Hidden structures of knot invariants
Sleptsov, A.

Citation for published version (APA):
Sleptsov, A. (2014). Hidden structures of knot invariants

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
Chapter 1

Introduction

1.1 Knots

1.1.1 How to distinguish knots

Knot theory is an area of low-dimensional topology. Topology studies properties of geometric objects preserved under continuous deformations.

Definition 1.1.1. ([1]) A knot is an embedding of the circle into the three-dimensional Euclidean space $\mathbb{K} : S^1 \rightarrow \mathbb{R}^3$.

In other words, a knot is a closed curve without self-intersections in the 3-space, see Figure 1.1 for particular examples of knots. Usually, one takes $\mathbb{R}^3$ for the ambient 3-space. We do not distinguish between a knot and any continuous deformations of this knot which can be performed without self-intersections. All

![Figure 1.1: Examples of knots](image)

Trefoil knot | Figure-eight knot | True-lovers' knot
---|---|---
$9_7$ | $9_{40}$ | Torus knot [15,-4]
these deformed curves are considered to be one and the same knot. We can think about a knot as if it is made from easily deformable rubber, which we cannot cut and glue. Such deformation are called ambient isotopies, which are a special type of homotopy. A homotopy of a space $X \subset \mathbb{E}^3$ is a continuous map $h : X \times [0, 1] \to \mathbb{E}^3$. If $h_t$ is one-to-one for all $t \in [0, 1]$, then $h$ is called an isotopy.

**Definition 1.1.2.** ([2]) Two knots $K_1$ and $K_2$ are ambient isotopic if there is an isotopy $h : \mathbb{E}^3 \times [0, 1] \to \mathbb{E}^3$ such that $h(K_1, 0) = K_1$ and $h(K_1, 1) = K_2$.

Thus, we consider ambient isotopy as an equivalence relation on knots, that is, two knots are equivalent if they can be deformed into one another. We refer to each equivalence class of knots as a knot type, and equivalent knots have the same type. However to avoid the abuse of terminology it is very common to apply the word ”knot” to mean the whole equivalence class i.e. to a knot type, or a particular representative member which we are interested in. For example, when we say that two knots are different, we actually mean that they are inequivalent, i.e. have different types.

The simplest knot of all is the unknotted circle, which we call the trivial knot or the unknot, see Figure 1.2. If a knot has the same type as the trivial knot, we say it is unknotted.

If you look carefully at Figure 1.2 and use physical intuition of deformable rubber, you will understand that these knots are equivalent. This simple exercise naturally leads us to the first significant scientific question in the knot theory:

**How to distinguish knots?**

Indeed, let us have two knots $K_1$ and $K_2$, for example, as in Figure 1.3. How do we know they are actually different or the same? Also we have not yet proved that there exist any other knots besides the unknot. Maybe every projection of a knot at the any figure above could simply be a messy projection of the unknot. To answer this question we shall find such properties of a knot, which depend only on the equivalence class of the knot. This idea gives rise to a theory of knot invariants, a major part of the knot theory.

Graphically we represent knots by means of knot diagrams. A knot diagram is a plane closed curve, which can have only double points (crossings) as singularities, together with the chosen overcrossing string and undercrossing string at each crossing. Thus, knot diagram can be considered as a projection of a knot along some ”vertical” direction, overcrossings and undercrossings indicate which string is
“higher” and which one is "lower". We refer to a deformation of a knot projection as a planar isotopy or an isotopy in the plane if it deforms the projection plane as if it were made of rubber with the projection drawn upon it Figure 1.4 [3]. We deform the knot only within the projection plane.

\[ \textbf{Figure 1.3: Two arbitrary knots} \]

Proposition 1.1.3. (Reidemeister, [1]) Two knots \( K_1 \) and \( K_2 \), are equivalent if and only if diagram of \( K_1 \) can be transformed into a diagram of \( K_2 \) by a sequence of ambient isotopies of the plane and local moves of the following three types:

\[ \textbf{Figure 1.5: Reidemeister moves} \]

Notice that although each of these moves changes the projection of the knot, it does not change the knot represented by the projection. Each such move is an ambient isotopy. For example, two projections in Figure 1.6 (taken from [3]), the left-most one and the right-most one, correspond to the same knot. Therefore, according to the Reidemeister theorem, there is a series of Reidemeister moves and planar isotopies that takes us from the first projection to the second. In Figure 1.6 we see one example of such series of moves, which demonstrates this equivalence.
The problem of determining whether two projections represent the same knot is not such an easy one as one might have hoped. We just check whether or not there is a series of Reidemeister moves to take us from the one projection to the other, but there is no limit on the number of Reidemeister moves. If the two given projections have 10 crossings each, it might happen that in the process of performing the Reidemeister moves the number of crossings necessarily increases to 57, before the projection is simplified back down to 10 crossings.

