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An alternative integral representation for the product of two parabolic cylinder functions

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ABSTRACT

Recently, [Veestraeten D. An integral representation for the product of parabolic cylinder functions. *Integral Transforms Spec Funct.* 2017;28(1):15–21] derived an integral representation for $D_\nu(x)D_\mu(y)$ with $\text{Re}(\nu + \mu) < 1$ that was expressed in terms of the Gaussian hypergeometric function. This paper obtains an alternative expression for $D_\nu(x)D_\mu(y)$ in which the integrand contains the parabolic cylinder function itself with the condition for convergence being $\text{Re}(\nu) < 0$. The latter property is subsequently used to generate a new integral representation for $D_\mu(y)$ in which restrictions on the order μ are absent.

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1. Introduction

The study of Nicholson-type integrals for the product of two parabolic cylinder functions has a long history. In the 1930s, Meijer [1] and Bailey [2] obtained representations for $D_\nu(z e^{(1/4)\pi i})D_\nu(z e^{-(1/4)\pi i})$ and $D_\nu(z)D_{-\nu-1}(z)$. In 2003, Malyshev [3] derived integral representations for $D_\nu(x)D_\nu(\pm x)$. Later, Glasser [4] and Veestraeten [5] obtained an integral expression for the product of two parabolic cylinder functions with identical orders but unrelated arguments, $D_\nu(x)D_\nu(y)$. Subsequently, Nasri [6] derived integral representations for unrelated orders but identical or opposite arguments, $D_\nu(\pm x)D_{\nu+\mu-1}(x)$. Recently, Veestraeten [7] used the convolution theorem of the Laplace transform to obtain an integral representation in which both the arguments and the orders are unrelated, i.e. $D_\nu(x)D_\mu(y)$. The integrand in the latter expression contained a Gaussian hypergeometric function or the associated Legendre function of the first kind with the condition for convergence for the orders being $\text{Re}(\nu + \mu) < 1$. However, that expression did not specialize into any of the aforementioned results in [3–6].

This paper obtains an alternative integral representation for $D_\nu(x)D_\mu(y)$ in which the integrand contains a parabolic cylinder function and for which the condition for convergence for the orders only refers to ν with $\text{Re}(\nu) < 0$. The paper starts from the Laplace transforms for products of two parabolic cylinder functions that were obtained in [8]. In the latter paper, Laplace transforms were obtained for $D_\nu(x)D_\nu(y)$ and $D_\nu(x)D_{\nu-1}(y)$ with

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x and y being real and $x+y > 0$. It was also shown in [8] that the recurrence relation of the parabolic cylinder function allowed to extend these results towards $D_\nu(x)D_{\nu-n}(y)$ with n being an integer. The present paper shows that the recursive structure in [8] can also be used to obtain the Laplace transform of $D_\nu(x)D_\mu(y)$ in which the orders of the two parabolic cylinder functions are completely unrelated and where the arguments can also be complex numbers. After obtaining this Laplace transform, the paper proceeds by illustrating that the Laplace transforms for single parabolic cylinder functions in [9] emerge as limiting cases.

The resulting Nicholson-type integral representation for $D_\nu(x)D_\mu(y)$ is first shown to specialize into the aforementioned expressions in [3–6]. As the integral representation for $D_\nu(x)D_\mu(y)$ converges for $\text{Re}(\nu) < 0$, it can straightforwardly be simplified into a novel integral representation for $D_\mu(y)$ in which no restrictions apply to the order μ . Also, the integral representation $D_\nu(x)D_\mu(y)$ can generate expressions for $D_\nu(x)$ in which the integrand alternatively contains the exponential function, the complementary error function or the modified Bessel function of order $\frac{1}{4}$.

