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# Finite-dimensional models of the Ginzburg–Landau equation

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**Abstract.** In this paper we study truncated finite dimensional models of the infinite-dimensional equation describing the evolution of even, space-periodic solutions of the Ginzburg–Landau equation. We derive estimates on the position of the global attractor of the flow, which yield that the magnitude of the  $M$ th mode of the global attractor decays faster than any algebraic power of  $M^{-1}$ . The estimates are independent of the dimension of the model. In a numerical section we simulate the flow for three radical low-dimensional models (of two, three and four complex modes); we analyse the influence of the number of modes on the global dynamics. The four-dimensional model exhibits the same intricate flow-characteristics as the 32-dimensional model studied by Keefe.

AMS classification scheme numbers: 34D05, 35A40, 35Q20, 58F12, 65M60

## 1. Introduction

The starting point of the investigations in this paper is the so-called Ginzburg–Landau equation:

$$\frac{\partial \Phi}{\partial \tau} = (\alpha + \beta |\Phi|^2) \Phi + \gamma \frac{\partial^2 \Phi}{\partial \xi^2}$$

with  $\alpha, \beta, \gamma \in \mathbb{C}$  and  $\Phi(\xi, \tau): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ . This equation appears as modulation equation describing the nonlinear evolution of disturbances of a basic pattern of a physical system, which are linear unstable. Problems of practical interest, such as Poiseuille flow, Rayleigh–Bénard convection and reaction–diffusion systems, can be modelled into one general description for which one can derive a modulation equation which governs the nonlinear evolution of patterns (for control parameter  $R$  close to the critical linear stability bound  $R_c$ ) (see DiPrima *et al* [2], Newell [11] or Doelman [3]). The Ginzburg–Landau equation can be considered as a model equation describing this nonlinear evolution: coefficients  $\alpha, \beta, \gamma$  can be computed for a given practical problem (see [3]). The derivation process yields that  $\text{Re } \alpha > 0$

and  $\operatorname{Re} \gamma > 0$ ; to balance the linear growth we choose  $\operatorname{Re} \beta < 0$ . Hence we find as rescaled Ginzburg–Landau equation:

$$\frac{\partial \psi}{\partial t} = (1 - (1 + ib) |\psi|^2) \psi + (1 + ia) \frac{\partial^2 \psi}{\partial z^2} \quad (1.1)$$

with  $a, b \in \mathbb{R}$ . The equation can be used to study the stability of periodic solutions of the basic equation, which are solutions of the type  $\psi(z, t) = \operatorname{Re} e^{i(kz + \omega t)}$  of (1.1). Stuart and DiPrima [16] established that all periodic solutions are unstable if

$$1 + ab < 0$$

(see also Ioos *et al* [6]). Instability of all periodic solutions suggests the presence of chaotic behaviour. To investigate (1.1) more thoroughly one can restrict oneself to a special class of solutions: the space-periodic, even solutions, i.e. one sets

$$\psi(z, t) = \sum_{n=-\infty}^{\infty} Z_n(t) e^{inqz} \quad (1.2)$$

for some  $q > 0$ , with  $Z_n: \mathbb{R} \rightarrow \mathbb{C}$  and  $Z_n(t) = Z_{-n}(t)$  for all  $n$ .

*Remark 1.1.* This spectral expansion is physically quite natural; it corresponds to the zero flux, i.e. homogeneous Neumann, boundary conditions,  $\partial \psi / \partial z = 0$ , at the ends of the interval  $[0, 2\pi/q]$ . Expression (1.2) is, from this point of view, the Fourier expansion of  $\psi(z, t)$  with respect to the natural base.

Substituting (1.2) into (1.1) yields

$$\dot{Z}_n = (1 - n^2 q^2 (1 + ia)) Z_n - (1 + ib) \sum_{k+l+m=n} Z_k Z_l Z_m^* \quad (1.3)$$

with  $Z_m^*$  the complex conjugate of  $Z_m$ . This equation, and also equation (1.1) for other boundary conditions, has been studied by Ghidaglia and Heron [5]. They derived upper and lower bounds on the dimension of the global attractor; see also Doering *et al* [4]. It appears that the infinite-dimensional system (1.3) has a finite-dimensional attractor of (low) dimension  $D_q(a, b)$  ( $D_q(a, b) \rightarrow \infty$  as  $q \downarrow 0$ ). Equation (1.3) has also been studied numerically: Kuramoto and Koga [8], who studied (1.3) as a model for chemical turbulence, fixed  $q$ , and took  $b$ , a measure for the nonlinear coupling between the reactants, as the bifurcation parameter. Moon *et al* [10] and Keefe [8] fixed the coefficients of the Ginzburg–Landau equation and used  $q$  as bifurcation parameter. Keefe [8] considered  $a = 4$ ,  $b = -4$  (thus  $1 + ab < 0$ ) and  $q \in [0.6, 1.4]$ , and found a global attractor of maximal (Lyapunov) dimension 3.05 (at  $q \approx 0.950$ ). Computations of the wavenumber spectra ([8], [10]) show that the relative energy content of a mode  $Z_n$ ,  $P_q(n)$  (see (3.19)), is a rapidly exponentially decaying function of  $n$  (for  $a$  fixed) and  $P_q(n) < 10^{-4}$  for  $n \geq 4$  for all  $q > 0.6$ .

It is of course impossible to study numerically an infinite-dimensional system. Moon *et al* [10] investigated a truncated model of (1.3) with 64 modes. They checked that increasing the number of modes does not alter the dynamics of the flow.

The outcome of the research in [4] and [5] and the numerical simulations carried into execution in [8] and [10] justify an exploration of the properties of the

truncation of (1.3):

$$\dot{Z}_n^{(N)} = (1 - n^2 q^2 (1 + ia)) Z_n^{(N)} - (1 + ib) \sum_{\substack{k+l+m=n \\ |k|, |l|, |m| \leq N}} Z_k^{(N)} Z_l^{(N)} Z_m^{(N)*} \quad (1.4)$$

for  $n = 0, 1, \dots, N$  and  $Z_m^{(N)} = Z_{-m}^{(N)}$  for all  $m, N$ .

In this paper, we study these finite-dimensional models of the Ginzburg–Landau equation. Note that system (1.4) can be considered as a (modified) Galerkin approximation of the Ginzburg–Landau equation with homogeneous Neumann boundary conditions.

As in [4, 5, 8, 10] we are mainly interested in the global attractor of the flow induced by (1.4): we want to determine its position in the  $(N + 1)$ -dimensional phase space. Moreover, we would like to investigate the influence of  $N$  on the global attractor (the numerical results of Moon *et al* [10] and Keefe [8] suggest that this influence is negligible).

Our first result is that there exists (for all  $N$ ) a region  $R_{\text{att}}$  which attracts the flow induced by (1.4):

$$R_{\text{att}} = \left\{ z \in \mathbb{C}^{N+1} : |z_0|^2 + 2 \sum_{n=1}^N |z_n|^2 \leq 1 \right\}.$$

Using this we derive our main result: for all  $\alpha > 0$  there exist constants  $K(\alpha)$  and  $M(\alpha)$ , independent of  $N$ , such that for  $M \geq M(\alpha)$

$$\limsup_{t \rightarrow \infty} |Z_M^{(N)}(t)| \leq \frac{K(\alpha)}{M^\alpha}. \quad (1.5)$$

Thus the maximal magnitude of a mode  $Z_M^{(N)}$  of the global attractor decays faster than any algebraic power of  $M^{-1}$  and we have estimated the position of the global attractor in phase space by bounds independent of  $N$ . Therefore, as was suggested by the numerical simulations, we may conclude that the influence of  $N$  on the global attractor is small. As a corollary of the analysis we can prove that there is a  $Q_0 = Q_0(b)$ , independent of  $N$ , such that the Stokes solution (i.e.  $Z_0^{(N)} = e^{-ibr}$ ,  $Z_j^{(N)} = 0 \ j = 1, \dots, N$ ) is a global attractor of system (1.4) for  $q > Q_0$ . Numerical simulations (see section 4, figure 1) show that the Stokes wave is a global attractor as long as it is linear stable (see section 2).

The results mentioned above are derived in section 3 (section 2 consists of some basic observations). In section 4 we investigate numerically the dynamics of (1.4) in cases of radical low-dimensional truncation:  $N = 1, 2$  and  $3$ . We study these systems by a projection of a Poincaré map into the plane spanned by  $\text{Re } Z_0^{(N)}$  and  $\text{Re } Z_1^{(N)}$  (see section 4). We compare in detail between these projected sections of the global attractor for  $N = 1, 2, 3$  and  $N = 31$  (the case studied by Keefe [8]). It should be remarked that the conclusions on the similarities between the global attractors are all based on the comparisons between the projected sections.