1.1.2 The problem of classification

We begin with a discussion of some types of knots, which can be useful for the present thesis.

Alternating knots. We refer to a knot with a projection that has crossings, which alternate between over and under on travelling along the knot as an alternating knot. Otherwise a knot is called non-alternating. The trefoil knot and the figure-eight knot are alternating.

Connected sum. If we have two knots, we can define a new knot obtained by removing a small arc from each knot and then connecting the four loose ends by two new arcs. It must be done with some caution: for projections we assume they do not overlap and we avoid removing or adding any crossings as in Figure 1.7 ([3]). We refer to the resulting knot as the composition of the two knots or the connected sum of the two knots, denoted by $K_1 \# K_2$.

We refer to a knot as composite knot if it can be represented as a composition of two knots, neither of which is the trivial knot. We refer to a knot as prime knot, if it is not a composition of any two nontrivial knots. Note that the composition of knot $K$ with the unknot is again $K$. From this point of view the knots are analogous to the positive integers, where we call an integer composite if it is a product of positive integers, neither of which is equal to 1 and we call it prime.
otherwise. If we multiply an integer by 1, we get the same integer back again. The knots which make up the composite knot are called factor knots. In the Figure both the trefoil knot and the figure-eight knot are prime knots.

The unknot is not a composite knot, because it is not possible to take the composition of two nontrivial knots and obtain the unknot. We can use integers analogy here again: this result is analogous to the fact that the integer 1 is not the product of two positive integers, each greater than 1. Moreover, just as an integer factors into unique set of prime numbers, a composite knot factors into unique set of prime knots. Tables of knots, e.g. the Rolfsen table [80], list only the prime knots and do not include any composite knots. They are similar to tables of prime numbers.

**Mirror knots.** A mirror knot is a knot obtained by changing every crossing in the given knot to the opposite crossing. A knot which is equivalent to its mirror image is called amphicheiral or achiral, otherwise non-amphicheiral or chiral. Despite a knot and its mirror image are distinct knots unless the knot is amphicheiral, knot tables do not list both a knot and its mirror image, only one from this pair. The simplest example of amphicheiral knot is figure-eight knot. One can prove it with the help of Reidemeister moves.

**Links.** Up to now we have considered embeddings of a single circle, i.e. restricted our attention to single knotted loops. However, there is a natural generalisation of this idea so that we consider embeddings of collections of circles and, hence, we obtain a set of knotted loops.

**Definition 1.1.4.** ([2]) A link is a finite disjoint union of knots: $L = \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_n$. Each knot $\mathcal{K}_i$ is called a component of the link. The number of components of a link $L$ is called the multiplicity of the link, and is denoted by $\mu(L)$. A subset of the components embedded in the same way is called a sublink.

We refer to a set of $l$ disjoint circles embedded in a plane as the trivial link of multiplicity $l$.

Some simple examples of links are shown in Figure 1.8. Each of the five 2-component links has two trivial knots as components, but these five links are different. Thus, the notation $\mathcal{K}_1 \cup \ldots \cup \mathcal{K}_n$ lists the component parts and does not indicate how they are put together, hence it is not enough to completely describe a link.

The Thistlethwaite link table [80] is the analog to the Rolfsen knot table. There the following notation is used: a label of the form $L_ia_j$ or $L_in_j$ indicates the $j$th link with $i$ crossings of its minimal plane projection; the label $a$ ($n$) indicates alternating (non-alternating) link. Also some other catalogues use labels of the form $N_\mu^N$, which indicate the $m$th link with $\mu$ components with $N$ crossings. Obviously, the set of links contains the set of knots.

