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# Minimal super- $W_N$ algebras in coset conformal field theories

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We show that a “minimal” supersymmetric extension of the  $W[A_{N-1}]$  algebra exists for central charge  $c_N = (3N+1)(N-1)/2(2N+1)$ . We argue that a specific coset model based on  $(A_{N-1}^{(1)} \oplus A_{N-1}^{(2)}, A_{N-1}^{(3)})$  at level  $(1, N; 1+N)$  corresponds to the diagonal modular invariant that can be constructed from the characters of this algebra. For  $N=3$  result leads to a proof of a conjecture recently put forward by Bilal.

A manifestly supersymmetric description of general supersymmetric rational conformal field theories (RCFTs), which play a role in string theory and in statistical mechanics, requires the notion of a “super-chiral algebra”. In a few especially simple models this superchiral algebra, is just the well-known super-Virasoro or Neveu–Schwarz–Ramond (NSR) algebra, but in the general case it will be a non-linear superalgebra, generalizing the bosonic  $W$  algebras. From the structure theory of RCFTs one knows that the torus partition function of every rational conformal field theory can be written, up to a possible permutation of characters, as a diagonal modular invariant of the (bosonic) chiral algebra. Of course, it would be interesting to establish similar results, in terms of the superchiral algebra, for supersymmetric RCFTs. In this letter we report on some progress in this direction. In particular, we will argue that specific coset conformal field theories correspond to the diagonal modular invariants of certain new superchiral algebras, which we call minimal super- $W_N$  algebras.

Before we come to these new results, we will first review briefly what is known about super- $W$  algebras in general and about the super- $W_3$  algebra in particular. In ref. [1], Inami et al. considered an extension of the  $N=1$  super-Virasoro algebra with a spin-3 bosonic and a spin- $\frac{5}{2}$  fermionic extra current, the two forming a multiplet under the  $N=1$  supersymmetry. They found that such an algebra, which turns out to have quadratic defining relations, can only be consistent if the central charge of the Virasoro algebra is  $c = \frac{10}{7}$  or  $c = -\frac{5}{2}$ . Furthermore, they observed that the bosonic  $W_3$  algebra [2] is a subalgebra of this superalgebra precisely for  $c = \frac{10}{7}$ . We will refer to this algebra as the “minimal” super- $W_3$  algebra.

One may observe that the value  $c = \frac{10}{7}$  is one of the values in the discrete unitary series of the super-Virasoro algebra ( $m=12$  in the series  $c_m = \frac{3}{2}(1 - 8/m(m+2))$ ). One can therefore look in the list of  $N=1$  modular invariant partition functions, which was given by Cappelli in ref. [3], and see if one of the possible unitary supersymmetric conformal field theories at  $c = \frac{10}{7}$  has super- $W_3$  invariance.

This was done by Bilal [4] who conjectured that the invariant

$$Z_{E_6, D_8}^{(N=1)} = \frac{1}{4} \sum_{s=1, \text{odd}}^{13} [ |\chi_{1s}^{NS} + \chi_{5s}^{NS} + \chi_{7s}^{NS} + \chi_{11s}^{NS}|^2 + (\chi^{NS} \rightarrow \tilde{\chi}^{NS}) + |\chi_{4s}^R + \chi_{8s}^R|^2 ] \tag{1}$$

is the diagonal modular invariant of the minimal super- $W_3$  algebra at  $c = \frac{10}{7}$ . In here we write  $\chi^{NS}$  and  $\tilde{\chi}^{NS}$  for the characters (without and with the  $(-1)^F$  insertion) in the Neveu–Schwarz sector and  $\chi^R$  for the characters in the Ramond sector of the  $N=1$  superconformal algebra (contrary to Cappelli, we do not absorb a factor of  $\sqrt{2}$  in the definition of  $\chi^R$ ). The labels  $(rs)$  label the various highest weight states in both sectors. Although Bilal

did not give a proof of his claim, he checked that the super- $W_3$  highest weights that one expects from (1) are compatible with bosonic  $W_3$  symmetry at  $c = \frac{10}{7}$ .

