k})}\left[\frac{e^2\boldsymbol{q}}{q^2} + \frac{e^2(\boldsymbol{k}-\boldsymbol{q})}{(\boldsymbol{k}-\boldsymbol{q})^2}\right]\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_{\alpha\beta}-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}\,. \tag{2.7}$$

Furthermore, (2.4) contains a consistent field term, which depends on the electric field generated by the charge density fluctuations $en(\boldsymbol{k})$; n is the equilibrium particle density N/V. The last term in (2.4) represents the influence of the uniform magnetic field, which points in the direction of the unit vector $\hat{\boldsymbol{B}}$, and has the strength B corresponding to the Larmor frequency $\omega_B = eB/mc$.

The energy density in Fourier language reads:

$$\varepsilon(\boldsymbol{k}) = \varepsilon^{\mathrm{kin}}(\boldsymbol{k}) + \varepsilon^{\mathrm{pot}}(\boldsymbol{k})\,, \tag{2.8}$$

with:

$$\varepsilon^{\mathrm{kin}}(\boldsymbol{k}) = \sum_{\alpha}\frac{p_\alpha^2}{2m}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}\,, \tag{2.9}$$

$$\varepsilon^{\mathrm{pot}}(\boldsymbol{k}) = -\frac{1}{2V}\sum_{\boldsymbol{q}(\neq\boldsymbol{0},\neq\boldsymbol{k})}\frac{e^2\boldsymbol{q}\cdot(\boldsymbol{k}-\boldsymbol{q})}{q^2(\boldsymbol{k}-\boldsymbol{q})^2}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_{\alpha\beta}-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}\,. \tag{2.10}$$

Its balance equation has the form:

$$\mathrm{i}L\varepsilon(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon(\boldsymbol{k})\,, \tag{2.11}$$

with the energy flow given by the sum of a kinetic and a potential term:

$$\boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon(\boldsymbol{k}) = \boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon^{\mathrm{kin}}(\boldsymbol{k}) + \boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon^{\mathrm{pot}}(\boldsymbol{k})\,, \tag{2.12}$$

$$\boldsymbol{k}\cdot\boldsymbol{j}_\varepsilon^{\mathrm{kin}}(\boldsymbol{k}) = \boldsymbol{k}\cdot\sum_{\alpha}\frac{\boldsymbol{p}_\alpha}{m}\,\frac{p_\alpha^2}{2m}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}\,, \tag{2.13}$$

$$
\begin{aligned}
\boldsymbol{k}\cdot\boldsymbol{j}_{\varepsilon}^{\mathrm{pot}}(\boldsymbol{k}) = {} & \frac{1}{V}\sum_{\boldsymbol{q}(\neq \boldsymbol{0})}\frac{e^2}{q^2}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\frac{\boldsymbol{q}\cdot\boldsymbol{p}_\alpha}{m}\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_{\alpha\beta}-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha} \\
& - \frac{1}{V}\sum_{\boldsymbol{q}(\neq \boldsymbol{0},\neq \boldsymbol{k})}\frac{e^2\boldsymbol{q}\cdot(\boldsymbol{k}-\boldsymbol{q})}{q^2(\boldsymbol{k}-\boldsymbol{q})^2}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\frac{(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{p}_\alpha}{m}\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_{\alpha\beta}-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_\alpha}\,.
\end{aligned}
\tag{2.14}
$$

It should be remarked that the potential parts of the pressure tensor and the energy flow have been defined here only in an implicit way, by (2.6) and (2.14). Explicit factors $\boldsymbol{k}$ may be extracted from the expressions at the right-hand sides of these formulae by expansion with respect to $\boldsymbol{k}$. The results, given earlier[6]), are not needed in the following.

The balance equations may be used to derive fluctuation formulae for the microscopic densities and flows, as has been shown before[6]). Several of these are essential in the derivation of the collective modes. They have been listed in the appendix.

3. Collective modes

The collective modes are particular linear combinations of the particle density, the momentum density and the energy density. Let $a_i(\boldsymbol{k})$ denote a set of five independent linear combinations of these quantities, with adjoints $\bar{a}_i(\boldsymbol{k})$ such that

$$\frac{1}{V}\langle \bar{a}_i^*(\boldsymbol{k})a_j(\boldsymbol{k})\rangle = \delta_{ij}\,. \tag{3.1}$$

Here the brackets denote a canonical ensemble average. In the course of time $a_i(\boldsymbol{k})$ evolves into $a_i(\boldsymbol{k}, t)$ of which the Laplace transform

$$a_i(\boldsymbol{k}, z) = -i\int_0^\infty \mathrm{d}t\, \mathrm{e}^{izt}a_i(\boldsymbol{k}, t) \tag{3.2}$$

satisfies the equation

$$(z + L)a_i(\boldsymbol{k}, z) = a_i(\boldsymbol{k})\,. \tag{3.3}$$

Introducing a projection operator P by writing

$$Pf(\boldsymbol{k}) = \sum_i \frac{1}{V} \langle \bar{a}_i^*(\boldsymbol{k}) f(\boldsymbol{k}) \rangle a_i(\boldsymbol{k}) \tag{3.4}$$

for an arbitrary function $f(\boldsymbol{k})$ in phase space, one derives an equation for the hydrodynamic propagators:

$$G_{ij}(\boldsymbol{k}, z) = \frac{1}{V} \left\langle \bar{a}_i^*(\boldsymbol{k}) \frac{1}{z + L} a_j(\boldsymbol{k}) \right\rangle \tag{3.5}$$

in the form:

$$\sum_l [z\delta_{il} - \Omega_{il}(\boldsymbol{k}, z)] G_{lj}(\boldsymbol{k}, z) = \delta_{ij} . \tag{3.6}$$

The frequency matrix is given by

$$\Omega_{ij}(\boldsymbol{k}, z) = \Omega_{ij}^{(1)}(\boldsymbol{k}, z) + \Omega_{ij}^{(2)}(\boldsymbol{k}, z) , \tag{3.7}$$

where the direct and the indirect parts are

$$\Omega_{ij}^{(1)}(\boldsymbol{k}, z) = -\frac{1}{V} \langle \bar{a}_i^*(\boldsymbol{k}) L a_j(\boldsymbol{k}) \rangle , \tag{3.8}$$

$$\Omega_{ij}^{(2)}(\boldsymbol{k}, z) = \frac{1}{V} \left\langle \bar{a}_i^*(\boldsymbol{k}) LQ \frac{1}{z + QLQ} QLa_j(\boldsymbol{k}) \right\rangle , \tag{3.9}$$

with $Q = 1 - P$.

The collective mode frequencies follow as the eigenfrequencies of the frequency matrix for small values of the wave number k. The modes themselves are the corresponding eigenvectors. To derive the modes we start by choosing as a basis set $k^{-1}n(\boldsymbol{k})$, $\boldsymbol{g}(\boldsymbol{k})$ and $\varepsilon(\boldsymbol{k})$. The reason for the additional factor k^{-1} accompanying the particle density becomes clear upon inspection of the fluctuation formulae of the appendix. In fact, since the particle density and the charge density are proportional in a one-component plasma, fluctuations in $n(\boldsymbol{k})$ are strongly suppressed and even vanish in the limit $\boldsymbol{k} \to \boldsymbol{0}$. Since $QLn(\boldsymbol{k}) = 0$, while $QL\boldsymbol{g}(\boldsymbol{k})$ and $QL\varepsilon(\boldsymbol{k})$ are both of first order in k, the matrix $\Omega^{(2)}$ is of order k^2. The eigenvalue problem in order k^0 may hence be written as

$$\det(z\delta_{ij} - \Omega_{ij}^{(1)}) = 0 , \tag{3.10}$$

or as

$$\det\left[\frac{z}{V} \langle a_i^*(\boldsymbol{k}) a_j(\boldsymbol{k}) \rangle + \frac{1}{V} \langle a_i^*(\boldsymbol{k}) L a_j(\boldsymbol{k}) \rangle \right] = 0 . \tag{3.11}$$

The averages may be evaluated by using the balance equations of section 2 and the fluctuation formulae of the appendix. The solutions of (3.11) in order k^0 are found as:

$$z_{\mathrm{T}}^{(0)} = 0\,, \tag{3.12}$$

$$z_{\lambda\rho}^{(0)} = \rho w_\lambda\,, \tag{3.13}$$

with $\lambda = \pm 1$, $\rho = \pm 1$. Here the frequencies w_λ are given by

$$w_\lambda = \tfrac{1}{2}(\omega_{\mathrm{p}}^2 + \omega_B^2 + 2\omega_{\mathrm{p}}\omega_B \hat{k}_{\|})^{1/2} + \tfrac{1}{2}\lambda(\omega_{\mathrm{p}}^2 + \omega_B^2 - 2\omega_{\mathrm{p}}\omega_B \hat{k}_{\|})^{1/2}\,, \tag{3.14}$$

with $\hat{k}_{\|} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{B}} = \boldsymbol{k} \cdot \hat{\boldsymbol{B}}/k$ and ω_{p} the plasma frequency. They are solutions of the equation

$$w_\lambda^4 - (\omega_{\mathrm{p}}^2 + \omega_B^2) w_\lambda^2 + \omega_{\mathrm{p}}^2 \omega_B^2 \hat{k}_{\|}^2 = 0\,. \tag{3.15}$$