**Torus knots.** Some of the simplest knots are torus knots, the ones which can be embedded into the surface of a standard torus in $\mathbb{R}^3$. They can be easily described parametrically.
The $T[m, n]$ torus knot is a knot obtained by winding a loop over one cycle of the torus $m$ times and over the other $n$ times. If integers $m$ and $n$ are coprime, than it is a knot, otherwise it is a link. A torus knot is trivial if and only if either $m$ or $n$ is equal to 1 or $-1$. The simplest nontrivial example is the $T[2, 3]$-torus knot, also known as the trefoil knot. The simplest nontrivial example of a link is the $T[2, 2]$-torus link, also known as the Hopf link. Each nontrivial torus knot is prime and chiral. The $T[m, n]$-torus knot is equivalent to the $T[n, m]$-torus knot. The $T[2, -n]$-torus knot is the mirror image of the $T[m, n]$-torus knot. We have already had some pictures of torus knots: the first and the last knots on Figure 1.1 are torus knots, on Figure 1.8 we can see the Hopf link.

**Classification.** The classification of objects of study is a basic problem in any branch of mathematics. Usually a classification is a list of objects, which contains all possibilities without repetition. We can use different criteria to create such a list and the usefulness of the list depends on the criteria. Usually we can produce an algorithm that will list all possible objects, but this list will contain duplicates. For a creature of the knot catalogue it lists knot diagrams with increasing numbers of crossings. There naturally arises the problem to identify which diagrams represented the same knot. Till 1980’s it was a big trouble, because there were no appropriate invariants to distinguish knot diagrams. The Reidemeister moves can be successfully used only in the case of positive solution, because there is a finite number of steps while in the case of negative solution the process will never terminate. As a special case one can consider the question about the knot triviality problem: find an algorithm which applies Reidemeister moves, simplifying the diagram at each step, and continue until it has no crossings; if there is no more possible simplification then the diagram cannot be trivial. However, Figure 1.9 (taken from [2]) shows diagrams of the trivial knot, which break this approach: any Reidemeister moves increases the number of crossings. Nevertheless there was invented an algorithm to solve the knot triviality problem by Wolfgang Haken, but it is based on the structure of the knot exterior – a compact 3-manifold. However the problem of detecting triviality with the help of
Reidemeister moves is still open.

Figure 1.9: Awkward diagrams of the trivial knot: any Reidemeister moves increase the number of crossings.

Anyway an algorithm for detecting knot equivalence gives us only a mere classification list with no underlying structure. In some sense we need a construction of the "moduli space" of knots, and for this reason we have to create the complete set of knot invariants.

1.1.3 Knot invariants

Knot invariants are central objects of the study in knot theory. Here we briefly discuss some invariants.

Link, unknotting and crossing numbers.

Definition 1.1.5. ([2]) A link invariant is a function from the set of links to some other set whose value depends only on the equivalence class of the link. Any representative from the class can be chosen to calculate the invariant. There is no restriction on the kind of objects in the target space. For example, they could be integers, polynomials, matrices or groups.

One of the the simplest link invariants is the multiplicity of a link denoted by $\mu(L)$, which is the number of components of $L$.

Many invariants are related to geometric and topological properties of links and measure their complexity in various ways. Some of them are easy to define and very hard to calculate. One of the oldest link invariants is the unknotting number.

Definition 1.1.6. ([2]) The unknotting number is the minimal number of times that a link must pass through itself to be transformed into a trivial link. This number is denoted by $u(L)$.

Although this invariant is one of the most obvious measures of knot complexity, it is very difficult to calculate it. A non-trivial knot $K$ which can be unknotted with only one pass has $u(K) = 1$ (see Figure 1.10 for an example of such knot).

Any knot can be represented by a plane diagram in infinitely many ways; for this reason the following invariant was introduced.
Figure 1.10: The knot $7_2$ becomes the unknot.

**Definition 1.1.7.** We refer to the minimal number of crossings in a plane diagram of $\mathcal{K}$ as the crossing number $c(\mathcal{K})$ of a knot $\mathcal{K}$.

If $c(\mathcal{K}) \leq 2$, then knot $\mathcal{K}$ is trivial. Therefore, to draw a diagram of a nontrivial knot the minimal number of crossings is required at least 3.

**Braid index.** A braid is a set of $n$ strings, all of which are attached to a horizontal bar at the top and at the bottom as in Figure 1.11 [3]. Each string intersects any horizontal plane between the two bars exactly once, i.e. each string always goes in a downward direction while we are moving along it from the top bar to the bottom bar.

![Figure 1.11: A braid.](image)

We can always pull the bottom bar around and glue it to the top bar, so that the resulting strings form a knot or link. It is called the closure of the braid (see Figure 1.12 [3]). Thus, every braid corresponds to a particular knot and we have a *closed braid representation* of the knot.