Of course, the value  $c = \frac{10}{7}$  is also in the unitary minimal series of the bosonic  $W_3$  algebra. Since the work in ref. [5] it has been known that this series is related to models based on the cosets  $(A_2^{(1)} \oplus A_2^{(1)}, A_2^{(1)})$  at level  $(1, m)$ , i.e. with one of the levels equal to 1. These same cosets at level  $(m, 3)$  are expected to be related to supersymmetric models (for the similar cosets based on  $A_{N-1}^{(1)}$  this is expected if one of the levels equals  $N$ , see e.g. ref. [6]). In recent work, Hornfeck and Ragoucy [7] have worked out the connection of the  $(A_2^{(1)} \oplus A_2^{(1)}, A_2^{(1)})$  coset models with level  $(m, 3)$  with the super- $W_3$  algebra. Following the analysis in ref. [5] and working in superspace, they constructed the  $N=1$  superconformal generators and a spin- $\frac{5}{2}$  superfield  $\mathcal{W}$  in terms of the current algebra of the coset  $(A_2^{(1)} \oplus A_2^{(1)}, A_2^{(1)})$  at level  $(m, 3)$ . For  $m=1$ , corresponding to  $c = \frac{10}{7}$ , they were able to prove that the defining relations of the “minimal” super- $W_3$  algebra are indeed reproduced. They also observed that for  $m > 1$  extra superprimary currents of dimension  $\frac{7}{2}$  and 4 are generated in the operator product expansions of the basic fields. However, even with these extra currents the operator algebra is not a closed algebra.

The full structure of the *generic* super- $W_3$  algebra, as opposed to the *minimal* super- $W_3$  algebra, was discussed by Ahn et al. in ref. [8]. This paper proposes an extension of the minimal super- $W_3$  algebra, which is found by means of a singlet counting technique. It is argued that this algebra is consistent for generic  $c$  and that it is common to all  $(A_2^{(1)} \oplus A_2^{(1)}, A_2^{(1)})$  coset models with levels  $(m, 3)$ .

Recently, other algebras of super- $W$  type have been proposed in the literature [9–12].

Our goal in this letter is twofold. First of all, we will prove the conjecture of Bilal. We hope that this proof clarifies some confusion on the issue that exists in the literature. The proof is relatively easy since the model in question has abundant symmetry: supersymmetry, bosonic  $W_3$  symmetry *and* super- $W_3$  symmetry. Secondly, we will observe that the structure of the minimal super- $W_3$  at  $c = \frac{10}{7}$  algebra can easily be generalized to minimal super- $W_N$  at  $c = (3N+1)(N-1)/2(2N+1)$ . We will construct the modular invariant partition functions that are diagonal with respect to these algebras.

In the rest of this letter, we focus on conformal field theories associated with the cosets

$$(A_{N-1}^{(1)} \oplus A_{N-1}^{(1)})/A_{N-1}^{(1)} \tag{2}$$

at level  $(1, N; 1+N)$ . On the one hand we know [6] that these models can be supersymmetric; on the other hand, they possess bosonic  $W_N$  symmetry [5]. It is therefore natural to expect that such models can be invariant under a superextension of the  $W_N$  algebra. We will investigate this issue by looking at the torus partition functions. Since the models in question are *minimal* models with respect to the bosonic  $W_N$  algebra, we know that their partition functions can be expressed in terms of finitely many characters of the  $W_N$  algebra. (Notice that for the description in terms of characters of the superconformal algebra this is only true for  $N=3$ .)