The outcome of the numerical simulations is summarized in figure 1 (section 4). There is a good agreement between the numerics and the analysis of section 3: system (1.4) exhibits for  $N = 3$  the same qualitative and quantitative features as the 32-dimensional complex system studied by Keefe [8], i.e. the modes  $Z_4^{(N)}, Z_5^{(N)}, \dots$ , etc, do not significantly influence the asymptotic dynamics of the flow of (1.4). The behaviour (for  $t \rightarrow \infty$ ) of the solutions of the three-dimensional system ( $N = 2$ ) differs only slightly from the  $N = 3$  system for values of  $q$  close to a chaotic region.

For  $N = 1$  system (1.4) exhibits already remarkable similarities with the higher dimensional systems.

*Remark 1.2.* In section 3 we do not use the symmetry  $Z_m^{(N)}(t) = Z_{-m}^{(N)}(t)$ , hence the estimates on the position of the global attractor are also valid in situations where one considers the same Galerkin truncation as (1.4) with homogeneous Dirichlet or periodic boundary conditions.

*Remark 1.3.* Although the higher-order modes appear to be very small (see section 3, 4 and [8, 10]) they do not become zero, i.e. they do not disappear as  $t \rightarrow \infty$ . Thus, the higher modes have no (significant) influence on the low-order ( $Z_0^{(N)}, Z_1^{(N)}, \dots$ ) modes of the attractor, but the attractor is on a manifold which has to be described by all  $N + 1$  axes of the  $(N + 1)$ -dimensional phase space. One could try to find a transformation of the base of the spectral expansion  $\{e^{inqz}\}_{n=-\infty}^{\infty}$  to determine a base such that the global attractor can be spanned by the first  $N'$  vectors of the new base (for some  $N'$ ). This has been done by Rodriguez and Sirovich by applying the Karhunen–Loeve expansion ([14, 15]): they determined a linear transformation by integrating (1.4) for  $N = 15$  with  $q$  fixed at 0.950, the most chaotic case, and solving (numerically) an eigenvalue problem. System (1.4) written down along this new base still excites all modes; however, truncated to a (complex) three-dimensional system, it exhibits the same dynamic behaviour as the system studied by Keefe [8]. It should be noted that it still takes a serious numerical effort to obtain good approximations of the coefficients of the terms of this truncation and that the truncation still excites eight modes of the standard base.

Another method to control the high-order modes is to construct an approximation of the inertial manifold. In [4] is shown that the Ginzburg–Landau equation has a finite-dimensional inertial manifold (see [1] for recent estimates on the dimension). A method to approximate the inertial manifold is developed in [17] and [7] (for the Navier–Stokes equations, respectively the Kuramoto–Sivashinsky equation). It should be possible to apply this approximation method to the Ginzburg–Landau equation. The Ginzburg–Landau equation can be a useful and important model problem to make a comparison, both analytical and numerical, between these three methods (i.e. constructing an inertial manifold, the Karhunen–Loeve expansion [14, 15] and the (modified) classical Galerkin truncation of this paper).

*Remark 1.4.* Ghidaglia and Heron [5] proved that the dimension  $D_q$  of the global attractor of untruncated system (1.4) becomes  $\infty$  as  $q \downarrow 0$ . The estimate of the constant  $K(\alpha)$  (see 1.5) also tends to infinity as  $q \downarrow 0$ . Moon *et al* [10] studied the dynamics of (1.4) for  $q < 0.6$ ; they observed that the influence of  $Z_4^{(N)}, Z_5^{(N)}$ , etc, grows as  $q$  decreases. Hence we expect that system (1.4) will not exhibit the ‘correct’ asymptotic dynamics as  $q$  decreases below 0.6.

## 2. Some basic observations

In this section we derive some elementary results which are useful in the remainder of this paper.

2.1. *The stability of the Stokes wave*

Equation (1.1), with the zero-flux boundary conditions, has a simple time-periodic solution:  $\psi(z, t) = e^{-ibt}$ , the so-called Stokes wave. This solution can also be regarded as a solution of finite-dimensional systems (1.4):

$$(Z_0^{(N)}, \dots, Z_N^{(N)}) = (e^{-ibt}, 0, \dots, 0). \tag{2.1}$$

The Stokes wave is a solution of (1.4) for any  $N$  (also  $N = \infty$ ).

*Property 2.1.* The Stokes solution (2.1) of  $N$ -dimensional systems (1.4) is stable for

$$q > q_0(a, b) = \sqrt{-2 \frac{1+ab}{1+a^2}}. \tag{2.2}$$

Thus, the critical (bifurcation) value of  $q$  does not depend on  $N$ . Remark that (2.1) is stable for all  $q$  if  $1+ab > 0$ . It appears in the numerical simulations that (2.1) is the only attractor for the flow induced by (1.4) for  $q > q_0$ , it is the global attractor. In corollary 3.5 we prove a weaker result. Equation (2.1) is a global attractor for  $q > Q_0 > q_0$ ,  $Q_0$  independent of  $N$ .

*Proof.* We linearize along the Stokes wave:

$$Z_0^{(N)}(t) = e^{-ibt} + z_0(t) \quad Z_k^{(N)}(t) = z_k(t) \quad k = 1, \dots, N.$$

The equations of (1.4) decouple; setting  $z_k = e^{-ibt} w_k$  yields

$$\dot{w}_k = -[(1 + k^2 q^2) + i(b + ak^2 q^2)] w_k - [1 + ib] w_k^*$$

for  $k = 0, 1, \dots, N$ . The zero solution of this system is stable if

$$q^2 \geq \frac{1 - 2(1+ab)}{k^2 (1+a^2)} \quad k = 1, \dots, N$$

( $w_0 = 0$  is stable for all  $q$ ). We remark that  $Z_1^{(N)}$  is the first mode which becomes unstable.

2.2. *Symmetries*

There are two transformations  $T_\phi$  and  $S_\sigma$  which carry over solutions of (1.4) to other solutions

$$T_\phi(Z_0^{(N)}, \dots, Z_N^{(N)}) = (e^{i\phi} Z_0^{(N)}, \dots, e^{i\phi} Z_N^{(N)}) \quad \phi \in [0, 2\pi) \tag{2.3}$$

$$S_\sigma(Z_0^{(N)}, \dots, Z_N^{(N)}) = ((-1)^{0+\sigma} Z_0^{(N)}, \dots, (-1)^{N+\sigma} Z_N^{(N)}) \quad \sigma = 0, 1. \tag{2.4}$$

We remark that, in a sense, we have already used  $T_\phi$  in the proof of property 2.1. Using  $T_\phi$  one derives the following.

*Property 2.2.* Isolated, periodic solutions of (1.4) have to be of the form  $\Gamma(t) = A^{(N)} e^{i\omega t}$  for some  $\omega \in \mathbb{R}$  and some complex vector  $A^{(N)} \in \mathbb{C}^{N+1}$ .

Although this result is quite classical we give a short proof.

*Proof.* Let  $\Gamma(t)$  be a general isolated periodic solution of (1.4);  $T_\phi$  generates a surface of solutions:

$$\{e^{i\phi}\Gamma(t), t \in \mathbb{R}, \phi \in [0, 2\pi)\}.$$

Thus,  $\Gamma(t)$  has to be invariant under  $T_\phi$ . Using the Fourier decomposition of  $\Gamma(t)$

$$\Gamma(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{iwn t}$$

$\alpha_n \in \mathbb{C}^{N+1}$ ,  $w \in \mathbb{R}$ . One easily derives that the only periodic solutions, invariant under  $T_\phi$  are  $\Gamma(t) = Ae^{iwt}$  for  $w \in \mathbb{R}$ ,  $A \in \mathbb{C}^{N+1}$ .

Property 2.2 yields that isolated, periodic solutions of (1.4) can be found by solving the  $(N + 1)$ -dimensional complex system

$$i w A_k = (1 - k^2 q^2 (1 + ia)) A_k - (1 + ib) \sum_{l+m+n=k} A_l A_m A_n^*. \tag{2.5}$$

Using  $T_\phi$  we can reduce (1.4) to a  $(2N + 1)$ -dimensional real system, by setting  $Z_k = R_k e^{i\phi_k}$  and replacing  $\phi_0, \dots, \phi_N$  by  $\phi_0 - \phi_N, \dots, \phi_{N-1} - \phi_N$ . The isolated periodic solutions of (1.4), solutions of (2.5), are stationary points of this system. The stability of the periodic solutions can now be computed by direct linear analysis.

*Remark.* The above results are also true for the untruncated ( $N = \infty$ ) system.

### 3. Estimates on the position of the global attractor

In this section we study the asymptotic dynamics of the flow induced by the  $(N + 1)$ -dimensional system (1.4). It is our goal to derive estimates on  $\limsup_{t \rightarrow \infty} |Z_m^{(N)}|$  for general  $N$ , and independent of  $N$ . The disappearance of the  $N$  in the constants of the estimates is achieved by a limit process of successive estimates. To start this process we need an estimate like

$$\limsup_{t \rightarrow \infty} |Z_M^{(N)}| \leq 1 \text{ for all } M.$$

This result is a direct consequence of proposition 3.1

*Proposition 3.1.* Let  $(Z_0^{(N)}, Z_1^{(N)}, \dots, Z_N^{(N)})$  be a solution of system (1.4). Then, for all  $N \geq 0$ ,

$$\limsup_{t \rightarrow \infty} \sum_{k=-N}^N |Z_k^{(N)}(t)|^2 \leq 1.$$

Thus the region  $R_{att} = \{z \in \mathbb{C}^{N+1} : |z_0|^2 + 2 \sum_{i=1}^N |z_i|^2 \leq 1\}$  is a domain of attraction for the flow induced by (1.4).

