# Spinon basis for higher level SU(2) WZW models

Bouwknegt, P.; Ludwig, A.W.W.; Schoutens, K.

**DOI**
10.1016/0370-2693(95)01078-5

**Publication date**
1995

**Document Version**
Final published version

**Published in**
Physics Letters B

ELSEVIER

12 October 1995

PHYSICS LETTERS B

Physics Letters B 359 (1995) 304-312

# Spinon basis for higher level $SU(2)$ WZW models

Peter Bouwknegt [a], Andreas W.W. Ludwig [b], Kareljan Schoutens [c]

[a] *Department of Physics and Astronomy, U.S.C., Los Angeles, CA 90089-0484, USA*
[b] *Department of Physics, University of California, Santa Barbara, CA 93106, USA*
[c] *Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA*

Received 3 April 1995
Editor: M. Dine

**Abstract**

We propose a spinon basis for the integrable highest weight modules of $\widehat{\mathfrak{sl}_2}$ at levels $k \geq 1$, and discuss the underlying Yangian symmetry. Evaluating the characters in this spinon basis provides new quasi-particle type expressions for the characters of these integrable modules, and explicitly exhibits the structure of an RSOS times a Yangian part, known e.g. from $S$-matrix results. We briefly discuss generalizations to other groups and more general conformal field theories.

## 1. Introduction

Conventionally, the Hilbert space of (rational) two dimensional conformal field theories (RCFT) is described in terms of a chiral algebra that acts on a finite set of fields that are primary with respect to this chiral algebra. This procedure leads to so-called Verma-module bases of RCFT's, and gives rise to 'bosonic type' formulas for the characters (i.e. torus partition functions) of such conformal field theories, reflecting this particular choice for the basis of the Hilbert space. On the other hand, in a conformal field theory (CFT) there are many bases of Hilbert space, none of which is a priori distinguished. Which of those different bases is most useful depends on the question that is being asked (see e.g. [1,2]). This is quite different in a massive field theory, where a basis of massive *particles* is naturally distinguished. In general, considering massive perturbations of CFT's, and their corresponding particle bases, one may generate, at least conceptually, "quasi-particle" bases of CFT's, by letting the mass tend to zero. This provides a way to define the notion of a massless "quasi-particle," which has recently proven to be very useful in cases where the massive perturbation, used to define such a basis, is Yang-Baxter integrable [3,2]. The so-generated bases inherit a particle ("Fock-space") like structure, which is not manifest in the Verma-module type basis of the *same* CFT. Each basis of the Hilbert space of a given CFT gives rise to a particular way of writing the partition function on a torus. The equality of the different ways to write the torus partition function, in different bases, gives rise to remarkable identities. Moreover, bases of massless quasiparticles appear to be deeply related to Yangian and affine quantum symmetries, that are often present even when conformal and scale invariance is broken. Therefore, we expect that a better understanding of such bases of CFT's will also provide valuable insights into perturbed CFT's.


*SSDI* 0370-2693(95)01078-5

Analysis of the thermodynamic Bethe Ansatz, arising from (Yang-Baxter) integrable perturbations of CFT's, for a variety of models, has recently led to a wealth of conjectures for so-called quasi-particle (or fermionic) type characters for conformal field theories (see, in particular [4,5]), some of which have been proven through $q$-analysis (see e.g. [6–8]). Most of these results still lack an interpretation (and/or proof) in terms of a corresponding structure of the Hilbert space of a conformal field theory, i.e. a characterization and/or construction of the corresponding "quasi-particle" basis of the Hilbert space (see however [9–15]).

A particular interesting model illustrating the issues above is the so-called Haldane-Shastry long-range spin chain [16], which is integrable and has Yangian symmetry (even for finite chains). Its low energy sector is identical to a well-known conformal field theory, namely the $SU(2)$ level-1 Wess-Zumino-Witten model [17,18]. While the description of the Hilbert space of the $SU(2)$ level-1 WZW model in terms of its chiral algebra, i.e. $\widehat{\mathfrak{sl}_2}$, is complicated due to the existence null vectors, it was found that the Hilbert space has a very simple structure [12,13], originating from the underlying Yangian symmetry: it may be viewed as a "Fock space" of massless 'spinon particles,' which satisfy generalized commutation relations [13] (and no other relations). No similar description has been known for higher levels. However, Bethe ansatz $S$-matrix calculations [19] suggest that the higher level models might be described by massless 'spinons' (similar to those at level-1) and 'kinks' (reflecting an RSOS structure). In this paper we propose such a basis of the Hilbert space of the higher level $SU(2)$ WZW conformal field theory in terms of the modes of the spin-1/2 affine primary chiral vertex operator, along the lines of Ref. [13]. The RSOS structure ('kinks') is provided by the sequence of fusions of the chiral vertex operator (which is absent at level-1), and represents the "non-abelian statistics" of the chiral vertex operators at level greater than one. We also argue that there is an underlying Yangian symmetry. Some justification for our proposal of the higher level spinon basis comes from the resulting quasi-particle type expressions for the characters, which we have been numerically verified to high order.

