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Charting the q-Askey scheme

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Dedicated to Jasper Stokman on the occasion of his fiftieth birthday, in admiration and friendship

Abstract

Following Verde-Star, Linear Algebra Appl. 627 (2021), we label families of orthogonal polynomials in the q-Askey scheme together with their q-hypergeometric representations by three sequences x_k, h_k, g_k of Laurent polynomials in q^k, two of degree 1 and one of degree 2, satisfying certain constraints. This gives rise to a precise classification and parametrization of these families together with their limit transitions. This is displayed in a graphical scheme. We also describe the four-manifold structure underlying the scheme.

1 Introduction

The Askey scheme [2, p.46], [8, p.184] and the q-Askey scheme [8, p.414] display in a graphical way the families of (q-)hypergeometric orthogonal polynomials as they occur as limit cases of the four-parameter top level families: Wilson and Racah polynomials for the Askey scheme and Askey–Wilson and q-Racah polynomials for the q-Askey scheme. By each arrow to the next lower level one parameter is lost. The bottom level families no longer depend on parameters. Since their introduction these schemes have been of great assistance to everybody who needs to do work with one or more of the families in the scheme.

These schemes are also expected and partially proven to exist in other contexts, parallel to the original schemes or generalizing them. These contexts are: (i) (q-)hypergeometric biorthogonal rational functions [4]; (ii) the nonsymmetric case [12], [13]; (iii) the $q = -1$ case starting with the Bannai–Ito polynomials [3, pp. 271–273], [18]; (iv) generalized (continuous) orthogonal systems [9]; (v) q-Askey scheme for root system BC_n, see, among others, Stokman [14] and references given there.

Still some questions can be posed about the original schemes which, in the author's opinion, have not been answered in a satisfactory way until now:

1. Are the schemes complete? For answering this question one first needs a precise criterium for inclusion of a family in the scheme. This criterium is usually that the orthogonal polynomials should satisfy a Bochner type property, i.e., that they are eigenfunctions of a second order linear differential or (q-)difference operator of certain type. However, earlier classifications [7], [20] arrive, in the continuous case, at the Askey–Wilson polynomials being the most general family satisfying the requirements, but do not give an exhaustive classification of all such families. A related question is if all limits or specializations from one level to the level below are present in the scheme.

2. Which families deserve an independent status in the scheme and which ones are just subfamilies of another family? Several families in the q-Askey scheme can be considered as a subfamily, obtained by restricting the parameters, of a family higher up in the scheme. Consider for instance the continuous dual q-Hahn polynomials [8, §14.3] and the continuous q-Jacobi polynomials [8, §14.10], both being subfamilies of the Askey–Wilson polynomials. Other subfamilies obtained by parameter restriction are not in the scheme, and do not even have a name. What makes the included subfamilies so particular?

3. Is there a suitable reparametrization of the top level polynomials in the schemes such that all families lower in the scheme can be obtained by specialization of parameters? Many arrows in the schemes correspond to taking a limit to 0 or ∞ of rescaled polynomials, involving parameter dependent dilation or translation of the independent variable. It would be nice to simplify this and make it more uniform. The author [11] made an attempt in this direction for the Askey scheme. However, there the formulas for the reparametrization were quite tedious and not very conceptual.

This paper presents, for the q-Askey scheme, one possible way to answer these three questions in a systematic way. Following the ideas by Vinet & Zhedanov [23] and Verde-Star [19] one can try to classify monic orthogonal polynomials u_n which not only satisfy the Bochner-type property that they are eigenfunctions of a second order linear q-difference operator L, so $Lu_n = h_n u_n$ with the h_n distinct, but the u_n should also have an expansion

$$u_n(x) = \sum_{k=0}^{n} c_{n,k}\, v_k(x), \qquad v_k(x) = (x-x_0)(x-x_1)\dots(x-x_{k-1}) \;\; (k\ge 1), \quad v_0(x)=1.$$

So the v_n are Newton type polynomials. Now replace the requirement on L to be a second order q-difference operator by the assumption that it acts on the basis of polynomials v_n as $Lv_n = h_n v_n + g_n v_{n-1}$. Then it follows that $c_{n,k} = \prod_{j=k}^{n-1} \dfrac{g_{j+1}}{h_n - h_j}$. Finally replace the orthogonality assumption by the property that $xu_n(x)$ is a linear combination of $u_{n+1}(x)$, $u_n(x)$ and (with nonvanishing coefficient) $u_{n-1}(x)$.

Verde-Star [19], whom we will follow in this paper, makes the Ansatz that h_k and x_k are Laurent polynomials in q^k of degree 1 and that g_k is a Laurent polynomial in q^k of degree 2. The corresponding $3+3+5=11$ Laurent coefficients then satisfy one trivial relation because $g_0 = 0$ and two further relations implied by the three-term recurrence for the u_n. All families in the q-Askey scheme [8, p.414] are caught by giving the 11 Laurent coefficients, and hence the x_k, h_k, g_k, suitable values. The only exception is the continuous q-Hermite polynomial [8, §14.26]. It does not have an explicit expansion which fits into our framework. Apart from this case our method gives a positive answer to the question whether the q-Askey scheme is complete.

It turns out that almost always the distinction between two families in the scheme can be read off from their different patterns of vanishing Laurent coefficients (although this does not distinguish between a discrete family and its continuous analogue). These different patterns correspond with different types of q-hypergeometric representations. This answers, in a sense, the second question. Furthermore, if we draw an arrow from one family to another in the case that a suitable nonzero Laurent coefficient for the first family becomes zero for the second family, we recover all arrows in the existing q-Askey scheme and find a few more.

Finally, the question about the reparametrization can be answered by starting with the 11 Laurent coefficients, reduce them by a number of identifications to a four-manifold, and distinguish lower dimensional submanifolds by putting one or more suitable Laurent coefficients to zero.

The contents of this paper are as follows. In Section 2 we describe the general set-up, following Verde-Star [19], and we illustrate this for the case of the Askey–Wilson polynomials. In Section 3, the heart of this paper, we give the resulting scheme. Section 4 describes the manifold structure associated with the scheme. Finally Section 5 gives some further perspectives. There are two Appendices. The first one gives explicit data for families in the scheme. The second one gives some limit transitions, which are partially missing in [8, p.414].

Acknowledgement I thank Paul Terwilliger and the referees for helpful comments.