The construction of modular invariant partition functions for models related to the cosets (2) is well understood (see e.g. refs. [13,5,14]). Such an invariant can be characterized by three modular invariants, at levels 1,  $N$  and  $N+1$ , of the affine Kac–Moody algebra  $A_{N-1}^{(1)}$ . The correct choice that leads to a supersymmetric theory is the following: for the levels 1 and  $N+1$  one chooses the diagonal modular invariants, and for level  $N$  one chooses the exceptional invariant based on the conformal embedding

$$\widehat{su}(N)_{\text{level}=N} \subset \widehat{so}(N^2-1)_{\text{level}=1} . \tag{3}$$

This choice, which will be justified below, can be understood by first considering a simple model that has  $N^2-1$  free fermions in the adjoint representation of  $A_{N-1}$ . This model, which can be described by both affine Kac–Moody algebras in the above embedding, is clearly supersymmetric, the supercurrent being given by  $G(z) = f^{abc} \psi_a \psi_b \psi_c(z)$ , where  $f^{abc}$  are the structure constants of  $A_{N-1}$ . Now this simple construction of the supercurrent can be mimicked (compare with refs. [15,7,8]) in the coset models (2) with level  $(m, N)$ , *provided* the level  $N$  factor in the coset can again be represented by  $N^2-1$  free fermions. Thus we see that the requirement of super-

symmetry of the coset model directly leads to the level  $N$  modular invariant of  $A_{N-1}$  based on (3).

The selected  $A_{N-1}^{(1)}$  modular invariant at level  $N$  can simply be written as follows in terms of characters of the algebra  $\widehat{so}(N^2 - 1)_{\text{level}=1}$ .

$$Z_{WZW} = |\chi_{\text{singlet}}|^2 + |\chi_{\text{vector}}|^2 + \lambda |\chi_{\text{spinor}}|^2, \tag{4}$$

where  $\lambda = 1$  (2) for  $N$  even (odd). It can be rewritten in terms of characters of the algebra  $A_{N-1}^{(1)}$  at level  $N$  by using branching rules such as  $\chi_{\text{singlet}} = \chi_{(0...0)} + \chi_{(20...010)} + \chi_{(010...02)} + \dots$  and  $\chi_{\text{vector}} = \chi_{(10...01)} + \dots$  (for describing representations of  $A_{N-1}$  we will alternatively use the Dynkin label notation and the notation indicating dimensions, hopefully without causing confusion). For  $N = 3, 4$  the complete result reads (see also refs. [16,14]).

$$Z_{WZW}^{(N=3)} = |\chi_1 + 2\chi_{10}|^2 + 3|\chi_8|^2, \quad Z_{WZW}^{(N=4)} = |\chi_1 + 2\chi_{45} + \chi_{105}|^2 + |\chi_{15} + 2\chi_{35} + \chi_{175}|^2 + 4|\chi_{64}|^2. \tag{5}$$

In order to construct the modular invariant partition function of the coset model, one needs the branching rule for representations of  $A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}$  at levels (1,  $N$ ) to representations of  $A_{N-1}^{(1)}$  at level  $N+1$  and representations of  $W_N$ . Such a branching rule was proposed in ref. [5]. Given this it is straightforward to compute, using the methods of ref. [13], the modular invariant partition function of  $W_N$  departing from the modular invariants of  $A_{N-1}^{(1)}$ . The results for  $N = 3$  and  $N = 4$ , in terms of the characters of the  $W_N$  algebra, read as follows:

$$Z(W_3) = \frac{1}{3} \sum_{q \in \mathbb{P}_+^4} (|\chi_{(00)q} + \chi_{(30)q} + \chi_{(03)q}|^2 + 3|\chi_{(11)q}|^2),$$

$$Z(W_4) = \frac{1}{4} \sum_{q \in \mathbb{P}_+^5} (|\chi_{(000)q} + \chi_{(210)q} + \chi_{(012)q} + \chi_{(040)q}|^2 + |\chi_{(101)q} + \chi_{(400)q} + \chi_{(004)q} + \chi_{(121)q}|^2 + 4|\chi_{(111)q}|^2), \tag{6}$$

where  $q$  runs over the set of integral dominant weights of  $A_2^{(1)}$  at level 4 and  $A_3^{(1)}$  at level 5, respectively. (This means that the projection  $\bar{q}$  to the weight lattice of the finite dimensional subalgebra  $A_{N-1}$  has Dynkin labels  $(q_1, q_2, \dots, q_{N-1})$ ,  $0 \leq q_i$  and  $\sum_{i=1}^{N-1} q_i \leq N+1$ .) We will now focus on  $N = 3$  and later on come back to  $N = 4$  and higher.