The paper is organized as follows. In Section 2 we will propose, along the lines of [13], a basis for the spinon Fock space at level $k$, and we will argue that this space decomposes into a direct sum of integrable highest weight modules (corresponding to $SU(2)$ level-$k$ affine primaries) - each integrable module occurring exactly once. In Section 3 we will argue that on the spinon Fock space (and hence on the integrable modules) we can define the action of the Yangian $Y(\mathfrak{sl}_2)$ under which the spinon Fock space is fully reducible. In Section 4 we will derive new (quasi-particle type) character formulas for the integrable highest weight modules of $(\widehat{\mathfrak{sl}_2})_k$ and, finally, in Section 5 we will make some remarks on the results and discuss generalizations to other groups and more general conformal field theories.

## 2. Spinon basis for $(\widehat{\mathfrak{sl}_2})_k$ integrable highest weight modules

The affine Lie algebra $\widehat{\mathfrak{sl}_2}$ at level-$k$ is defined by means of the commutators

$$[J_m^a, J_n^b] = f^{ab}{}_c J_{m+n}^c + km\delta_{m+n} d^{ab}, \tag{2.1}$$

where $m,n \in \mathbb{Z}$ and the adjoint index takes values $a = (++), 3, (--)$. The metric is determined by $d^{(++)(--)} = 1$, $d^{33} = 2$ and the structure constants follow from $f^{(++)(--)}{}_3 = 1$.

The integrable highest weight modules of $(\widehat{\mathfrak{sl}_2})_k$ will be denoted by $L_j$. Here, the spin $j$ can take values $j = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2}$. For the chiral vertex operators (CVO's) $L_{j_1} \to L_{j_2}$ transforming in the irreducible $\mathfrak{sl}_2$ representation of spin $j_3$, we will use the notation of [20], i.e. $\Phi\begin{pmatrix} & j_3 & \\ j_2 & & j_1 \end{pmatrix}(z)$. They are nonvanishing iff $j_2$ occurs in the fusion rule $j_1 \times j_3$, i.e. $j_2 \in \{|j_1 - j_3|, \ldots, \min(j_1 + j_3, k - (j_1 + j_3))\}$. The spinons correspond to $j_3 = \frac{1}{2}$, and transform according to

$$[J_m^a, \phi^\alpha\begin{pmatrix} & \frac{1}{2} & \\ j_2 & & j_1 \end{pmatrix}(z)] = z^m\,(t^a \cdot \phi)^\alpha\begin{pmatrix} & \frac{1}{2} & \\ j_2 & & j_1 \end{pmatrix}(z) \equiv z^m\,(t^a)^\alpha{}_\beta \phi^\beta\begin{pmatrix} & \frac{1}{2} & \\ j_2 & & j_1 \end{pmatrix}(z), \tag{2.2}$$

where $(t^{(++)})^{-}{}_{+} = (t^{(--)})^{+}{}_{-} = 1$, $(t^3)^{\pm}{}_{\pm} = \pm 1$.

The basic relations for the CVO's are the so-called braiding relations (see e.g. [20])

$$\Phi\begin{pmatrix} k_1 \\ j_1 \; p \end{pmatrix}(z_1)\Phi\begin{pmatrix} k_2 \\ p \; j_2 \end{pmatrix}(z_2) = \sum_{p'} B_{pp'}\begin{bmatrix} k_1 & k_2 \\ j_1 & j_2 \end{bmatrix}\Phi\begin{pmatrix} k_2 \\ j_1 \; p' \end{pmatrix}(z_2)\Phi\begin{pmatrix} k_1 \\ p' \; j_2 \end{pmatrix}(z_1), \tag{2.3}$$

and fusion relations

$$\Phi\begin{pmatrix} k_1 \\ j_1 \; p \end{pmatrix}(z_1)\Phi\begin{pmatrix} k_2 \\ p \; j_2 \end{pmatrix}(z_2) = \sum_{p'} F_{pp'}\begin{bmatrix} k_1 & k_2 \\ j_1 & j_2 \end{bmatrix}\sum_{Q\in L_{p'}} \Phi^Q\begin{pmatrix} p' \\ j_1 \; j_2 \end{pmatrix}(z_2)\langle Q|\Phi\begin{pmatrix} k_1 \\ p' \; k_2 \end{pmatrix}(z_1 - z_2)|k_2\rangle, \tag{2.4}$$

where $Q \in L_{p'}$ denotes primary and descendant states in the module $L_{p'}$. The braiding and fusion matrices, $B_{pp'}\begin{bmatrix} k_1 & k_2 \\ j_1 & j_2 \end{bmatrix}$ and $F_{pp'}\begin{bmatrix} k_1 & k_2 \\ j_1 & j_2 \end{bmatrix}$, satisfy the pentagon and hexagon identities [20].

