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Brandts, J.; Cihangir, A.

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COUNTING TRIANGLES THAT SHARE THEIR VERTICES WITH THE UNIT N-CUBE

Jan Brandts, Apo Cihangir

Korteweg-de Vries Institute, University of Amsterdam
Science Park 904, 1098 XH Amsterdam, The Netherlands
janbrandts@gmail.com

Abstract

This paper is about 0/1-triangles, which are the simplest nontrivial examples of 0/1-polytopes: convex hulls of a subset of vertices of the unit n-cube I^n. We consider the subclasses of right 0/1-triangles, and acute 0/1-triangles, which only have acute angles. They can be explicitly counted and enumerated, also modulo the symmetries of I^n.

1. Introduction

A 0/1-polytope [3] is the convex hull of a subset of the set of vertices $\mathbb{B}^n$ of the unit n-cube I^n. Since I^n has 2^n vertices, the number of subsets of $\mathbb{B}^n$ equals $2^{(2^n)}$, a number that grows so quickly that the practical study of 0/1-polytopes is a complicated matter. Therefore, it is convenient to consider two 0/1-polytopes as equivalent if there exists an n-cube symmetry that maps one onto the other. The group H_n of symmetries of I^n is called the hyperoctahedral group. It is generated by the reflections in the n hyperplanes that orthogonally intersect the coordinate axes at their midpoints, and the transposition of labels of coordinate axes. The number of elements of H_n, its order, is $n!2^n$. An orbit of a 0/1-polytope under the action of H_n, or in other words, the set of images of the polytope under each of the cube's symmetries, can therefore contain at most $n!2^n$ elements. Under the proposed equivalence, the number of equivalence classes of 0/1-polytopes can, in principle, be counted using Pólya's Enumeration Theorem. This requires the explicit computation of the so-called cycle index of H_n. In [2], it is described how to compute this cycle index, but the procedure is nontrivial and does not lead to a general formula in n. Also, it does not distinguish between 0/1-polytopes whose dimension equals n, and the ones that are less-dimensional. This explains why only for $n \leq 6$ it is known how many equivalence classes of n-dimensional 0/1-polytopes exist.

1.1. Goal and outline of this paper

In this paper, we will fully characterize the 0/1-polytopes that are the convex hull of three different vertices of I^n, the 0/1-triangles. Next to individual vertices and line segments, these are the simplest 0/1-polytopes. We will count the number of 0/1-triangles in I^n, and also the number of elements in the disjoint subsets of right and acute 0/1-triangles. This will be done in Section 2. In Section 3 we will count the number of 0/1-equivalence classes of such triangles. We will also enumerate them, by which we mean that we list from each equivalence class a unique member.

2. Counting and enumerating all 0/1-triangles in I^n

Let $n \geq 2$. A 0/1-triangle is the convex hull of three distinct vertices of the unit n-cube I^n. We will write Δ_n for the set of 0/1-triangles in I^n. A first observation is that no three vertices of I^n lie on the same line, and thus that each $\mathcal{T} \in \Delta_n$ is nondegenerate. A second observation is that each $\mathcal{T} \in \Delta_n$ is nonobtuse, by which we mean that all its angles are less than or equal to 90°. This is true because the inner product between two vectors $u, v \in \mathbb{B}^n$, the set of 0/1-vectors of length n representing the vertices of I^n, is nonnegative. Thus, any angle between two edges that meet at the origin is nonobtuse. By symmetry, this also holds for angles located at other vertices of I^n. This leads to the following proposition.

Proposition 2.1 *The number $|\Delta_n|$ of elements of the set Δ_n of 0/1-triangles in I^n equals*

$$|\Delta_n| = \binom{2^n}{3} = \frac{1}{6} 2^n \left(2^n - 1\right)\left(2^n - 2\right). \tag{1}$$

and each $\mathcal{T} \in \Delta_n$ is nondegenerate, and moreover nonobtuse.