2. A Laplace transform for the product of two parabolic cylinder functions with unrelated orders and arguments

The following Laplace transforms for real arguments x and y with $x+y > 0$, $\beta > 0$, $c \geq 0$ and $\text{Re}(s) > 0$ are taken from Table 1 in [8]

$$\begin{aligned} &\Gamma\left(\frac{s+c}{\beta}\right) D_{-(s+c)/\beta}(x)D_{-(s+c)/\beta}(y) \\ &= \int_0^\infty \exp(-st)\beta \frac{\exp(-ct)}{(1-\exp(-2\beta t))^{1/2}} \exp\left(\frac{y^2-x^2}{4}\right) \\ &\quad \times \exp\left(-\frac{(y+x\exp(-\beta t))^2}{2(1-\exp(-2\beta t))}\right) dt \\ &\Gamma\left(\frac{s+c}{\beta}\right) D_{-(s+c)/\beta}(x)D_{-1-(s+c)/\beta}(y) = \int_0^\infty \exp(-st)\beta \exp(-ct)\sqrt{\frac{\pi}{2}} \exp\left(\frac{y^2-x^2}{4}\right) \\ &\quad \times \operatorname{erfc}\left(\frac{y+x\exp(-\beta t)}{\sqrt{2(1-\exp(-2\beta t))}}\right) dt \\ &\Gamma\left(\frac{s+c}{\beta}\right) D_{-(s+c)/\beta}(x)D_{-2-(s+c)/\beta}(y) = \int_0^\infty \exp(-st)\beta \exp(-ct) \exp\left(\frac{y^2-x^2}{4}\right) \\ &\quad \times \left\{ (1-\exp(-2\beta t))^{1/2} \exp\left(-\frac{(y+x\exp(-\beta t))^2}{2(1-\exp(-2\beta t))}\right) \right. \\ &\quad \left. - (y+x\exp(-\beta t))\sqrt{\frac{\pi}{2}} \operatorname{erfc}\left(\frac{y+x\exp(-\beta t)}{\sqrt{2(1-\exp(-2\beta t))}}\right) \right\} dt, \end{aligned}$$

where $\Gamma(z)$ and $\operatorname{erfc}(z)$ denote the gamma function and the complementary error function, respectively. Equations (9.253), (9.254.1) and (9.254.2) in [10] give the following

specializations for the parabolic cylinder function

$$\begin{aligned}
 D_0(z) &= \exp\left(-\frac{z^2}{4}\right), \\
 D_{-1}(z) &= \sqrt{\frac{\pi}{2}} \exp\left(\frac{z^2}{4}\right) \operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right), \\
 D_{-2}(z) &= \exp\left(-\frac{z^2}{4}\right) - z\sqrt{\frac{\pi}{2}} \exp\left(\frac{z^2}{4}\right) \operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right).
 \end{aligned}$$

The latter relations allow to express the integrands in the above Laplace transforms in terms of the parabolic cylinder function

$$\begin{aligned}
 \Gamma\left(\frac{s+c}{\beta}\right) D_{-(s+c)/\beta}(x) D_{-(s+c)/\beta}(y) &= \int_0^\infty \exp(-st)\beta \exp(-ct)(1 - \exp(-2\beta t))^{-1/2} \\
 &\quad \times \exp\left(-\frac{(x+y\exp(-\beta t))^2}{4(1 - \exp(-2\beta t))}\right) D_0\left(\frac{y+x\exp(-\beta t)}{\sqrt{1 - \exp(-2\beta t)}}\right) dt \\
 \Gamma\left(\frac{s+c}{\beta}\right) D_{-(s+c)/\beta}(x) D_{-1-(s+c)/\beta}(y) &= \int_0^\infty \exp(-st)\beta \exp(-ct) \\
 &\quad \times \exp\left(-\frac{(x+y\exp(-\beta t))^2}{4(1 - \exp(-2\beta t))}\right) D_{-1}\left(\frac{y+x\exp(-\beta t)}{\sqrt{1 - \exp(-2\beta t)}}\right) dt \\
 \Gamma\left(\frac{s+c}{\beta}\right) D_{-(s+c)/\beta}(x) D_{-2-(s+c)/\beta}(y) &= \int_0^\infty \exp(-st)\beta \exp(-ct)(1 - \exp(-2\beta t))^{1/2} \\
 &\quad \times \exp\left(-\frac{(x+y\exp(-\beta t))^2}{4(1 - \exp(-2\beta t))}\right) D_{-2}\left(\frac{y+x\exp(-\beta t)}{\sqrt{1 - \exp(-2\beta t)}}\right) dt.
 \end{aligned}$$