The CVO's $\Phi\begin{pmatrix} j_3 \\ j_2 \; j_1 \end{pmatrix}(z)$ have conformal dimension $\Delta(j_3)$ where

$$\Delta(j) \equiv \frac{j(j+1)}{k+2}, \tag{2.5}$$

and their mode expansion is given by

$$\Phi\begin{pmatrix} j_3 \\ j_2 \; j_1 \end{pmatrix}(z) = \sum_{n\in\mathbb{Z}} \Phi\begin{pmatrix} j_3 \\ j_2 \; j_1 \end{pmatrix}_{-n-(\Delta(j_2)-\Delta(j_1))} z^{n+(\Delta(j_2)-\Delta(j_1)-\Delta(j_3))}. \tag{2.6}$$

In general, due to the non-locality of the CVO's, the modes will not satisfy simple relations. If, however, for a given $p$, the braiding and fusion matrices $B_{pp'}$ and $F_{pp'}$ are nonvanishing for one choice of $p'$ only, the Eqs. (2.3) and (2.4) can be combined into so-called generalized commutation relations. For example,

$$\sum_{l\geq 0} C_l^{(2\Delta-1)} \left(\phi^\alpha\begin{pmatrix} \frac{1}{2} \\ 0 \; \frac{1}{2} \end{pmatrix}_{-m-l+\Delta} \phi^\beta\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \; 0 \end{pmatrix}_{-n+l-\Delta} - \begin{pmatrix} m \leftrightarrow n+1 \\ \alpha \leftrightarrow \beta \end{pmatrix}\right) = \epsilon^{\alpha\beta}\delta_{m+n,0}, \tag{2.7}$$

where $C_l^{(\alpha)}$ is defined by

$$(1-x)^\alpha = \sum_{l\geq 0} C_l^{(\alpha)} x^l, \tag{2.8}$$

and $\Delta = \Delta(\frac{1}{2})$. Algebras of the type (2.7) are known as 'generalized vertex algebras' [21]. Other examples include the so-called parafermion algebras [22] or $Z$-algebras [9]. Unfortunately, when the braiding and/or fusion matrix is not one-dimensional (i.e. the case of 'non-abelian statistics'), we do not know how to translate the content of Eqs. (2.3) and (2.4) into generalized commutation relations between the modes.

Let $\mathcal{F}$ denote the spinon Fock space, i.e. the collection of states of the form

$$\phi^{\alpha_N}\begin{pmatrix} \frac{1}{2} \\ j_N \; j_{N-1} \end{pmatrix}_{-\Delta_N-n_N} \cdots \phi^{\alpha_2}\begin{pmatrix} \frac{1}{2} \\ j_2 \; j_1 \end{pmatrix}_{-\Delta_2-n_2} \phi^{\alpha_1}\begin{pmatrix} \frac{1}{2} \\ j_1 \; 0 \end{pmatrix}_{-\Delta_1-n_1} |0\rangle, \tag{2.9}$$

with $n_i + \Delta_i \geq 0$, and where the spins $\{j_1, \ldots, j_N\}$ run over the set of spins allowed by the fusion rules. Also we have put $\Delta_k = \Delta(j_k) - \Delta(j_{k-1})$.

Clearly, because of the relations (2.3) and (2.4), the basis (2.9) is overcomplete. In sub-channels $0 \to \frac{1}{2} \to 0$ we can 'straighten' the basis by means of the generalized commutation relation (2.7). Furthermore, by empirically matching the low lying energy states for low levels $k$ against the known irreducible characters for arbitrary fusion channels $(\ldots, j_3, j_2, j_1, 0)$, we found evidence that a set of (independent) basis vectors is provided by the states

$$\begin{aligned}&\phi^-\begin{pmatrix} & \frac{1}{2} & \\ j_{M+N} & & j_{M+N-1}\end{pmatrix}_{-\Delta_{M+N}-n_{M+N}} \cdots \phi^-\begin{pmatrix} & \frac{1}{2} & \\ j_{M+1} & & j_M\end{pmatrix}_{-\Delta_{M+1}-n_{M+1}} \\ &\times \phi^+\begin{pmatrix} & \frac{1}{2} & \\ j_M & & j_{M-1}\end{pmatrix}_{-\Delta_M-n_M} \cdots \phi^+\begin{pmatrix} & \frac{1}{2} & \\ j_1 & & 0\end{pmatrix}_{-\Delta_1-n_1} |0\rangle\,, \end{aligned} \tag{2.10}$$

where the modes $n_i \equiv n_{i,\min} + \tilde{n}_i$ satisfy $\tilde{n}_M \geq \tilde{n}_{M-1} \geq \ldots \geq \tilde{n}_1 \geq 0$, $\tilde{n}_{M+N} \geq \tilde{n}_{M+N-1} \geq \ldots \geq \tilde{n}_{M+1} \geq 0$, and where $n_{1,\min}, \ldots, n_{M+N,\min}$ is a 'minimal allowed mode sequence' corresponding to the given fusion channel (Bratteli diagram) constructed as follows

$$\begin{aligned} n_{1,\min} &= 0\,, \\ n_{i+1,\min} &= \begin{cases} n_{i,\min} + 1 & \text{if } j_{i+1} = j_{i-1} < j_i \\ n_{i,\min} & \text{otherwise}\,. \end{cases} \end{aligned} \tag{2.11}$$

The rule (2.11) is consistent with (2.7), but at this point we do not know how to derive (2.11) from (2.3) and (2.4). Clearly, this deserves further investigation.