Note For definition and notation of q-shifted factorials and q-hypergeometric series we follow [5, §1.2]. We will only need terminating series:

$$ {}_r\phi_s\left(\begin{matrix} q^{-n}, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z\right) := \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{(a_2, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} \left((-1)^k q^{\frac{1}{2}k(k-1)}\right)^{s-r+1} z^k. $$

Here $(b_1, \dots, b_s; q)_k := (b_1; q)_k \dots (b_s; q)_k$ with $(b; q)_k := (1-b)(1-qb) \dots (1-q^{k-1}b)$ the q-shifted factorial.

For formulas on orthogonal polynomials in the q-Askey scheme we refer to [8, Chapter 14].

2 Askey–Wilson polynomials and Verde-Star's theorem

Let $u_n(x)$ be an Askey–Wilson polynomial, normalized such that it is monic in $x = z + z^{-1}$:

$$ u_n(x) = p_n\left(\tfrac{1}{2}x; a, b, c, d \,|\, q\right) = \frac{(ab, ac, ad; q)_n}{a^n\,(q^{n-1}abcd; q)_n}\; {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right). \tag{2.1} $$

We will write some properties of these polynomials in a conceptual form which we can next use more generally.

Formula (2.1) can be rewritten as

$$ u_n(x) = \sum_{k=0}^{n} c_{n,k}\, v_k(x), \tag{2.2} $$

where

$$ v_k(x) = (x - x_0)(x - x_1) \dots (x - x_{k-1}) \;\; (k \ge 1), \quad v_0(x) = 1, \tag{2.3} $$

$$ x_k = aq^k + a^{-1}q^{-k}, \tag{2.4} $$

and

$$c_{n,k}=\prod_{j=k}^{n-1}\frac{g_{j+1}}{h_n-h_j}, \tag{2.5}$$

$$h_k=q^{-n}(1-q^n)(1-abcdq^{n-1}), \tag{2.6}$$

$$g_k=a^{-1}q^{-2k+1}(1-abq^{k-1})(1-acq^{k-1})(1-adq^{k-1})(1-q^k). \tag{2.7}$$

Note that (2.2) expands the Askey–Wilson polynomial in terms of Newton type polynomials (2.3) with nodes (2.4). The expansion coefficients (2.5) are expressed in terms of sequences h_k and g_k given by (2.6) and (2.7). Since we will not consider orthogonality, the only constraints to be imposed on $q,a,b,c,d\in\mathbb{C}$ are

$$q\neq 0,\quad 1\notin q^{\mathbb{Z}},\quad a\neq 0,\quad 1\notin abcdq^{\mathbb{Z}_{\geq 0}}.$$

These constraints make x_k,h_k,g_k well-defined and they let $h_n\neq h_j$ for $n>0$, $0\leq j<n$.

According to [8, (14.1.7)] there is an explicit second order q-difference operator L such that

$$Lu_n=h_nu_n,\quad n\geq 0. \tag{2.8}$$

By (2.2) we can also characterize L by its action on the basis of polynomials v_n:

$$Lv_0=h_0v_0,\qquad Lv_n=h_nv_n+g_nv_{n-1},\quad n>0. \tag{2.9}$$

The h_k, x_k, g_k have the form

$$\begin{aligned}&h_k=a_0+a_1q^k+a_2q^{-k},\qquad x_k=b_0+b_1q^k+b_2q^{-k},\\&g_k=d_0+d_1q^k+d_2q^{-k}+d_3q^{2k}+d_4q^{-2k},\quad \sum_{i=0}^4 d_i=0.\end{aligned} \tag{2.10}$$

Furthermore, we see from (2.4), (2.6), (2.7) and (2.10) that

$$d_3=q^{-1}a_1b_1,\quad d_4=qa_2b_2. \tag{2.11}$$

Now consider arbitrary sequences h_k,x_k,g_k $(k\geq 0)$. Assume $g_0=0$ and $h_n\neq h_j$ for $n>0$, $0\leq j<n$. Let monic polynomials v_n be given by (2.4) and let monic polynomials u_n of degree n be expanded in terms of the v_k by (2.2) for certain coefficients $c_{n,k}$. Let L be a linear operator on the space of polynomials. Then any two of the three formulas (2.5), (2.8) and (2.9) implies the third formula.

The q-case of a recent more general result by Verde-Star [19, Theorem 6.1] can be formulated as follows.

Theorem 2.1 *Let $q\neq 0$, $1\notin q^{\mathbb{Z}}$. Let h_k,x_k,g_k have the form (2.10). Assume that $h_n\neq h_j$ for $n>0$, $0\leq j<n$, or equivalently $a_2\notin a_1q^{\mathbb{Z}_{>0}}$. Let the Newton type polynomials v_k have the form (2.3) ($v_k(x)=(x-b_0)^k$ allowed) and let the monic polynomials u_n of degree n be defined by (2.2) and (2.5). Then the polynomials u_n satisfy a three-term recurrence relation*

$$xu_n(x)=u_{n+1}(x)+A_nu_n(x)+B_nu_{n-1}(x),\quad n\geq 1. \tag{2.12}$$

iff (2.11) holds.

By [19, (5.5), (5.6)] the coefficients A_n and B_n in (2.12) are given by

$$A_n = x_n + \frac{g_{n+1}}{h_n - h_{n+1}} - \frac{g_n}{h_{n-1} - h_n}\,, \tag{2.13}$$

$$B_n = \frac{g_n}{h_{n-1} - h_n}\left(\frac{g_{n-1}}{h_{n-2} - h_n} - \frac{g_n}{h_{n-1} - h_n} + \frac{g_{n+1}}{h_{n-1} - h_{n+1}} + x_n - x_{n-1}\right). \tag{2.14}$$

For $n = 0$ (2.12) and (2.13) degenerate to

$$u_1(x) = x - A_0, \qquad A_0 = x_0 - \frac{g_1}{h_1 - h_0}\,.$$

Verde-Star claims that all families in the q-Askey scheme [8, Chapter 14], except for the continuous q-Hermite polynomials, can be obtained in this way. We will make this concrete in the next section.

In Theorem 2.1 it is allowed that $B_n = 0$ for all n. This degenerate case will certainly happen if $g_n = 0$ for all n. Then $A_n = x_n$ and $u_n = v_n$, clearly not belonging to a family of orthogonal polynomials. We will not include this case in our classification.

It is also possible that the B_n are zero because the second factor on the right-hand side of (2.14) is zero. This case will be included in our classification.

Finally we may have that g_n vanishes only for some values of n. Let then $n = N + 1$ the lowest value of n for which $g_n = 0$. Then $c_{n,k} = 0$ if $N < k < n$. If we only consider u_n for $n \le N$ we obtain one of the finite systems of orthogonal polynomials in the q-Askey scheme.