We can divide the triangles in Δ_n into two subsets, the subset R_n of right triangles, and the subset A_n of acute triangles, which are the triangles that have three acute angles,

$$\Delta_n = A_n \cup R_n \quad \text{and} \quad A_n \cap R_n = \emptyset,$$

with as immediate consequence that

$$|\Delta_n| = |A_n| + |R_n|. \tag{2}$$

It is possible to count the number $|R_n|$ of right triangles, and thus to count $|A_n|$ as well.

Theorem 2.2 *The number $|R_n|$ of right 0/1-triangles in I^n equals*

$$|R_n| = 2^{n-1}\left(3^n - 2^{n+1} + 1\right). \tag{3}$$

Proof. We will first count the right triangles $\mathcal{T} \in R_n$ that have their right angle at the origin. The other two vertices $u, v \in \mathbb{B}^n$ of such a $\mathcal{T}$ are nonzero and orthogonal. If u has $k < n$ zero entries, there are $2^k - 1$ different $v \neq 0$ such that $u \perp v$. The number of $u \in \mathbb{B}^n$ with k zero entries is $\binom{n}{k}$, leading to a total of

$$\frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} (2^k - 1)$$

right triangles with right angle at the origin, where the factor of a half is due to the fact that the roles of u and v can be interchanged. As a consequence,

$$|R_n| = 2^n \cdot \frac{1}{2}\sum_{k=1}^{n-1} \binom{n}{k}(2^k - 1), \tag{4}$$

because the right angle can be located at any of the 2^n vertices of I^n. Using the binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

with $x = 2$ and $y = 1$, and also with $x = y = 1$, the expression in (4) can be easily simplified until (3) remains. This proves the theorem. $\square$

Corollary 2.3 *The number $|A_n|$ of acute 0/1-triangles in I^n equals*

$$|A_n| = \frac{1}{6}2^n\left(4^n - 3^{n+1} + 3\cdot 2^n - 1\right). \tag{5}$$

Proof. Substitute expressions (1) and (3) into (2) and rearrange some terms. $\square$

The following table gives the values of $|\Delta_n|, |R_n|$ and $|A_n|$ for small values of n. The asymptotic behavior of $|R_n| = \mathcal{O}(n^6)$ and $|A_n| = \mathcal{O}(n^8)$ is clearly visible.

n	$\lvert R_n\rvert$	$\lvert A_n\rvert$	$\lvert\Delta_n\rvert$
2	4	0	4
3	48	8	56
4	400	160	560
5	2880	2080	4960
6	19264	22400	41664
7	123648	217728	341376
8	774400	1989120	2763520
9	4776960	17461760	22238720
10	29185024	149248000	178433024

Neither $|R_n|$ nor $|A_n|$ is mentioned in the Online Encyclopedia of Integer Sequences (OEIS). But the scaled sequence $|R_n|/2^{n-1}$ can be found under label A028243 and has annotation *essentially Stirling numbers of second kind*, whereas $|A_n|/2^n$ has label A000453, *Stirling numbers of the second kind, $S(n, 4)$*.

3. Counting and enumerating modulo cube symmetries

In the previous section we described how to generate and count right and acute 0/1-triangles. We did not take into account 0/1-equivalence, as described in Section 1. This will be done here. We will count the number of 0/1-equivalence classes of right and acute 0/1-triangles, and explicitly give one representative for each equivalence class.