The conformal dimensions  $h_{pq}$  of the  $W_3$  primary fields  $\phi_{pq}$  follow from the Kac formula for the  $W_3$  minimal models as given in refs. [17,5]. Switching the notation  $\chi_{pq}$  to  $\chi_h$  (i.e. suppressing the second  $W_3$  quantum number  $w$ ), we find the following result for  $Z(W_3)$ :

$$Z(W_3) = (|\chi_0 + 2\chi_4|^2 + |\chi_{3/2}|^2) + (|\chi_{1/14}|^2 + |\chi_{4/7} + 2\chi_{11/7}|^2) + 2(|\chi_{1/7} + \chi_{15/7} + \chi_{22/7}|^2 + |\chi_{9/14}|^2) + (|\chi_{5/14}|^2 + |\chi_{6/7} + \chi_{20/7}|^2) + 2(|\chi_{1/14}|^2 + |\chi_{5/14}|^2 + 2|\chi_{9/14}|^2 + |\chi_{3/2}|^2). \tag{7}$$

We now claim that this modular invariant is (i) the diagonal modular invariant of the minimal super- $W_3$  algebra and (ii) identical to the invariant (1), which was given in terms of characters of the  $N = 1$  superconformal algebra. Together these statements imply Bilal's original conjecture in ref. [4]. Although the second statement is not obvious by inspection, one easily checks that various multiplicities of low-lying states come out the same in both expressions.

In order to work out the statements (i) and (ii), we have to consider the highest weight states of the super- $W_3$  algebra that give rise to unitary representations. One easily convinces oneself that the conformal dimensions of these states should occur in the Kac table for unitary representations of both the superconformal algebra (in the NS or R sectors, respectively) and the  $W_3$  algebra. However, not all states that are selected by this criterion do give rise to unitary models (this point was missed in ref. [7]). After throwing out those highest weight states which do not lead to unitary representations, we find the following list of bona fide highest weight states for unitary representations (we give their  $W_3$  quantum numbers  $(h, w)$ ): in the NS sector we have five states with quantum number  $(0, 0)$ ,  $(\frac{1}{14}, 0)$ ,  $(\frac{1}{7}, \pm \frac{1}{7}\sqrt{\frac{2}{51}})$  and  $(\frac{5}{14}, 0)$  and in the R sector five state with quantum numbers  $(\frac{1}{14}, 0)$ ,  $(\frac{5}{14}, 0)$ ,  $(\frac{9}{14}, \pm \frac{8}{7}\sqrt{\frac{2}{51}})$  and  $(\frac{3}{2}, 0)$ . The fact that these representations do give rise to unitary representations will be confirmed below.

We now propose the following branching rules for the characters of the super- $W_3$  (s $W_3$ ) algebra (which we

will denote by  $\text{ch}^{\text{NS}}$ , etc.) and those of the  $N=1$  superconformal algebra and of the bosonic  $W_3$  algebra.

1.  $W_3 \subset sW_3$ , NS sector:

$$\begin{aligned} \text{ch}_0^{\text{NS}} &= (\chi_0 + 2\chi_4) + \chi_{3/2}, & \text{ch}_{1/14}^{\text{NS}} &= \chi_{1/14} + (\chi_{4/7} + 2\chi_{11/7}), \\ \text{ch}_{1/7}^{\text{NS}} &= (\chi_{1/7} + \chi_{15/7} + \chi_{22/7}) + \chi_{9/14}, & \text{ch}_{5/14}^{\text{NS}} &= \chi_{5/14} + (2\chi_{6/7} + \chi_{20/7}). \end{aligned} \quad (8)$$