These relations then can be summarized into the following general expression:

$$\begin{aligned}
 &\Gamma\left(\frac{s+c}{\beta}\right) D_{-(s+c)/\beta}(x) D_{-q-(s+c)/\beta}(y) \\
 &= \int_0^\infty \exp(-st)\beta \exp(-ct)(1 - \exp(-2\beta t))^{q/2-1/2} \\
 &\quad \times \exp\left(-\frac{(x+y\exp(-\beta t))^2}{4(1 - \exp(-2\beta t))}\right) D_{-q}\left(\frac{y+x\exp(-\beta t)}{\sqrt{1 - \exp(-2\beta t)}}\right) dt,
 \end{aligned}$$

which also holds for non-integer real values as well as complex values of q . This relation will be rewritten by using the following property of the Laplace transform:

$$\bar{f}(\beta s) = \frac{1}{\beta} L\left\{f\left(\frac{t}{\beta}\right)\right\} \quad \text{with } \beta > 0,$$

see Equation (29.2.13) in [11], where $\bar{f}(s)$ is the Laplace transform of the original function $f(t)$. Simplifying notation then gives the following Laplace transform in which the

arguments may be complex or real:

$$\begin{aligned} \Gamma(s+c)D_{-c-s}(x)D_{q-s}(y) &= \int_0^\infty \exp(-st) \exp(-ct)(1-\exp(-2t))^{-(1/2)(1+c+q)} \\ &\times \exp\left(-\frac{(x+y\exp(-t))^2}{4(1-\exp(-2t))}\right) D_{q+c}\left(\frac{y+x\exp(-t)}{\sqrt{1-\exp(-2t)}}\right) dt \\ &\left[\operatorname{Re}(s+c) > 0, \operatorname{Im}(x) \neq 0, \operatorname{Im}(y) \neq 0, |\arg(x)| < \frac{\pi}{2}, |\arg(y)| < \frac{\pi}{2}, \right. \\ &|\arg(x) + \arg(y)| < \frac{\pi}{2} \text{ or } \operatorname{Re}(s+c) > 0, \operatorname{Im}(x) = \operatorname{Im}(y) = 0, x+y > 0 \text{ or} \\ &\left. \operatorname{Re}(s+c) > 0, \operatorname{Re}(c+q) < 1, \operatorname{Im}(x) = \operatorname{Im}(y) = 0, x+y = 0 \right]. \end{aligned} \tag{2.1}$$

The Laplace transform (2.1) can be specialized into the four expressions for single parabolic cylinder functions in [9] in which the Laplace parameter figured in the order. Hereto, the following expression for the parabolic cylinder function with zero argument will be used:

$$D_\nu(0) = \frac{2^{\nu/2}\sqrt{\pi}}{\Gamma(\frac{1-\nu}{2})},$$

see [12]. Also, the recurrence and duplication properties of the gamma function are to be used

$$\begin{aligned} \Gamma(1+z) &= z\Gamma(z), \\ \Gamma(2z) &= (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+\frac{1}{2}), \end{aligned}$$

see Equations (6.1.15) and (6.1.18) in [11]. Plugging $x=0$ into the Laplace transform (2.1) and simplifying gives

$$\begin{aligned} 2^s\Gamma(s+c)D_{q-2s}(y) &= \int_0^\infty \exp(-st)2^{-c} \exp(-ct)(1-\exp(-t))^{-(1/2)(1+2c+q)} \\ &\times \exp\left(\frac{y^2}{4(1-\exp(t))}\right) D_{q+2c}\left(\frac{y}{\sqrt{1-\exp(-t)}}\right) dt \\ &\left[\operatorname{Re}(s+c) > 0, \operatorname{Im}(y) \neq 0, |\arg(y)| < \frac{\pi}{4} \text{ or } \operatorname{Re}(s+c) > 0, \operatorname{Im}(y) = 0, y > 0 \right]. \end{aligned} \tag{2.2}$$

Equation (2.2) corresponds with the Laplace transform (3.11.6.1) in [9] and its specializations in Equations (3.11.6.2) and (3.11.6.3).