Note that a classification according to Theorem 2.1 does not use the usual Bochner type criterium [20] of finding all families of orthogonal polynomials which are eigenfunctions of a suitable second order q-difference operator. Instead it classifies families of polynomials satisfying a three-term recurrence relation which have an expansion of specific type in terms of Newton type polynomials of a specific type. Then there is also an eigenvalue equation (2.9), involving an operator L defined by (2.8). For each family it can be shown in an ad hoc way that this operator L can be written as the second order q-difference operator given in [8, Chapter 14]. But without having done this computation one already sees that the obtained numbers h_n are the eigenvalues of L given in [8].

Remark 2.2 As sketched in [23, §3.3], if we assume that the u_n satisfy a three-term recurrence relation (2.12) and if we assume (2.10) only for the h_k, then (2.10) for the x_k and g_k will follow.

Remark 2.3 The polynomials u_n can be renormalized (under assumptions on the x_k) as polynomials U_n given by (3.6). In this form, and with $g_{N+1} = 0$ for some N, these polynomials also occur in some of Terwilliger's papers, in particular, [16, (10)], [17, (85)]. Our x_i, h_i, g_i correspond to Terwilliger's $\theta_i, \theta_i^*, \varphi_i$, respectively. By [17, Defs. 7.1, 8.1, 14.1, Theor. 23.2] any Leonard system gives rise to a three-term recurrence relation, of which renormalized solutions have the mentioned form. See [16, §5] for explicit values of $\theta_i, \theta_i^*, \varphi_i$.

Remark 2.4 Geronimus raised the problem to classify orthogonal polynomials u_n and Newton polynomials v_k which satisfy (2.2) with $c_{n,k} = a_{n-k} b_k$. For an exposition and follow-up of this problem see [1, §§3, 4]

3 The q-Verde-Star scheme

Let us again give the data leading to polynomials u_n in the q-Askey scheme according to Theorem 2.1:

$$u_n(x) = \sum_{k=0}^{n} c_{n,k}\, v_k(x), \quad v_k(x) = \prod_{j=0}^{k-1}(x - x_j), \quad c_{n,k} = \prod_{j=k}^{n-1} \frac{g_{j+1}}{h_n - h_j}\,, \tag{3.1}$$

$$\begin{aligned} x_k &= b_2 q^{-k} + b_0 + b_1 q^k, \quad h_k = a_2 q^{-k} + a_0 + a_1 q^k, \\ g_k &= d_4 q^{-2k} + d_2 q^{-k} + d_0 + d_1 q^k + d_3 q^{2k}, \end{aligned} \tag{3.2}$$

$$\sum_{i=0}^{4} d_i = 0, \quad d_3 = q^{-1} a_1 b_1, \quad d_4 = q a_2 b_2, \tag{3.3}$$

$$a_2 \neq a_1 q^{\mathbb{Z}_{>0}}, \quad \text{in particular,} \quad a_1 \text{ or } a_2 \neq 0, \qquad d_i \neq 0 \text{ for some } i. \tag{3.4}$$

So everything is determined by the 11 parameters $a_0, a_1, a_2, b_0, b_1, b_2, d_0, d_1, d_2, d_3, d_4$. There are several invariances:

1. If $a_0 \to a_0 + \tau$ then $h_k \to h_k + \tau$.
2. If a_0, a_1, a_2 and d_0, d_1, d_2, d_3, d_4 are multiplied by $\mu \neq 0$ then h_k, g_k are multiplied by μ.
3. If $b_0 \to b_0 + \sigma$ and $x \to x + \sigma$ then $x_k \to x_k + \sigma$ and $u_n(x) \to u_n(x + \sigma)$.
4. If b_0, b_1, b_2 and d_0, d_1, d_2, d_3, d_4 are multiplied by $\rho \neq 0$ then x_k, g_k are multiplied by ρ, $v_k(x) \to \rho^k v_k(\rho^{-1}x)$ and $u_n(x) \to \rho^n u_n(\rho^{-1}x)$.

In each case, what is not mentioned remains unchanged. In items 1 and 2 there is no effect on the $u_n(x)$. Also the translations and dilations of the independent variable of u_n by items 3 and 4 are not considered as essential changes of a family of orthogonal polynomials. So the above four items give rise to four degrees of freedom in the 11 parameters. Together with the three constraints (3.3) on the parameters, there are four essential parameters left, in agreement with the number of four parameters of the Askey–Wilson polynomials.

There are two further remarkable operations which can be performed on the 11 parameters:

$q \leftrightarrow q^{-1}$ *exchange*: $a_1 \leftrightarrow a_2$, $b_1 \leftrightarrow b_2$, $d_1 \leftrightarrow d_2$, $d_3 \leftrightarrow d_4$.

$x \leftrightarrow h$ *duality*: $a_0 \leftrightarrow b_0$, $a_1 \leftrightarrow b_1$, $a_2 \leftrightarrow b_2$; assume also that $b_2 \neq b_1 q^{\mathbb{Z}_{>0}}$, in particular, b_1 or $b_2 \neq 0$. This relates u_n given by (3.1) to its *dual* $\widetilde{u}_n$ given by

$$\widetilde{u}_n(x) = \sum_{k=0}^{n} \widetilde{c}_{n,k}\, \widetilde{v}_k(x), \quad \widetilde{v}_k(x) = \prod_{j=0}^{k-1}(x - h_j), \quad \widetilde{c}_{n,k} = \prod_{j=k}^{n-1} \frac{g_{j+1}}{x_n - x_j}\,. \tag{3.5}$$

If we put

$$U_n(x) = \prod_{j=0}^{n-1} \frac{h_n - h_j}{g_{j+1}} \times u_n(x) = \sum_{k=0}^{n} \frac{\prod_{j=0}^{k-1}(h_n - h_j) \times \prod_{j=0}^{k-1}(x - x_j)}{\prod_{j=1}^{k} g_j}\,, \tag{3.6}$$

$$\widetilde{U}_m(x) = \prod_{j=0}^{m-1} \frac{x_m - x_j}{g_{j+1}} \times \widetilde{u}_m(x) = \sum_{k=0}^{m} \frac{\prod_{j=0}^{k-1}(x_m - x_j) \times \prod_{j=0}^{k-1}(x - h_j)}{\prod_{j=1}^{k} g_j} \tag{3.7}$$

then (see also [23, (1.9)])

$$U_n(x_m) = \widetilde{U}_m(h_n). \tag{3.8}$$

For classification purposes we arrange the 11 parameters in an array

$$\begin{array}{ccccc} & b_2 & b_0 & b_1 & \\ d_4 & d_2 & d_0 & d_1 & d_3 \\ & a_2 & a_0 & a_1 & \end{array} \tag{3.9}$$

It will turn out that only the vanishing of some of these parameters determines the families in the scheme. Let ● denote any parameter value (which may be zero) and ○ a zero parameter value. So we can represent Askey–Wilson by (3.9) with all entries given by ●. The distribution of ● and ○ in an array (3.9) has to satisfy the following rules:

1. If b_1 or a_1 is ○ then d_3 is ○; if b_2 or a_2 is ○ then d_4 is ○
   (because of the second and third formula in (3.3)).

