Exploring jet properties in magnetohydrodynamics with gravity
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The model developed in this work belongs to the class of axisymmetric, radially self-similar wind models. Every wind model is fully specified by the so-called wind equation. By integrating this wind equation, with possible several supplementary equations, we can obtain a full description of the density, velocity and temperature of the material and, if present, the magnetic and electric field configuration and it is therefore of primary importance to this thesis. In this chapter we will describe this class of models in detail and introduce the important concepts associated with them. We also will give an overview of the steps taken to extend the model to include gravity. Since the basics of the models are the same, we will give only a cursory list of steps in the following chapters.

2.1 History

There has been a long history of wind\(^1\) models, from spherically symmetric hydrodynamic, to axisymmetric magnetohydrodynamic ones. They all share elements in their derivation and also have several concepts in common. By giving a chronological overview, from relatively simple to more advanced, we will introduce the important concepts and show how the different assumptions at the basis of them influence their properties.

\(^1\)While the word wind originally denoted outflowing plasma from a star, the models have been generalised to such an extent that now in the most general sense it can be an (un)collimated outflow or inflow, gravitationally bound or unbound, with velocities ranging from subsonic to relativistic, consisting of neutral matter or a plasma, anchored on any surface, such as a star or a disc.
2 Background and Methodology

2.1.1 Non-relativistic spherically symmetric HD wind

The first wind model was developed in the 1950’s by Parker (1958). By observing the direction of comet tails, Biermann (1951) proposed that the interplanetary gas was moving away from the Sun, which led Parker (1958) to consider the spherically symmetric case. Due to the fact that only the square of the velocity enters the equations, this non-relativistic hydrodynamic model actually describes both a stellar wind and accretion onto a star. While it is relatively simple, it already incorporates some general concepts, which are clear to see from the governing equation. The equations this model and all others are based on are the continuity equation, describing conservation of mass:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \]  

(2.1)

where \( \rho \) is the matter density, \( t \) is time, and \( \mathbf{V} \) is the velocity vector; the Euler equation, describing the conservation of momentum:

\[ \rho \frac{\partial \mathbf{V}}{\partial t} + \rho \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla P + \mathbf{f}, \]

(2.2)

where \( P \) is the gas pressure, and \( \mathbf{f} \) is the external force density, which in the case of the Parker wind is the gravitational force density, \( \mathbf{f}_g = -G M \rho / r^2 \), where \( G \) is the gravitational constant, \( M \) is the mass of the star, and \( r \) is the spherical radius; the ideal gas law, which relates the temperature, pressure and density of the gas:

\[ P = \frac{\rho k T}{\mu m_H}, \]

(2.3)

where \( k \) is Boltzmann’s constant, \( T \) is the gas temperature, \( m_H \) is the mass of a hydrogen atom, and \( \mu \) is the mean molecular weight of the gas in units of \( m_H \); and finally a polytropic energy equation, relating the gas pressure and density:

\[ P = Q \rho^\Gamma, \]

(2.4)

where \( Q \) is the adiabat, an \( \Gamma \) is the polytropic index. The assumptions going into the model are time independence and spherical symmetry \((r, \theta, \phi)\). Combining the above mentioned equations, and taking all the assumptions into account, it is possible to derive an equation for the derivative of the velocity squared with respect to the radius:

\[ \frac{dV^2}{dr} = -2 \frac{G M}{r^2} \left( \frac{1}{1 - \frac{V^2}{c_s^2}} \right), \]

(2.5)

where \( V \) is the radial velocity, and \( c_s \) the sound speed, which is usually a function of radius. Since this equation fully describes the outflow, it is called a wind equation. It
is this wind equation that we can integrate, with possible several supplementary equations, to obtain a full description of the wind, making it the most important equation in this thesis. The wind equation also divides the solutions into different families (see figure 2.1). This can be understood by looking at equation (2.5) a bit more closely. The overall form of the right hand side is that of a numerator and a denominator. When the flow velocity crosses the sound speed, the denominator goes through zero. If the numerator still has a finite value at that point, the acceleration of the flow becomes infinite. So in order to cross this point smoothly, the numerator has to be zero there as well. This constitutes an internal boundary condition, called a ‘regularity condition’, and therefore specifies the radius of this singular point, namely:

\[ r_S = \frac{GM}{2c_S^2}, \]  