The eigenvectors corresponding to the eigenvalues (3.12) and (3.13) have the form:

$$a_{\mathrm{T}}^{(0)}(\boldsymbol{k}) = C_{\mathrm{T}}\, \varepsilon(\boldsymbol{k})\,, \tag{3.16}$$

$$a_{\lambda\rho}^{(0)}(\boldsymbol{k}) = C_\lambda \left[\frac{k_{\mathrm{D}}}{k}\, n(\boldsymbol{k}) + \frac{1}{(m k_B T)^{1/2}}\, \boldsymbol{v}_{\lambda\rho}(\boldsymbol{k}) \cdot \boldsymbol{g}(\boldsymbol{k}) \right], \tag{3.17}$$

where k_{D} is the Debye wave vector and where $\boldsymbol{v}_{\lambda\rho}(\boldsymbol{k})$ is given by

$$\boldsymbol{v}_{\lambda\rho}(\boldsymbol{k}) = \frac{\rho w_\lambda \omega_{\mathrm{p}}}{w_\lambda^2 - \omega_B^2}\, \hat{\boldsymbol{k}}_\perp + \frac{\rho \omega_{\mathrm{p}}}{w_\lambda}\, \hat{\boldsymbol{k}}_{\|} - \frac{\mathrm{i}\omega_{\mathrm{p}}\omega_B}{w_\lambda^2 - \omega_B^2}\, \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}}\,, \tag{3.18}$$

with $\hat{\boldsymbol{k}}_{\|} = \hat{k}_{\|}\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{k}}_\perp = \hat{\boldsymbol{k}} - \hat{\boldsymbol{k}}_{\|}$. The normalization constants are chosen such that (3.1) is satisfied up to order k^0. It should be noted here that the adjoints $\overline{a}_i^{(0)}(\boldsymbol{k})$ are equal to $a_i^{(0)}(\boldsymbol{k})$ (up to terms of order k^0), since the Liouville operator L is hermitian. Using the fluctuation formulae of the appendix one gets:

$$C_{\mathrm{T}} = \frac{1}{(n k_B c_V)^{1/2} T}\,, \tag{3.19}$$

$$C_\lambda = \frac{\omega_{\mathrm{p}}}{(2n)^{1/2}} \left[\frac{w_\lambda^2 - \omega_B^2 \hat{k}_{\|}^2}{w_\lambda^2(\omega_{\mathrm{p}}^2 + \omega_B^2) - 2\omega_{\mathrm{p}}^2\omega_B^2\hat{k}_{\|}^2} \right]^{1/2}. \tag{3.20}$$

Choosing the set (3.16), (3.17) as a new basis set one may solve the eigenvalue problem (3.11) in first order of k. The eigenvalues are then still given by (3.12) and (3.13). The eigenvectors, however, get additional terms, so that the modes up to first order in k read:

$$a_{\mathrm{T}}(\boldsymbol{k}) = C_{\mathrm{T}}[\varepsilon(\boldsymbol{k}) - hn(\boldsymbol{k})] , \tag{3.21}$$

$$a_{\lambda\rho}(\boldsymbol{k}) = C_{\lambda}\left[\frac{k_{\mathrm{D}}}{k}\, n(\boldsymbol{k}) + \frac{1}{k_{\mathrm{B}}Tc_V}\,(\tfrac{1}{3}c_V + \tfrac{1}{2}k_{\mathrm{B}})\,\frac{k}{k_{\mathrm{D}}}\,\varepsilon(\boldsymbol{k}) + \frac{1}{(mk_{\mathrm{B}}T)^{1/2}}\,\boldsymbol{v}_{\lambda\rho}(\boldsymbol{k})\cdot \boldsymbol{g}(\boldsymbol{k})\right] . \tag{3.22}$$

Here h is the enthalpy per particle, which is related to the specific heat c_V and the isothermal compressibility κ_T of the plasma:

$$h = -k_{\mathrm{B}}T + \tfrac{1}{3}c_V T + 3/(n\kappa_T) . \tag{3.23}$$

The corrections to the mode frequencies that are of order k^2 follow directly by applying perturbation theory. In this way one obtains up to order k^2:

$$z_i = \left[-\frac{1}{V}\,\langle a_i^*(\boldsymbol{k})La_i(\boldsymbol{k})\rangle + \frac{1}{V}\left\langle a_i^*(\boldsymbol{k})LQ\,\frac{1}{z_i^{(0)} + QLQ}\,QLa_i(\boldsymbol{k})\right\rangle\right] \times \left[\frac{1}{V}\,\langle a_i^*(\boldsymbol{k})a_i(\boldsymbol{k})\rangle\right]^{-1} , \tag{3.24}$$

where the modes (3.21) and (3.22) in first order in k, and the mode frequencies (3.12) and (3.13) in zeroth order should be inserted. Evaluation of the right-hand side up to terms of order k^2 gives:

$$z_{\mathrm{T}} = \frac{1}{V}\left\langle a_{\mathrm{T}}^*(\boldsymbol{k})LQ\,\frac{1}{z + QLQ}\,QLa_{\mathrm{T}}(\boldsymbol{k})\right\rangle , \tag{3.25}$$

$$z_{\lambda\rho} = \rho w_{\lambda}\left[1 + \tfrac{1}{2}k^2c_{\mathrm{s}}^2\,\frac{w_{\lambda}^2 - \omega_B^2\hat{k}_{\parallel}^2}{w_{\lambda}^2(\omega_{\mathrm{p}}^2 + \omega_B^2) - 2\omega_{\mathrm{p}}^2\omega_B^2\hat{k}_{\parallel}^2}\right] + \frac{1}{V}\left\langle a_{\lambda\rho}^*(\boldsymbol{k})LQ\,\frac{1}{z + QLQ}\,QLa_{\lambda\rho}(\boldsymbol{k})\right\rangle . \tag{3.26}$$

Here c_{s} is the sound velocity, which is given by

$$\frac{mc_{\mathrm{s}}^2}{k_{\mathrm{B}}T} = \frac{1}{nk_{\mathrm{B}}T\kappa_T} + \frac{1}{k_{\mathrm{B}}c_V}\,(\tfrac{1}{3}c_V + \tfrac{1}{2}k_{\mathrm{B}})^2 . \tag{3.27}$$

The dependence of the indirect parts of the mode frequencies (3.25) and (3.26) on the wave vector $\boldsymbol{k}$ can be analyzed by using the balance equations and the symmetry properties of the system. By substituting (3.21) with (3.19) and using the balance equations (2.2) and (2.11) the frequency of the thermal mode gets the form:

$$z_{\mathrm{T}} = \frac{k^2}{nk_{\mathrm{B}}c_V T^2} \lim_{z\to \mathrm{i}0} \lim_{k\to 0} \frac{1}{k^2}\, \frac{1}{V} \left\langle \boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon^*(\boldsymbol{k}) Q\, \frac{1}{z + QLQ}\, Q\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon(\boldsymbol{k}) \right\rangle . \tag{3.28}$$

As a consequence of rotation invariance the average at the right-hand side depends on $\boldsymbol{k}$ and $\hat{\boldsymbol{B}}$ only through the scalar product $\boldsymbol{k}\cdot\hat{\boldsymbol{B}}$, while parity implies that the average is invariant under the interchange $\boldsymbol{k} \leftrightarrow -\boldsymbol{k}$. Hence we may write up to second order in k

$$\begin{aligned} &\frac{1}{V} \left\langle \boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon^*(\boldsymbol{k}) Q\, \frac{1}{z + QLQ}\, Q\boldsymbol{k}\cdot \boldsymbol{j}_\varepsilon(\boldsymbol{k}) \right\rangle \\ &\quad = -\mathrm{i}k_{\mathrm{B}}T^2[k_\perp^2 \lambda_\perp(z) + k_\|^2 \lambda_\|(z)] , \end{aligned} \tag{3.29}$$

with dynamical (generalized) longitudinal and transverse thermal conductivity coefficients $\lambda_\|(z)$ and $\lambda_\perp(z)$. These fulfil the reality conditions

$$\lambda_i^*(z) = \lambda_i(-z^*) , \quad \mathrm{Im}\, z > 0 . \tag{3.30}$$

At constant z the coefficients $\lambda_i(z)$ depend parametrically on the field strength or on the Larmor frequency ω_B. One may show from (3.29) that $\lambda_i(z, \omega_B)$ is even in ω_B; this property will be used later.