![Figure 1.12: The closure of a braid.](image)

Every knot or link is a closed braid as was proven by Alexander. As in the case of plane diagrams, we are interested in representing the link by the braid with as few strings as possible.
Definition 1.1.8. The braid index of a link is the minimal number of strings in a braid corresponding to a closed braid representation of the link.

For example, the braid index of the unknot is 1 and of the trefoil is 2. The braid index is an invariant for knots and links.

Polynomial invariants. The most important knot invariants, from my point of view, are polynomial invariants taking values in the rings of polynomials in one or several variables with integer coefficients. The first discovered polynomial invariant was the Alexander polynomial \( \Lambda(K) \) introduced in 1928. Then in 1970 Conway found a simple recursive construction of the Alexander polynomial. Then in 1985 Jones invented the Jones polynomial, which generalises Alexander polynomial. Very soon the HOMFLY polynomial (sometimes called HOMFLY-PT) was discovered which generalises the Jones polynomial. There is much speculation that the HOMFLY polynomial is the central object in knot theory of the present days. My point of view is the same, hence, we have devoted our efforts to studies of the HOMFLY polynomials and actually this thesis is devoted to them.

Definition 1.1.9. The HOMFLY polynomial is the Laurent polynomial in two variables \( A \) and \( q \) with integer coefficients satisfying the following skein relation and the initial condition:

\[
AH(\begin{array}{c}
\Tilde{\bowtie} \\
\end{array}) - A^{-1}H(\begin{array}{c}
\Tilde{\bowtie} - 1 \\
\end{array}) = (q - q^{-1})H(\begin{array}{c}
\Tilde{\bowtie} - 1 \\
\end{array});
\]

\[
H(\begin{array}{c}
\circ \\
\end{array}) = 1 \quad \text{or} \quad H(\begin{array}{c}
\circ \\
\end{array}) = \frac{A - A^{-1}}{q - q^{-1}}.
\]

The first initial condition corresponds to the so-called normalized HOMFLY polynomial, while the second one corresponds to the non-normalized HOMFLY. For normalized HOMFLY polynomial we use notation \( H^K \), while for non-normalized we use \( \mathcal{H}^K \). The HOMFLY polynomial unifies the quantum \( \mathfrak{sl}_N \) polynomial invariants of \( K \) which are denoted by \( H_K^N(q) \) or \( \mathcal{H}^K_N(q) \) and are equal to \( H^K(A = q^N, q) \) or \( \mathcal{H}^K(A = q^N, q) \). To abuse the terminology we below use only notation "HOMFLY polynomial" considering \( A = q^N \) correspondingly.

The HOMFLY polynomial is not a complete invariant for knots, because it cannot distinguish all knots. In particular, a pair of mutant knots always have the same HOMFLY polynomial (Figure 1.13 from [3]).

![Figure 1.13: Two mutant knots have the same polynomial.](image)
After the HOMFLY polynomials were introduced, it soon became clear that they are the first members of a whole family of knot polynomial invariants called quantum invariants. The original idea of quantum invariants was proposed by A. Schwarz [4] and E. Witten in the paper [5] in 1989. This approach came from physics, namely from quantum field theory, and was not completely justified from the mathematical point of view. However Reshetikhin and Turaev soon gave mathematically impeccable definition of quantum invariants of knots [74, 75, 76]. They used quantum groups, which were introduced shortly before by Drinfeld in [73, 72]. Actually, a quantum group is a family of Hopf algebras, depending on a complex parameter $q$ and satisfying certain axioms. The quantum group $U_q\mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ is a deformation of the universal enveloping algebra of $\mathfrak{g}$ (corresponding the value $q = 1$) in the class of Hopf algebras [1]. HOMFLY (Jones) polynomial coincides, up to normalization, with the quantum invariant corresponding to the Lie algebra $\mathfrak{g} = sl_N(sl_2)$ in its standard two-dimensional representation. However this approach allows to consider any irreducible representations and construct corresponding polynomial invariants. For the Lie algebra $\mathfrak{su}_N$ they are called colored HOMFLY polynomials.

1.1.4 Relation with Quantum Field Theory

It is believed that the path-integral representation for knot invariants arising from topological quantum field theory (TQFT) gives the most profound and general description of knot invariants. Ideologically, it means that all possible descriptions of knot invariants can be derived from this representation by utilizing different methods of path-integral calculus. For example, the usage of certain non-perturbative methods leads to the well known description of polynomial knot invariants through the "skein relations" [5]. The perturbative computations naturally lead to the numerical Vassiliev Invariants [60]. In the last case we obtain the formulae for Vassiliev invariants in the form of "Feynman integrals". For a recent comprehensive treatise on Vassiliev invariants see [1].