(2.6)

where \( r_S \) is the spherical radius of this so-called sonic point. While it is usually called the sonic point because the wind equation is one-dimensional, because of the spherical symmetry, it is actually a spherical surface surrounding the star. It is convenient to express the radius \( r \) in units of the radius of the sonic point radius \( r_S \) and the velocity in units of the sound speed, that is to say, as the Mach number:

\[ M_S = \frac{V}{c_S}. \]  

(2.7)

There is also a physical way to approach this issue. Since the flow is hydrodynamic, there is a characteristic velocity, the sound speed. As the flow velocity exceeds the sound speed, the flow downstream can no longer affect the flow upstream. There is no signal that can travel fast enough to propagate upstream to the star. In other words the flow upstream is causally disconnected from the flow downstream of the sonic point and the surface it defines is called a separatrix surface, since it separates two modes of behaviour, the subsonic and the supersonic regime. This is a general feature of the wind equation: where there are mathematical singularities, there are corresponding physical separatrix surfaces.

The velocity in the Parker wind equation is squared everywhere, so the wind equation describes an outflow, or wind, as well as an inflow, or accretion. There are thus two solutions crossing the sonic point, labelled I for the wind and II for the accretion mode. These two lines divide the diagram in four regions. Regions III and IV are double-valued, a result of the numerator not being zero, where the denominator is, causing infinite acceleration and turning the flow back on itself. This behaviour is unphysical and thus it cannot be a proper solution. Regions V and VI have flow velocities that are supersonic or subsonic throughout the domain, with a minimum and maximum velocity respectively. These extrema are realised when the
Figure 2.1: Families of solutions of the Parker wind equation. The velocity in units of the sound speed is plotted against the radius in units of the sonic point radius. The thick lines labelled I and II are the wind and accretion solution respectively, dividing the plot into four separate regions. Every region, denoted by III – VI, has one typical solution plotted for that region, but is actually completely filled with solutions. See the text for details on the different solution families.

The numerator is zero, while the denominator is still finite, since the flow velocity is never equal to the sound speed. While region V is unrealistic with supersonic velocities far away from the central object, solutions in region VI could be possible and are called breeze solutions. Although all regions can be part of the solutions in shock transitions due to their discontinuous nature, in this thesis we will focus on winds with smooth transitions of the singular point(s), called type I solutions in figure 2.1.

2.1.2 Non-relativistic cold cylindrically symmetric MHD wind

The first non-relativistic magnetohydrodynamic wind model was derived by Blandford & Payne (1982). As the aim was to describe a jet, a cylindrical geometry ($\sigma, \phi, z$) was adopted. The matter is cold and the acceleration is caused by the centrifugal force as it is scooped up by the field lines. Since the general MHD equations are too complicated to solve for the general case, several assumptions were made in order to reduce the number of free parameters to be solved for. These are time independence; ideal MHD, giving

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0,$$

(2.8)
corresponding to a high magnetic Reynolds number, meaning the matter strictly follows the magnetic field lines; axisymmetry around the z-axis; and self-similarity, meaning all field lines have the same shape and can be obtained by scaling one reference field line, although the magnetic field strength and density have separate scalings (see figure 2.2). For a flow that is steady and axisymmetric, the poloidal velocity and magnetic field are parallel to each other

\[ \mathbf{V}_p = \frac{\Psi_A(A)}{4\pi\rho} \mathbf{B}_p, \]  

(2.9)

where \( \Psi_A \) is the mass-to-magnetic flux function, \( A \) is the magnetic flux function, and \( \mathbf{B}_p \) is the poloidal magnetic field. The magnetic flux function can be thought of as a circular surface centred on the z-axis, with a value corresponding to the number of magnetic field lines crossing this surface, being zero for zero radius. This value changes depending on the radius of the surface and its height, but due to the axisymmetry and since field lines do not cross, the value is the same along a particular field line, and thus a field line can be labelled by the value of this function. The toroidal velocity is provided by the angular velocity of the field line (\( \Omega \)), which is also constant.