2.  $(N=1) \subset sW_3$ , NS sector:

$$\begin{aligned} \text{ch}_0^{\text{NS}} &= \chi_0^{\text{NS}} + \chi_{5/2}^{\text{NS}} + \chi_{11/2}^{\text{NS}} + \chi_{15}^{\text{NS}}, & \text{ch}_{1/14}^{\text{NS}} &= \chi_{1/14}^{\text{NS}} + \chi_{15/14}^{\text{NS}} + \chi_{11/7}^{\text{NS}} + \chi_{46/7}^{\text{NS}}, \\ \text{ch}_{1/7}^{\text{NS}} &= \chi_{1/7}^{\text{NS}} + \chi_{51/14}^{\text{NS}}, & \text{ch}_{5/14}^{\text{NS}} &= \chi_{5/14}^{\text{NS}} + \chi_{6/7}^{\text{NS}} + \chi_{20/7}^{\text{NS}} + \chi_{145/14}^{\text{NS}}. \end{aligned} \quad (9)$$

3.  $W_3 \subset sW_3$ , R sector:

$$\text{ch}_h^{\text{R}} = 4\chi_h, \quad \text{for all } h. \quad (10)$$

4.  $(N=1) \subset sW_3$ , R sector:

$$\text{ch}_{1/14}^{\text{R}} = 2(\chi_{1/14}^{\text{R}} + \chi_{29/14}^{\text{R}}), \quad \text{ch}_{5/14}^{\text{R}} = 2(\chi_{5/14}^{\text{R}} + \chi_{61/14}^{\text{R}}), \quad \text{ch}_{9/14}^{\text{R}} = 2\chi_{9/14}^{\text{R}}, \quad \text{ch}_{3/2}^{\text{R}} = 2(\chi_{3/2}^{\text{R}} + \chi_{15/2}^{\text{R}}). \quad (11)$$

Using these one can rewrite the partition function (7) as follows in terms of characters of the super- $W_3$  algebra:

$$\begin{aligned} 2Z(W_3) &= (|\text{ch}_0^{\text{NS}}|^2 + |\text{ch}_{1/14}^{\text{NS}}|^2 + |\text{ch}_{5/14}^{\text{NS}}|^2 + |\text{ch}_{1/7,+}^{\text{NS}}|^2 + |\text{ch}_{1/7,-}^{\text{NS}}|^2) + (\text{ch}^{\text{NS}} \rightarrow \overline{\text{ch}^{\text{NS}}}) \\ &+ \frac{1}{4} (|\text{ch}_{1/14}^{\text{R}}|^2 + |\text{ch}_{5/14}^{\text{R}}|^2 + |\text{ch}_{3/2}^{\text{R}}|^2 + |\text{ch}_{9/14,+}^{\text{R}}|^2 + |\text{ch}_{9/14,-}^{\text{R}}|^2). \end{aligned} \quad (12)$$

This is precisely a ‘‘diagonal’’ combination of all characters of the  $c = \frac{10}{7}$  super- $W_3$  algebra! By using the branching rules of super- $W_3$  to  $N=1$  one reproduces the modular invariant in (1).

Thus we learn that the correctness of both the statements (i) and (ii) follows from the branching rules (8)–(11). Although we lack a rigorous proof of these rules, we performed various checks which all confirm that these rules are indeed correct. In checking the correctness of the branching rules one should keep in mind the existence of various low-lying null states. For example, for the embedding  $W_3 \subset \text{super-}W_3$  one naively expects

$$\text{ch}_h^{\text{NS}} = \chi_h + 2\chi_{h+1/2} + \dots, \quad (13)$$

because there are two current modes  $G_{-1/2}$  and  $U_{-1/2}$  (where  $G$  and  $U$  are the spin- $\frac{3}{2}$  and spin- $\frac{5}{2}$  supercurrents in the model), which, when acting on a highest weight state of the super- $W_3$  algebra, can both create a state which is a new highest weight state with respect to the bosonic  $W_3$  algebra. This naive rule is clearly in conflict with the rules (8). However, there are null states