The spinon Fock space $\mathcal{F}$ is, in fact, an $(\widehat{\mathfrak{sl}_2})_k$ module but, a priori, this module does not have to be integrable (or even a direct sum of integrable modules). However, in this particular case, we claim that we do have $\mathcal{F} \cong \bigoplus_{j=0}^{k/2} L_j$. To prove that $\mathcal{F}$ is indeed a direct sum of integrable modules one would have to show that the operator $S_N^{(k+1)} = \sum_{i_1+\ldots i_{k+1}=N} J_{i_1}^{(++)} \cdots J_{i_{k+1}}^{(++)}$ annihilates all states in $\mathcal{F}$ for all $N \in \mathbb{Z}$ [11]. The highest weight vectors $|j\rangle$ of $L_j$ are then given by

$$|j\rangle = \phi^+\begin{pmatrix} & \frac{1}{2} & \\ j & & j-\frac{1}{2}\end{pmatrix}_{-\Delta(j)+\Delta(j-\frac{1}{2})} \cdots \phi^+\begin{pmatrix} & \frac{1}{2} & \\ \frac{1}{2} & & 0\end{pmatrix}_{-\Delta(\frac{1}{2})} |0\rangle\,. \tag{2.12}$$

Clearly, these are $\mathfrak{sl}_2$ highest weight vectors. That they are in fact $\widehat{\mathfrak{sl}_2}$ highest weight vectors follows from

$$(J_{-1}^{(++)})^N |j\rangle = 0\,, \tag{2.13}$$

for all $N \geq k + 1 - 2j$. Assuming the correctness of the basis (2.10), (2.11), Eq. (2.13) is a simple consequence of the fact that the basis in $\mathcal{F}$ does not contain any states at that particular $+$-spinon number and energy eigenvalue $L_0$.

To summarize this section, we conjecture that the spinon Fock space $\mathcal{F}$ has a basis given by (2.10) and (2.11), and is precisely the sum of all integrable highest weight modules $L_j$ of $(\widehat{\mathfrak{sl}_2})_k$. For level $k = 1$ this conjecture has been proven in [13]. In Section 4 we will, assuming the validity of the conjecture, compute a quasi-particle type expression for the character of $L_j$. The correctness of these characters, verified numerically for low level, isospin and energy, strongly supports the validity of the conjecture.

## 3. Yangian and Hecke symmetry

We recall that the level-1 $\widehat{\mathfrak{sl}_2}$ integrable modules carry a (fully reducible) representation of the Yangian $Y(\mathfrak{sl}_2)$ [17,12,13]. The Yangian generators were suggested by taking the low energy limit of an XXX spin chain model with long-range interactions, the so-called Haldane-Shastry model [16]. This spin chain has exact Yangian symmetry and goes to the level-1 $\widehat{\mathfrak{sl}_2}$ conformal field theory in the low energy limit, as for the usual nearest neighbour Heisenberg XXX chain. Explicitly, we have [17]

$$Q_0^a = J_0^a\,, \quad Q_1^a = \tfrac{1}{2} f^a{}_{bc} \sum_{m>0} J_{-m}^b J_m^c\,. \tag{3.1}$$

To prove that (3.1) satisfy the relations of $Y(\mathfrak{sl}_2)$ [23] is a straightforward, albeit tedious, excercise. In practice, one finds that the Yangian relations are satisfied modulo fields which are null if and only if $k = 1$. On the other hand, there are results (in particular those of [19]) suggesting that the Yangian symmetry *does* generalize to higher level.

It proves to be convenient to work out the action of (3.1) on the spinon basis. Moreover, to describe the action on the spinon basis it is convenient to work with generating series for the $N$-spinon basis elements, i.e.

$$\phi^{\alpha_N}\begin{pmatrix} & \frac{1}{2} & \\ j_N & & j_{N-1}\end{pmatrix}(w_N) \cdots \phi^{\alpha_1}\begin{pmatrix} & \frac{1}{2} & \\ j_1 & & 0\end{pmatrix}(w_1)|0\rangle \in \mathcal{F}((w_1,\ldots,w_N))\,. \tag{3.2}$$

On $\mathcal{F}((w_1,\ldots,w_N))$, the action of (3.1) is given by the following differential operators [12]

$$\begin{aligned} Q_0^a &= \sum_i t_i^a\,, \\ Q_1^a &= -2\sum_i D_i t_i^a + \tfrac{1}{2} f^a{}_{bc} \sum_{i\neq j} \theta_{ij} t_i^b t_j^c\,, \end{aligned} \tag{3.3}$$

where

$$D_i = w_i \partial_{w_i}\,, \quad \theta_{ij} = \frac{w_i}{w_i - w_j}\,. \tag{3.4}$$

In fact, recognizing (3.3) as the Yangian generators of the $\mathfrak{sl}_2$ Calogero-Sutherland-Moser model for coupling value $\lambda = -\frac{1}{2}$ (see e.g. [18]) gives an easy proof of the fact that, for $k = 1$, the generators (3.1) satisfy the relations of $Y(\mathfrak{sl}_2)$ [12].