*Proof.* Set

$$\Psi^{(N)}(z, t) = \sum_{k=-N}^N Z_k^{(N)}(t) e^{ikqz}. \tag{3.1}$$

It is natural to expect that  $\Psi^{(N)}$  is an approximation of  $\Psi$ , a solution of (1.1). Since  $Z_k^{(N)}$  satisfies (1.4),  $\Psi^{(N)}$  is a solution of a partial differential equation resembling (1.1):

$$\frac{\partial \Psi^{(N)}}{\partial t} = (1 - (1 + ib) |\Psi^{(N)}|^2) \Psi^{(N)} + (1 + ia) \frac{\partial^2 \Psi^{(N)}}{\partial z^2} + R^{(N)} \tag{3.2}$$

with

$$R^{(N)} = (1 + ib) \sum_{N+1 \leq |n| \leq 3N} \left( \sum_{\substack{k+l+m=n \\ |k|, |l|, |m| \leq N}} Z_k^{(N)} Z_l^{(N)} Z_m^{(N)*} \right) e^{inqz}, \tag{3.3}$$

In corollary 3.9 we will prove that  $|R^{(N)}| = O(1/N^\beta) \forall \beta > 1$ .

We now derive an equation for  $|\Psi^{(N)}|^2$ ; by integrating this equation over one (spatial) period we obtain

$$\frac{\partial}{\partial t} \int_0^{2\pi/q} |\Psi^{(N)}|^2 dz = 2 \int_0^{2\pi/q} |\Psi^{(N)}|^2 dz - 2 \int_0^{2\pi/q} |\Psi^{(N)}|^4 dz - 2 \int_0^{2\pi/q} \left| \frac{\partial}{\partial z} \Psi^{(N)} \right|^2 dz \tag{3.4}$$

(we remark that  $\int_0^{2\pi/q} R^{(N)} \Psi^{(N)*} dz = 0$ ). Since

$$\begin{aligned} \int_0^{2\pi/q} |\Psi^{(N)}|^2 dz &= \frac{2\pi}{q} \sum_{k=-N}^N |Z_k^{(N)}|^2 \\ \int_0^{2\pi/q} |\Psi^{(N)}|^4 dz &\geq \frac{2\pi}{q} \left( \sum_{k=-N}^N |Z_k^{(N)}|^2 \right)^2 \end{aligned}$$

we obtain, with  $\sum_{k=-N}^N |Z_k^{(N)}|^2 = I(t)$ ,

$$\frac{dI}{dt} \leq 2I - 2I^2 \tag{3.5}$$

which proves the proposition.

*Remark.* A similar result, for the untruncated system ( $R^{(N)} = 0$ ) has been obtained by DiPrima *et al* [2] and Newton and Sirovich [12].

We want to derive upper bounds on  $|Z_k^{(N)}(t)|$  for  $t \rightarrow \infty$ , i.e. we are interested in the position of the  $\omega$ -limit set of a solution of (1.4) with arbitrary initial data. We have now proved that an  $\omega$ -limit set of (1.4) will be always inside  $R_{att}$ . In the following we choose the initial data such that

$$I(t) = \sum_{k=-N}^N |Z_k^{(N)}(t)|^2 \leq 1 \quad \text{for all } t \tag{3.6}$$

which can be achieved by choosing the initial data inside (or on the boundary of)  $R_{att}$ . *This is no restriction.* It is *a priori* possible that a solution of (1.4) remains outside  $R_{att}$  for all finite  $t$ . However, such a solution approaches, arbitrarily accurate, one or more solutions of (1.4) which satisfy (3.6). This is due to the fact that an  $\omega$ -limit set consists of a collection of solutions of the differential equation (see, for instance, Verhulst [18]). Thus, estimates on the behaviour for  $t \rightarrow \infty$  for solutions satisfying (3.6) are valid for all solutions.

We proceed by estimating individual modes  $|Z_k^{(N)}|$  for  $t \rightarrow \infty$  (assuming (3.6)). The estimates we derive are valid for all values of parameters  $a, b$  and  $q (q > 0)$ . We are mainly concerned with estimates which bound  $|Z_M^{(N)}|$  by functions of  $M$  which decay as  $M$  grows and which are independent of  $N$ .



Theorem 3.2. For  $M \geq M_0(q) \geq 2$ :

$$\limsup_{t \rightarrow \infty} |Z_M^{(N)}(t)| \leq \frac{\mathcal{L}_\infty^2}{M^2} \tag{3.7}$$

$\mathcal{L}_\infty$  does not depend on  $N$ ,  $\mathcal{L}_\infty = \max(M_0 - 1, \bar{K}\sqrt{1 + b^2/q^2})$ , for some  $\bar{K} = \bar{K}(M_0)$ .

Thus, for any  $N$ , a mode  $|Z_M^{(N)}(t)|$  will become smaller than  $K/M^2$ , for some  $K$ , as  $t \rightarrow \infty$ . The proof of this theorem is based on an iterative estimation process: first we derive a (rough) estimate for  $|Z_M^{(N)}|$  depending on  $N$ ; the  $N$  disappears after ‘ $\infty$ ’ iteration steps.

Proof of theorem 3.2. Define

$$F_M^{(N)} = \sum_{\substack{k+l+m=M \\ |l|, |l|, |m| \leq N}} Z_k^{(N)} Z_l^{(N)} Z_m^{(N)*}$$

and

$$G_M^{(N)} = -(1 + ib)Z_M^{(N)*} F_M^{(N)} - (1 - ib)Z_M^{(N)} F_M^{(N)*}.$$

Then, by equation (1.4),

$$\frac{\partial}{\partial t} |Z_M^{(N)}|^2 = 2(1 - M^2 q^2) |Z_M^{(N)}|^2 + G_M^{(N)}. \tag{3.8}$$

We observe, using proposition 3.1 (or (3.6)) that

$$\begin{aligned} |F_M^{(N)}| &\leq \sum_{k=-N}^N |Z_k^{(N)}| \sum_{\substack{l+m=M-k \\ |l|, |m| \leq N}} |Z_l^{(N)}| |Z_m^{(N)}| \\ &\leq \sum_{k=-N}^N |Z_k^{(N)}| \sum_{\substack{l=-N \\ |M-k-l| \leq N}}^N (\frac{1}{2} |Z_l^{(N)}|^2 + \frac{1}{2} |Z_{M-k-l}^{(N)}|^2) \\ &\leq \sum_{k=-N}^N |Z_k^{(N)}| \end{aligned}$$

and

$$\left( \sum_{k=-L}^L |Z_k^{(N)}| \right)^2 = \sum_{k,l=-L}^L |Z_k^{(N)}| |Z_l^{(N)}| \leq \frac{1}{2} \sum_{l,k=-L}^L (|Z_k^{(N)}|^2 + |Z_l^{(N)}|^2) \leq 2L + 1$$

for all  $L \leq N$ . Thus:

$$|F_M^{(N)}| \leq \sqrt{2N + 1}. \tag{3.9}$$

Expression  $G_M^{(N)}$  can be estimated by

$$|G_M^{(N)}| \leq 2\sqrt{1 + b^2} |F_M^{(N)}| |Z_M^{(N)}| \tag{3.10}$$

From (3.8) we deduce: if  $1 - M^2 q^2 < 0$  then

$$\limsup_{t \rightarrow \infty} |Z_M^{(N)}(t)|^2 \leq \sup_t |G_M^{(N)}| \frac{1}{2|1 - M^2 q^2|} \tag{3.11}$$

which yields, using  $|Z_M^{(N)}| \leq 1$ , (3.9) and (3.10),

$$\limsup_{t \rightarrow \infty} |Z_M^{(N)}(t)|^2 \leq \frac{\sqrt{(1 + b^2)(2N + 1)}}{|1 - M^2 q^2|}. \tag{3.12}$$

This situation is exactly as above: since we are only interested in estimates for the  $\omega$ -limit set ( $t \rightarrow \infty$ ) we may from now on assume that (3.12) is satisfied for all  $t$ . This idea will again be used (frequently) in this proof: strictly speaking we derive estimates on the lim sup for  $t \rightarrow \infty$  of  $|Z_M^{(N)}|$ , but we immediately assume that the inequality holds for all  $t$ . Inequality (3.12) can be reformulated into:

$$|Z_M^{(N)}(t)| \leq \frac{\mathcal{L}_1}{M} \quad \text{for all } t \tag{3.13}$$

with  $M$  such that  $1 - M^2 q^2 < 0$ , i.e.  $M \geq M_0(q) > 1/q$  and  $\mathcal{L}_1 = N^{1/4} L_1(q, b)$ . Constant  $L_1$  depends on the choice of  $M_0$ , if for instance  $M_0$  is such that  $M_0^2 q^2 - 1 \geq a M_0^2 q^2$  for some  $a > 0$  then one may choose

$$L_1 = K_a \sqrt[4]{3} \frac{\sqrt[4]{1+b^2}}{q} \tag{3.14}$$

(estimating  $\sqrt{2N+1}$  by  $\sqrt{N}\sqrt{3}$ ), with  $K_a = 1/\sqrt{a}$ . Remark that  $K_a$ , and thus  $L_1$ , becomes larger as one chooses  $M_0$  smaller. Substituting (3.13) into (3.10) and (3.11) we obtain

$$|Z_M^{(N)}| \leq \frac{\mathcal{L}_1^{3/2}}{M^{3/2}} \text{ for } M > M_0 \tag{3.15}$$

(and, as mentioned above, we assume that (3.15) holds for all  $t$ ). We can now start a process of successive estimates.