3.1. Matrix representation and $0/1$-equivalence

Apart from the empty set, we will represent a $0/1$-polytope $\mathcal{P} \subset I^n$ by a $0/1$-matrix P of size $n \times p$ whose columns are the p coordinate vectors in $\mathbb{B}^n$ of its vertices. Since we do not allow multiple vertices, this can be done in exactly $p!$ different ways. If we assign to each $n \times p$ $0/1$-matrix U an integer vector

$$\nu(U) = \mathcal{S}(v_n^\top U), \quad \text{where} \quad v_n^\top = \left(1, 2, 4, \ldots, 2^{n-1}\right), \tag{6}$$

and where $\mathcal{S}$ sorts the integer vector in its argument in increasing order, we see that each matrix representation P of $\mathcal{P}$ has the same vector value $\nu(P)$. Moreover, if the $0/1$ polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ are distinct subsets of I^n, then their vertex sets are distinct [3], and hence for given matrix representations P_1 of $\mathcal{P}_1$ and P_2 of $\mathcal{P}_2$ we have that $\nu(P_1) \neq \nu(P_2)$. Therefore, with a slight abuse of notation, we will also consider ν as an injective map on the set of all nonempty $0/1$-polytopes into the set consisting of all vectors up to length $2^n - 1$.

Let P_1 be a matrix representing a $0/1$-polytope $\mathcal{P}_1$. Then $\mathcal{P}_2$ is a $0/1$-polytope that is $0/1$-equivalent to $\mathcal{P}_1$ if and only if $\mathcal{P}_2$ has a matrix representation P_2 that can be transformed into P_1 by permuting and negating some rows of P_2. A row negation is to replace the zeros by ones, and the ones by zeros within a row. The negation of row j corresponds to the reflection of I^n into the hyperplane with equation $2x_j = 1$, whereas the exchange of rows i and j corresponds to the relabeling of coordinate axes i and j.

Definition 3.1 The minimal representative within the $0/1$-equivalence class $\mathcal{E}(\mathcal{P})$ of a given $0/1$-polytope $\mathcal{P}$ is the unique element $\mathcal{P}^* \in \mathcal{E}(\mathcal{P})$ for which $\nu(\mathcal{P}^*)$ is lexicographically smaller than $\nu(\mathcal{P})$ for all $\mathcal{P} \in \mathcal{E}(\mathcal{P}), \mathcal{P} \neq \mathcal{P}^*$. The minimal matrix representation P^* of the equivalence class $\mathcal{E}(\mathcal{P})$ is the matrix representation P^* for $\mathcal{P}^*$ for which $v^\top P^*$ is increasing.

In the following section we will see some examples of matrix representations and of geometrical invariants under cube symmetries.

3.2. Congruence versus $0/1$-equivalence

If two $0/1$-polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ are $0/1$-equivalent, $\mathcal{P}_1$ can be transformed into $\mathcal{P}_2$ by a cube symmetry, which is a congruence. Conversely, it is well known that congruent $0/1$-polytopes need not be $0/1$-equivalent. An example, adapted from [3], is given by the two full-dimensional 5-simplices in I^5 represented by the matrices

$$P_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

If for $j \in \{1, 2\}$ we write R_j for P_j with its first column removed, then $R_1^\top R_1 = R_2^\top R_2$ and since R_2 is invertible, $(R_1 R_2^{-1})^\top (R_1 R_2^{-1}) = I$. Thus $Q = R_1 R_2^{-1}$ is orthogonal, and we conclude that $R_1 = QR_2$ and hence $P_1 = QP_2$, proving the congruence. To disprove 0/1-equivalence, consider the effect of cube symmetries on the vector of row sums of a matrix. Row permutations do not alter the values, only permute them, whereas a row negation replaces a row sum s by $p - s$, where p is the number of columns. Since P_2 has two row sums equal to 1, whereas P_1 has only one row sum equal to one and no row sum equal to $6 - 1 = 5$, we see that $\mathcal{P}_1 \notin \mathcal{E}(\mathcal{P}_2)$. Geometrically speaking, P_2 has two exterior facets, which are facets that lie in a facet of I^5, and P_1 has only one. Obviously, cube symmetries preserve such exterior facets.