The analogous expression for $k > 1$ on the 2-spinon states

$$\phi^{\alpha_2}\begin{pmatrix} & \frac{1}{2} & \\ 0 & & \frac{1}{2}\end{pmatrix}(w_2)\, \phi^{\alpha_1}\begin{pmatrix} & \frac{1}{2} & \\ \frac{1}{2} & & 0\end{pmatrix}(w_1)|0\rangle\,, \tag{3.5}$$

is again given by (3.3), but now with

$$D_i = w_i \partial_{w_i} - \tfrac{1}{4}(1 - 4\Delta) \sum_{j\neq i} (\theta_{ij} - \theta_{ji})\,, \tag{3.6}$$

where $\Delta \equiv \Delta(\frac{1}{2}) = 3/4(k+2)$. Note, in particular, that for $k > 1$ it is impossible to express (3.3) in terms of affine currents.

The rationale behind (3.3) is as follows. The one-spinon states transform in an irreducible (2-dimensional) representation of $\mathfrak{sl}_2$. This representation can be extended to a representation of $Y(\mathfrak{sl}_2)$, the so-called 'evaluation representation' [23,24] (the evaluation parameter is given by $2n$, where $n$ is the mode of the spinon). This gives Eq. (3.3) for $N = 1$ (one spinon). The action of $Y(\mathfrak{sl}_2)$ on multi-spinon states is, in principle, given by a 'co-multiplication.' The multi-spinon Fock space is, however, not a direct product of one-spinon Fock spaces but is constrained by the generalized commutation relations (2.7) as well as the constraints on the allowed fusion channels. The correct co-multiplication is the one that descends to the multi-spinon Fock space, i.e. leaves the generalized commutation relations invariant. It is straightforward to show that (3.3) satisfies this property on the 2-spinon states (3.5). In fact, the improvement term in $D_i$ (3.6) simply corresponds to a 'gauge transformation' [18] that effectively transforms the generalized commutation relations (2.7) at level $k$ into the ones for $k = 1$. Since the gauge transformation does not affect the commutation relations $[D_i, D_j] = 0$, $[D_i, w_j] = \delta_{ij} w_i$, the generators (3.3) again satisfy the relations of $Y(\mathfrak{sl}_2)$.

The commutant of $Y(\mathfrak{sl}_2)$ on integrable $(\widehat{\mathfrak{sl}_2})_k$ modules for level $k = 1$ contains an infinite set of mutually commuting 'Hamiltonians.' The first two are explicitly given by

$$H_1 = L_0 = \sum_i (D_i + \Delta) ,$$
$$H_2 = 2\sum_i \left((D_i)^2 + 2\Delta D_i\right) - d_{ab}\sum_{i\neq j} \theta_{ij}\theta_{ji} t_i^a t_j^b . \tag{3.7}$$

where $D_i$ and $\theta_{ij}$ are as in (3.4). Note that (for $k = 1$) the Calogero-Sutherland-Moser Hamiltonian $H_2$ can also be expressed as (see [12])

$$H_2 = d_{ab}\sum_{m>0} m J^a_{-m} J^b_m . \tag{3.8}$$

For $k > 1$ it is easily seen that on the 2-spinon states (3.5) the Hamiltonians (3.7), but now with $D_i$ given by (3.6), are again in the commutant of $Y(\mathfrak{sl}_2)$.

Although we have not been able to prove this, we expect the Yangian symmetry at $k > 1$ to generalize to multi-spinon states. We are lacking explicit expressions for the Yangian generators on $N$-spinon states with $N > 2$ (expressions of the form (3.3) will not do), but we have observed that the decomposition of $\mathcal{F}$ into irreducible representations of $Y(\mathfrak{sl}_2)$ seems to be completely analogous to the one for level-1, explained in [13]. In particular, it seems that the highest weight vectors of $Y(\mathfrak{sl}_2)$ are given by the fully polarized states, i.e. containing only $\phi^+$ fields. For generic mode sequences $n_1, \ldots, n_M$, the corresponding (irreducible) Yangian representation is $2^M$ dimensional and, as an $\mathfrak{sl}_2$ representation, is isomorphic to the $M$-fold tensorproduct of the spin-$\frac{1}{2}$ representation. For sequences $n_1, \ldots, n_M$ containing pairs $(n_i, n_{i+1})$ with the minimal possible increment as allowed by the rules below (2.10), the corresponding (irreducible) Yangian representation is smaller, namely, one has to project onto the symmetric part (e.g. the triplet for $M = 2$) of the corresponding tensor product of doublet representations.

A new feature, as compared to $k = 1$, is that in addition to the Yangian we have the action of a Hecke algebra on the $N$-spinon Fock space. Its generators essentially correspond to the exchange of spinons in (3.2) by means of the braiding matrices. Clearly, this defines a representation of the Braid group $\mathcal{B}_N$ but, as shown in e.g. [25], this representation actually factors through the Hecke algebra $\mathcal{H}_N(q)$ for $q = \exp(2\pi i/(k+2))$.