$$G_{-1/2}\phi_0^{\text{NS}}, \quad U_{-1/2}\phi_0^{\text{NS}}, \quad U_{-1/2}\phi_{1/14}^{\text{NS}}, \quad (G_{-1/2} \mp \sqrt{102} U_{-1/2})\phi_{1/7,\pm}^{\text{NS}}, \quad (14)$$

as can be checked by explicitly computing the norm squared of these states using the commutation relations of the super- $W_3$  algebra, as for example given in ref. [7]. With these null states taken into account (13) confirms the branching rule (8).

In a similar way, one can test the branching rules to the  $N=1$  superconformal subalgebra. For example, the spin- $\frac{11}{2}$  primary field that occurs in the branching rule of the identity character  $\text{ch}_0^{\text{NS}}$ , is found to be created by the operator  $(U_{-5/2}W_{-3} + \dots)$ , where the dots stand for a combination of modes  $L_{-m}$  and  $G_{-r}$  of the currents in the  $N=1$  vacuum module.

The multiplicities in the branching rules in the Ramond sector can be understood from the existence of the supercurrent zero modes  $G_0$  and  $U_0$ . Both these modes commute with  $L_0$  and  $W_0$ , i.e. they respect the quantum numbers  $h$  and  $w$ . Together they give rise to a four-fold degeneracy of states. (We did not find any null states that could reduce these degeneracies.)

Let us now come to generalization of these results to general values of  $N$ . We already indicated how one constructs a modular invariant partition function (in terms of characters of the bosonic  $W_N$  algebra) for the cosets (2) at level  $(1, N; 1+N)$ , corresponding to central charge  $c = (3N+1)(N-1)/2(2N+1)$ , which is ex-

pected to be supersymmetric. For  $N \geq 4$ , the corresponding model is no longer minimal with respect to the  $N=1$  superconformal algebra. However, we can still study how the  $W_N$  representations can be combined into larger modules by using the supersymmetry. (Of course, the coset partition function precisely tells us how to do this!) These larger modules are then irreducible representations of a supersymmetric extension of the  $W_N$  algebra, which we call the “minimal” super- $W_N$  algebra, and the coset partition function is nothing else than the diagonal modular invariant of this superalgebra. We expect that this “minimal” super- $W_N$  algebra has generating supercurrents of dimensions  $\frac{3}{2}, \frac{5}{2}, \dots, N - \frac{1}{2}$ .

A few things can be said for general  $N$ . For example, by using the  $W_N$  Kac formula formula one can compute the conformal dimensions of some of the  $W_N$  primary fields that are present in the NS vacuum module of the super- $W_N$  algebra. One finds that  $\phi_{(0\dots 0)(0\dots 0)}$  has dimension 0,  $\phi_{(10\dots 01)(0\dots 0)}$  has dimension  $\frac{3}{2}$ ,  $\phi_{(20\dots 010)(0\dots 0)}$  has dimension 4, independent of the value of  $N$ . The branching of the NS super- $W_N$  vacuum module into  $W_N$  representation is thus given by

$$\text{ch}_0^{\text{NS}} = \chi_0 + \chi_{3/2} + 2\chi_4 + \dots \tag{15}$$

This can be verified by simply counting low-lying states.