Evaluating the limit of the transform (2.1) for $y = 0$ yields

$$\begin{aligned}
 2^{-s} \frac{\Gamma(2s-c)}{\Gamma(s+q)} D_{c-2s}(x) &= \int_0^\infty \exp(-st) \frac{2^{q-3/2}}{\sqrt{\pi}} \exp\left(\frac{c}{2}t\right) (1 - \exp(-t))^{q+c/2-1} \\
 &\times \exp\left(-\frac{x^2}{4(1-\exp(-t))}\right) D_{1-2q-c}\left(\frac{x}{\sqrt{\exp(t)-1}}\right) dt \\
 &\left[\operatorname{Re}(2s-c) > 0, \operatorname{Im}(x) \neq 0, |\arg(x)| < \frac{\pi}{4} \text{ or } \operatorname{Re}(2s-c) > 0, \operatorname{Im}(x) = 0, x > 0 \right],
 \end{aligned} \tag{2.3}$$

which corresponds with the Laplace transform (3.11.6.4) in [9].

3. Integral representations of (products of) parabolic cylinder functions

Plugging $\nu = -c - s$ and $\mu = q - s$ in the Laplace transform (2.1) gives

$$\begin{aligned}
 D_\nu(x)D_\mu(y) &= \frac{1}{\Gamma(-\nu)} \int_0^\infty \exp(\nu t) (1 - \exp(-2t))^{-(1/2)(1-\nu+\mu)} \\
 &\times \exp\left(-\frac{(x+y\exp(-t))^2}{4(1-\exp(-2t))}\right) D_{\mu-\nu}\left(\frac{y+x\exp(-t)}{\sqrt{1-\exp(-2t)}}\right) dt \\
 &\left[\operatorname{Re}(\nu) < 0, \operatorname{Im}(x) \neq 0, \operatorname{Im}(y) \neq 0, |\arg(x)| < \frac{\pi}{2}, |\arg(y)| < \frac{\pi}{2}, \right. \\
 &|\arg(x) + \arg(y)| < \frac{\pi}{2} \text{ or } \operatorname{Re}(\nu) < 0, \operatorname{Im}(x) = \operatorname{Im}(y) = 0, x + y > 0 \text{ or} \\
 &\left. \operatorname{Re}(\nu) < 0, \operatorname{Re}(\mu - \nu) < 1, \operatorname{Im}(x) = \operatorname{Im}(y) = 0, x + y = 0 \right].
 \end{aligned} \tag{3.1}$$

Note that the integrand in the integral representation (3.1) contains the parabolic cylinder function, whereas the representation for $D_\nu(x)D_\mu(y)$ in Equation (3.1) in [7] was expressed in terms of the Gaussian hypergeometric function or the associated Legendre function of the first kind. The restrictions on x and y in both expressions are identical but the representations differ considerably in terms of the condition for convergence for the orders, namely $\operatorname{Re}(\nu + \mu) < 1$ in [7] versus $\operatorname{Re}(\nu) < 0$ in Equation (3.1). This property of integral representation (3.1) has interesting consequences as will be noted below.