In spite of the above, it is known that equivalence does indeed hold for all full dimensional 0/1-polytopes of dimension $n \leq 4$. The full dimensionality cannot be omitted, as is shown by the following counter example,

$$P_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Both matrices represent a regular tetrahedron in I^4, but the tetrahedron at the left has a zero row, and hence lies in a cube facet (basically in I^3), whereas the right one has neither a zero row nor a row with ones. Thus, they are not 0/1-equivalent. This also shows that for 0/1-tetrahedra in I^n, congruence and 0/1-equivalence are not the same. For 0/1-triangles however, they are.

Theorem 3.2 *If* 0/1*-triangles* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are congruent, then they are* 0/1*-equivalent.*

Proof. Let $\mathcal{T}_1, \mathcal{T}_2 \in \Delta_n$ be congruent. Then their edge lengths and angles are equal. Therefore, it is possible to apply a cube symmetry S_1 to $\mathcal{T}_1$ such that the origin is a vertex of $S_1(\mathcal{T}_1)$, while its remaining vertices are $v_1, w_1 \in \mathbb{B}^n$, and then to apply a cube symmetry S_2 to $\mathcal{T}_2$ such that the origin is a vertex of $S_2(\mathcal{T}_2)$ and its remaining vertices are $v_2, w_2 \in \mathbb{B}^n$, such that

$$\|v_1\| = \|v_2\| = \sqrt{p}, \quad \|w_1\| = \|w_2\| = \sqrt{q}, \quad \text{and} \quad v_1^\top w_1 = v_2^\top w_2 = r, \tag{7}$$

for certain integers p, q, r. Due to (7), the $3 \times n$ matrices $P_1 = (0|v_1|w_1)$ representing $\mathcal{T}_1$ and $P_2 = (0|v_2|w_2)$ representing $\mathcal{T}_2$, both have r rows equal to $(0, 1, 1)$, and consequently, $p - r$ rows equal to $(0, 1, 0)$ and $q - r$ rows equal to $(0, 0, 1)$. And since P_1 and P_2 have the same rows, $\mathcal{T}_1$ and $\mathcal{T}_2$ are 0/1-equivalent. □

3.3. The minimal matrix representation for each 0/1-equivalence class

We will now formulate necessary and sufficient conditions under which a matrix is a minimal matrix representation of an equivalence class $\mathcal{E}(\mathcal{T})$. The necessity of the block form of the matrix in 8 was already described in [1] in a more general context.

Theorem 3.3 *An $n \times 3$ matrix P^* is a minimal matrix representation of an equivalence class $\mathcal{E}(\mathcal{T})$ of $0/1$-triangles in I^n if and only if*

$$P^* = \left[\begin{array}{ccc} 0 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 1 \\ \hline 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array}\right], \quad (P^*)^\top P^* = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & p & r \\ 0 & r & q \end{array}\right], \quad 1 \leq p \leq q \leq p{-}r{+}q{-}r \leq n{-}r \leq n. \tag{8}$$

Note that p, q and $p - r + q - r$ are the squares of the lengths of the edges of the triangle.

Proof. Suppose that P^* is a minimal matrix representation. Then the block form given in (8) is necessary for the following reasons. Firstly, any vertex of any triangle can be mapped onto the origin by a cube symmetry, hence the first column of P^* needs to be zero. Secondly, if there is a zero entry in the second column in row i and an entry equal to one in row j with $j > i$, interchanging rows i and j would decrease the second value in $v_n^\top P^*$ while the first value of $v_n^t P^*$ remains zero, contradicting the minimality. Further, if in the third column there is a zero entry in row i with $i < p$ and an entry equal to one in row j with $i < j \leq p$, then interchanging rows i and j would decrease the third value in $v_n^t P$ while the first value of $v_n^t P$ remains zero, and the second also remains the same because in the second column, two entries equal to one are swapped, contradicting the minimality. Finally, if in the third column there is a zero entry in row i with $i \geq p$ and an entry equal to one in row j with $i < j$, then interchanging rows i and j would decrease the third value in $v_n^\top P$ while the first value of $v_n^\top P$ remains zero, and the second also remains the same because in the second column, two entries equal to zero are swapped. This shows the necessity of the block form in (8).