The mode frequency (3.28) becomes upon insertion of (3.29):

$$z_{\mathrm{T}} = \frac{-\mathrm{i}k^2}{nc_V} (\hat{k}_\perp^2 \lambda_\perp^{\mathrm{s}} + \hat{k}_\|^2 \lambda_\|^{\mathrm{s}}) , \tag{3.31}$$

with $\lambda_i^{\mathrm{s}} = \lim_{z\to \mathrm{i}0} \lambda_i(z)$ the static thermal conductivities.

The indirect part of the mode frequencies $z_{\lambda\rho}$ can be analyzed in a similar way. From (3.22) and (2.4) one obtains up to second order in k:

$$\begin{aligned} &\frac{1}{V} \left\langle a_{\lambda\rho}^*(\boldsymbol{k}) LQ\, \frac{1}{z + QLQ}\, QLa_{\lambda\rho}(\boldsymbol{k}) \right\rangle \\ &\quad = \frac{C_\lambda^2}{mk_{\mathrm{B}}T}\, \frac{1}{V} \left\langle \left[\boldsymbol{k}\cdot \boldsymbol{\tau}(\boldsymbol{k})\cdot \boldsymbol{v}_{\lambda\rho}(\boldsymbol{k}) \right]^* Q\, \frac{1}{z + QLQ}\, Q\boldsymbol{k}\cdot \boldsymbol{\tau}(\boldsymbol{k})\cdot \boldsymbol{v}_{\lambda\rho}(\boldsymbol{k}) \right\rangle . \end{aligned} \tag{3.32}$$

Employing again the invariance with respect to rotations and parity, and using moreover time reversal invariance, one may write up to second order in k:

$$\frac{1}{V}\left\langle \boldsymbol{k}\cdot\boldsymbol{\tau}^*(\boldsymbol{k})Q\,\frac{1}{z+QLQ}\,Q\boldsymbol{k}\cdot\boldsymbol{\tau}(\boldsymbol{k})\right\rangle = -\mathrm{i}k_\mathrm{B}Tk^2\boldsymbol{T}(\hat{\boldsymbol{k}}, z)\,, \tag{3.33}$$

with

$$\begin{aligned} T_{ij}(\hat{\boldsymbol{k}}, z) &= f_1(z)\delta_{ij} + f_2(z)\hat{k}_i\hat{k}_j + f_3(z)(\hat{k}_i\hat{B}_j + \hat{k}_j\hat{B}_i)\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}} \\ &\quad + f_4(z)[\hat{B}_i\hat{B}_j + \delta_{ij}(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}})^2] + f_5(z)\hat{B}_i\hat{B}_j(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}})^2 \\ &\quad + f_6(z)[\hat{k}_i(\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}})_j - (\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}})_i\hat{k}_j - \varepsilon_{ijm}\hat{B}_m] \\ &\quad + f_7(z)[\hat{B}_i(\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}})_j - (\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}})_i\hat{B}_j - \varepsilon_{ijm}\hat{B}_m\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}}]\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}}\,. \end{aligned} \tag{3.34}$$

The coefficients f_i depend on z and on the Larmor frequency ω_B. They satisfy the reality conditions

$$f_i^*(z) = f_i(-z^*)\,, \quad \operatorname{Im} z > 0\,. \tag{3.35}$$

Furthermore, for $i = 1, \ldots, 5$ the coefficients are even in ω_B, at fixed z, whereas they are odd in ω_B for $i = 6, 7$. Instead of $f_i(z)$ one may introduce dynamical viscosity coefficients by writing[1])

$$\begin{aligned} &f_1 = -\eta_1 + 2\eta_2\,, \quad f_2 = \tfrac{1}{3}\eta_1 + \eta_V - 2\zeta\,, \quad f_3 = -\eta_1 + \eta_3 + 3\zeta\,, \\ &f_4 = \eta_1 - 2\eta_2 + \eta_3\,, \quad f_5 = 2\eta_1 + 2\eta_2 - 4\eta_3\,, \quad f_6 = \tfrac{1}{2}\eta_4\,, \\ &f_7 = -\tfrac{1}{2}\eta_4 - \eta_5\,. \end{aligned} \tag{3.36}$$

The coefficients $\eta_1, \ldots, \eta_5$ are the shear viscosities, η_V is the volume viscosity, while ζ describes a cross effect between shear stresses and volume strains and vice versa[7]). The reality conditions for the viscosity coefficients are similar to (3.35). At fixed z the viscosities η_1, η_2, η_3, η_V, ζ are even in ω_B, whereas η_4, η_5 are odd.

Substituting (3.18) and (3.33), with (3.34) and (3.36), into (3.26) with (3.32) we obtain the mode frequencies:

$$
\begin{aligned}
z_{\lambda\rho} = \rho w_\lambda \Big[1 &+ \tfrac{1}{2} k^2 c_s^2 \frac{w_\lambda^2 - \omega_B^2 \hat{k}_\parallel^2}{w_\lambda^2(\omega_p^2 + \omega_B^2) - 2\omega_p^2 \omega_B^2 \hat{k}_\parallel^2} \Big] \\
&+ \frac{k^2}{2nm[w_\lambda^2(\omega_p^2 + \omega_B^2) - 2\omega_p^2 \omega_B^2 \hat{k}_\parallel^2]} \\
&\times \{ \rho w_\lambda^3 \omega_B [2(\eta_4 + \eta_5)\hat{k}_\parallel^2 - 2\eta_4] \\
&+ \mathrm{i} w_\lambda^2 \{ \omega_p^2 [-2(\eta_1 + \eta_2 - 2\eta_3)\hat{k}_\parallel^4 \\
&+ 2(2\eta_2 - 2\eta_3 - 3\zeta)\hat{k}_\parallel^2 + \tfrac{2}{3}\eta_1 - 2\eta_2 - \eta_V + 2\zeta] \\
&+ \omega_B^2 [(-\tfrac{5}{3}\eta_1 + 4\eta_2 - 2\eta_3 + \eta_V - 2\zeta)\hat{k}_\parallel^2 \\
&+ \tfrac{5}{3}\eta_1 - 4\eta_2 - \eta_V + 2\zeta]\} \\
&+ \rho w_\lambda \omega_p^2 \omega_B \hat{k}_\parallel^2 [-2(\eta_4 + 2\eta_5)\hat{k}_\parallel^2 + 2(\eta_4 + \eta_5)] \\
&+ \mathrm{i} \omega_p^2 \omega_B^2 \hat{k}_\parallel^2 [(3\eta_1 - 4\eta_2 + \eta_3 + 6\zeta)\hat{k}_\parallel^2 \\
&- \tfrac{5}{3}\eta_1 + 4\eta_2 + \eta_3 + \eta_V - 2\zeta]\} , \qquad (3.37)
\end{aligned}
$$

where the viscosities η_i, η_V, ζ are to be evaluated at $z = \rho w_\lambda$.

The expressions (3.31) and (3.37) for the mode frequencies may be compared to those obtained by alternative methods[1]). A macroscopic magnetohydrodynamical treatment leads to expressions for the mode frequencies of a similar form. However, they contain phenomenological transport coefficients that are static real quantities, defined at frequency zero. On the other hand, in (3.37) dynamical complex-valued viscosities at the finite frequency ρw_λ show up.

Another method to derive the mode spectrum is furnished by the kinetic theory for time correlation functions. The results obtained in this way contain frequency-dependent transport coefficients that are given in terms of matrix elements of a kinetic kernel. The connexion with the results obtained here is somewhat more subtle; it will be discussed in section 7.