*Step A: a new estimate for  $|F_M^{(N)}|$ .* Inequality (3.9) can be sharpened using (3.15):

$$\begin{aligned} |F_M^{(N)}| &\leq \sum_{k=-N}^N |Z_k^{(N)}| \\ &\leq \sum_{k=-[\mathcal{L}_1]}^{[\mathcal{L}_1]} |Z_k^{(N)}| + 2 \sum_{k=[\mathcal{L}_1+1]}^N |Z_k^{(N)}| \end{aligned} \tag{3.16}$$

in which we use  $[\mathcal{L}_1]$  = integral part of  $\mathcal{L}_1$ . By proposition 3.1 we have (see above)

$$\sum_{k=-[\mathcal{L}_1]}^{[\mathcal{L}_1]} |Z_k^{(N)}| \leq \sqrt{2[\mathcal{L}_1] + 1} \leq \sqrt{2\mathcal{L}_1 + 1}.$$

We want to apply (3.15) to the second term of (3.16); to this end it is necessary that  $[\mathcal{L}_1] + 1 \geq M_0$ . If this is not the case we replace  $\mathcal{L}_1$  by  $\mathcal{L}'_1 = \mathcal{L}_1 + l_1$ ,  $l_1 \in \mathbb{N}$  such that  $[\mathcal{L}'_1] + 1 \geq M_0$ . We immediately skip the accent of  $\mathcal{L}'_1$ . (Remark that  $\mathcal{L}_1$  is of order  $N^{1/4}$ , hence large for large  $N$ .) Thus, by (3.15)

$$\sum_{k=[\mathcal{L}_1+1]}^N |Z_k^{(N)}| \leq \frac{\mathcal{L}_1^\alpha}{([\mathcal{L}_1] + 1)^\alpha} + \int_{[\mathcal{L}_1+1]}^N \frac{\mathcal{L}_1^\alpha}{k^\alpha} dk \leq 1 + \frac{1}{\alpha - 1} ([\mathcal{L}_1] + 1)$$

with, here,  $\alpha = \frac{3}{2}$ ; in the following we use other values of  $\alpha \in (\frac{3}{2}, 2)$ . Hence, we deduce (using  $\mathcal{L}_1 > M_0 - 1$  and assuming  $M_0 \geq 2$ )

$$|F_M^{(N)}| \leq \left[ 2 + 2 \left( 1 + \frac{2}{\alpha - 1} \right) \right] \mathcal{L}_1 = C_1 \mathcal{L}_1 \tag{3.17}$$

with  $C_1 = 12$  as  $\alpha = \frac{3}{2}$ :  $|F_M^{(N)}|$  is bounded by a term of order  $N^{1/4}$ , while the previous estimate of  $|F_M^{(N)}|$ , (3.9), is only of order  $N^{1/2}$ .

*Step B:* a new estimate for  $|Z_M^{(N)}|$ . Using estimate (3.15) of  $|Z_M^{(N)}|$  and new estimate (3.17) of  $(|F_M^{(N)}|)$  in (3.10) and (3.11) we find:

$$|Z_M^{(N)}| \leq \frac{\mathcal{L}_2^{7/4}}{M^{7/4}} \quad \text{for } M \geq M_0 \tag{3.18}$$

with

$$\mathcal{L}_2^{7/4} = K_a \frac{\sqrt[4]{1+b^2}}{q} \sqrt{C_1} \mathcal{L}_1^{5/4}$$

and thus  $\mathcal{L}_2 = N^{(1/4)(5/7)} L_2(q, b)$ ,  $L_2 = K_2 (\sqrt[4]{1+b^2}/q)^{9/7}$ , with  $K_2$  a certain constant.

It is clear that we can continue this process by using (3.18) instead of (3.15) in steps A and B:

$$|Z_M^{(N)}| \leq \frac{\mathcal{L}_3^{15/8}}{M^{15/8}}$$

with  $\mathcal{L}_3 = N^{(1/4)(5/7)(11/15)} K_3 (\sqrt[4]{1+b^2}/q)^{\gamma_3}$ , for some exponent  $\gamma_3$  (one may have been forced to adapt  $\mathcal{L}_2$  in step A to obtain  $[\mathcal{L}_2] + 1 \geq M_0$ ). We can now again return to step A, etc. We obtain, for the general  $n$ th step:

$$|Z_M^{(N)}| \leq \frac{\mathcal{L}_n^{2-2^{-n}}}{M^{2-2^{-n}}}$$

$$\mathcal{L}_n^{2-2^{-n}} = K_a \frac{\sqrt[4]{1+b^2}}{q} \sqrt{C_{n-1}} \mathcal{L}_{n-1}^{(3/2)-2^{-n}}$$

with  $K_\alpha$  as in (3.14) and  $C_n = 4\alpha_n/(\alpha_n - 1)$ ,  $\alpha_n = 2 - 2^{-n}$  (see (3.17)). By basic calculations one can show:

$$\mathcal{L}_n \rightarrow \frac{8K_a^2 \sqrt{1+b^2}}{q^2} \quad \text{as } n \rightarrow \infty.$$

(This can be done by deriving the exact expressions for the exponents of  $(K_a \sqrt[4]{1+b^2}/q)$  and observing that  $C_n \rightarrow 8$  as  $n \rightarrow \infty$ ).

We remark that any time step A is used it is necessary that  $[\mathcal{L}_i] + 1 \geq M_0$ . Hence, we define  $\mathcal{L}_\infty = \max(M_0 - 1, 8K_a^2 \sqrt{1+b^2}/q^2)$  and deduce

$$\limsup_{t \rightarrow \infty} |Z_M^{(N)}(t)| \leq \frac{\mathcal{L}_\infty^2}{M^2} \quad \text{for } M \geq M_0$$

which proves theorem 3.2.

*Remark 3.3.* One can see directly that  $\mathcal{L}_\infty$  is independent of  $N$ : let  $\beta_n$  be the exponent of  $N$  in  $\mathcal{L}_n$ :  $\mathcal{L}_n = N^{\beta_n} L_n(q, b)$ . Applying, once, step A and step B yields:

$$\beta_{n+1} = \left( \frac{3(2^n) - 1}{4(2^n) - 1} \right) \beta_n$$

hence  $\beta_n \rightarrow 0$  as  $n \rightarrow \infty$ .

*Remark 3.4.* The essential result of this proposition is the fact that  $|Z_M^{(N)}|$  can be bounded by  $K/M^2$  for  $t \rightarrow \infty$ ,  $K$  independent of  $N$ . Of course  $K$  can be sharpened by a more sophisticated procedure using less rough estimates.

Numerical simulations demonstrate that for  $q > q_0(a, b)$  (see (2.2)) Stokes solution  $Z_0^{(N)} = e^{-ibt}$ ,  $Z_k^{(N)} \equiv 0$   $k = 1, \dots, N$  attracts all solutions of (1.4), it is the global attractor (see section 4). This result does not follow from our analysis of (the estimates are not sharp enough); however:

*Corollary 3.5.* There exists a  $Q_0 = Q_0(b) > q_0(a, b)$ ,  $Q_0$  independent of  $N$ , such that for  $q > Q_0$ :

$$\begin{aligned} \lim_{t \rightarrow \infty} |Z_k^{(N)}| &= 0 && \text{for } k = 1, \dots, N \\ \lim_{t \rightarrow \infty} |Z_0^{(N)}| &= 1. \end{aligned}$$

Hence, the Stokes solution is a global attractor for the flow induced by (1.4) as  $q > Q_0$ . We remark that this result cannot be a direct consequence of theorem 3.2 since  $\mathcal{L}_\infty \geq M_0 - 1 \geq 1$ .