Additionally, the set of inequalities $1 \leq p \leq q \leq p - r + q - r \leq n - r$ is necessary for the following reasons. Firstly, the second column of P needs to be nonzero, hence $1 \leq p$. Secondly, $p \leq q$ or otherwise swapping the block with rows (0 1 0) with the block with rows (0 0 1) followed by swapping the second and third column, would result in a zero first column, and a second column with $q < p$ entries equal to one, and this would reduce the second value of $v_n^\top P$ while the first remains zero. Thirdly, $q \leq p - r + q - r$, or equivalently $r \leq p - r$ or otherwise negating all rows that have a one in the second column, followed by interchanging the first and second column, followed by restoring the block form by interchanging the block with rows (0 1 1) with the block with rows (0 1 0), would result in a matrix with zero first and second column and also the third and fourth block of rows unchanged. However, there would be $p-r$ ones at the top of the third column instead of r, and if $r > p-r$, this would reduce the third value of $v_n^\top P^*$ while the first and second remain unchanged, contradicting the minimality. The next inequality, equivalent to $p+q-r \le n$, is necessary because $p+q-r$ is the number of nonzero rows of P^*, which must, of course, be bounded by n. Finally, the rightmost inequality is necessary because the other ones do not yet guarantee that r is nonnegative.

Now we prove that the given conditions in (8) are sufficient. Firstly, since the first entry of $v_n^\top P^*$ equals zero, this value cannot be reduced. Secondly, since the triangle has no edge with length less than $\sqrt{p}$, also the second entry of $v_n^\top P^*$ cannot be reduced. The third column of P^* represents one of the two remaining edges of the triangle. The third entry of $v_n^\top P^*$ is minimal for the edge whose inner product with the second column of p is maximal, because this minimizes the number of rows equal to (0 0 1). This follows from the requirement $r \le p - r$. $\square$

As a consequence, we can directly characterize the equivalence classes of right triangles in I^n.

Corollary 3.4 *An $n \times 3$ matrix P^* is a minimal matrix representation of an equivalence class $\mathcal{E}(\mathcal{T})$ of right 0/1-triangles in I^n if and only if (8) holds with $r = 0$.*

Proof. If (8) holds with $r = 0$, the matrix P^* in (8) obviously represents a right triangle, and due to Theorem 3.2, this representation is minimal. Conversely, suppose that P^* is a minimal representation of a right triangle. Then (8) holds due to Theorem 3.2. We will prove that additionally, $r = 0$. Writing $P^* = (0\ u\ v)$ with $u, v \in \mathbb{B}^n$, we that either $u \perp v$ or $u - v \perp u$ or $u - v \perp v$. The second of these options, $u - v \perp u$, implies that P^* has no rows equal to (0 1 0), or in other words, that $p = r$. But due to the inequality $q \le p - r + q - r$ from (8), this implies that $r = 0$. Consequently, also $p = r = 0$, contradicting $p \ge 1$. The third option $u - v \perp v$ similarly implies that P^* has no rows equal to (0 0 1), hence $q = r$, hence the inequality $p \le p - r + q - r$ from (8) implies that $r = 0$. Therefore $q = r = 0$, contradicting $q \ge 1$. The only option left is $u \perp v$, which indeed implies $r = 0$. $\square$

Corollary 3.5 *An $n \times 3$ matrix P^* is a minimal matrix representation of an equivalence class $\mathcal{E}(\mathcal{T})$ of acute 0/1-triangles in I^n if and only if (8) holds with $r > 0$.*

Proof. Follows immediately from Theorem 3.2 and Corollary 3.4. $\square$

3.4. Counting the 0/1-equivalence classes of right and acute 0/1-triangles

In order to count the number of equivalence classes of 0/1-triangles in I^n, by Theorem 3.3 we only have to count the number of triples (p, q, r) such that

$$1 \le p \le q \le p + q - 2r \le n - r \le n. \tag{9}$$

We will do this by fixing a value for r and counting the tupels (p, q) that satisfy the resulting equation. The following lemmas will be of use.