2. b_0 and a_0 are always ●
   (because $h_k \to h_k + \tau$ and $x_k \to x_k + \sigma$ are allowed).

3. In the second row there are no ○ ones between two ● ones
   (because it will turn out that only the most left and the most right nonzero d_i determines the family).

4. In the third row there are at least two ● ones
   (because of rule 2 and the first part of (3.4)).

5. In the second row there are at least two ● ones
   (because of the first part of (3.3) and the second part of (3.4)).

6. Flipping a ● into a ○ causes an arrow between the symbols.
   (This determines a limit case where a parameter tends to zero. If b_2 or a_2 becomes white, then also d_4. If b_1 or a_1 becomes white, then also d_3.)

7. Reflection with respect to the central column in the black-white array of form (3.9) means $q \leftrightarrow q^{-1}$ exchange.

8. Reflection with respect to the middle row means $x \leftrightarrow h$ duality (only possible if there are at least two ● ones in the first row).

In Figure 1 we give half of the scheme according to these rules. It has to be complemented with the scheme obtained from the present one by reflecting each diagram with respect to its middle column and preserving all arrows.

Let us number the rows in the scheme from top to bottom by 1 to 5. In each row list the successive diagrams from left to right by a, b, ... Adding a prime to this notation means a $q \to q^{-1}$ exchange for the corresponding diagram. For instance **3c'** is diagram **3c** reflected with respect to its central column. Note that **1a** = **1a'** and **3e** = **3e'**. For all other diagrams in Figure 1 the primed counterpart is different and not in Figure 1.

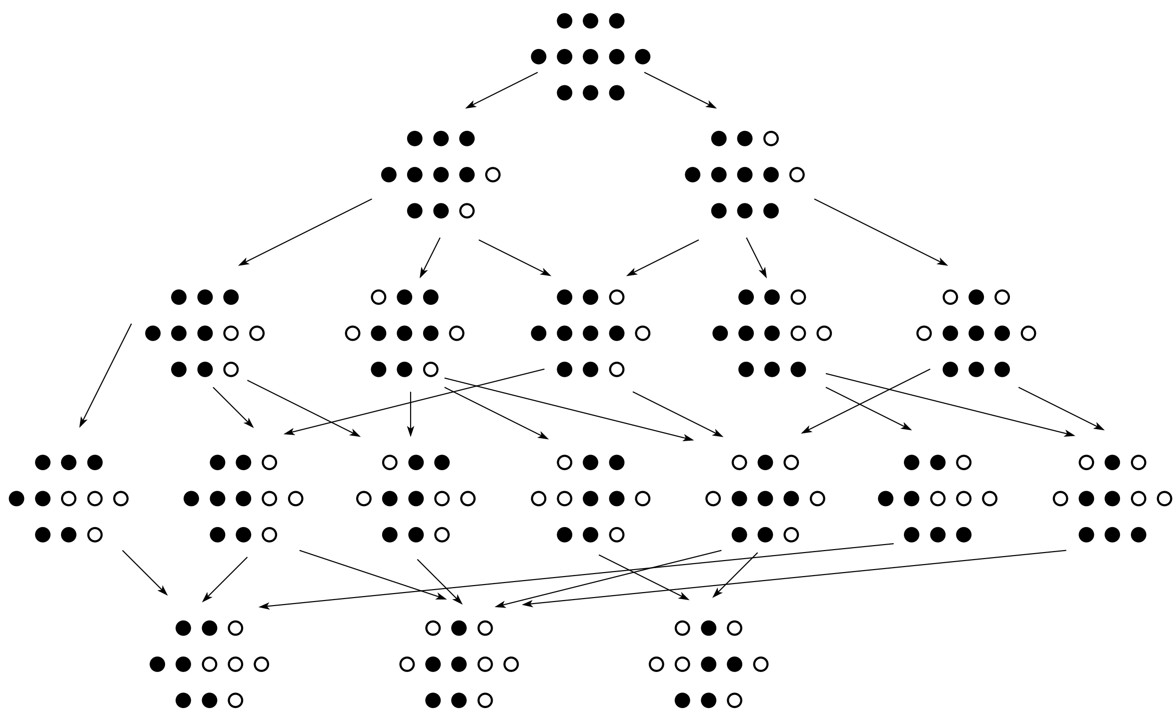


Figure 1: The q-Verde-Star scheme

Note also the following $x \leftrightarrow h$ dualities:
1a, **3c**, **4b**, **5a** are self-dual;
2a $\leftrightarrow$ **2b**, **3a** $\leftrightarrow$ **3d**, **3b** $\leftrightarrow$ **3b'**, **4a** $\leftrightarrow$ **4f**, **4c** $\leftrightarrow$ **4d'** are dual pairs.

The diagrams in Figure 1 correspond with families in the q-Askey scheme as given in the list below (numbers given with these families apply to the corresponding section numbers in [8, Chapter 14]).

1a. Askey–Wilson (1), q-Racah (2)

2a. continuous dual q-Hahn (3), dual q-Hahn (7)

2b. big q-Jacobi (5), q-Hahn (6)

3a. Al-Salam–Chihara (8), dual q-Krawtchouk (17)

3b. big q-Laguerre (11), q^{-1}-Meixner (13), affine q-Krawtchouk (16), quantum q^{-1}-Krawtchouk (14) with $v_k(x) = \prod_{j=0}^{k-1}(x - b_1 q^j)$.

3c. big q-Laguerre (11), q^{-1}-Meixner (13), affine q-Krawtchouk (16), quantum q^{-1}-Krawtchouk (14) with $v_k(x) = \prod_{j=0}^{k-1}(x - b_2 q^{-j})$.