As an illustration of the general structure, we explicitly worked out the structure of the modular invariant corresponding to the  $N=4$  minimal superconformal algebra with central charge  $c = \frac{13}{6}$ . In this case the vacuum module of the super- $W_4$  algebra decomposes according to

$$\text{ch}_0^{\text{NS}} = \chi_0 + \chi_{3/2} + 2\chi_4 + \chi_{13/2} + 2\chi_{15/2} + \chi_{10}, \tag{16}$$

in accord with the general result given above. Both in the NS and in the R sector of the super- $W_4$  algebra one finds six highest weight states with multiplicity 1 and four with multiplicity 2, where the multiplicity is 1 (2) if the quantum number  $w_3$  is (is not) equal to 0. The naive branching rule for the embedding  $W_4 \subset \text{super-}W_4$  in the NS sector (compare with (13)) is given by

$$\text{ch}_h^{\text{NS}} = \chi_h + 3\chi_{h+1/2} + \dots, \tag{17}$$

since we now have three modes  $X_{-1/2}$  (for the spin- $\frac{3}{2}, \frac{5}{2}$  and  $\frac{7}{2}$  currents, respectively) available This generic behavior only occurs for one of the NS highest weight states, of dimension  $\frac{5}{6}$ , where we have

$$\text{ch}_{5/6}^{\text{NS}} = \chi_{5/6} + 3\chi_{4/3} + 2\chi_{11/6} + \chi_{10/3} + \chi_{29/6}. \tag{18}$$

In all other cases, the behavior is different due to the presence of null states. It is interesting to note that the supercurrent  $G(z)$  carries a non-zero  $w_4$  charge. The coset partition function  $Z(W_4)$  has the following expression in terms of  $W_4$  characters, denoted here by the eigenvalue of  $L_0$ ,

$$\begin{aligned} Z(W_4) = & (|\chi_0 + 2\chi_4 + \chi_{10}|^2 + |\chi_{3/2} + \chi_{13/2} + 2\chi_{15/2}|^2 + 4|\chi_{55/16}|^2) \\ & + 2(|2\chi_1 + \chi_3 + \chi_7|^2 + |\chi_{1/2} + \chi_{5/2} + \chi_{7/2} + \chi_{13/2}|^2 + 4|\chi_{23/16}|^2) \\ & + (|\chi_{5/9} + 2\chi_{14/9} + \chi_{59/9}|^2 + |\chi_{1/18} + \chi_{55/18} + 2\chi_{73/18}|^2 + 4|\chi_{143/144}|^2) \\ & + 2(|\chi_{1/9} + 2\chi_{19/9} + \chi_{37/9}|^2 + |\chi_{11/18} + 2\chi_{29/18} + \chi_{101/18}|^2 + 4|\chi_{79/144}|^2) \\ & + (|\chi_{1/6} + \chi_{7/6} + 2\chi_{19/6}|^2 + |2\chi_{2/3} + \chi_{5/3} + \chi_{11/3}|^2 + 4|\chi_{5/48}|^2) \\ & + 2(|\chi_{1/6} + 2\chi_{13/6} + \chi_{49/6}|^2 + |\chi_{2/3} + 2\chi_{14/3} + \chi_{20/3}|^2 + 4|\chi_{101/48}|^2) \\ & + (|\chi_{2/9} + 2\chi_{20/9} + \chi_{56/9}|^2 + |\chi_{13/18} + \chi_{67/18} + 2\chi_{103/18}|^2 + 4|\chi_{239/144}|^2) \\ & + 2(|\chi_{1/3} + \chi_{7/3} + \chi_{10/3} + \chi_{16/3}|^2 + |2\chi_{5/6} + \chi_{11/6} + \chi_{29/6}|^2 + 4|\chi_{37/48}|^2) \\ & + (|\chi_{7/18} + \chi_{25/18} + 2\chi_{43/18}|^2 + |2\chi_{8/9} + \chi_{26/9} + \chi_{44/9}|^2 + 4|\chi_{47/144}|^2) \\ & + (|\chi_{5/6} + 2\chi_{11/6} + \chi_{29/6}|^2 + |3\chi_{4/3} + \chi_{10/3}|^2 + 4|\chi_{37/48}|^2). \end{aligned} \tag{19}$$