4. The collective modes with a wave vector perpendicular to the magnetic field

The derivation of the collective modes, as given in the previous section, breaks down if the wave vector and the magnetic field are orthogonal. In fact, in that case $k_\parallel$ vanishes, so that one gets $w_\lambda = 0$ for $\lambda = -1$. As a consequence, the zeroth-order mode frequencies (3.12)–(3.13) form a degenerate spectrum: three eigenvalues vanish, while the remaining two are given by

$$z_\rho^{(0)} = \rho(\omega_\mathrm{p}^2 + \omega_B^2)^{1/2}\,, \tag{4.1}$$

with $\rho = \pm 1$. The modes corresponding to the latter are

$$a_\rho^{(0)}(\boldsymbol{k}) = C\left\{\frac{k_\mathrm{D}}{k}\, n(\boldsymbol{k}) + \frac{1}{(mk_\mathrm{B}T)^{1/2}}\left[\frac{\rho(\omega_\mathrm{p}^2 + \omega_B^2)^{1/2}}{\omega_\mathrm{p}}\,\hat{\boldsymbol{k}} - \frac{\mathrm{i}\omega_B}{\omega_\mathrm{p}}\,\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}}\right]\cdot \boldsymbol{g}(\boldsymbol{k})\right\}, \tag{4.2}$$

up to zeroth order in k; the normalization constant follows from (3.20) by putting $\lambda = 1$ and $\hat{k}_\parallel = 0$. As an independent set of eigenvectors for the eigenvalue $z^{(0)} = 0$ one may choose:

$$a_\mathrm{T}^{(0)}(\boldsymbol{k}) = C_\mathrm{T}\,\varepsilon(\boldsymbol{k})\,, \tag{4.3}$$

$$a_\mathrm{V}^{(0)}(\boldsymbol{k}) = C_\mathrm{V}\, g_\parallel(\boldsymbol{k})\,, \tag{4.4}$$

$$a_\mathrm{C}^{(0)}(\boldsymbol{k}) = C_\mathrm{C}\left[\frac{k_\mathrm{D}}{k}\, n(\boldsymbol{k}) + \frac{\mathrm{i}}{(mk_\mathrm{B}T)^{1/2}}\,\frac{\omega_\mathrm{p}}{\omega_B}\,(\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}})\cdot \boldsymbol{g}(\boldsymbol{k})\right], \tag{4.5}$$

with normalization constants given by (3.19) and

$$C_\mathrm{V} = \frac{1}{(nmk_\mathrm{B}T)^{1/2}}\,, \tag{4.6}$$

$$C_\mathrm{C} = \frac{1}{n^{1/2}}\,\frac{\omega_B}{(\omega_\mathrm{p}^2 + \omega_B^2)^{1/2}}\,. \tag{4.7}$$

In first order in k the eigenvalue problem still leads to a degenerate spectrum. The eigenvalues remain unchanged. However, the eigenvectors (4.2) and (4.3) acquire additional terms of first order in k:

$$\delta a_\rho(\boldsymbol{k}) = C\,\frac{1}{k_\mathrm{B}Tc_V}\,(\tfrac{1}{3}c_V + \tfrac{1}{2}k_\mathrm{B})\,\frac{k}{k_\mathrm{D}}\,\varepsilon(\boldsymbol{k})\,, \tag{4.8}$$

$$\delta a_\mathrm{T}(\boldsymbol{k}) = C_\mathrm{T}\left[-h\,\frac{\omega_\mathrm{p}^2}{\omega_\mathrm{p}^2 + \omega_B^2}\, n(\boldsymbol{k}) + \mathrm{i}h\,\frac{\omega_B k}{m(\omega_\mathrm{p}^2 + \omega_B^2)}\,(\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}})\cdot \boldsymbol{g}(\boldsymbol{k})\right]. \tag{4.9}$$

The degeneracy is removed completely, if second order terms in k are included. The mode frequencies are then found as:

$$z_\mathrm{T} = -\,\frac{\mathrm{i}k^2}{nc_V}\,\lambda_\perp^\mathrm{s}\,, \tag{4.10}$$

$$z_V = -\frac{ik^2}{nm}\,\eta_3^s\,, \tag{4.11}$$

$$z_C = -\frac{ik^2}{nm}\,\frac{\omega_p^2}{\omega_p^2+\omega_B^2}\,(-\eta_1^s + 2\eta_2^s)\,, \tag{4.12}$$

$$\begin{aligned} z_\rho = {} & \rho(\omega_p^2+\omega_B^2)^{1/2}\Big(1 + \frac{1}{2}\,\frac{k^2c_s^2}{\omega_p^2+\omega_B^2} - \frac{k^2\omega_B}{nm(\omega_p^2+\omega_B^2)}\,\eta_4\Big) \\ & - \frac{ik^2}{2nm}\Big[-\tfrac{2}{3}\eta_1 + 2\eta_2 + \eta_V - 2\zeta + \frac{\omega_B^2}{\omega_p^2+\omega_B^2}\,(-\eta_1+2\eta_2)\Big]. \end{aligned} \tag{4.13}$$

Both the transverse heat conductivity in (4.10) and the viscosities occurring in (4.11), (4.12) are static transport coefficients. The viscosity coefficients in (4.13), however, are dynamical quantities to be evaluated at the frequency $z = \rho(\omega_p^2 + \omega_B^2)^{1/2}$.

Since the degeneracy in the spectrum is removed in second order in k, the eigenvectors corresponding to the frequencies (4.10)–(4.12) are uniquely determined. These eigenvectors are, up to first order in k, linear combinations of the expressions (4.3)–(4.5) (with the correction (4.9) included). In fact, we get:

$$a_T(\boldsymbol{k}) = C_T[\varepsilon(\boldsymbol{k}) - hn(\boldsymbol{k})]\,, \tag{4.14}$$

$$a_V(\boldsymbol{k}) = C_V g_\parallel(\boldsymbol{k})\,, \tag{4.15}$$

$$\begin{aligned} a_C(\boldsymbol{k}) = C_C\Big[& \frac{k_D}{k}\,n(\boldsymbol{k}) + \frac{1}{k_BTc_V}\,(\tfrac{1}{3}c_V + \tfrac{1}{2}k_B)\,\frac{k}{k_D}\,\varepsilon(\boldsymbol{k}) \\ & + \frac{i}{(mk_BT)^{1/2}}\,\frac{\omega_p}{\omega_B}\,(\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}})\cdot\boldsymbol{g}(\boldsymbol{k})\Big]\,, \end{aligned} \tag{4.16}$$

$$\begin{aligned} a_\rho(\boldsymbol{k}) = C\Big\{ & \frac{k_D}{k}\,n(\boldsymbol{k}) + \frac{1}{k_BTc_V}\,(\tfrac{1}{3}c_V + \tfrac{1}{2}k_B)\,\frac{k}{k_D}\,\varepsilon(\boldsymbol{k}) \\ & + \frac{1}{(mk_BT)^{1/2}}\Big[\rho\,\frac{(\omega_p^2+\omega_B^2)^{1/2}}{\omega_p}\,\hat{\boldsymbol{k}} - i\,\frac{\omega_B}{\omega_p}\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}}\Big]\cdot\boldsymbol{g}(\boldsymbol{k})\Big\}\,. \end{aligned} \tag{4.17}$$

The modes (4.15) and (4.16), with the frequencies (4.11) and (4.12), cannot be obtained from the results of the preceding section by taking the limit $k_\parallel \to 0$. The mode spectrum of a magnetized plasma thus shows a discontinuity, as has been found earlier[1]). The mode (4.15), with the frequency (4.11), is a purely viscous mode depending on the momentum density in the direction of the field (and hence perpendicular to the wave vector, as should be the case for a viscous mode). The mode (4.16), with the frequency (4.12), is the so-called convective cell mode[8]). This purely dissipative mode, which is present only for wave vectors orthogonal to the magnetic field, has been the object of much discussion.

5. Modes for plasmas in a strong magnetic field

The expressions for the modes given in the previous sections simplify considerably, if the magnetic field is so strong that the Larmor frequency ω_B is much bigger than the plasma frequency ω_p. The frequencies w_λ (3.14) then reduce to:

$$w_\lambda \simeq \omega_B\,, \qquad \lambda = 1\,, \tag{5.1}$$

$$w_\lambda \simeq \omega_p \hat{k}_\parallel\,, \quad \lambda = -1\,. \tag{5.2}$$

In the case $\lambda = 1$ the modes $a_{1,\rho}(\boldsymbol{k})$ as given by (3.22) get the form:

$$a_{1,\rho}(\boldsymbol{k}) = \frac{1}{(nmk_B T)^{1/2}}\, \boldsymbol{e}_{B,\rho}(\boldsymbol{k}) \cdot \boldsymbol{g}(\boldsymbol{k})\,. \tag{5.3}$$

Here $\boldsymbol{e}_{B,\rho}(\boldsymbol{k})$ are two mutually orthogonal unit vectors in the plane perpendicular to the magnetic field:

$$\boldsymbol{e}_{B,\rho}(\boldsymbol{k}) = \frac{1}{\sqrt{2}} \left(\rho\, \frac{\boldsymbol{k}_\perp}{k_\perp} - \mathrm{i}\, \frac{\boldsymbol{k}_\perp}{k_\perp} \wedge \hat{\boldsymbol{B}} \right). \tag{5.4}$$

The frequencies of the modes (5.3) follow from (3.37) as:

$$z_{1,\rho} = \rho\omega_B - \rho\, \frac{k^2}{nm}\, [-(\eta_4 + \eta_5)\hat{k}_\parallel^2 + \eta_4] + \frac{\mathrm{i}k^2}{2nm}\, [(-\tfrac{5}{3}\eta_1 + 4\eta_2 - 2\eta_3 + \eta_V - 2\zeta)\hat{k}_\parallel^2 + \tfrac{5}{3}\eta_1 - 4\eta_2 - \eta_V + 2\zeta]\,. \tag{5.5}$$

Hence, the modes are oscillating at a frequency near the Larmor frequency. According to (5.3) the momentum density field is circularly polarized in a plane perpendicular to the magnetic field.