Lemma 3.6 *Let $m \geq 1$ be an integer. The number of integer tupels (a, b) satisfying*

$$1 \leq a \leq b \leq m - a \tag{10}$$

equals

$$\left\lfloor \frac{m}{2} \right\rfloor \left\lceil \frac{m}{2} \right\rceil, \tag{11}$$

where $\lfloor \cdot \rfloor$ is the floor-operator and $\lceil \cdot \rceil$ the ceil-operator.

Proof. Only for values of a with $1 \leq a \leq \lfloor m/2 \rfloor$, we have that $a \leq m - a$. The number of integers between such an a and $m - a$ equals $m + 1 - 2a$. This leads to a total of

$$\sum_{a=1}^{\lfloor m/2 \rfloor} m + 1 - 2a = \left\lfloor \frac{m}{2} \right\rfloor (m + 1) - 2 \cdot \frac{1}{2} \left\lfloor \frac{m}{2} \right\rfloor \left(\left\lfloor \frac{m}{2} \right\rfloor + 1 \right) \tag{12}$$

tupels (a, b) that satisfy (10). Using the relation

$$m = \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor, \tag{13}$$

together with Lemma 3.10, this leads, after some simplifications, to the statement. □

Corollary 3.7 *The number of $0/1$-equivalence classes of right triangles in I^n equals*

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil. \tag{14}$$

Proof. According to Corollary 3.4, we need to count to number of tupels (p, q) satisfying

$$1 \leq p \leq q \leq p + q \leq n. \tag{15}$$

Since the inequality $q \leq p + q$ is always valid, it can be removed. Thus, we only need to count the number of tupels (p, q) such that $1 \leq p \leq q \leq n - p$, which was done in Lemma 3.6. □

In the next lemma we will count equivalence classes of triangles for fixed values of $r \geq 1$. It will turn out that if $3r > n$, there are no solutions. Moreover, substituting $r = 0$ in (16) below does not yield the result of Corollary 3.7. After its proof it is explained why not.

Lemma 3.8 *For given $r \geq 1$ with $3r \leq n$, the number of tupels (p, q) satisfying (9) equals*

$$\left\lfloor \frac{n - 3r + 2}{2} \right\rfloor \left\lceil \frac{n - 3r + 2}{2} \right\rceil. \tag{16}$$

Proof. Let $r \geq 1$ be fixed. If $p < 2r$, there are no integers q that satisfy the third inequality $q \leq p + q - 2r$ in (9). If $p \geq 2r$, this inequality holds for all q and can thus be removed. Thus, we only need to count the tupels (p, q) for which

$$2r \leq p \leq q \leq n + r - p. \tag{17}$$

For such tupels to exist, we need that $2r \leq n+r-p$, but since $p \geq 2r$ this translates into $2r \leq n + r - 2r$. This explains the requirement $3r \leq n$ in the statement of this lemma. To count the tupels, subtract $2r - 1$ from each term in (17), and define $a = p - (2r - 1), b = q - (2r - 1)$, and $m = n - 3r + 2$, then

$$1 \leq a \leq b \leq n + r - a - 2(2r - 1) = n - 3r + 2 - a = m - a. \tag{18}$$

Applying Lemma 3.6 gives the number of tupels (a, b) satisfying these inequalities in terms of m, and substituting back $m = n - 3r + 2$ proves the statement. $\square$

Remark 3.9 Choosing $r = 0$ in (16) does not give (14). This is because setting $r = 0$ in (17) does not imply $1 \leq p$, as is required in Theorem 3.3, whereas for $r \geq 1$, it does.

We will now count the number of equivalence classes of acute triangles. First another lemma.