3d. little q-Jacobi (12), q-Krawtchouk (15) with $v_k(x) = \prod_{j=0}^{k-1}(x - b_2 q^{-j})$.

3e. little q-Jacobi (12), q-Krawtchouk (15) with $v_k(x) = x^k$.

4a. continuous big q-Hermite (18)

4b. $u_n(x) = x^n(bx^{-1};q)_n, \quad v_k(x) = (-1)^k q^{\frac{1}{2}k(k-1)}(x;q)_k.$

4c. Al-Salam–Carlitz I (24), q^{-1}-Al-Salam–Carlitz II (25)

4d. little q-Laguerre (20), q^{-1}-Laguerre (21), q^{-1}-Charlier (23), $\quad v_k(x) = x^k(x^{-1};q)_k.$

4e. little q-Laguerre (20), q^{-1}-Laguerre (21), q^{-1}-Charlier (23), $\quad v_k(x) = x^k.$

4f. q^{-1}-Bessel (22), $\quad v_k(x) = (-1)^k q^{\frac{1}{2}k(k-1)}(x;q)_k.$

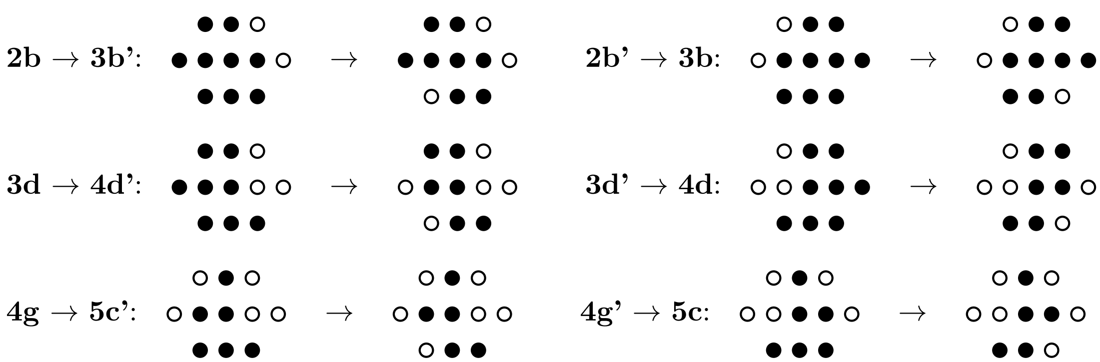
4g. q-Bessel (22), $\quad v_k(x) = x^k.$

5a. $u_n(x) = x^n, \quad v_k(x) = (-1)^k q^{\frac{1}{2}k(k-1)}(x;q)_k.$

5b. $u_n(x) = x^n(x^{-1};q)_n, \quad v_k(x) = x^k.$

5c. q^{-1}-Stieltjes–Wigert (27)

Figure 1 should be complemented with a similar scheme, where each diagram is replaced by its primed counterpart and the arrows are preserved. There are a few arrows from a diagram in the one scheme to a diagram in the other scheme:



Renarks

1. For each diagram in Figure 1 the explicit expressions of x_k, h_k, g_k for one (continuous) family belonging to this diagram or its primed counterpart are given in Appendix A.

2. Since, the first row of a diagram determines the kind of the Newton type polynomials v_k involved, it can happen that one family occurs twice in the scheme because it can be expanded in two different kinds of v_k. See **3b**, **3c** (big q-Laguerre, affine q-Krawtchouk), **3d**, **3e** (little q-Jacobi, q-Krawtchouk), **4d**, **4e** (little q-Laguerre, q^{-1}-Laguerre, q^{-1}-Charlier), **4f'**, **4g** (q-Bessel). A family may also have expansions in different v_k, which are still of the same kind. This will not be recognized by our scheme. For instance, with Askey–Wilson polynomials we may exchange the parameter a with one of the three other parameters b, c, d.

3. The cases **4b**, **5a**, **5b** are degenerate in the sense that u_n turns out to be a Newton type polynomial itself which is expanded in terms of different Newton type polynomials v_k.

4. Most diagrams in the scheme correspond to both a continuous and a discrete family of orthogonal polynomials.

5. Figure 1, when compared with the q-Askey scheme on [8, p.414], misses some families. The reason is that they are special cases of larger families, obtained by restriction of parameter values, but such that our black-white diagrams do not recognize these restrictions. This concerns (numbers mean again section numbers in [8, Chapter 14]) continuous q-Jacobi (10) as subfamily of Askey–Wilson, continuous q-Laguerre (19) as subfamily of Al-Salam–Chihara, and discrete q-Hermite I, II (28, 29) as subfamilies of Al-Salam–Carlitz I, II. Similarly, continuous q-Hahn (4) (Askey–Wilson $p_n(x;a,b,c,d\,|\,q)$ with a,c and b,d pairs of complex conjugates such that $\arg a = \arg c$) and q-Meixner–Pollaczek (9) (Al-Salam–Chihara $Q_n(x;a,b\,|\,q)$ with a,b as a pair of complex conjugates) are not in Figure 1. (The notation in [8, §§14.4, 14.9] for these two classes of polynomials is confusing.)

6. The continuous q-Hermite polynomials are also missing in our scheme because their expansion falls outside the scope of Theorem 2.1.

7. If, in the q-Askey scheme on [8, p.414], the families mentioned in the previous two items are omitted, together with the arrows to and from those families, then all further arrows in that scheme are also present in our scheme. However, we have some more arrows whcih are missing in [8, p.414]. These are (see Appendix B):

   2a $\to$ **3b** and **2a** $\to$ **3c**: continuous dual q-Hahn $\to$ big q-Laguerre,

   3a $\to$ **4c**: Al-Salam–Chihara $\to$ Al-Salam–Carlitz I

4 The q-Verde-Star scheme as a four-manifold

Fix $q \neq 0$ such that $1 \notin q^{\mathbb{Z}}$. We will sketch how the q-Verde-Star scheme can be made into a (complex) four-manifold having specific submanifolds of lower dimension 3, 2, 1 and 0. We will ignore the case of a finite system, where $g_{N+1} = 0$ for some N.