*Proof.* Since the Stokes solution is stable for  $q > q_0(a, b)$  one can find a neighbourhood of the orbit of this solution which is attracting, i.e. there exists an  $\varepsilon_0 (= \varepsilon_0(q))$  such that if  $|1 - Z_0^{(N)}(t)| < \varepsilon_0$  and  $|Z_k^{(N)}(t)| < \varepsilon_0$ ,  $k = 1, \dots, N$ , for some  $t$  then  $\lim_{t \rightarrow \infty} |Z_0^{(N)}| = 1$  and  $\lim_{t \rightarrow \infty} |Z_k^{(N)}| = 0$ . This  $\varepsilon_0$  can be chosen independent of  $N$  (due to the structure of the linear stability problem, see section 2). Define  $N_0$  by  $N_0 = \lceil \mathcal{L}_\infty / \sqrt{\varepsilon_0} \rceil$ , then by theorem 3.2 we have: for  $M > N_0$ ,  $\limsup_{t \rightarrow \infty} |Z_M^{(N)}| < \varepsilon_0$ . Substituting this into (1.4) yields for  $Z_0^{(N)}, \dots, Z_{N_0}^{(N)}$  a perturbed version of (1.4) with  $N = N_0$ . The perturbation of the equation for  $Z_n^{(N)}$  can be estimated:

$$\begin{aligned} & \left| (1 + ib) \sum_{\substack{k+l+m=n \\ \min(|k|, |l|, |m|) > N_0}} Z_k^{(N)} Z_l^{(N)} Z_m^{(N)*} \right| \\ & \leq 6\sqrt{1 + b^2} \sum_{k=N_0+1}^N |Z_k^{(N)}| \left( \sum_{l+m=n-k} |Z_l^{(N)}| |Z_m^{(N)}| \right) \\ & \leq 6\sqrt{1 + b^2} \sum_{k=N_0+1}^N \frac{\mathcal{L}_\infty^2}{k^2} \cdot 1 && \text{by proposition 3.1} \\ & \leq 6\sqrt{1 + b^2} \frac{\mathcal{L}_\infty^2}{N_0} \\ & \leq 6\sqrt{1 + b^2} \mathcal{L}_\infty \sqrt{\varepsilon_0}. \end{aligned}$$

Thus, the perturbation is  $\mathcal{O}(\sqrt{\varepsilon_0})$ , independent of  $N$  (the perturbation is zero if  $N \leq N_0$ ). This perturbation has no significant influence on the derivation of estimate (3.12), with  $N$  replaced by  $N_0$ . The ‘new’ estimate (3.12) can now be used to determine a  $Q_0$ , independent of  $N$ , such that  $\limsup_{t \rightarrow \infty} |Z_k^{(N)}| < \varepsilon_0$  for  $q > Q_0$ ,  $k = 1, \dots, N_0$ . Similarly one then shows that  $Z_0^{(N)}(t)$  satisfies

$$\dot{Z}_0^{(N)} = Z_0^{(N)} - (1 + ib)Z_0^{(N)} |Z_0^{(N)}|^2 + \mathcal{O}(\sqrt{\varepsilon_0}).$$

Hence  $|Z_0^{(N)}| \rightarrow 1$  for  $t \rightarrow \infty$ ; this concludes the proof of the corollary.

The result of theorem 3.2 can be improved by using another type of estimate for  $|F_M^{(N)}|$ . We observe that  $|F_M^{(N)}|$  has to be small for large  $M$ : in proposition 3.1 and theorem 3.2 we only used estimates on  $F_M^{(N)}$  independent of  $M$ . Although it requires a lot of ‘bookkeeping’, it is not difficult to prove, by basic calculations, the following lemma.

*Lemma 3.6.* Let  $a_k > 0$ ,  $k = -N, \dots, N$  with

$$a_k = a_{-k} \text{ for all } k$$

$$\sum_{|k| \leq N} a_k^2 \leq 1$$

$$a_k \leq \frac{L}{|k|^\alpha} \text{ for some } \alpha > 1, |k| > k_0.$$

Then

$$\left| \sum_{k+l+m=M} a_k a_l a_m \right| \leq \frac{G}{|M|^\alpha}$$

for some  $G = G(\alpha, L)$ ,  $|M| > M_1 = M_1(\alpha, L, k_0)$ ,  $G$  and  $M_1$  independent of  $N$ . Hence, since  $|F_M^{(N)}| \leq \sum_{k+l+m=M} |Z_k^{(N)}| |Z_l^{(N)}| |Z_m^{(N)}|$ , we establish, using theorem 3.2 and lemma 3.6

$$|F_M^{(N)}| \leq \frac{G}{M^2} \quad \text{for some } G, M > M_1$$

$G$  and  $M_1$  independent of  $N$ . This new bound for  $F_M^{(N)}$  can be used to estimate  $G_M^{(N)}$ . Thus, as in the proof of theorem 3.2 we deduce, using (3.10) and (3.11),

$$\limsup_{t \rightarrow \infty} |Z_M^{(N)}| \leq \frac{L'}{M^3} \quad \text{for } M > M_1$$

and some  $L'$  independent of  $N$ . We can now again start a process of successive estimates similar to the process used in the proof of theorem 3.2: step A has to be replaced by an argument as above, using lemma 3.6. This yields:

*Theorem 3.7.* For all  $\alpha \geq 2$  there exist a  $K(\alpha)$  and a  $M(\alpha)$ ,  $K(\alpha)$  and  $M(\alpha)$  are independent of  $N$ , such that for  $M > M(\alpha)$

$$\limsup_{t \rightarrow \infty} |Z_M^{(N)}(t)| \leq \frac{K(\alpha)}{M^\alpha}.$$

*Remark 3.8.* Hence we have proved that  $\limsup_{t \rightarrow \infty} |Z_M^{(N)}|$  declines faster to zero (as function of  $M$ ) than any power of  $M^{-1}$  as  $M$  increases. However, we did not prove that this decay is exponential. The wavenumber spectra computed numerically by Moon *et al* [10] (with  $N = 63$ , i.e. 64 modes) and Keefe [8] ( $N = 31$ ) exhibit an exponential decay of

$$P_q^{(n)} = \lim_{t \rightarrow \infty} \frac{\int_{T_0}^t |Z_n^{(N)}(\tau)|^2 d\tau}{\sum_{|k| \leq N} \int_{T_0}^t |Z_k^{(N)}(\tau)|^2 d\tau} \tag{3.19}$$

as a function of  $n$  (for  $q$  fixed and  $T_0$  ‘large’).

In the proof of proposition 3.1 we introduced the ‘trigonometric polynomial’  $\Psi^{(N)}(z, t)$ , see (3.1). The function  $\Psi^{(N)}$  is a solution of a ‘perturbed’ Ginzburg–Landau equation:

$$\frac{\partial \Psi^{(N)}}{\partial t} = (1 - (1 + ib) |\Psi^{(N)}|^2) \Psi^{(N)} + (1 + ia) \frac{\partial^2 \Psi^{(N)}}{\partial z^2} + R^{(N)}(z, t).$$

We can now estimate  $R^{(N)}$ , using theorem 3.7 and lemma 3.6:

$$\begin{aligned} |R^{(N)}(z, t)| &\leq 2\sqrt{1+b^2} \sum_{k=N+1}^{3N} |F_k^{(N)}| && \text{by (3.3)} \\ &\leq 2\sqrt{1+b^2} L(\alpha) \sum_{k=N+1}^{3N} \frac{1}{k^\alpha} && \text{as } t \rightarrow \infty, \text{ for some } L(\alpha) \\ &\leq \frac{L'(\alpha)}{N^{\alpha-1}} && \text{for some } L'. \end{aligned}$$

Hence,

*Corollary 3.9.*  $\limsup_{t \rightarrow \infty} |R^{(N)}(z, t)| = O(1/N^\alpha)$  for all  $\alpha > 0$  uniformly in  $z$ .

*Remark 3.10.* As was noted in remark 1.2: the symmetry  $Z_m^{(N)}(t) = Z_{-m}^{(N)}(t)$  is not used in this section, the results obtained here are also valid for other than homogeneous Neumann boundary conditions. This is with the exception of corollary 3.5: the Stokes wave is no solution of (1.1) with homogeneous Dirichlet boundary conditions. However, one can check that corollary 3.5 and property 2.1 are true in the case of periodic boundary conditions.

#### 4. Bifurcation histories for $N = 1, 2$ and $3$

In this section we study, numerically, three low-dimensional truncated models of the Ginzburg–Landau equation: system (1.4) for  $N = 1, 2$  and  $3$ . We compare these models with each other and with the outcome of the  $N = 31$  model studied by Keefe [8]. As Keefe did, we fix  $a = 4$  and  $b = -4$  and use wavenumber  $q$  as a bifurcation parameter in the range  $0.6 < q < \infty$ . It should be remarked that system (1.4) with respectively  $N = 1, 2$  or  $3$  is a two-, three- or four-dimensional complex differential equation which can be reduced, using symmetry  $T_\phi$  (section 2), to, respectively, a three-, five- or seven-dimensional real system.