The modes with $\lambda = -1$ are given, for $\omega_B \gg \omega_p$, by

$$a_{-1,\rho}(\boldsymbol{k}) = \frac{1}{(2n)^{1/2}} \left[\frac{k_{\mathrm{D}}}{k}\, n(\boldsymbol{k}) + \frac{1}{k_{\mathrm{B}} T c_V} \left(\tfrac{1}{3} c_V + \tfrac{1}{2} k_{\mathrm{B}}\right) \frac{k}{k_{\mathrm{D}}}\, \varepsilon(\boldsymbol{k}) \right.$$
$$\left. + \frac{\rho}{(m k_{\mathrm{B}} T)^{1/2}}\, g_{\parallel}(\boldsymbol{k}) \right], \tag{5.6}$$

with the frequencies

$$z_{-1,\rho} = \rho \omega_{\mathrm{p}} \hat{k}_{\parallel} \left(1 + \frac{1}{2} \frac{k^2 c_{\mathrm{s}}^2}{\omega_{\mathrm{p}}^2}\right) - \frac{\mathrm{i} k^2}{2nm} \left[\left(\tfrac{4}{3}\eta_1 - \eta_3 + \eta_V + 4\zeta\right) \hat{k}_{\parallel}^2 + \eta_3 \right]. \tag{5.7}$$

In these modes only the component of the momentum density in the direction of the magnetic field contributes; the transverse components of $\boldsymbol{g}(\boldsymbol{k})$ have dropped out.

In the special case $k_{\parallel} = 0$ the modes (5.6)–(5.7) mix up to yield a viscous and a convective cell mode, as before. For strong fields the latter becomes:

$$a_{\mathrm{C}}(\boldsymbol{k}) = \frac{1}{n^{1/2}} \left[\frac{k_{\mathrm{D}}}{k}\, n(\boldsymbol{k}) + \frac{1}{k_{\mathrm{B}} T c_V} \left(\tfrac{1}{3} c_V + \tfrac{1}{2} k_{\mathrm{B}}\right) \frac{k}{k_{\mathrm{D}}}\, \varepsilon(\boldsymbol{k}) \right]. \tag{5.8}$$

The corresponding frequency

$$z_{\mathrm{C}} = -\frac{\mathrm{i} k^2 \omega_{\mathrm{p}}^2}{n m \omega_B^2} \left(-\eta_1^{\mathrm{s}} + 2\eta_2^{\mathrm{s}}\right) \tag{5.9}$$

tends to zero for $\omega_B/\omega_{\mathrm{p}} \to \infty$ (at least if the viscosities remain bounded), so that this mode becomes long-lived in the limit of strong fields.

In a recent paper[9]) expressions for the frequencies of the oscillating modes of a strongly magnetized plasma have been presented. However, the terms of order k^2 are not given explicitly in terms of the seven anisotropic viscosity coefficients, so that a comparison is difficult. Expressions for the modes that might be compared to (5.3) and (5.6) are not given either.

6. Weak magnetic field

The spectrum of modes for a plasma in a magnetic field differs qualitatively from that of an unmagnetized plasma. In the latter case three of the five modes, viz. the heat mode and two viscous modes, are purely dissipative, while the remaining two modes are oscillating plasma modes. If a magnetic field is turned on the viscous modes disappear; instead of them two oscillating modes, with a frequency determined by the Larmor frequency, show up. The heat mode retains its character of a purely dissipative mode, with a real damping coefficient proportional to k^2. To analyze the nature of the change in the mode spectrum we shall determine the modes for weak magnetic fields.

Developing the frequencies w_λ (3.14) for small values of ω_B/ω_p we get:

$$w_\lambda \simeq \omega_p\left(1 + \frac{\omega_B^2}{2\omega_p^2}\,\hat{k}_\perp^2\right), \qquad \lambda = 1\,, \tag{6.1}$$

$$w_\lambda \simeq \omega_B \hat{k}_\parallel\left(1 - \frac{\omega_B^2}{2\omega_p^2}\,\hat{k}_\perp^2\right), \quad \lambda = -1\,. \tag{6.2}$$

For $\lambda = 1$ the modes (3.22) become upon expansion in powers of ω_B/ω_p:

$$\begin{aligned} a_{1,\rho}(\boldsymbol{k}) = {} & \frac{1}{(2n)^{1/2}}\left[\frac{k_\mathrm{D}}{k}\,n(\boldsymbol{k}) + \frac{1}{k_\mathrm{B}Tc_V}\,(\tfrac{1}{3}c_V + \tfrac{1}{2}k_\mathrm{B})\,\frac{k}{k_\mathrm{D}}\,\varepsilon(\boldsymbol{k})\right]\left(1 - \frac{\omega_B^2}{2\omega_p^2}\,\hat{k}_\perp^2\right) \\ & + \frac{1}{(2nmk_\mathrm{B}T)^{1/2}}\left[\rho\hat{\boldsymbol{k}}\cdot\boldsymbol{g}(\boldsymbol{k}) - \frac{\mathrm{i}\omega_B}{\omega_p}\,(\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}})\cdot\boldsymbol{g}(\boldsymbol{k})\right. \\ & \left. + \frac{\rho\omega_B^2}{\omega_p^2}\,(\hat{k}_\parallel^2\hat{\boldsymbol{k}}_\perp - \hat{k}_\perp^2\hat{\boldsymbol{k}}_\parallel)\cdot\boldsymbol{g}(\boldsymbol{k})\right]. \end{aligned} \tag{6.3}$$

These modes are perturbed plasma oscillations. Their frequencies are, up to terms of order ω_B^2/ω_p^2:

$$\begin{aligned} z_{1,\rho} = {} & \rho\omega_p\left[1 + \frac{1}{2}\,\frac{k^2c_s^2}{\omega_p^2} + \frac{1}{2}\,\frac{\omega_B^2}{\omega_p^2}\,\hat{k}_\perp^2\left(1 - \frac{1}{2}\,\frac{k^2c_s^2}{\omega_p^2}\right)\right] \\ & + \frac{\mathrm{i}k^2}{2nm}\left\{2(-\eta_1 - \eta_2 + 2\eta_3)\hat{k}_\parallel^4\right. \\ & + 2(2\eta_2 - 2\eta_3 - 3\zeta)\hat{k}_\parallel^2 + \tfrac{2}{3}\eta_1 - 2\eta_2 - \eta_V + 2\zeta \\ & + 2\mathrm{i}\rho\,\frac{\omega_B}{\omega_p}\,[(\eta_4 + 2\eta_5)\hat{k}_\parallel^4 - 2(\eta_4 + \eta_5)\hat{k}_\parallel^2 + \eta_4] \\ & + \frac{\omega_B^2}{\omega_p^2}\,[4(-\eta_1 - \eta_2 + 2\eta_3)\hat{k}_\parallel^6 \\ & + (5\eta_1 + 6\eta_2 - 11\eta_3 - 6\zeta)\hat{k}_\parallel^4 \\ & \left. + (-2\eta_1 + 3\eta_3 + 6\zeta)\hat{k}_\parallel^2 + \eta_1 - 2\eta_2]\right\}, \end{aligned} \tag{6.4}$$

with dynamical viscosity coefficients at a finite frequency, given by $z_{1,\rho}$ for $\boldsymbol{k}=\boldsymbol{0}$.