Let us start with a six-manifold with seven coordinates $a_1, a_2, b_1, b_2, d_0, d_1, d_2$ such that $d_0 + d_1 + d_2 + q^{-1}a_1b_1 + qa_2b_2 = 0$. Also assume that $a_2 \notin a_1 q^{\mathbb{Z}_{>0}}$ and $(d_0, d_1, d_2, a_1b_1, a_2b_2) \neq (0,0,0,0,0)$. Now we make two one-parameter identifications. Let nothing change if a_1, a_2, d_0, d_1, d_2 are multiplied by the same nonzero constant or if b_1, b_2, d_0, d_1, d_2 are multiplied by the same nonzero constant. Then there are several possibilities to put two out of the five coordinates a_1, a_2, b_1, b_2, d_0 equal to 1. The three untouched coordinates among these five, together with d_1 or d_2 will then provide local coordinates for our four-manifold. In the generic case all choices are allowed. However, we can regard the six families in the bottom row of Figure 1 and its $q \leftrightarrow q^{-1}$ complement as points in our four-manifold. In a neighbourhood of each of these points we can make a special choice of four coordinates such that the point has all coordinates zero and such that any family in the scheme from which that point is reachable via arrows is a submanifold obtained by putting some of the coordinates equal to zero. Below we give the details for the three families in the bottom row of Figure 1.

$a_2 = b_2 = 1$

	a_1	b_1	d_0	d_1
1a	●	●	●	●
2a	○	●	●	●
2b	●	○	●	●
3a	○	●	●	○
3c	○	○	●	●
3d	○	●	○	●
4a	○	●	○	○
4b	○	○	●	○
4f	●	○	○	○
5a	○	○	○	○

$a_2 = d_0 = 1$

	a_1	b_1	b_2	d_1
1a	●	●	●	●
2a	○	●	●	●
2b	●	○	●	●
2b’	●	●	○	●
3a	○	●	●	○
3b	○	●	○	●
3c	○	○	●	●
3e	●	○	○	●
4b	○	○	●	○
4c	○	●	○	○
4e	○	○	○	●
4g	●	○	○	○
5b	○	○	○	○

$a_2 = d_0 = 1$

	a_1	b_1	b_2	d_2
1a	●	●	●	●
2a	○	●	●	●
2b	●	○	●	●
2b’	●	●	○	●
3b	○	●	○	●
3c	○	○	●	●
3e	●	○	○	●
3d’	●	○	●	○
4d	○	●	○	○
4e	○	○	○	●
4g’	●	○	○	○
5e	○	○	○	○

5 Further perspectives

In a next paper the author will express the coefficients in the relations defining the Zhedanov algebra associated with a family in the q-Askey scheme (see [6, (3.2)] with $R = 1 - \frac{1}{2}(q + q^{-1})$) in terms of the 11 parameters $a_0, a_1, a_2, b_0, b_1, b_2, d_0, d_1, d_2, d_3, d_4$. It will turn out that vanishing properties of these coefficients are also a way to distinguish between the families, although the resulting scheme is slightly different from the scheme in Figure 1.

Verde-Star [19] introduces polynomials u_n and v_k associated with sequences x_k, h_k, g_k as in our §2, but only assuming that x_k and h_k are solutions of a certain four-term difference equation and g_k is a solution of a certain six-term difference equation. As special cases he has the q-case, where x_k, h_k, g_k have the form (2.10), the $q = 1$ case, and the $q = -1$ case. Earlier, in a somewhat different approach, these three cases were examined by Vinet & Zhedanov [23]. The author is also planning to write a paper where the $q = 1$ case will be treated systematically and in full detail, just as the q-case is treated in the present paper. We will also deal there with the corresponding Zhedanov algebra (see [6, (3.2)] with $R = 1$). There will also be need of a systematic and detailed treatment of the $q = -1$ case. Much material about this is already available in papers by Vinet & Zhedanov and coauthors, see for instance [21], [22].

Since the labeling of orthogonal polynomials in the (q-)Askey scheme is by the sequences x_k, h_k, g_k or by the parameters occurring in their expansions is so clean, these data may be helpful for recognizing polynomials in these schemes from the coefficients in the three-term recurrence relation, assuming that it would be possible to obtain these data from these coefficients. See Tcheutia [15] for recent work on this recognition problem by different methods.

Finally, an approach as in the present paper may be tried in other situations where (part of) a q-Askey scheme occurs, see the examples mentioned in the second paragraph of the Introduction.

A Explicit data for the families in Figure 1

For each diagram in Figure 1 we give the data of one (continuous) family belonging to that diagram or its primed counterpart. Bold numbers like **1a** follow the convention explained in connection with Figure 1. Numbers in brackets apply to the corresponding section numbers in [8, Chapter 14].

1a. Askey–Wilson (1): $u_n(x) = k_n^{-1} p_n(\frac{1}{2}x; a, b, c, d \,|\, q)$,

$x_k = aq^k + a^{-1}q^{-k}, \quad h_k = q^{-k}(1-q^k)(1-abcdq^{k-1})$,

$g_k = q^{-2k+1}a^{-1}(1-abq^{k-1})(1-acq^{k-1})(1-adq^{k-1})(1-q^k), \quad k_n = (q^{n-1}abcd; q)_n$.

2a. continuous dual q-Hahn (3): $u_n(x) = p_n(\frac{1}{2}x; a, b, c \,|\, q)$,

$x_k = aq^k + a^{-1}q^{-k}, \quad h_k = q^{-k} - 1, \quad g_k = q^{-2k+1}a^{-1}(1-abq^{k-1})(1-acq^{k-1})(1-q^k)$.

2b. big q-Jacobi (5): $u_n(x) = k_n^{-1} P_n(x; a, b, c; q)$,

$x_k = q^{-k}, \quad h_k = (1-q^{-k})(-1+q^{k+1}ab), \quad g_k = q^{1-2k}(1-aq^k)(1-cq^k)(1-q^k)$,

$$k_n = \frac{(q^{n+1}ab; q)_n}{(qa; q)_n (qc; q)_n}\,.$$

3a. Al-Salam–Chihara (8): $u_n(x) = Q_n(\frac{1}{2}x; a, b \,|\, q)$,

$x_k = aq^k + a^{-1}q^{-k}, \quad h_k = q^{-k} - 1, \quad g_k = q^{-2k+1}a^{-1}(1-abq^{k-1})(1-q^k)$.

3b. big q-Laguerre (11): $u_n(x) = k_n^{-1} P_n(x; a, b; q), \quad v_k(x) = x^k (qax^{-1}; q)_k$

$$x_k = aq^{k+1}, \quad h_k = q^{-k} - 1, \quad g_k = -q^{1-k}b(1-aq^k)(1-q^k), \quad k_n = \frac{1}{(qa; q)_n (qb; q)_n}\,.$$

3c. idem, $v_k(x) = (-1)^k q^{-\frac{1}{2}k(k-1)} (x; q)_k$

$x_k = q^{-k}, \quad h_k = q^{-k} - 1, \quad g_k = q^{1-2k}(1-aq^k)(1-bq^k)(1-q^k)$.