We determine numerically the asymptotic dynamics ( $t \rightarrow \infty$ ) of the flow induced by (1.4), for arbitrary initial data (inside  $R_{\text{an}}$ , see section 3), using a Poincaré map: we plot the projection in the  $\text{Re}(Z_0^{(N)})$ ,  $\text{Re}(Z_1^{(N)})$  plane of the section of a solution  $Z(t)$  of (1.4) at  $\text{Im}(Z_0^{(N)}) = 0$  with  $(d/dt)\text{Im}(Z_0^{(N)}) > 0$ . The set of points in the  $\text{Re}(Z_0^{(N)})$ ,  $\text{Re}(Z_1^{(N)})$  plane, depending on the value of parameter  $q$ , is called  $\omega_q$ .

The set of points  $S_0\omega_q$ , induced by solution  $S_0Z(t)$ ,  $S_0$  as in (2.4), is the mirror image of  $\omega_q$  in the  $\text{Re}(Z_0^{(N)})$ -axis. Thus  $S_0\omega_q = \omega_q$  if  $\omega_q$  is symmetrical. (There is an interplay of merging and splitting between  $\omega_q$  and  $S_0\omega_q$  which causes many bifurcations.)

In this section we will sometimes refer to Keefe’s results ([8],  $N = 31$ ) as the untruncated  $N = \infty$  model. The magnitudes of the various modes ( $N = 1, 2, 3$ ) are very similar to the magnitudes of the modes in the  $N = 31$ , or  $\infty$  case:

$\max(|Z_0^{(N)}|) \leq 1$  (and decreases as  $q$  decreases) and  $\max(|Z_1^{(N)}|) \leq 0.4$  for any  $N$ ;  $|Z_2^{(N)}|$  and  $|Z_3^{(N)}|$  are very small compared with  $|Z_0^{(N)}|$  and  $|Z_1^{(N)}|$ :  $|Z_2^{(N)}| > 0.01$  and  $|Z_3^{(N)}| \sim 0.001$  (although they grow as  $q$  decreases).

The results are summarized in figure 1. Here we plotted, for  $N = 1, 2, 3$  and  $\infty$  the bifurcation value of  $q$ . Lines connecting the ' $N$ -beams' represent similar bifurcations. Parts of the  $N$ -beams with the same character in it represent similar structures  $\omega_q$  in the Poincaré sections, for different  $N$ . We did not plot all bifurcation values of  $q$ : in regions (z) and (y), E, F and G, more bifurcations can be found.

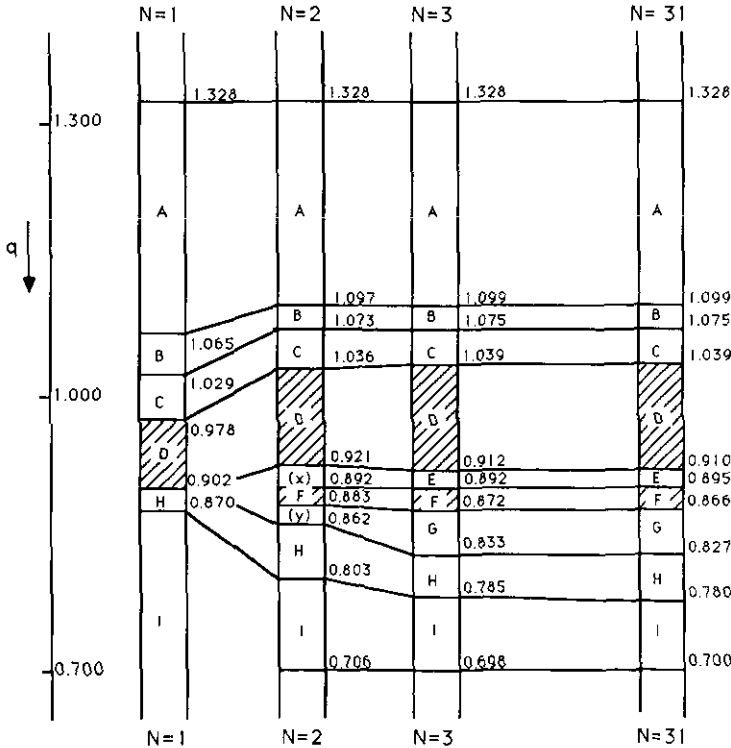
As was remarked (and partly proved) in section 3, the Stokes wave is a global attractor for the flow as  $q > q_0(a, b)$ . For  $a = 4, b = -4$  we have  $q_0(4, -4) = \sqrt{\frac{30}{17}} = 1.328 \dots$

We discuss in detail the differences and similarities between the bifurcation histories of (the asymptotic dynamics) of solutions of (1.4) for  $N = 1, 2$  and with bifurcation parameter  $q$ .

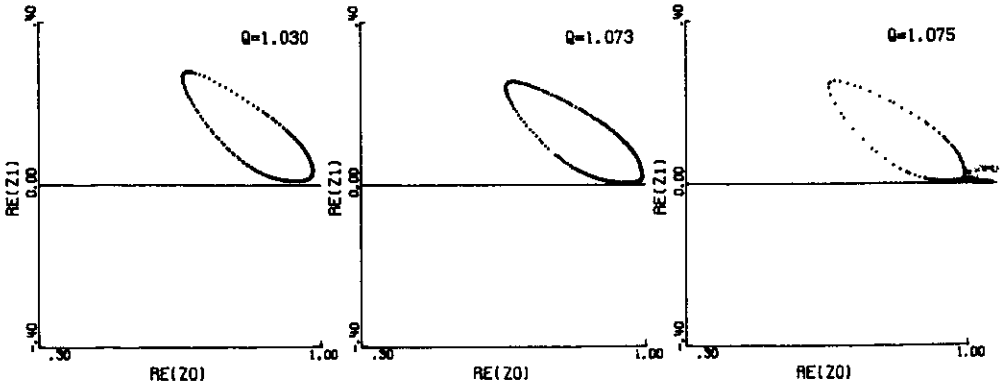
*Regions A and B.  $N = 1, 2, 3$  (figure 2).* The Stokes wave corresponds to a critical point in the  $(R_k, \phi_k - \phi_N)$  system:

this critical point undergoes a pitchfork bifurcation, due to symmetry  $S_\sigma$ , as  $q$  passes through  $q_0(-4, 4) = 1.328 \dots$ ;

as  $q$  decreases further the two new stable critical points become unstable again by a Hopf bifurcation, at values of  $q$ , now dependent on  $N$ .



**Figure 1.** Bifurcation values for  $q$  decreasing from 1.4 to 0.6. Shaded regions stand for  $q$  values at which (1.4) has a chaotic attractor.



**Figure 2.** Region B. (a)  $N = 1$ , (b)  $N = 2$ , (c)  $N = 3$ .

Translating this back to solutions of (1.4) yields:

entering region A: two new stable periodic solutions split off from the Stokes wave;

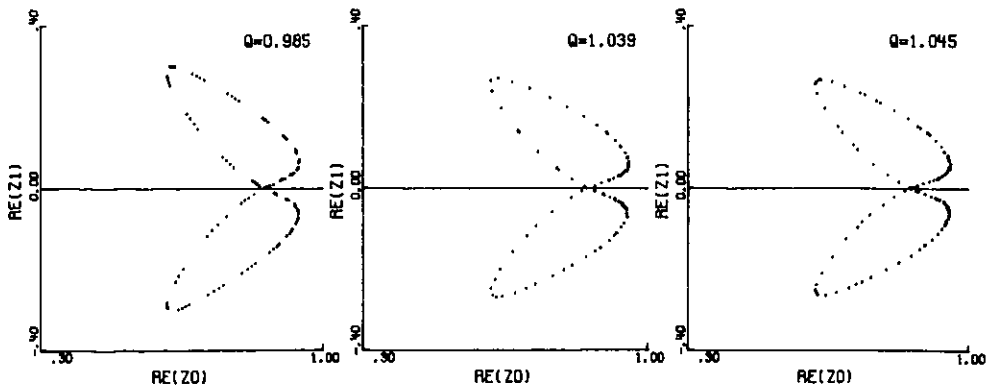
entering region B: the two periodic solutions bifurcate into invariant tori, by a Hopf bifurcation; see figure 2.

These two bifurcations can be analysed locally in the  $(R_k, \phi_k - \phi_N)$  system.

*Remark.* Newton and Sirovich [12, 13] analysed the untruncated equation (1.1) near the Stokes wave  $\Psi(z, t) = e^{-ibr}$  using perturbation techniques. For  $q$  close to  $q_0(a, b)$  they found two, stable, symmetric, even periodic solutions, the same as we find in the truncated system. In [13] they proposed, but did not prove, a Hopf behaviour similar to the behaviour we observed at the next bifurcation value.

*Region C.*  $N = 1, 2, 3$  (figure 3). The tori grow larger and melt together ( $\omega_q = S_0\omega_q$ ).