Lemma 3.10 *For nonnegative integers k we have that $(k \bmod 2)^2 = k \bmod 2$, and hence*

$$\left\lfloor \frac{k}{2} \right\rfloor \left\lceil \frac{k}{2} \right\rceil = \left(\frac{k - k \bmod 2}{2} \right) \left(\frac{k + k \bmod 2}{2} \right) = \frac{1}{4}(k^2 - k \bmod 2). \tag{19}$$

Moreover,

$$\sum_{k=1}^{n} k \bmod 2 = \left\lfloor \frac{n+1}{2} \right\rfloor, \quad \text{and} \quad \left\lfloor \frac{n - \lfloor \frac{n}{3} \rfloor}{2} \right\rfloor = \left\lfloor \frac{n+1}{3} \right\rfloor. \tag{20}$$

Proof. Elementary, and thus left to the reader. $\square$

Theorem 3.11 *The number of $0/1$-equivalence classes of acute triangles in I^n equals*

$$\left\lfloor \frac{2n^3 + 3n^2 - 6n + 9}{72} \right\rfloor. \tag{21}$$

Proof. We need to sum the expression in (16) over all $r \geq 1$ satisfying $3r \leq n$. Now, since $(n - 3r + 2) \bmod 2 = (n - r) \bmod 2$, we find using Lemma 3.10 that

$$\sum_{r=1}^{\lfloor \frac{n}{3} \rfloor} \left\lfloor \frac{n - 3r + 2}{2} \right\rfloor \left\lceil \frac{n - 3r + 2}{2} \right\rceil = \frac{1}{4} \sum_{r=1}^{\lfloor \frac{n}{3} \rfloor} (n - 3r + 2)^2 - \frac{1}{4} \sum_{r=1}^{\lfloor \frac{n}{3} \rfloor} (n - r) \bmod 2. \tag{22}$$

The first sum in the right-hand side of (22) can be evaluated using standard expressions for sums of squares as

$$\sum_{r=1}^{\lfloor \frac{n}{3} \rfloor}(n-3r+2)^2 = \left\lfloor \frac{n}{3} \right\rfloor (n+2)\left(n-1-3\left\lfloor \frac{n}{3} \right\rfloor\right) + \frac{3}{2}\left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor + 1\right)\left(2\left\lfloor \frac{n}{3} \right\rfloor + 1\right). \tag{23}$$

Using Lemma 3.10 again, the second sum in the right-hand side of (22) evaluates to

$$\sum_{r=1}^{\lfloor \frac{n}{3} \rfloor}(n-r) \bmod 2 = \sum_{r=1}^{n-1} r \bmod 2 - \sum_{r=1}^{n-\lfloor \frac{n}{3} \rfloor - 1} r \bmod 2 = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n+1}{3} \right\rfloor. \tag{24}$$

Combining (22), (23) and (24), the number of equivalence classes of acute 0/1-triangles equals

$$\frac{1}{4}\left(\left\lfloor \frac{n}{3} \right\rfloor (n+2)\left(n-1-3\left\lfloor \frac{n}{3} \right\rfloor\right) + \frac{3}{2}\left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor + 1\right)\left(2\left\lfloor \frac{n}{3} \right\rfloor + 1\right) - \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor\right). \tag{25}$$

To verify that this expression equals (21) is a tedious task, but can be done as follows. First, we substitute $n = 6k + \ell$ with $\ell \in \{0, \ldots, 5\}$ into (21), which after simplifications results in

$$6k^3 + \frac{3}{2}(2\ell+1)k^2 + \frac{1}{2}\left(\ell^2 + \ell - 1\right)k + \left\lfloor \frac{1}{36}\ell^3 + \frac{1}{24}\ell^2 - \frac{1}{12}\ell + \frac{1}{8} \right\rfloor, \tag{26}$$

where we have used that $2\ell + 1$ and $\ell^2 + \ell - 1 = \ell(\ell + 1) - 1$ are both odd, which implies that the sum of the first three terms in (26) is indeed an integer for all k and ℓ.