## 4. Character formulas

Given the spinon basis (2.10) the computation of the character $\mathrm{ch}_{L_j}(z;q) = \mathrm{Tr}_{L_j}(q^{L_0} z^{J_0^3})$ is now straightforward. Consider the states with $(M, N)$ number of $(+,-)$-spinons. The sum over the spinon modes $\tilde{n}_1, \ldots, \tilde{n}_{M+N}$ contributes the usual factor

$$\mathcal{S}_{M,N}(z,q) = \frac{1}{(q)_M (q)_N} z^{\frac{1}{2}(M-N)} , \tag{4.1}$$

which, in terms of the total spinon number $m_1 = M + N$ and isospin $j' = \frac{1}{2}(M - N)$, can also be written as

$$\mathcal{S}_{M,N}(z,q) = \frac{1}{(q)_{\frac{1}{2}m_1+j'}(q)_{\frac{1}{2}m_1-j'}} z^{j'} = \frac{1}{(q)_{m_1}} \begin{bmatrix} m_1 \\ \frac{1}{2}m_1 - j' \end{bmatrix} z^{j'} \equiv \mathcal{S}^{j'}_{m_1}(z;q) . \tag{4.2}$$

Here we have introduced, as usual, the $q$-numbers and $q$-binomial by

$$(q)_N = \prod_{k=1}^{N} (1 - q^k) , \quad \begin{bmatrix} M \\ N \end{bmatrix} = \frac{(q)_M}{(q)_N (q)_{M-N}} . \tag{4.3}$$

The combinatorics involved in performing the sum over fusion channels of length $m_1 = M+N$, with the minimal mode sequence $n_{1,\min}, \ldots, n_{M+N,\min}$, and such that $j_{M+N} = j$ is formally similar to the combinatorics that we used in [13] to compute spinon contributions to the Virasoro characters. The only difference, as compared to [13], is that we are now dealing with a truncated Bratteli diagram (as dictated by the fusion rules). This reflects itself in the fact that now there only exists a *finite* set $\{m_2, \ldots, m_k\}$ labelling the 'ghost excitations.' [Here, $k$ is the level of $(\widehat{\mathfrak{sl}_2})_k$.] Provided $\frac{m_1}{2} - j \in \mathbb{Z}_{\geq 0}$, we find the following contribution

$$q^{-\frac{j}{2}-\frac{1}{4}m_1^2} \sum_{m_2,\ldots,m_k} q^{\frac{1}{2}(m_1^2+m_2^2+\cdots+m_k^2-m_1m_2-m_2m_3-\cdots-m_{k-1}m_k)} \prod_{i=2}^{k} \begin{bmatrix} \frac{1}{2}(m_{i-1}+m_{i+1}+\delta_{i,2j+1}) \\ m_i \end{bmatrix}, \tag{4.4}$$

where the summation is over all odd positive integers for $m_{2j}, m_{2j-2}, m_{2j-4}, \ldots$ and over the even positive integers for the remaining ones (we set $m_{k+1} \equiv 0$). Recall that $m_1$ (not summed) is the total spinon number. For $q = 1$ the expression (4.4) gives the number of spin-$j$ representations in the $m_1$-fold (truncated) tensor product of the spin-$\frac{1}{2}$ representation of $U_q(\mathfrak{sl}_2)$ at $q = \exp(2\pi i/(k+2))$.

Let us define for an arbitrary (symmetric) $k \times k$-matrix $K$ and $k$-vector $u$, the $q$-series

$$\Phi_K^{m_1}(u;q) = \sum_{m_2,m_3,\ldots,m_k} q^{\frac{1}{4}m\cdot K\cdot m} \prod_{i\geq 2} \begin{bmatrix} \frac{1}{2}((2-K)\cdot m+u)_i \\ m_i \end{bmatrix}, \tag{4.5}$$

then (4.4) can be written as

$$q^{-\frac{j}{2}-\frac{1}{4}m_1^2}\, \Phi_{A_k}^{m_1}(u_j;q)\,, \tag{4.6}$$

where $A_k$ is the Cartan matrix of the Lie algebra $A_k \cong \mathfrak{sl}_{k+1}$ and $u_j$ is the unit vector $(u_j)_i = \delta_{i,2j+1}$.

Combining the ingredients, we have obtained the following expression for the characters of the $(\widehat{\mathfrak{sl}_2})_k$ integrable modules $L_j$ $(j = 0, \frac{1}{2}, \ldots, \frac{k}{2})$

$$\mathrm{ch}_{L_j}(z,q) = q^{\Delta(j)-j/2} \sum_{M,N\geq 0} q^{-\frac{1}{4}(M+N)^2} \Phi_{A_k}^{M+N}(u_j;q) S_{M,N}(z;q)\,. \tag{4.7}$$

For $k = 1$ this reproduces the result of [5,12,13]. For $k > 1$ this quasi-particle form of the character, as well as the expressions for the string functions [26] $c_{2j'}^{2j}(q)$ that can immediately be read off using (4.2), appear to be new. For low level and isospin, we have verified numerically (typically up to $\mathcal{O}(q^{25})$) that these string functions do indeed coincide with the ones in e.g. [26]. The correctness of the characters (4.7) strongly supports the claims made in Sections 2 and 3.