We expect that it can be rewritten as follows in terms of the characters of the super- $W_4$  algebra:

$$\begin{aligned}
2Z(W_4) = & (|\text{ch}_0^{\text{NS}}|^2 + |\text{ch}_{1/2,+}^{\text{NS}}|^2 + |\text{ch}_{1/2,-}^{\text{NS}}|^2 + |\text{ch}_{1/18}^{\text{NS}}|^2 + |\text{ch}_{1/9,+}^{\text{NS}}|^2 + |\text{ch}_{1/9,-}^{\text{NS}}|^2 + |\text{ch}_{1/6}^{\text{NS}}|^2 + |\text{ch}_{1/6,+}^{\text{NS}}|^2 \\
& + |\text{ch}_{1/6,-}^{\text{NS}}|^2 + |\text{ch}_{2/9}^{\text{NS}}|^2 + |\text{ch}_{1/3,+}^{\text{NS}}|^2 + |\text{ch}_{1/3,-}^{\text{NS}}|^2 + |\text{ch}_{7/18}^{\text{NS}}|^2 + |\text{ch}_{5/6}^{\text{NS}}|^2) + (\text{ch}^{\text{NS}}, \widetilde{\text{ch}}^{\text{NS}}) \\
& + \frac{1}{8} (|\text{ch}_{55/16}^{\text{R}}|^2 + |\text{ch}_{23/16,+}^{\text{R}}|^2 + |\text{ch}_{23/16,-}^{\text{R}}|^2 + |\text{ch}_{43/144}^{\text{R}}|^2 + |\text{ch}_{79/144,+}^{\text{R}}|^2 + |\text{ch}_{79/144,-}^{\text{R}}|^2 + |\text{ch}_{5/48}^{\text{R}}|^2 \\
& + |\text{ch}_{101/48,+}^{\text{R}}|^2 + |\text{ch}_{101/48,-}^{\text{R}}|^2 + |\text{ch}_{239/144}^{\text{R}}|^2 + |\text{ch}_{37/48,+}^{\text{R}}|^2 + |\text{ch}_{37/48,-}^{\text{R}}|^2 + |\text{ch}_{47/144}^{\text{R}}|^2 + |\text{ch}_{37/48}^{\text{R}}|^2) . \quad (20)
\end{aligned}$$

Let us comment on the multiplicities of states and characters in the Ramond sector. We expect that the multiplicity of those  $W_N$  characters in  $Z(W_N)$  that correspond to states in the R sector of the super- $W_N$  algebra is  $2^{N-2}$  (we explicitly checked this for  $N=2, \dots, 6$ ). Assuming the absence of null states, the basic degeneracy of states in the R sector of the super  $W_N$  algebra is  $2^{N-1}$ , leading to a multiplying factor of  $2^{-N}$  in the R term in the super- $W_N$  "diagonal" partition function.

In order to obtain explicit and rigorous results for the modular invariant partition functions for other models with super- $W_N$  symmetry, a different approach is called for. The chiral structure (e.g. representations, characters, ...) of the supersymmetric coset theories based on  $A_N^{(1)} \oplus A_{N-1}^{(1)} / A_N^{(1)}$  at levels  $(m, N; m+N)$ , can be worked out along the lines of ref. [18]. These theories will have a finite description in terms of characters of an extension of the minimal super- $W_N$  algebra. By comparison with the situation for super- $W_3$  [8], we expect the following structure. The lowest model in the above series of coset models has the minimal super- $W_N$  algebra, consisting of supercurrents of dimension  $\frac{3}{2}, \frac{5}{2}, \dots, N - \frac{1}{2}$ , as its superchiral algebra. As one goes to higher models in the series, additional currents start to appear till at a certain point an algebra is obtained which is associative for all values of  $c$ . This algebra is most easily found by considering the limit model of the series, which is a simple model describing  $N^2 - 1$  fermions in the adjoint representation of  $A_{N-1}$ .

Precise statements on *all* possible modular invariants for a given (maximal) superchiral algebra would require a generalization of the work of ref. [19] to the case of chiral algebras that include currents of half-integer spins. We leave this point for further investigation.

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