For $\lambda=-1$ the modes (3.22) become:

$$a_{-1,\rho}(\boldsymbol{k})=\frac{1}{(nmk_{\mathrm{B}}T)^{1/2}}\,\rho\boldsymbol{e}_{\rho}(\boldsymbol{k})\cdot\boldsymbol{g}(\boldsymbol{k})$$
$$+\frac{1}{(2n)^{1/2}}\,\frac{\omega_B}{\omega_{\mathrm{p}}}\,\hat{k}_{\perp}\left[\frac{k_{\mathrm{D}}}{k}n(\boldsymbol{k})+\frac{1}{k_{\mathrm{B}}Tc_V}\,(\tfrac{1}{3}c_V+\tfrac{1}{2}k_{\mathrm{B}})\,\frac{k}{k_{\mathrm{D}}}\,\varepsilon(\boldsymbol{k})\right]$$
$$+\frac{1}{(2nmk_{\mathrm{B}}T)^{1/2}}\,\frac{\omega_B^2}{\omega_{\mathrm{p}}^2}\,\hat{k}_{\perp}[\rho\hat{k}_{\parallel}\hat{\boldsymbol{k}}-\tfrac{1}{2}\mathrm{i}(\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}})]\cdot\boldsymbol{g}(\boldsymbol{k})\,,\qquad(6.5)$$

with $\boldsymbol{e}_{\rho}$ two mutually orthogonal unit vectors in the plane perpendicular to $\boldsymbol{k}$:

$$\boldsymbol{e}_{\rho}(\boldsymbol{k})=\frac{1}{\sqrt{2}}\left(\frac{\hat{k}_{\perp}}{\hat{k}_{\parallel}}\,\hat{\boldsymbol{k}}_{\parallel}-\frac{\hat{k}_{\parallel}}{\hat{k}_{\perp}}\,\hat{\boldsymbol{k}}_{\perp}+\frac{\mathrm{i}\rho}{\hat{k}_{\perp}}\,\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}}\right).\qquad(6.6)$$

The frequencies of these modes read, up to second order in $\omega_B/\omega_{\mathrm{p}}$:

$$z_{-1,\rho}=\rho\omega_B\hat{k}_{\parallel}+\frac{\mathrm{i}k^2}{2nm}\left\{2(\eta_1+\eta_2-2\eta_3)\hat{k}_{\parallel}^4-2\mathrm{i}\rho(\eta_4+2\eta_5)\hat{k}_{\parallel}^3\right.$$
$$+3(-\eta_1+\eta_3)\hat{k}_{\parallel}^2+2\mathrm{i}\rho(\eta_4+\eta_5)\hat{k}_{\parallel}+\eta_1-2\eta_2-\eta_3$$
$$+\frac{\omega_B^2}{\omega_{\mathrm{p}}^2}\,[4(\eta_1+\eta_2-2\eta_3)\hat{k}_{\parallel}^6+3\mathrm{i}\rho(-\eta_4-2\eta_5)\hat{k}_{\parallel}^5$$
$$+(-5\eta_1-6\eta_2+11\eta_3+6\zeta)\hat{k}_{\parallel}^4+\mathrm{i}\rho(6\eta_4+7\eta_5)\hat{k}_{\parallel}^3$$
$$\left.+(2\eta_1-3\eta_3-6\zeta)\hat{k}_{\parallel}^2+\mathrm{i}\rho(-3\eta_4-\eta_5)\hat{k}_{\parallel}-\eta_1+2\eta_2]\right\}.\qquad(6.7)$$

For vanishing magnetic field the modes (6.5) reduce to viscous modes with associated transverse directions given by the unit vectors $\boldsymbol{e}_{\rho}$. The momentum density of these modes is circularly polarized in the plane orthogonal to the wave vector. The frequencies (6.7) reduce to $-\mathrm{i}k^2\eta/nm$ for $\omega_B=0$, since $\eta_1=\eta_2=\eta_3=\eta$ and $\eta_4=\eta_5=\zeta=0$ for vanishing magnetic field. For small ω_B the viscosities in (6.7) should be taken at the frequency $\rho\omega_B\hat{k}_{\parallel}$. From the reality conditions (3.35) and the even (or odd) character of the parametric field strength dependence of the viscosities η_1, η_2, η_3, η_V, ζ (or η_4, η_5) it follows that the real parts of η_1, η_2, η_3, $\mathrm{i}\eta_4$, $\mathrm{i}\eta_5$, η_V, ζ are even in ω_B, whereas their imaginary parts are odd in ω_B. As a consequence the imaginary parts of the frequencies (6.7), which represent the damping of the modes, are even in ω_B. On the other hand, the real parts of (6.7), and in particular the dispersive contributions proportional to k^2, are odd in ω_B.

In the special case of a wave vector orthogonal to the magnetic field the expressions (6.5) and (6.7) again loose their validity, since then a viscous and a convective cell mode show up. For small ω_B the convective cell mode (4.16) with (4.7) gets the form:

$$a_{\mathrm{C}}(\boldsymbol{k}) = \frac{\mathrm{i}}{(nmk_{\mathrm{B}}T)^{1/2}} \left(1 - \frac{\omega_B^2}{2\omega_{\mathrm{p}}^2}\right)(\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}}) \cdot \boldsymbol{g}(\boldsymbol{k}) + \frac{\omega_B}{\sqrt{n}\omega_{\mathrm{p}}} \left[\frac{k_{\mathrm{D}}}{k} n(\boldsymbol{k}) + \frac{1}{k_{\mathrm{B}}Tc_V} \left(\tfrac{1}{3}c_V + \tfrac{1}{2}k_{\mathrm{B}}\right) \frac{k}{k_{\mathrm{D}}} \varepsilon(\boldsymbol{k}) \right], \tag{6.8}$$

up to second order in $\omega_B/\omega_{\mathrm{p}}$. The corresponding mode frequency is given by the expanded form of (4.12). It should be noted that in the limit $\omega_B \to 0$ the convective cell mode becomes a purely viscous mode, with a linearly polarized momentum density. This is in contrast to the circularly polarized momentum density that was obtained in the $\omega_B \to 0$ limit of the modes with $k_\parallel \neq 0$.

7. Green–Kubo relations and the connexion with kinetic theory

The indirect parts of the mode frequencies (3.25) and (3.26) contain the resolvent operator $(z + QLQ)^{-1}$. Likewise, the heat conductivities and the viscosities are defined, in (3.29) and (3.33)–(3.36), in terms of this operator. To establish the connexion with the Green–Kubo expressions for the transport coefficients one may use an equality[10]) that relates the matrices $\boldsymbol{\Omega}^{(1)}$ and $\boldsymbol{\Omega}^{(2)}$, given in (3.8) and (3.9), to the matrix:

$$\bar{\Omega}_{ij}^{(2)}(\boldsymbol{k}, z) = \frac{1}{V} \left\langle \bar{a}_i^*(\boldsymbol{k}) LQ \frac{1}{z + L} QLa_j(\boldsymbol{k}) \right\rangle . \tag{7.1}$$

In fact, one has:

$$\bar{\boldsymbol{\Omega}}^{(2)} = (z - \boldsymbol{\Omega}^{(1)}) \cdot (z - \boldsymbol{\Omega}^{(1)} - \boldsymbol{\Omega}^{(2)})^{-1} \cdot \boldsymbol{\Omega}^{(2)} . \tag{7.2}$$

Since for $\boldsymbol{k} \to \boldsymbol{0}$ the matrix $\boldsymbol{\Omega}^{(2)}$ is of order k^2, while $\boldsymbol{\Omega}^{(1)}$ has elements of order k^0, this relation implies that $\bar{\boldsymbol{\Omega}}^{(2)}$ is of order k^2 as well and that:

$$\lim_{\boldsymbol{k} \to \boldsymbol{0}} \frac{1}{k^2} \bar{\Omega}_{ij}^{(2)} = \lim_{\boldsymbol{k} \to \boldsymbol{0}} \frac{1}{k^2} \Omega_{ij}^{(2)} . \tag{7.3}$$

For the thermal conductivities this formula gives in view of (3.29):

$$\begin{aligned}
&\hat{k}_{\perp}^{2}\lambda_{\perp}(z)+\hat{k}_{\parallel}^{2}\lambda_{\parallel}(z)\\
&\quad=\frac{\mathrm{i}}{k_{\mathrm{B}}T^{2}}\lim_{k\to 0}\frac{1}{V}\left\langle[Q\hat{k}\cdot j_{\varepsilon}(k)]^{*}\,\frac{1}{z+L}\,Q\hat{k}\cdot j_{\varepsilon}(k)\right\rangle\\
&\quad=\frac{1}{k_{\mathrm{B}}T^{2}}\int_{0}^{\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}zt}F^{\lambda}(\hat{k},t)\,,
\end{aligned}\tag{7.4}$$

with the time correlation function:

$$F^{\lambda}(\hat{k},t)=\lim_{k\to 0}\frac{1}{V}\left\langle[Q\hat{k}\cdot j_{\varepsilon}(k)]^{*}\,\mathrm{e}^{\mathrm{i}Lt}Q\hat{k}\cdot j_{\varepsilon}(k)\right\rangle.\tag{7.5}$$

The projected heat current is given by

$$Q\hat{k}\cdot j_{\varepsilon}(k)=\hat{k}\cdot j_{\varepsilon}(k)-(h/m)\hat{k}\cdot g(k)\,,\tag{7.6}$$

since one has for small k:

$$\begin{aligned}
&\frac{1}{V}\left\langle g^{*}(k)k\cdot j_{\varepsilon}(k)\right\rangle\\
&\quad=\frac{1}{V}\left\langle\left[k\cdot\tau^{*}(k)+ne^{2}\,\frac{k}{k^{2}}\,n^{*}(k)-\mathrm{i}\omega_{B}g^{*}(k)\wedge\hat{B}\right]\varepsilon(k)\right\rangle\\
&\quad=nk_{\mathrm{B}}Thk\,,
\end{aligned}\tag{7.7}$$

where use has been made of (2.11), (2.4), (A.4), (A.6) and (3.23).