Similarly, expressions for the generating series $\Psi_{j,j'}(q)$ of the number of $\mathfrak{sl}_2$ representations of spin $j'$ in $L_j$ can be obtained from (4.7), (4.2) and the identity

$$\begin{bmatrix} m_1 \\ \frac{m_1}{2}-j' \end{bmatrix} - \begin{bmatrix} m_1 \\ \frac{m_1}{2}-j'-1 \end{bmatrix} = q^{-\frac{m_1^2}{4}-\frac{j'}{2}}\, \Phi_{A_\infty}^{m_1}(u_{j'};q)\,, \tag{4.8}$$

which follows from the analysis in [13]. We find

$$\Psi_{j,j'}(q) = q^{\Delta(j)-\frac{j+j'}{2}} \sum_{m_1} \Phi_{A_k}^{m_1}(u_j;q)\, \frac{q^{-\frac{1}{2}m_1^2}}{(q)_{m_1}}\, \Phi_{A_\infty}^{m_1}(u_{j'};q)\,. \tag{4.9}$$

This generalizes the result for $k = 1$ (see Eq. (35) in [13]) in which case (4.9) corresponds to the irreducible Virasoro character $\mathrm{ch}_{h=j'^2}^{\mathrm{Vir}}(q)$, where $j = 0, \frac{1}{2}$ for $j'$ integer or halfinteger, respectively. Note that (4.9) coincides with the $l \to \infty$ limit of the conjectured branching functions for the coset $(\widehat{\mathfrak{sl}_2})_k \oplus (\widehat{\mathfrak{sl}_2})_l/(\widehat{\mathfrak{sl}_2})_{k+l}$ in [5] (for $j = j' = 0$), as it should.

## 5. Concluding remarks

In this paper we have proposed a generalization of the results of [13] for level-1 $\widehat{\mathfrak{sl}_2}$ to higher level. We have seen that both the spinon description of the integrable modules as well as the Yangian symmetry are likely to pertain at higher levels. In addition to the Yangian symmetry, we have seen the occurrence of a Hecke symmetry $\mathcal{H}_N(q = \exp(2\pi i/k+2))$ on the $N$-spinon subspace. Using the proposed spinon description of the spectrum we have derived new character formulas for the level-$k$ integrable modules.

From the characters (4.7) it can be seen that the $N$-spinon subspace naturally factorizes into a 'Yangian part,' corresponding to $S_{M,N}(z;q)$, and a 'Hecke (or RSOS) part,' corresponding to $\Phi^{m_1}_{A_k}(u_j;q)$. A similar factorization was observed in the $S$-matrix for higher level $SU(2)$ WZW-models, as well as in the analysis of the TBA for the generalized (i.e. higher $\mathfrak{sl}_2$ representations) integrable XXX spin chain models which, in the low energy limit, are known to give rise to higher level $SU(2)$ WZW-models [19]. The spinon basis in this paper gives a natural explanation of these aforementioned results. An interesting problem that immediately comes to mind is whether there exists a corresponding generalization of the Haldane-Shastry model, i.e. a spin chain model (with long-range interactions) that has exact Yangian symmetry and gives rise to higher level $\widehat{\mathfrak{sl}_2}$ in the low energy limit. A natural candidate, obviously, would be to extend the usual Haldane-Shastry model with additional 'local height variables.'

Generalization of the results in this paper to $(\widehat{\mathfrak{sl}_N})_k$ is, in principle, straightforward. Again we expect to find that the Fock space of the primary field in the vector representation of $\mathfrak{sl}_N$ decomposes into a direct sum of integrable representations - each integrable representation occurring exactly once - and that the integrable representations carry a (fully reducible) representation of $Y(\mathfrak{sl}_N)$ (see [27] for level $k=1$), whose action on the spinon basis is related to the $\mathfrak{sl}_N$ Calogero-Sutherland-Moser Yangian.

In general, i.e. for other Lie algebras $\mathfrak{g}$ and/or other representations of $\mathfrak{g}$ as well as for more general conformal field theories such as coset models, the situation is more complicated. For instance, for groups $\mathfrak{g}$ other than $\mathfrak{sl}_N$ the integrable highest weight modules of $\widehat{\mathfrak{g}}$ do not carry a representation of $Y(\mathfrak{g})$, simply because finite-dimensional irreducible representations of $\mathfrak{g}$ do, in general, not extend to representations of $Y(\mathfrak{g})$. Another, more important, difference is that the Fock spaces corresponding to some chiral primary field in the theory do, in general, not decompose into a direct sum of irreducible (integrable) modules. Rather, in terms of quasi-particle language, one would say that the corresponding conformal field theory corresponds to a theory of quasi-particles with other than purely statistical interactions.

That these two issues are sometimes closely related can be seen in the following example. Consider $\mathfrak{g} \cong \mathfrak{so}_N$. The spinor representation of $\mathfrak{so}_N$ extends to a representation of $Y(\mathfrak{so}_N)$ [23]. This then defines an action of $Y(\mathfrak{so}_N)$ on the one-spinon Fock space of the spinon field of $(\widehat{\mathfrak{so}_N})_k$, which can presumably be extended to an action on the entire spinon Fock space by co-multiplication (as outlined in Section 3). However, for $N \neq 3$, this Fock space is *bigger* than just a direct sum of integrable modules. Upon projection to the integrable modules this Yangian symmetry gets lost.


### Acknowledgements

K.S. thanks the Physics Department of UC Santa Barbara, where part of this work was done, for hospitality. The research of K.S. was supported in part by the National Science Foundation under grant PHY90-21984. A.W.W.L. is a Fellow of the A.P. Sloan Foundation.