*Region D.*  $N = 1, 2, 3$  (figures 4 and 5). The attracting forces of the 8-shaped  $\omega_q$  become weaker as  $q$  decreases. At the next bifurcation value a chaotic attractor,



**Figure 3.** Region C. (a)  $N = 1$ , (b)  $N = 2$ , (c)  $N = 3$ .



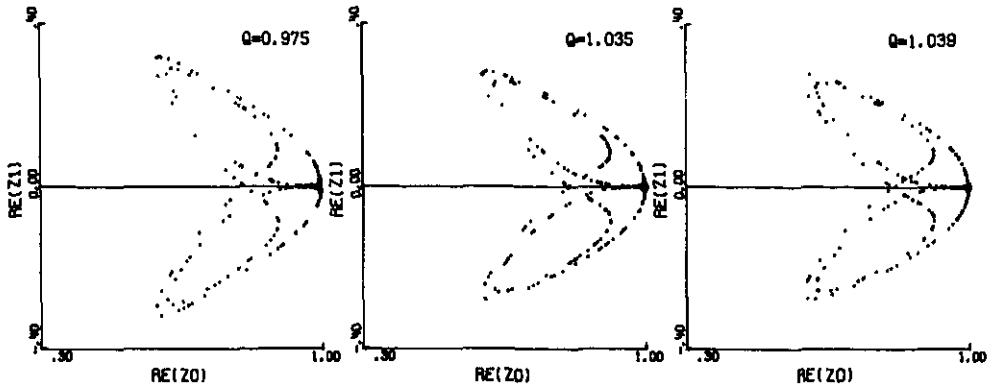


Figure 4. Just after entering chaotic region  $D$ . (a)  $N = 1$ , (b)  $N = 2$ , (c)  $N = 3$ .

enveloping the 8-shaped  $\omega_q$ , appears (although, for a small range of  $q$  both attractors seem coexistent). Note that, although for different values of  $q$ , the topological features of the Poincaré sections for  $N = 1, 2, 3$  (and ‘ $\infty$ ’) are still similar. Keefe [8] computed the Lyapunov dimension  $L_q$  of this chaotic attractor as a function of  $q$ :  $L_q$  is larger than 3 for  $q \in [0.910, 1.030]$ . Hence the dimension of the governing system has to be at least 4, thus  $N = 1$ , which is essentially a three-dimensional  $(R_k, \phi_k - \phi_N)$  system, can no longer exhibit the same Poincaré sections as  $N = 2, 3$  and ‘ $\infty$ ’. The maximum value of  $L_q$ ,  $L_q = 3.05$ , is attained at  $q = 0.950$ , hence  $N = 2, 3$  can still give the same pictures as  $N = \infty$ . These pictures lack every kind of structure: they are two-dimensional projections of a structure with dimension larger than 3; they need to be examined using a different approach.

*Remark.* The  $\omega$ -limits sets are necessarily inside  $R_{att}$  (see proposition 3.1). However, the numerics show that a solution of (1.4) in the  $\omega$ -limit set passes the Stokes solution (and thus the boundary of  $R_{att}$ ) frequently very close (for  $q$  in chaotic region  $D$ ). This is possible (and not surprising) since for these values of  $q$  the Stokes wave still has an  $N$ -dimensional attracting manifold and a one-

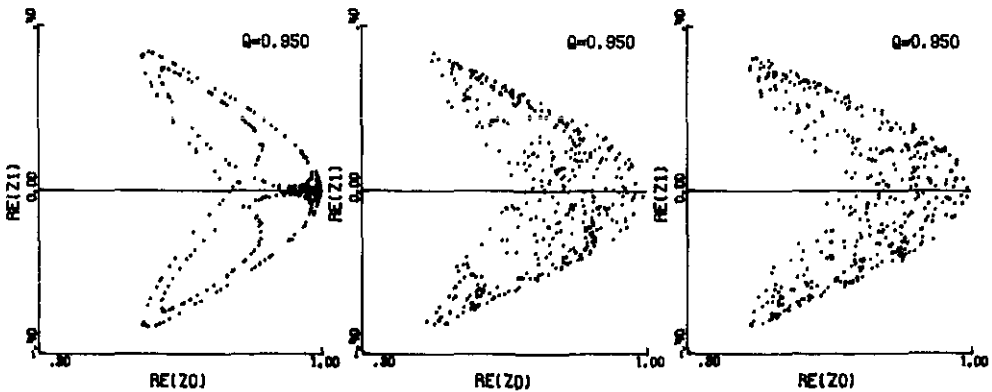


Figure 5. (a)  $N = 1$ , the chaotic attractor remains as in figure 4,  $L_q < 3$ ; (b), (c):  $N = 2, 3$ :  $L_q > 3$ .

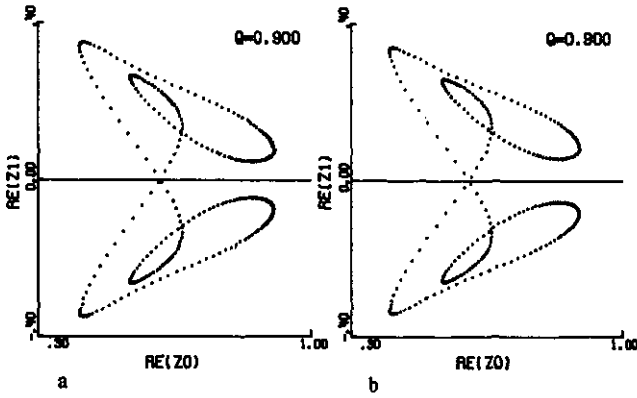


Figure 6. Between regions *D* and *F*: (a)  $N = 2$ , (b)  $N = 3$ .

dimensional unstable manifold: solutions may flow close along unstable solutions of saddle type.

*Region E.  $N = 3$  (figure 7).* In figure 7 we show how the  $\omega_q$ 's leave the chaotic region for  $N = 3$ . Although the bifurcation values are not identical (they differ by  $\approx 0.002$ ), the topological structure of the  $\omega_q$ 's is the same as observed in the  $N = 31(\infty)$  case. There is a different bifurcation sequence for  $N = 2$  at the boundary of chaotic region *D*, see region (*x*). The  $\omega_q$ 's are again identical, for  $N = 2, 3$ , at  $q = 0.900$  (see figure 6).

We now describe the bifurcation history for  $N = 3$  for increasing  $q$ , starting at  $q = 0.900$ :

$q = 0.905$ :  $\omega_q$  becomes asymmetrical:  $\omega_q \neq S_0 \omega_q$ ;

$q = 0.908$ : the loops of  $\omega_q$  split in two ('period-doubling');

$q = 0.910$ : another 'loop-splitting': the onset of a sequence of loop-splittings;

$q = 0.911$ : chaos,  $\omega_q \neq S_0 \omega_q$ : the big loop of  $\omega_q$  does not intersect the  $\text{Re}(Z_0^{(3)})$ -axis; it grows nearer as  $q$  grows;

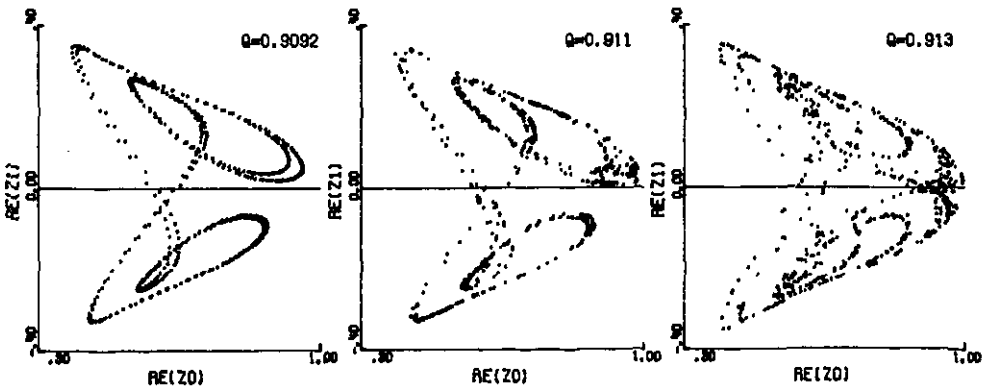


Figure 7. Region *E*,  $N = 3$ .

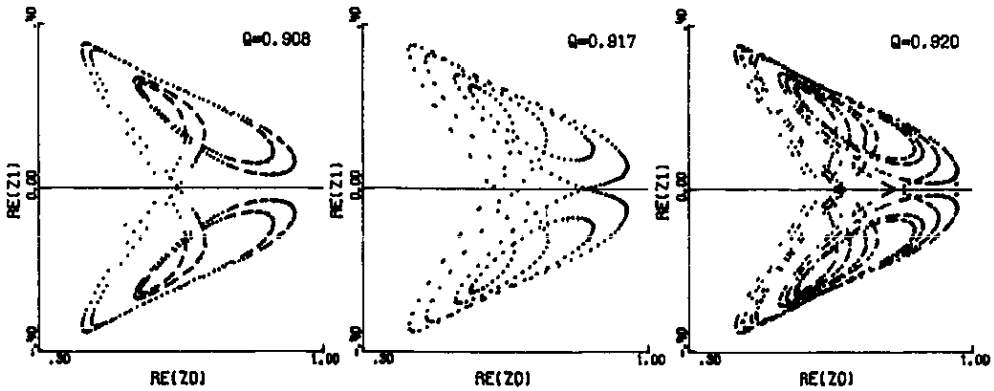


Figure 8. Region (x),  $N = 2$ , (a)  $\omega_q \cup S_0\omega_q$ .