Despite beauty and simplicity of this picture many problems in the theory of knot invariants remain unsolved. Currently more mathematical descriptions of knot invariants are known than can be derived from path-integral. One such problem is the derivation of quantum group invariants from the path-integral representation. The main ingredient in the theory of these invariants is the universal quantum $R$-matrix defined for integrable quantum deformation of a Lie group. The appearance of quantum groups in the path-integral representation looks mysterious and we lack the derivation of the corresponding $R$-matrix (object with noncommutative matrix elements should appear from classical integral) from path-integral. These problems were discussed in details in [61, 62].

The second interesting problem (which as we believe is closely connected to the first one) concerns the combinatorial description of the numerical Vassiliev invariants of knots. At the moment, there exist three different descriptions of Vassiliev invariants: through the generalized Gauss integrals [63], through Kontsevich integral [64], and finally there are combinatorial formulae for Vassiliev invariants of orders 2, 3 and 4, [65]-[68]. The first two descriptions can be easily derived from
the path-integral but (surprisingly) we lack such a derivation for combinatorial formulae. The Gauss integral representation for Vassiliev invariants comes from the perturbative computations of path-integral in the covariant Lorentz gauge. Similarly, the usage of non-covariant holomorphic gauge (which is sometimes referred to as light-cone gauge) computation of path-integral leads to Kontsevich integral [70].

The main difference between the non-covariant gauges (temporal or holomorphic gauge) and the covariant Lorentz gauge is that the Feynman integrals in the former case can be naturally "localized", in the sense that only a finite number of special points on the knots contribute to the integrals. In the Lorentz gauge the Feynman integrals have a form of multiple 3-d integrals (see (3.36),(3.37) as an example) and all points of the knot enter the integral equally. On the contrary, in the holomorphic gauge only special points contribute to the Feynman integrals. Summed in all orders of perturbation theory these contributions lead to definition of polynomial knot invariants through simple crossing operator and Drinfeld associator with rational zeta-function coefficients [71]. We discuss this localization process in detail in section 3.1.1.

The temporal gauge is distinguished among all gauges. The Feynman integrals here have ultra-local form – only crossing points of two-dimensional projection of the knot contribute to the answer. This is exactly what happens in the quantum group description of knot invariants where the crossing points contribute as a universal quantum $R$-matrix. On the other hand, the combinatorial formulae for Vassiliev invariants also are based on the information from these crossing points only. All these observations make it natural to argue that the Feynman integrals arising from the general path-integral representation in the temporal gauge should give the universal combinatorial formulae for Vassiliev invariants, and summed in all orders of perturbation theory they should give the perturbative expansion of the universal quantum $R$-matrix for $\hbar$-deformed gauge group of the path integral. These questions and structures arising from the path-integral representation of knot invariants in the temporal gauge are discussed in section 3.5.

Recently new polynomial invariants of knots appeared, the so-called "super-polynomials" [24], which are generalizations of HOMFLY polynomials. However, there is still no related QFT-like formulation. Probably this interpretation should involve something like "Chern-Simons theory with a quantum group for its gauge group" which is still not constructed.
1.2 Main results

In this section we overview our main results presented in the thesis without going into details. The theorem number corresponds to the section number, where this theorem is introduced and considered; same for conjectures.

**Theorem 2.3.5.** Colored HOMFLY polynomial of a knot $K$ belongs to the algebra of shifted symmetric functions $\Lambda^*$. 

**Theorem 2.4.7.** Ooguri-Vafa partition function of a HOMFLY polynomial is a Hurwitz partition function.

**Theorem 3.4.1.** (Conditional) Higher special polynomials are related with Vassiliev invariants as follows:

$$
\sigma^K_\Delta(g) = \sum_{k \geq 0} a^k \sum_{m=1}^{N_{u+k}} c_{u+k,m,k}(\Delta) V^c_{u+k,m},
$$

The theorem is proven under assumption on the explicit form of genus expansion for HOMFLY polynomial (conjecture 2.4.3 below).