3d. little q-Jacobi (12): $u_n(x) = k_n^{-1} p_n(x; a, b; q), \quad v_k(x) = (-b)^{-k} q^{-\frac{1}{2}k(k+1)} (qbx; q)_k$,

$x_k = q^{-k-1}b^{-1}, \quad h_k = (1-q^{-k})(-1+q^{k+1}ab), \quad g_k = (1-q^{-k})(1-b^{-1}q^{-k})$,

$$k_n = (-1)^n q^{-\frac{1}{2}n(n-1)} \frac{(abq^{n+1}; q)_n}{(aq; q)_n}\,.$$

3e. idem, $v_k(x) = x^k$,

$x_k = 0, \quad h_k = (1-q^{-k})(-1+q^{k+1}ab), \quad g_k = (1-q^{-k})(1-aq^k)$.

4a. continuous big q-Hermite (18): $u_n(x) = H_n(\frac{1}{2}x; a \,|\, q)$,

$x_k = aq^k + a^{-1}q^{-k}, \quad h_k = q^{-k} - 1, \quad g_k = q^{1-2k}a^{-1}(1-q^k)$.

4b. $u_n(x) = x^n (bx^{-1}; q)_n\,, \quad x_k = q^{-k}, \quad h_k = q^{-k} - 1, \quad g_k = (1-q^{-k})(b - q^{1-k})$.

4c. Al-Salam–Carlitz I (24): $u_n(x) = U_n^{(a)}(x; q)$,

$x_k = q^k, \quad h_k = q^{-k} - 1, \quad g_k = a(1-q^{-k})$.

4d. little q-Laguerre (20): $u_n(x) = k_n^{-1} p_n(x; a; q), \quad v_k(x) = x^k (x^{-1}; q)_k$,

$$x_k = q^k, \quad h_k = 1 - q^{-k}, \quad g_k = a(q^k - 1), \quad k_n = \frac{(-1)^n q^{-\frac{1}{2}n(n-1)}}{(aq; q)_n}\,.$$

Note that q^{-1}-Laguerre and little q-Laguerre can be essentially identified with each other by [8, p.521].

4e. idem, $v_k(x) = x^k, \quad x_k = 0, \quad h_k = 1 - q^{-k}, \quad g_k = q^{-k}(1 - aq^k)(1 - q^k).$

4f'. q-Bessel (22): $u_n(x) = k_n^{-1} y_n(x; a; q) \quad v_k(x) = x^k (x^{-1}; q)_k,$

$x_k = q^k, \quad h_k = (1 - q^{-k})(1 + aq^k), \quad g_k = aq^{k-1}(q^k - 1), \quad k_n = (-1)^n q^{-\frac{1}{2}n(n-1)}(-aq^n; q)_n\,.$

4g. idem, $v_k(x) = x^k, \quad x_k = 0, \quad h_k = (1 - q^{-k})(1 + aq^k), \quad g_k = q^{-k} - 1.$

5a. $u_n(x) = x^n, \quad x_k = q^{-k}, \quad h_k = q^{-k} - 1, \quad g_k = q^{1-2k}(1 - q^k).$

5b. $u_n(x) = k_n^{-1}\,{}_1\phi_0(q^{-n};\,;q, qx) = x^n (x^{-1}; q)_n\,,$

$x_k = 0, \quad h_k = q^{-k} - 1, \quad g_k = 1 - q^{-k}, \quad k_n = (-1)^n q^{-\frac{1}{2}n(n-1)}.$

5c'. Stieltjes–Wigert (27): $u_n(x) = k_n^{-1} S_n(x; q),$

$$x_k = 0, \quad h_k = q^k - 1, \quad g_k = q^{-k} - 1, \quad k_n = \frac{(-1)^n q^{n^2}}{(q; q)_n}\,.$$

B Some explicit limit transitions

2a $\to$ **3b**: continuous dual q-Hahn $\to$ big q-Laguerre (missing in [8, §§14.3, 14.11]).

$$\lim_{a\to 0} a^n p_n\big(\tfrac{1}{2}a^{-1}x; a, a^{-1}bq, a^{-1}cq \,|\, q\big) = (bq, cq; q)_n\, P_n(x; b, c; q), \tag{B.1}$$

where continuous dual q-Hahn ([8, (14.3.1)] together with symmetry in a, b, c) and monic big q-Laguerre [8, (14.11.1)] are respectively represented as

$$p_n\big(\tfrac{1}{2}x; a, b, c \,|\, q\big) = \frac{(ab, bc; q)_n}{b^n}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, bz, bz^{-1} \\ ab, bc \end{matrix}; q, q\right), \quad x = z + z^{-1}, \quad ab, ac, bc < 1, \tag{B.2}$$

$$(bq, cq; q)_n\, P_n(x; b, c; q) = (-c)^n q^{\frac{1}{2}n(n+1)} (bq; q)_n \times {}_2\phi_1\left(\begin{matrix} q^{-n}, bqx^{-1} \\ bq \end{matrix}; q, c^{-1}x\right), \quad 0 < bq < 1,\ c < 0. \tag{B.3}$$

Here and elsewhere in this Appendix, when we mention conditions on the parameters, these are such that the coefficient B_n in (2.12) is positive, also assuming A_n real. This assures that the polynomials are orthogonal. The conditions above, where the parameters are assumed real, can be obtained from [8, (14.3.5), (14.11.4)]. By these conditions the passage to the limit in (B.1) can be made while keeping the polynomials orthogonal.