along a particular field line, leading to an expression for the full velocity

$$V = \frac{\Psi_A}{4\pi \rho} B + \sigma \Omega \hat{\phi}. \quad (2.10)$$

There are two other constants of motion for a particular field line, namely the specific energy:

$$E = \frac{V^2}{2} + h + \Phi - \frac{\Omega \sigma B_{\phi}}{\Psi_A}, \quad (2.11)$$

where $E$ is the energy density, $h$ is the enthalpy per unit mass, and $\Phi$ is the gravitational potential, and the specific angular momentum:

$$L = \sigma V_{\phi} - \frac{\sigma B_{\phi}}{\Psi_A}. \quad (2.12)$$

Using these expressions it is possible to define three dimensionless parameters that are constant along a field line, describing the energy density:

$$\epsilon_{BP} \equiv \frac{E}{(GM/\sigma_i)}, \quad (2.13)$$

the angular momentum:

$$\lambda_{BP} \equiv \frac{L}{(GM/\sigma_i)^{1/2}}, \quad (2.14)$$

and the mass-to-magnetic flux ratio:

$$\kappa_{BP} \equiv \Psi_A \left[1 + \frac{1}{\tan^2(\psi_i)}\right]^{1/2} \frac{(GM/\sigma_i)^{1/2}}{B_i}, \quad (2.15)$$

where $\psi$ is the angle the field line makes with the disc, and a subscript $i$ indicates a quantity evaluated at the disc surface. When the flow is adiabatic with polytropic index $\Gamma$, a fourth dimensionless constant can be defined:

$$\mu_{BP} \equiv \frac{P}{(B_i^2/4\pi)} \left(\frac{B_i^2/4\pi GM}{\sigma_i}\right)^{\Gamma}, \quad (2.16)$$

describing the thermal energy density. Since the flow is cold, this constant is equal to zero in the article.

Since we are now dealing with magnetic fields, a new characteristic velocity enters the problem. This Alfvén velocity, in the poloidal direction given by:

$$V_{A,p} = \frac{B_p}{\sqrt{4\pi \rho}} \quad (2.17)$$
plays a similar role for magnetic fields as the sound speed does for gas, and analogously we define a poloidal Alfvénic Mach number:

$$M = \sqrt{4\pi \rho \frac{V_p}{B_p}}.$$ (2.18)

There is another velocity present in MHD flow, the fast magnetosonic velocity, with a Mach number given by:

$$M_f = \sqrt{4\pi \rho \frac{V_p}{B}}.$$ (2.19)

Again using the continuity equation (2.1) and the z-component of the momentum equation (2.2), it is possible to derive a wind equation. Thanks to the presence of magnetic fields, this wind equation has many more terms, but the overall form, a single numerator and denominator, is preserved. The denominator in this case consists of the product of two subtractions, \((M^2 - 1)(M^2_{f,\theta} - 1)\), where \(M_{f,\theta}\) is the fast magnetosonic Mach number in the \(\theta\)-direction, towards the \(z\)-axis. This product means the denominator can go to zero at two points, where the flow velocity reaches the Alfvén velocity, called the Alfvén point (AP), and where the flow velocity in the \(\theta\)-direction attains the fast magnetosonic velocity, called the modified fast point (MFP, so called because the name fast magnetosonic point is reserved for the location where the poloidal velocity reaches the fast magnetosonic velocity). While they are called points because they occur on a one-dimensional field line, due to the self-similarity and axisymmetry assumption they are actually surfaces, more specifically cones around the \(z\)-axis. The reason for these specific conditions lie in the same assumptions. Because according to self-similarity all flow properties must be the same along a radial line and due to axisymmetry they are the same along a toroidal line, the only direction a wave can propagate is the \(\theta\)-direction. Since a separatrix surface forms at the location where the flow velocity matches the wave velocity in the direction of propagation, it is the flow velocity in the \(\theta\)-direction, not the poloidal direction, that has to be compared to the different characteristic velocities. As Alfvén waves are purely magnetic and \(V_p \parallel B_p\), they can travel in any direction in the meridional plane and therefore also \(V_p = V_{A,p}\) at the AP.

Beyond the AP Alfvén waves cannot propagate backwards towards the central object, so all the APs indeed form a separatrix surface. However the fast magnetosonic speed exceeds the Alfvén velocity, so it is still possible to transmit information from beyond the Alfvén point to the central object and the flow upstream of the AP is not necessarily causally disconnected from the flow downstream.

Since the fast magnetosonic velocity is the fastest velocity at which any signal can travel, the flow does become causally disconnected at the fast magnetosonic separatrix surface (FMSS), formed by all the MFPs, where \(M^2_{f,\theta} = 1\).
Blandford & Payne (1982) were interested in solutions where the flow became collimated parallel to the z-axis. This constraint meant that the flow velocity in the θ-direction never became very large and consequently that \( M^2_{\theta,\theta} < 1 \) throughout the flow. Therefore the second singular point was not crossed. The flow was also cold, which means that the third singular point of MHD flows was not treated by these authors.