To analyze the consequences of (7.3) for the viscosity coefficients we start from (3.32). Defining a tensor $\bar{T}(k,z)$ analogous to (3.33), with $(z+QLQ)^{-1}$ replaced by $(z+L)^{-1}$, we get:

$$v_{\lambda\rho}^{*}(k)\cdot T(\hat{k},z)\cdot v_{\lambda'\rho'}(k)=v_{\lambda\rho}^{*}(k)\cdot\bar{T}(\hat{k},z)\cdot v_{\lambda'\rho'}(k)\,.\tag{7.8}$$

Upon inserting (3.18) and choosing independently $\rho=\pm 1$ and $\rho'=\pm 1$ we obtain:

$$V_{\lambda}^{(i)}\cdot T(\hat{k},z)\cdot V_{\lambda'}^{(i')}=V_{\lambda}^{(i)}\cdot\bar{T}(\hat{k},z)\cdot V_{\lambda'}^{(i')}\,,\tag{7.9}$$

with the vectors:

$$V_{\lambda}^{(1)}=w_{\lambda}^{2}\hat{k}_{\perp}+(w_{\lambda}^{2}-\omega_{B}^{2})\hat{k}_{\parallel}\,,\tag{7.10}$$

$$V_\lambda^{(2)} = \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}} \ . \tag{7.11}$$

Since for fixed orientation of $\boldsymbol{k}$ the combinations

$$w_\lambda^2 w_{\lambda'}^2 \ , \tag{7.12}$$

$$w_\lambda^2 (w_{\lambda'}^2 - \omega_B^2) \ , \quad w_{\lambda'}^2 (w_\lambda^2 - \omega_B^2) \ , \tag{7.13}$$

$$(w_\lambda^2 - \omega_B^2)(w_{\lambda'}^2 - \omega_B^2) \ , \tag{7.14}$$

with $\lambda = \pm 1$, $\lambda' = \pm 1$, are independent, the relation (7.9) with $i = i' = 1$ yields:

$$\boldsymbol{e} \cdot \boldsymbol{T} \cdot \boldsymbol{e}' = \boldsymbol{e} \cdot \bar{\boldsymbol{T}} \cdot \boldsymbol{e}' \ , \tag{7.15}$$

with $\boldsymbol{e}$ and $\boldsymbol{e}'$ equal to $\hat{\boldsymbol{k}}_\perp$ or $\hat{\boldsymbol{k}}_\parallel$. Furthermore, by choosing in (7.9) $(i, i') = (1, 2)$, $(2, 1)$ and $(2, 2)$, respectively, one proves that (7.15) is valid generally, for $\boldsymbol{e}$ and $\boldsymbol{e}'$ any of the vectors $\hat{\boldsymbol{k}}_\perp$, $\hat{\boldsymbol{k}}_\parallel$ and $\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}}$. Hence one has:

$$\boldsymbol{T} = \bar{\boldsymbol{T}} \ , \tag{7.16}$$

so that the viscosities $\eta_i(z)$ $(i = 1, \ldots, 5, V)$ and $\zeta(z)$ are given by

$$\begin{aligned} \frac{1}{k_B T} \int_0^\infty \mathrm{d}t \, \mathrm{e}^{\mathrm{i}zt} F_{ij}^\eta(\hat{\boldsymbol{k}}, t) = {} & (-\eta_1 + 2\eta_2)\delta_{ij} + (\tfrac{1}{3}\eta_1 + \eta_V - 2\zeta)\hat{k}_i\hat{k}_j \\ & + (-\eta_1 + \eta_3 + 3\zeta)(\hat{k}_i\hat{B}_j + \hat{k}_j\hat{B}_i)\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{B}} \\ & + (\eta_1 - 2\eta_2 + \eta_3)[\hat{B}_i\hat{B}_j + \delta_{ij}(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{B}})^2] \\ & + (2\eta_1 + 2\eta_2 - 4\eta_3)\hat{B}_i\hat{B}_j(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{B}})^2 \\ & + \tfrac{1}{2}\eta_4[\hat{k}_i(\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}})_j - (\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}})_i\hat{k}_j - \varepsilon_{ijm}\hat{B}_m] \\ & + (-\tfrac{1}{2}\eta_4 - \eta_5)[\hat{B}_i(\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}})_j - (\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}})_i\hat{B}_j \\ & - \varepsilon_{ijm}\hat{B}_m\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{B}}]\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{B}} \ , \end{aligned} \tag{7.17}$$

with

$$F_{ij}^\eta(\hat{\boldsymbol{k}}, t) = \lim_{\boldsymbol{k} \to \boldsymbol{0}} \frac{1}{V} \langle [Q\hat{\boldsymbol{k}} \cdot \boldsymbol{\tau}(\boldsymbol{k})]_i^* \, \mathrm{e}^{\mathrm{i}Lt} [Q\hat{\boldsymbol{k}} \cdot \boldsymbol{\tau}(\boldsymbol{k})]_j \rangle \ . \tag{7.18}$$

The projected pressure tensor follows by writing

$$P\hat{\boldsymbol{k}}\cdot\boldsymbol{\tau}(\boldsymbol{k})=\boldsymbol{\alpha}(\boldsymbol{k})\varepsilon(\boldsymbol{k})+\boldsymbol{\beta}(\boldsymbol{k})n(\boldsymbol{k})\,. \tag{7.19}$$

The coefficients $\boldsymbol{\alpha}(\boldsymbol{k})$ and $\boldsymbol{\beta}(\boldsymbol{k})$ may be determined by requiring

$$\frac{1}{V}\langle\varepsilon^*(\boldsymbol{k})P\hat{\boldsymbol{k}}\cdot\boldsymbol{\tau}(\boldsymbol{k})\rangle=\frac{1}{V}\langle\varepsilon^*(\boldsymbol{k})\hat{\boldsymbol{k}}\cdot\boldsymbol{\tau}(\boldsymbol{k})\rangle \tag{7.20}$$

and an analogous condition, with $\varepsilon(\boldsymbol{k})$ replaced by $n(\boldsymbol{k})$. With the use of the fluctuation formulae of the appendix two equations for $\boldsymbol{\alpha}(\boldsymbol{k})$ and $\boldsymbol{\beta}(\boldsymbol{k})$ are found, the solution of which gives

$$Q\hat{\boldsymbol{k}}\cdot\boldsymbol{\tau}(\boldsymbol{k})=\hat{\boldsymbol{k}}\cdot\boldsymbol{\tau}(\boldsymbol{k})-\hat{\boldsymbol{k}}\left[\frac{1}{c_V}\left(\tfrac{1}{3}c_V+\tfrac{1}{2}k_{\mathrm{B}}\right)\varepsilon(\boldsymbol{k})\right.$$
$$\left.+\frac{3k_{\mathrm{B}}T}{2c_V}\left(\tfrac{1}{3}c_V+\tfrac{1}{2}k_{\mathrm{B}}-\frac{1}{nT\kappa_T}\right)n(\boldsymbol{k})\right], \tag{7.21}$$

up to first order in k.