2a $\to$ **3c**: The same limit (B.1) also holds with other q-hypergeometric representations [8, (14.3.1), (14.11.1)]:

$$p_n\big(\tfrac{1}{2}x; a, b, c \,|\, q\big) = \frac{(ab, ac; q)_n}{a^n}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, az, az^{-1} \\ ab, ac \end{matrix}; q, q\right), \quad x = z + z^{-1}, \tag{B.4}$$

$$(bq, cq; q)_n\, P_n(x; b, c; q) = (bq, cq; q)_n\,{}_3\phi_2\left(\begin{matrix} q^{-n}, 0, x \\ bq, cq \end{matrix}; q, q\right). \tag{B.5}$$

3a → **4c**: Al-Salam–Chihara → Al-Salam–Carlitz I (missing in [8, §§14.8, 14.24]).

$$\lim_{a\to\infty}(a)^{-n}Q_n(\tfrac{1}{2}ax;a,ab\,|\,q)=U_n^{(b)}(x;q), \tag{B.6}$$

where Al-Salam–Chihara [8, (14.8.1)] and Al-Salam–Carlitz I [8, (14.24.1)] are respectively represented as

$$Q_n(\tfrac{1}{2}x;a,b\,|\,q)=\frac{(ab;q)_n}{a^n}\,{}_3\phi_2\left(\begin{matrix}q^{-n},az,az^{-1}\\ ab,0\end{matrix};q,q\right),\quad x=z+z^{-1},\ ab<1, \tag{B.7}$$

$$U_n^{(b)}(x;q)=(-b)^nq^{\frac{1}{2}n(n-1)}\,{}_2\phi_1\left(\begin{matrix}q^{-n},x^{-1}\\ 0\end{matrix};q,qb^{-1}x\right),\quad b<0. \tag{B.8}$$

3a → **4b**: Al-Salam–Chihara → $x^n(bx^{-1};q)_n$.

$$\lim_{a\to0}a^nQ\big((2a)^{-1}x;a,a^{-1}b\,|\,q\big)=(b;q)_n\,{}_2\phi_1\left(\begin{matrix}q^{-n},x\\ b\end{matrix};q,q\right)=x^n(bx^{-1};q)_n, \tag{B.9}$$

where Al-Salm–Chihara is given by (B.7) and the second equality in (B.9) is [5, (II.6)].

2b → **3d**: big q-Jacobi → little q-Jacobi [8, p.442, Remarks].

$$\lim_{d\to0}(qa)^{-n}\frac{(qa;q)_n(-qad;q)_n}{(q^{n+1}ab;q)_n}P_n(x;a,b,1,d;q)=(-1)^nq^{\frac{1}{2}n(n-1)}\frac{(qb;q)_n}{(q^{n+1}ab;q)_n}p_n(x;b,a;q), \tag{B.10}$$

where big and little q-Jacobi are respectively represented by [8, p.442, (14.5.1) and Remarks]

$$P_n(x;a,b,c,d;q)=P_n(ac^{-1}qx;a,b,-ac^{-1}d;q)={}_3\phi_2\left(\begin{matrix}q^{-n},q^{n+1}ab,qac^{-1}x\\ qa,-qac^{-1}d\end{matrix};q,q\right),$$
$$c,d>0,\ -q^{-1}cd^{-1}<a<q^{-1},\ -q^{-1}c^{-1}d<b<q^{-1}, \tag{B.11}$$

$$p_n(x;a,b;q)=(-qb)^{-n}q^{-\frac{1}{2}n(n-1)}\frac{(qb;q)_n}{(qa;q)_n}\,{}_3\phi_2\left(\begin{matrix}q^{-n},q^{n+1}ab,qbx\\ qb,0\end{matrix};q,q\right),$$
$$0<a<q^{-1},\ b<q^{-1}. \tag{B.12}$$

2b → **3e**: The same limit (B.10) also holds with other q-hypergeometric representations [10, (2.39), (2.37)], [8, (14.12.1)]:

$$P_n(x;a,b,c,d;q)=\left(-\frac{ad}{bc}\right)^n\frac{(qb;q)_n(-qbcd^{-1};q)_n}{(qa;q)_n(-qac^{-1}d;q)_n}\,{}_3\phi_2\left(\begin{matrix}q^{-n},q^{n+1}ab,-qbd^{-1}x\\ qb,-qbcd^{-1}\end{matrix};q,q\right), \tag{B.13}$$

$$p_n(x;a,b;q)={}_2\phi_1\left(\begin{matrix}q^{-n},q^{n+1}ab\\ qa\end{matrix};q,qx\right). \tag{B.14}$$

3e → **4g**: little q-Jacobi → q-Bessel [8, (14.12.14)].

$$\lim_{b\to-\infty}p_n(x;-q^{-1}ab^{-1},b;q)=y_n(x;a;q), \tag{B.15}$$

where the little q-Jacobi polynomial is given by (B.14) and the q-Bessel function by [8, (14.22.1)]

$$y_n(x;a;q)={}_2\phi_1\left(\begin{matrix}q^{-n},-aq^n\\ 0\end{matrix};q,qx\right),\qquad a>0. \tag{B.16}$$

3d' → 4f': The same limit (B.15) also holds with other q-hypergeometric representations for little q-Jacobi and q-Bessel:

$$p_n(x;a,b;q) = (-1)^n q^{\frac{1}{2}n(n+1)} a^n \,\frac{(bq;q)_n}{(aq;q)_n}\, {}_3\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1}, x^{-1} \\ qb \end{matrix}; q, a^{-1}x\right), \tag{B.17}$$

$$y_n(x;a;q) = (-1)^n q^{n^2} a^n \, {}_3\phi_0\left(\begin{matrix} q^{-n}, -aq^n, x^{-1} \\ - \end{matrix}; q, -a^{-1}x\right). \tag{B.18}$$

Formula(B.17) folllows from (B.14) by [5, (III.8)] and formula (B.18) follows from (B.16) by taking the limit $c \to 0$ in [5, (III.8)].

4a → 5a: continuous big q-Hermite $\to x^n$.

$$\lim_{a\to 0} a^n H_n\big((2a)^{-1}x; a\,|\,q\big) = {}_2\phi_1\left(\begin{matrix} q^{-n}, x \\ 0 \end{matrix}; q, q\right) = x^n, \tag{B.19}$$

where continuous big q-Hermite is given by [8, (14.18.1)]

$$H_n\big(\tfrac{1}{2}x; a\,|\,q\big) = a^{-n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, az, az^{-1} \\ 0, 0 \end{matrix}; q, q\right), \quad x = z + z^{-1}, \tag{B.20}$$

and the second equality in (B.19) is [5, (II.6)].

4e → 5b: little q-Laguerre $\to x^n(x^{-1};q)_n$.

$$\lim_{a\to 0} (-1)^n q^{\frac{1}{2}n(n-1)} (aq;q)_n\, p_n(x;a;q) = (-1)^n q^{\frac{1}{2}n(n-1)}\, {}_1\phi_0\left(\begin{matrix} q^{-n} \\ - \end{matrix}; q, qx\right) = x^n(x^{-1};q)_n. \tag{B.21}$$

Here little q-Laguerre is given by [8, (14.20.1)]

$$p_n(x;a;q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ qa \end{matrix}; q, qx\right) \tag{B.22}$$

and the second equality in (B.22) follows from [5, (II.4)].

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