**Theorem 3.4.2.** Vassiliev invariants of a knot $K$ with $r$ strands satisfy the following relation

$$
\det M_{j_1 \ldots j_{r+1}} = 0,
$$

where $M$ is a matrix of Vassiliev invariants and any positive integers $j_k$.

Beyond these points there are some numerical results in the thesis. We have calculated Vassiliev invariants up to order 6 (inclusive) for knots with crossing numbers $\leq 14$. There are around 60,000 such knots, complete results are available on the website [80]. One can download packages with these Vassiliev invariants from this website or refer to the section ”Vassiliev invariants” for particular knots from the Rolfsen table. In section 3.4.2 for illustrative purpose we list Vassiliev invariants for knots with crossing numbers $\leq 7$. Furthermore, with the help of these results we check that there are no universal relations on the Vassiliev invariants with order $\leq 6$.

Also we have some conjectures based on explicit calculations done for a lot of particular knots.

**Conjecture 2.4.3.** Large $N$ expansion (or genus expansion) for HOMFLY
polynomials is given by

\[ H^K_R(q, A) = \exp \left( \sum_{\Delta} h^{\Delta|+l(\Delta)-2} S^K_\Delta(A^2, h^2) \varphi_R(\Delta) \right), \]

\[ S^K_\Delta(A^2, h^2) = \sum_g \sigma^K_\Delta(g) h^{2g}, \]

where the sum goes over all Young diagrams \( \Delta \) and \( q = \exp(h/2) \).

**Conjecture 4.3.1.** DAHA-superpolynomials belong to the \( \Lambda_\beta^* \) algebra of functions symmetric in variables \( \mu_i = R_i - \beta i \).

**Conjecture 4.3.2.** As for multiplicative basis \( T^\beta_k(R) \) of the algebra \( \Lambda_\beta^* \) we consider

\[ T^\beta_k(R) = \sum_{i,j} \left( (j - 1) - \beta(i - 1) \right)^{k-1}. \]

**Conjecture 4.3.3.** Large \( N \) expansion for DAHA-superpolynomials is given by the following expression:

\[ P^K_R(q, t, A) = \exp \left\{ \sum_{\Delta} h^{\Delta|+l(\Delta)-2} \cdot S^K_\Delta(h^2, \beta, A) \cdot T^\beta(R) \right\}, \]

\[ S^K_\Delta(h^2, \beta, A) = \sum_{n=0}^{\infty} h^{2n} s^K_\Delta(n). \]

For loop expansion of superpolynomials we consider thin knots and thick knots separately.

**Conjecture 4.3.5.** In the case of thin knots the loop expansion of superpolynomials has the form

\[ P^K_R(A, q, t) = \sum_{i=0}^{\infty} h^i \sum_{j=1}^{N_i^\beta} D^{(R)}_{i,j} V^K_{i,j}, \]

where \( D^{(R)}_{i,j} \) are beta-deformations of trivalent diagrams, \( V^K_{i,j} \) are exactly the same Vassiliev invariants as for the loop expansion of HOMFLY polynomials.

**Conjecture 4.3.6.** In the case of thick knots the loop expansion of superpolynomials has the form

\[ P^K_R(A, q, t) = \sum_{i=0}^{\infty} h^i \sum_{j=1}^{M_i^\beta} D^{(R)}_{i,j} V^K_{i,j} + (\beta - 1) \sum_{i=0}^{\infty} h^i \sum_{j=1}^{M_i^\beta} \zeta^{(R)}_{i,j} \rho^K_{i,j}, \]

15
where the first sum is the same as for the thin knots, while the second sum is different: $\Xi_{i,j}^{(R)}$ are some other group factors and $\kappa_{i,j}$ are some other numbers different from the Vassiliev invariants of the first sum.

These results are mostly based on the following papers by the author of the thesis:


Particular examples of higher special polynomials and superpolynomials were taken from:


### 1.3 Acknowledgments

I am very grateful to the Korteweg-de Vries Institute for Mathematics at the University of Amsterdam, where I did my PhD research. The atmosphere created by the people from the institute made my work very pleasant and productive.

I am truly indebted and thankful to my promotor Sergey Shadrin for the opportunity, support, help, numerous fruitful discussions and attention to my thesis. Also I would like to thank my copromotor Alexey Morozov for the opportunity, problem formulations, patient explanations and many helpful discussions.

This thesis would not have been possible without successful collaborations with Petr Dunin-Barkowski, Andrey Smirnov and especially Andrey Mironov, to whom I owe my deepest gratitude.