Having established the connexion with the Green–Kubo formulae we wish to return to a discussion of the results obtained by means of formal kinetic theory[1]). Employing the vectors $\boldsymbol{v}_{\lambda\rho}$ as defined in (3.18) we may rewrite the mode frequencies given in (I.5.1) with (I.5.2) and (I.5.3) of ref. 1 as:

$$z_{\lambda\rho}=\rho w_\lambda\left[1+\tfrac{1}{2}k^2\frac{1}{nm\kappa_T}\,\frac{w_\lambda^2-\omega_B^2\hat{k}_\parallel^2}{w_\lambda^2(\omega_{\mathrm{p}}^2+\omega_B^2)-2\omega_{\mathrm{p}}^2\omega_B^2\hat{k}_\parallel^2}\right]$$
$$-\frac{\mathrm{i}nk_{\mathrm{B}}T}{m}k^2C_\lambda^2\left(\boldsymbol{v}_{\lambda\rho}\cdot\boldsymbol{\alpha}\cdot\boldsymbol{v}_{\lambda\rho}^*+\frac{\mathrm{i}\boldsymbol{v}_{\lambda\rho}\cdot\bar{\boldsymbol{\beta}}\bar{\boldsymbol{\beta}}\cdot\boldsymbol{v}_{\lambda\rho}^*}{z^{(0)}+\mathrm{i}\gamma}\right). \tag{7.22}$$

Here, $\boldsymbol{\alpha}$, $\bar{\boldsymbol{\beta}}$ and γ are defined as:

$$\alpha_{ij}(\hat{\boldsymbol{k}},z^{(0)})=\frac{\mathrm{i}m}{k_{\mathrm{B}}T}\lim_{\boldsymbol{k}\to\boldsymbol{0}}\frac{1}{k^2}\Omega_{ij}(\boldsymbol{k},z^{(0)})\,, \tag{7.23}$$

$$\bar{\beta}_i(\hat{\boldsymbol{k}},z^{(0)})=\left(\frac{m}{k_{\mathrm{B}}T}\right)^{1/2}\lim_{\boldsymbol{k}\to\boldsymbol{0}}\frac{1}{k}\Omega_{i4}(\boldsymbol{k},z^{(0)})+\sqrt{\tfrac{2}{3}}\hat{k}_i\,, \tag{7.24}$$

$$\gamma(\hat{\boldsymbol{k}},z^{(0)})=\mathrm{i}\lim_{\boldsymbol{k}\to\boldsymbol{0}}\Omega_{44}(\boldsymbol{k},z^{(0)})\,. \tag{7.25}$$

The frequency matrix $\Omega_{\mu\nu}(\boldsymbol{k},z)$ is given in terms of the memory kernel Σ of the kinetic equation:

$$\Omega_{\mu\nu}(\boldsymbol{k},z)=\langle\mu|\Sigma|\nu\rangle+\langle\mu|\Sigma\bar{Q}\,\frac{1}{z+\bar{Q}\Sigma\bar{Q}}\,\bar{Q}\Sigma|\nu\rangle \tag{7.26}$$

(see (I.3.6) with (I.3.2) and (I.3.4) for the notation). As the frequency in (7.22)–(7.25) one should take $z^{(0)} = \rho w_\lambda$. However, for strongly coupled plasmas collisions are expected to dominate the collective behaviour[11]). Then the coefficients $\boldsymbol{\alpha}$, $\bar{\boldsymbol{\beta}}$ and γ are well approximated by their values near $z = \mathrm{i}0$. In particular, one has in this approximation:

$$\bar{\beta}_i \simeq \sqrt{\tfrac{2}{3}}\left(\frac{1}{3}\,\frac{c_V}{k_\mathrm{B}} + \frac{1}{2}\right)\hat{k}_i\,, \tag{7.27}$$

$$\gamma \simeq \mathrm{i}z^{(0)}\left(-\frac{2}{3}\,\frac{c_V}{k_\mathrm{B}} + 1\right). \tag{7.28}$$

Inserting these expressions in (7.22) and employing (3.27) one gets

$$z_{\lambda\rho} \simeq \rho w_\lambda\left[1 + \tfrac{1}{2}k^2c_\mathrm{s}^2\,\frac{w_\lambda^2 - \omega_B^2\hat{k}_\parallel^2}{w_\lambda^2(\omega_\mathrm{p}^2 + \omega_B^2) - 2\omega_\mathrm{p}^2\omega_B^2\hat{k}_\parallel^2}\right] - \frac{\mathrm{i}nk_\mathrm{B}T}{m}\,k^2C_\lambda^2\boldsymbol{v}_{\lambda\rho}(\boldsymbol{k})\cdot\boldsymbol{\alpha}(\hat{\boldsymbol{k}}, z^{(0)})\cdot\boldsymbol{v}_{\lambda\rho}^*(\boldsymbol{k})\,. \tag{7.29}$$

On the other hand, the result (3.26) becomes upon insertion of (3.32) with (3.33):

$$z_{\lambda\rho} = \rho w_\lambda\left[1 + \tfrac{1}{2}k^2c_\mathrm{s}^2\,\frac{w_\lambda^2 - \omega_B^2\hat{k}_\parallel^2}{w_\lambda^2(\omega_\mathrm{p}^2 + \omega_B^2) - 2\omega_\mathrm{p}^2\omega_B^2\hat{k}_\parallel^2}\right] - \frac{\mathrm{i}}{m}\,k^2C_\lambda^2\boldsymbol{v}_{\lambda\rho}^*(\boldsymbol{k})\cdot\boldsymbol{T}(\hat{\boldsymbol{k}}, z^{(0)})\cdot\boldsymbol{v}_{\lambda\rho}(\boldsymbol{k})\,. \tag{7.30}$$

Both $\boldsymbol{\alpha}(\hat{\boldsymbol{k}}, z)$ and $\boldsymbol{T}(\hat{\boldsymbol{k}}, z)$ may be analyzed in terms of invariants; the coefficients in these expansions determine a set of generalized viscosity coefficients η_i, ζ, as in (3.34) with (3.36). By comparison of (7.29) and (7.30) it follows that for strongly coupled plasmas these sets of viscosity coefficients agree (the trivial factor $nk_\mathrm{B}T$ is absorbed in the definition of η_i, ζ). However, for intermediate coupling the reduced expression (7.29) is not valid and one should return to (7.22). In that case there is no simple relation between $\boldsymbol{\alpha}$ and $\boldsymbol{T}$, so that the sets of generalized viscosity coefficients, at the frequency $z^{(0)} = \rho w_\lambda$, have no simple connexion either. In particular, this means that an (approximate) evaluation of the kinetic expressions for the viscosity coefficients at finite frequency will not yield results that can be compared directly to those obtained by means of a molecular dynamics evaluation of the Green–Kubo integrals. It should be remarked that this complication does not arise for static transport coefficients. Both the heat conductivities $\lambda_\perp^\mathrm{s}$, $\lambda_\parallel^\mathrm{s}$ that occur in the heat mode and the viscosities η_3^s, $\eta_1^s - 2\eta_2^s$ that determine the damping of the viscous mode and the convective cell mode for $k_\parallel = 0$, are uniquely defined quantities, which may be evaluated either in kinetic theory or by means of the appropriate Green–Kubo expressions.

As a final remark we notice that for an unmagnetized plasma similar conclusions may be drawn. In that case dynamical transport coefficients only occur in the damping and the dispersive terms of the plasma mode frequencies. Both the shear viscosity and the heat conductivity appear as uniquely defined static quantities in the damping terms of purely diffusive modes.

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Appendix

Fluctuation formulae

Fluctuation formulae may be derived from the balance equations[6]) or alternatively from the hierachy of equations for the distribution functions[12]). In the main text the following formulae have been used:

$$\frac{1}{V}\langle n^*(\boldsymbol{k})n(\boldsymbol{k})\rangle = n\,\frac{k^2}{k_D^2}\left(1 - \frac{1}{nk_BT\kappa_T}\frac{k^2}{k_D^2}\right) + \mathcal{O}(k^6)\,, \tag{A.1}$$

$$\frac{1}{V}\langle \boldsymbol{g}^*(\boldsymbol{k})\boldsymbol{g}(\boldsymbol{k})\rangle = nmk_BT\boldsymbol{U}\,, \tag{A.2}$$

$$\frac{1}{V}\langle \varepsilon^*(\boldsymbol{k})\varepsilon(\boldsymbol{k})\rangle = nk_BT^2c_V + \mathcal{O}(k^2)\,, \tag{A.3}$$

$$\frac{1}{V}\langle n^*(\boldsymbol{k})\varepsilon(\boldsymbol{k})\rangle = 3nk_BT\left(\frac{1}{nk_BT\kappa_T} - \frac{1}{2}\right)\frac{k^2}{k_D^2} + \mathcal{O}(k^4)\,, \tag{A.4}$$

$$\frac{1}{V}\langle n^*(\boldsymbol{k})\boldsymbol{\tau}(\boldsymbol{k})\rangle = \frac{k^2}{\kappa_T k_D^2}\,\boldsymbol{U} + \mathcal{O}(k^4)\,, \tag{A.5}$$

$$\frac{1}{V}\langle \varepsilon^*(\boldsymbol{k})\boldsymbol{\tau}(\boldsymbol{k})\rangle = nk_BT^2(\tfrac{1}{3}c_V + \tfrac{1}{2}k_B)\boldsymbol{U} + \mathcal{O}(k^2)\,. \tag{A.6}$$

Here k_D is the Debye wave vector ($k_D^2 = ne^2/k_B T$, with n the density, e the charge and T the temperature), κ_T is the isothermal compressibility, c_V the specific heat (per particle) at constant volume and $\boldsymbol{U}$ the second-rank unit tensor.

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