Next, substitute $n = 6k + \ell$ with $\ell \in \{0, 1, 2\}$ in (25), and note that it simplifies to

$$6k^3 + \frac{3}{2}(2\ell+1)k^2 + \frac{1}{2}\left(\ell^2 + \ell - 1\right)k, \tag{27}$$

which equals the expression in (26) because for $\ell \in \{0, 1, 2\}$ the floor results in zero. Finally, set $n = 6k + \ell$ with $\ell \in \{3, 4, 5\}$ in (25). After simplification there remains

$$6k^3 + \frac{3}{2}(2\ell+1)k^2 + \frac{1}{2}\left(\ell^2 + \ell - 1\right)k + \frac{1}{4}\left(\ell^2 - 2\ell + 1 - \left\lfloor \frac{\ell}{2} \right\rfloor + \left\lfloor \frac{\ell+1}{3} \right\rfloor\right). \tag{28}$$

Comparing (26) with (28), it can be easily verified that for $\ell \in \{3, 4, 5\}$,

$$\frac{1}{4}\left(\ell^2 - 2\ell + 1 - \left\lfloor \frac{\ell}{2} \right\rfloor + \left\lfloor \frac{\ell+1}{3} \right\rfloor\right) = \left\lfloor \frac{1}{36}\ell^3 + \frac{1}{24}\ell^2 - \frac{1}{12}\ell + \frac{1}{8} \right\rfloor. \tag{29}$$

And this proves the theorem. $\square$

Below are listed the numbers r_n and a_n of 0/1-equivalence classes of right and acute 0/1-triangles and their sum d_n for small values of n.

n	2	3	4	5	6	7	8	9	10
r_n	1	2	4	6	9	12	16	20	25
a_n	0	1	2	4	7	11	16	23	31
d_n	1	3	6	10	16	23	32	43	56

In the OEIS, the sequence r_n has label A002620, sequence a_n has label A181120, and d_n has label A034198. Only the latter has as description "number of distinct triangles on vertices of n-dimensional cube", the other two are not associated with counting triangles in I^n.

On a personal note

I met Karel Segeth for the first time in the beginning of October 1997, when I was 29 years old. Karel was director of the Mathematical Institute of the Academy of Sciences of the Czech Republic, and I had just arrived to take up a one year visiting position at his Institute. He invited me to his director's office for a cup of tea, and to welcome me. I recall being impressed and a bit nervous, and listened to what Professor Segeth had to say, in his typical (although at that time, of course, I did not know this) calm and amiable tone of voice. He seemed to be the type of person taking his responsibilities seriously; the greater was my surprise when he good-humouredly laughed at my humble wish to take up a Czech language course now that I had arrived in Prague, and actually rather cheekily added: "Pardon me, but I'm afraid you will never learn to speak Czech!". Notwithstanding cheekiness, he immediately organized for me to be enroled in a Czech language course provided by the Academy of Sciences, and until this day I still get goose bumps when I recall the teacher, a strict lady who asked me questions when, and only when, I had completely lost track of things. It was the beginning of my personal quest to prove Karel wrong, a quest that still goes on today, and which, of course, I can never complete. It was also the beginning of a wonderful year in Prague. When I left the institute, Karel spoke the words "Please, come again!".

And so I did. In the almost fifteen years since my first stay in Prague, I have visited the Institute many times a year. Instead of -or maybe better, next to- being an impressive director, Karel became a fellow mathematician, a trustworthy source of Czech culture and history, a fixed point in the audience of my mathematical presentations, and a good companion in not always politically correct jokes and a celebrational glass of spirit. And each time when I left, he spoke the words "Please, come again!". What choice do I have, than to follow his advice?

I wish Karel all the best, and hope to see him regularly at the Institute; at the seminar, the corridor, at Michal's office, the printer room, and to enjoy his typical humor and wisdom for many years to come. Happy seventieth birthday!

Jan Brandts

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