$q = 0.912$ : the big loop of  $\omega_q$  becomes tangent to the  $\text{Re}(Z_0^{(3)})$ -axis:  $\omega_q$  merges with  $S_0\omega_q$ .

Region (x).  $N = 2$  (figure 8). Again we discuss the bifurcation behaviour for increasing  $q$ , starting at  $q = 0.900$ :

$q = 0.905$ : as  $N = 3$ :  $\omega_q$  becomes asymmetrical;

$q = 0.915$ : different from  $N = 3$ : the big loop of  $\omega_q$  touches the  $\text{Re}(Z_0^{(2)})$ -axis:  $\omega_q$  and  $S_0\omega_q$  merge. Hence,  $\omega_q$  and  $S_0\omega_q$  merge before getting chaotic, while  $\omega_q$  gets chaotic before merging with  $S_0\omega_q$  for  $N = 3$ . In figure 8(a) we plotted  $\omega_q$  and  $S_0\omega_q$  in one picture to show what happens when  $\omega_q$  and  $S_0\omega_q$  merge;

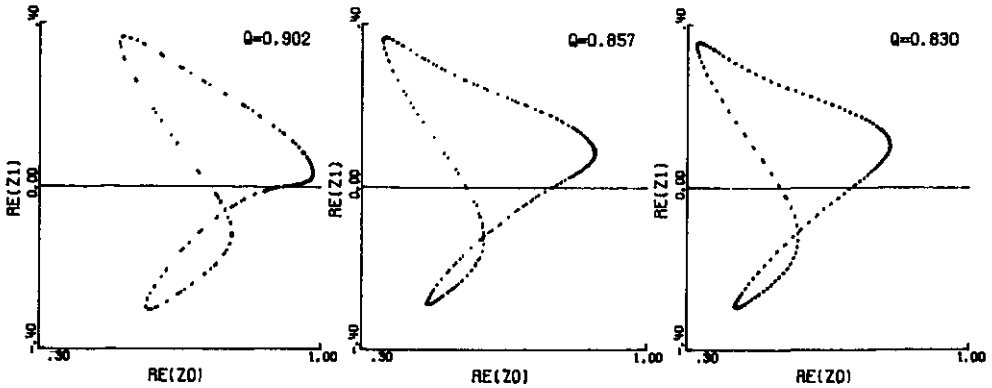
$q = 0.918$ :  $\omega_q$  becomes symmetrical again;

$q = 0.920$ :  $\omega_q$  and  $S_0\omega_q$  merge again: one could call this step a period-doubling;

$q = 0.921$ : chaos, after a sequence of period-doublings.

As  $q$  decreases further system (1.4) enters for  $N = 2$  and  $N = 3$  a second chaotic region,  $F$ . The routes to and from chaos are similar to the situation around region  $D$ : a chaotic attractor appears 'suddenly' as  $q$  decreases into region  $F$ ; it disappears by a sequence of period-halvings. Also, as in regions (x) and  $E$ , there is a difference between the case  $N = 2$  and  $N = 3$ : for  $N = 2$  (region (y))  $\omega_q$  and  $S_0\omega_q$  melt together before getting chaotic, for  $N = 3$  (region  $G$ )  $\omega_q$  becomes chaotic before merging with  $S_0\omega_q$  (both cases for increasing  $q$ ). The bifurcations for  $N = 3$  are again similar to  $N = \infty$ .

We observe that the  $\omega_q$  are again topologically the same for  $N = 1, 2$  and  $3$  as  $q$  has entered region  $H$ , see figure 9. Chaotic attractor  $\omega_q$  originates from the  $\omega_q$  of figure 9(a),  $N = 1$ , by a process which had been observed before: as  $q$  increases the intersection point of  $\omega_q$  with the  $\text{Re}(Z_0^{(1)})$  axis becomes tangent,  $\omega_q$  and  $S_0\omega_q$  merge into one chaotic attractor. It should be remarked that one can find only one period-doubling in region (y) ( $N = 2$ ,  $q$  increases). It seems that there is no complete sequence of period-doublings: chaos appears suddenly after  $\omega_q$  has become asymmetrical again (just as in the case  $N = 1$ ). These observations may be caused by the numerical inaccuracy.



**Figure 9.** Region *H*. (a)  $N=1$ , (b)  $N=2$ , (c)  $N=3$ .

Regions *H*, *I* and *J*.  $N=1, 2, 3$  (figures 9 and 10). The  $\omega_q$ 's are similar for  $N=1, 2, 3$  and  $31(\infty)$  for  $q$  in regions *H* and *I*. At the bifurcation between *H* and *I* the asymmetrical 8-shaped  $\omega_q$  becomes symmetrical (i.e. merges with  $S_0\omega_q$ ). It should be remarked that in region *I*, for instance at  $q=0.750$ , the  $\omega_q$ 's are not only topologically equivalent, but also their proportions and the position in the plane are very much alike. Keefe [8], i.e.  $N=31(\infty)$ , found the next bifurcation at  $q=0.700$ : all even modes (i.e.  $Z_0, Z_2, \dots$ ) disappear, and an attracting periodic solution, with only non-zero odd modes appears. We remark that due to symmetry  $S_0$  solutions with only non-trivial odd modes are possible. Hence we search for an isolated periodic solution with  $Z_0^{(N)} = Z_2^{(N)} = \dots = 0$ .

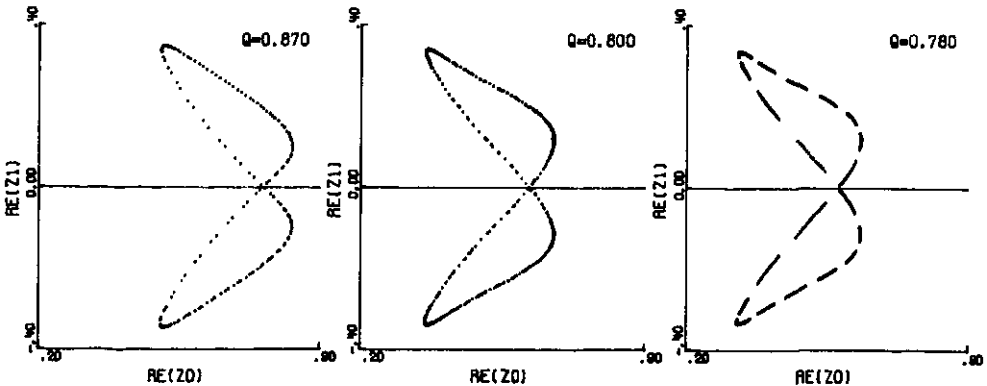
$N=1$ . Setting  $Z_0^{(1)}=0$  we find

$$\dot{Z}_1^{(1)} = (1 - q^2(1 + ia))Z_1^{(1)} - 3(1 + ib)Z_1^{(1)}|Z_1^{(1)}|^2 \tag{4.1}$$

which has a periodic solution

$$W_q(t) = Q(q)e^{i\theta(q)t} \quad \text{with} \quad Q(q) = \sqrt{\frac{1}{3}(1 - q^2)} \quad \theta(q) = (b - a)q^2 - b. \tag{4.2}$$

Straightforward computation shows that this solution cannot be stable. (The next bifurcation appears at  $q=0.513$ :  $\omega_q$  collapses into an isolated orbit with  $Z_0^{(1)} \neq 0$ .)



**Figure 10.** Region *I*. (a)  $N=1$ , (b)  $N=2$ , (c)  $N=3$ .

$N = 2$ . Setting  $Z_0^{(2)} = Z_2^{(2)} = 0$  we again obtain (4.1) with periodic solution (4.2).  $W_q(t)$  is a stable stationary point in the  $(R_k, \phi_k - \phi_2)$ -system for  $q \in [0.574, 0.706]$ . This is in agreement with the numerical observations: the even modes disappear in this  $q$  region; solutions tend towards  $W_q$ , i.e.  $W_q$  is a global attractor.

$N = 3$ . We now obtain a two-dimensional  $Z_1^{(3)}, Z_3^{(3)}$  system, after setting  $Z_0^{(3)} = Z_2^{(3)} = 0$ , with an isolated periodic solution

$$Z_1^{(3)} = \Gamma_q^1(t), Z_3^{(3)} = \Gamma_q^3(t).$$

This solution resembles  $W_q$ :  $||W_q| - |\Gamma_q^3||$  and  $|\Gamma_q^3|$  are small, the periods of  $W_q$  and  $\Gamma_q$  are almost the same, etc. Using the  $(R_k, \phi_k - \phi_3)$  system we compute that  $\Gamma_q$  is stable for  $q \in [0.574, 0.698]$ . We observe numerically a disappearance of the even modes on this region;  $(0, \Gamma_q^1, 0, \Gamma_q^3)$  is also a global attractor.

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