The integral representation (3.1) specializes into the expressions there were obtained in [3–6]. Setting $\mu = \nu$ in Equation (3.1) gives

$$\begin{aligned}
 D_\nu(x)D_\nu(y) &= \frac{1}{\Gamma(-\nu)} \exp\left(\frac{y^2 - x^2}{4}\right) \int_0^\infty \frac{\exp(\nu t)}{\sqrt{1 - \exp(-2t)}} \\
 &\times \exp\left(-\frac{(y+x\exp(-t))^2}{2(1-\exp(-2t))}\right) dt \\
 &\left[\operatorname{Re}(\nu) < 0, \operatorname{Im}(x) \neq 0, \operatorname{Im}(y) \neq 0, |\arg(x)| < \frac{\pi}{2}, |\arg(y)| < \frac{\pi}{2}, \right. \\
 &\left. |\arg(x) + \arg(y)| < \frac{\pi}{2} \text{ or } \operatorname{Re}(\nu) < 0, \operatorname{Im}(x) = \operatorname{Im}(y) = 0, x + y \geq 0 \right].
 \end{aligned} \tag{3.2}$$

This expression is equivalent to Equation (2.1) in [5] that subsequently was shown to also yield the expressions for $D_\nu(x)D_\nu(\pm x)$ and $D_\nu(x)D_\nu(y)$ in [3,4], respectively. Setting $y=x$ in the integral representation (3.1), using the substitution $u = \frac{1}{2}t$ and employing the identities $\sinh(2u) = (1 - \exp(-4u))/2 \exp(-2u)$ and $\coth(u) = (1 + \exp(-2u))/(1 - \exp(-2u))$ gives

$$\begin{aligned}
 D_\nu(x)D_\mu(x) &= 2^{(1/2)(1+\nu-\mu)} \frac{1}{\Gamma(-\nu)} \int_0^\infty \exp\left((1 + \nu + \mu)u - \frac{x^2}{4} \coth(u)\right) \\
 &\quad \times D_{\mu-\nu}(x\sqrt{\coth(u)}) \sinh(2u)^{(1/2)(\nu-\mu-1)} \, du \\
 &\quad \left[\operatorname{Re}(\nu) < 0, \operatorname{Re}(\mu - 2\nu) < 0, \operatorname{Im}(x) \neq 0, |\arg(x)| < \frac{\pi}{4} \text{ or} \right. \\
 &\quad \left. \operatorname{Re}(\nu) < 0, \operatorname{Re}(\mu - 2\nu) < 0, \operatorname{Im}(x) = 0, x > 0 \right], \tag{3.3}
 \end{aligned}$$

which corresponds with Equation (2.36) in [6].

The integral representation (3.1) can be used to generate a wide variety of novel integral representations for single parabolic cylinder functions by using $x = 0, y = 0, \nu = 0, \mu = 1$, etc. For instance, plugging $\nu = -1$ into Equation (3.1), setting x at 0 and using $\operatorname{erfc}(0) = 1$ (see Equations (8.250.1) and (8.250.4) in [10]) gives

$$\begin{aligned}
 D_\mu(y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \exp(-t)(1 - \exp(-2t))^{-(1/2)(2+w)} \exp\left(-\frac{y^2 \exp(-2t)}{4(1 - \exp(-2t))}\right) \\
 &\quad \times D_{\mu+1}\left(\frac{y}{\sqrt{1 - \exp(-2t)}}\right) \, dt \\
 &\quad \left[\operatorname{Im}(y) \neq 0, |\arg(y)| < \frac{\pi}{4} \text{ or } \operatorname{Im}(y) = 0, y > 0 \right]. \tag{3.4}
 \end{aligned}$$

Note that this new integral representation for $D_\mu(y)$ has the property of converging for all values of the order μ .

Equation (3.1) can also be used to generate integral representations for single parabolic cylinder functions in which the integrand is simplified into the limiting functions of the parabolic cylinder function. Using $y=0$ in Equation (3.1) gives

$$\begin{aligned}
 D_\nu(x) &= \frac{\Gamma(\frac{1-\mu}{2})}{\sqrt{\pi}\Gamma(-\nu)} 2^{-\mu/2} \int_0^\infty \exp(\nu t)(1 - \exp(-2t))^{-(1/2)(1-\nu+\mu)} \\
 &\quad \times \exp\left(-\frac{x^2}{4(1 - \exp(-2t))}\right) D_{\mu-\nu}\left(\frac{x \exp(-t)}{\sqrt{1 - \exp(-2t)}}\right) \, dt \\
 &\quad \left[\operatorname{Re}(\nu) < 0, \operatorname{Im}(x) \neq 0, |\arg(x)| < \frac{\pi}{4} \text{ or } \operatorname{Re}(\nu) < 0, \operatorname{Im}(x) = 0, x > 0 \right]. \tag{3.5}
 \end{aligned}$$