## References


[1] I. Affleck and A.W.W. Ludwig, Nucl. Phys. B 360 (1991) 641; Phys. Rev. Lett. 68 (1992) 1046;
A.W.W. Ludwig and I. Affleck, Nucl. Phys. B 428 (1994) 545.

[2] P. Fendley, A.W.W. Ludwig and H. Saleur, Exact conductance through point contacts in the $\nu = 1/3$ fractional quantum Hall effect, Phys. Rev. Lett. (1995), in print, (cond-mat/9408068).

[3] P. Fendley, H. Saleur and N. Warner, Nucl. Phys. B 430 (1994) 577.

[4] A. Kuniba, T. Nakanishi and J. Suzuki, Mod. Phys. Lett. A8 (1993) 1649.

[5] R. Kedem, T. Klassen, B. McCoy and E. Melzer, Phys. Lett. B 304 (1993) 263; B 307 (1993) 68;
S. Dasmahapatra, R. Kedem, T. Klassen, B. McCoy and E. Melzer, J. Stat. Phys. 74 (1994) 239;
R. Kedem, B. McCoy and E. Melzer, The sums of Rogers, Schur and Ramanujan and the Bose-Fermi correspondence in $1+1$-dimensional quantum field theory, (hep-th/9304056);
E. Melzer, Lett. Math. Phys. 31 (1994) 233.

[6] A. Berkovich, Nucl. Phys. B 431 (1994) 315;
A. Berkovich and B. McCoy, Continued fractions and fermionic representations for characters of $M(p,p')$ minimal models, (hep-th/9412030).

[7] A.N. Kirillov, Dilogarithm identities, (hep-th/9408113).

[8] O. Foda and Y.-H. Quano, Polynomial identities of the Rogers–Ramanujan type, (hep-th/9407191); Virasoro character identities from the Andrews–Bailey construction, (hep-th/9408086).

[9] J. Lepowsky and R.L. Wilson, Proc. Natl. Acad. Sci. USA 78 (1981) 7254; Adv. Math. 45 (1982) 21; Inv. Math. 77 (1984) 199; Inv. Math. 79 (1985) 417.

[10] B. Feigin, T. Nakanishi and H. Ooguri, Int. J. Mod. Phys. A7 Suppl. 1A (1992) 217.

[11] B. Feigin and A. Stoyanovsky, Quasi-particle models for the representations of Lie algebras and geometry of flag manifold, (hep-th/9308079).

[12] D. Bernard, V. Pasquier and D. Serban, Nucl. Phys. B 428 (1994) 612.

[13] P. Bouwknegt, A. Ludwig and K. Schoutens, Phys. Lett. B 338 (1994) 448; Affine and Yangian Symmetries in $SU(2)_1$ Conformal Field Theory, to appear in the proceedings of the 1994 Trieste Summer School on "High Energy Physics and Cosmology," Trieste, July 1994, (hep-th/9412199).

[14] S. Iso, Anyon basis of $c=1$ conformal field theory, (hep-th/9411051).

[15] G. Georgiev, Combinatorial constructions of modules for infinite-dimensional Lie algebras, I. Principal subspace, (hep-th/9412054).

[16] F.D.M. Haldane, Phys. Rev. Lett. 60 (1988) 635;
B.S. Shastry, Phys. Rev. Lett. 60 (1988) 639.

[17] F.D.M. Haldane, Z.N.C. Ha, J.C. Talstra, D. Bernard and V. Pasquier, Phys. Rev. Lett. 69 (1992) 2021.

[18] D. Bernard, M. Gaudin, F.D.M. Haldane and V. Pasquier, J. Phys. A 26 (1993) 5219.

[19] L. Faddeev and N. Reshetikhin, Ann. Phys. 167 (1986) 227;
N. Reshetikhin, J. Phys. A 24 (1991) 3299;
P. Fendley, Phys. Rev. Lett. 71 (1993) 2485.

[20] G. Moore and N. Seiberg, Comm. Math. Phys. 123 (1989) 177; Lectures on RCFT, in "Superstrings '89," Proceedings Trieste 1989.

[21] C. Dong and J. Lepowsky, Generalized vertex algebras and relative vertex operators, Prog. in Math. 112 (Birkhäuser, Boston, 1993).

[22] A.B. Zamolodchikov and V.A. Fateev, Sov. Phys. JETP 62 (1985) 215; 63 (1986) 913.

[23] V.G. Drinfel'd, Sov. Math. Dokl. 32 (1985) 254; 36 (1988) 212; Quantum Groups, in Proc. of the International Congress of Mathematicians (Berkeley, 1986).

[24] V. Chari and A. Pressley, L'Enseignement Mathématique 36 (1990) 267; Z. reine angew. Math. 417 (1991) 87.

[25] L. Alvarez-Gaumé, C. Gomez and G. Sierra, Nucl. Phys. B 319 (1989) 155; B 330 (1990) 347.

[26] V. Kac and D. Peterson, Adv. Math. 53 (1984) 125.

[27] K. Schoutens, Phys. Lett. B 331 (1994) 335.