The integrand in Equation (3.5) can be expressed in terms of the exponential function by choosing $\mu = \nu$ and $\mu = \nu + 1$, whereas choosing $\mu = \nu - 1$ gives the complementary

error function, see Equations (9.253) and (9.254.1) in [10]. The modified Bessel function of order $\frac{1}{4}$, $K_{1/4}(z)$, emerges for $\mu = \nu - \frac{1}{2}$ given $D_{-1/2}(z) = \sqrt{z/2\pi}K_{1/4}(\frac{1}{4}z^2)$, see [12]. Using these relations gives

$$D_\nu(x) = \frac{2^{1+\nu/2}}{\Gamma(-\frac{\nu}{2})} \int_0^\infty \exp(\nu t)(1 - \exp(-2t))^{-1/2} \\ \times \exp\left(-\frac{x^2(1 + \exp(-2t))}{4(1 - \exp(-2t))}\right) dt \\ \left[\operatorname{Re}(\nu) < 0, \operatorname{Im}(x) \neq 0, |\arg(x)| < \frac{\pi}{4} \text{ or } \operatorname{Re}(\nu) < 0, \operatorname{Im}(x) = 0, x > 0 \right], \quad (3.6)$$

$$D_\nu(x) = x \frac{2^{(1+\nu)/2}}{\Gamma(\frac{1-\nu}{2})} \int_0^\infty \exp((\nu - 1)t)(1 - \exp(-2t))^{-3/2} \\ \times \exp\left(-\frac{x^2(1 + \exp(-2t))}{4(1 - \exp(-2t))}\right) dt \\ \left[\operatorname{Re}(\nu) < 1, \operatorname{Im}(x) \neq 0, |\arg(x)| < \frac{\pi}{4} \text{ or } \operatorname{Re}(\nu) < 1, \operatorname{Im}(x) = 0, x > 0 \right], \quad (3.7)$$

$$D_\nu(x) = -\frac{\nu 2^{\nu/2} \sqrt{\pi} \exp\left(-\frac{x^2}{4}\right)}{\Gamma(\frac{1-\nu}{2})} \int_0^\infty \exp(\nu t) \operatorname{erfc}\left(\frac{x \exp(-t)}{\sqrt{2(1 - \exp(-2t))}}\right) dt \\ \left[\operatorname{Re}(\nu) < 0, \operatorname{Im}(x) \neq 0, |\arg(x)| < \frac{\pi}{4} \text{ or } \operatorname{Re}(\nu) < 0, \operatorname{Im}(x) = 0, x > 0 \right] \quad (3.8)$$

$$D_\nu(x) = \sqrt{x} \frac{\Gamma\left(\frac{3}{4} - \frac{\nu}{2}\right) 2^{-\nu/2-1/4}}{\pi \Gamma(-\nu)} \int_0^\infty \exp\left(\left(\nu - \frac{1}{2}\right)t\right) (1 - \exp(-2t))^{-1/2} \\ \times \exp\left(-\frac{x^2}{4(1 - \exp(-2t))}\right) K_{1/4}\left(\frac{x^2 \exp(-2t)}{4(1 - \exp(-2t))}\right) dt \\ \left[\operatorname{Re}(\nu) < 0, \operatorname{Im}(x) \neq 0, |\arg(x)| < \frac{\pi}{4} \text{ or } \operatorname{Re}(\nu) < 0, \operatorname{Im}(x) = 0, x > 0 \right]. \quad (3.9)$$

Alternative expressions in which these integrands emerge can be obtained from Equation (3.1) by setting $x=0$ and subsequently varying ν .

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No potential conflict of interest was reported by the author.

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