### 2.1.3 Relativistic cold cylindrically symmetric MHD wind

Ten years later Li et al. (1992) constructed a class of self-similar solutions for relativistic winds. The relativistic treatment introduces a natural length scale, the light cylinder radius. If field lines were to rotate as a spoked wheel, dragging the matter with them, the matter would rotate with the velocity of light at a certain radius given by:

\[
\omega_L = \frac{c}{\Omega_i},
\]

where \( c \) is the velocity of light, and \( \Omega_i \) is the angular velocity of the part of the disc where the field line is anchored. Since this is physically impossible, the field lines have to bend backwards with respect to the rotation, increasing the azimuthal magnetic field.

By filling in the identity of the Lorentz factor, it is possible to obtain an equation that describes the partition of energy in the system. This can be referred to as the energy equation, but is also sometimes called the Bernoulli equation. If the geometry is known, this equation immediately gives the acceleration.

Another relativistic effect is the importance of an electric field. The direction of the electric field is in the poloidal direction and perpendicular to the poloidal velocity streamlines, called the transfield direction, and therefore the electric field does not cause any additional acceleration. It does however affect the collimation of the flow. By projecting the relativistic momentum equation onto a unit vector in the transfield direction it is possible to write down the full transfield force balance equation, which determines the geometry of the field lines. Therefore together with the energy equation this equation completely describes the system and from them it is possible to derive the wind equation.

In this article solutions were sufficiently far away from the central object for gravity and gas pressure to be neglected and it is actually these simplifications that allow self-similar solutions to be found in the relativistic case. The downside is that the region near the compact object is not very well described. Another side effect of the self-similarity assumption is that the angular velocity of the disc falls of as \( \omega^{-1} \), which is more restrictive than the non-relativistic case, as Keplerian accretion discs can no longer be modelled.
2.1.4 Non-relativistic warm cylindrically symmetric MHD wind

Another eight years later the cold Blandford & Payne model was generalised by Vlahakis et al. (2000) to allow the flow to be hot. They assumed a polytropic relationship between the gas pressure and the density

\[ Q = \frac{P}{\rho^\Gamma}, \]  

(2.21)

where \( Q \) is the specific entropy, and a parameter proportional to the gas entropy, describing the ratio of the gas pressure to the magnetic energy density at the Alfvén point:

\[ \mu_{\text{VTST}} = \frac{8\pi P_A}{B_A^2}. \]  

(2.22)

Both \( Q \) and \( \mu_{\text{VTST}} \) are constant along a field line.

While a similar parameter was present in Blandford & Payne (1982), in this article it is used for the first time in a calculation. The effect of a warm flow is that a third singular point appears in the wind equation. When the denominator of the wind equation becomes zero, the velocity \( V_\theta \) satisfies the quartic:

\[ V_\theta^4 - V_\theta^2 \left( c_s^2 + V_A^2 \right) + c_s^2 V_{A,\theta}^2 = 0, \]  

(2.23)

in other words, where \( V_\theta^2 \) equals:

\[ V_s^2 = \frac{1}{2} \left\{ c_s^2 + V_A^2 - \left[ \left( c_s^2 + V_A^2 \right) - 4 c_s^2 V_{A,\theta}^2 \right]^{\frac{1}{2}} \right\}, \]  

(2.24)

called the slow magnetosonic velocity, or:

\[ V_f^2 = \frac{1}{2} \left\{ c_s^2 + V_A^2 + \left[ \left( c_s^2 + V_A^2 \right)^2 - 4 c_s^2 V_{A,\theta}^2 \right]^{\frac{1}{2}} \right\}, \]  

(2.25)

called the fast magnetosonic velocity. The first equality corresponds to the slow magnetosonic separatrix surface (SMSS), or the modified slow point (MSP). In a cold flow the sound speed is zero, and consequently the slow magnetosonic velocity is 0 as well. For a cold flow the MSP thus lies at zero height, which is why Blandford & Payne (1982) were unable to cross it.

By starting the integration at the Alfvén point and integrating inward towards the MSP and outward towards the MFP, it was possible to cross all three singular points in the flow, leading to a credible jet solution spanning from below the MSP to beyond the MFP. All that remained was to allow the flow to become relativistic.
2.1.5 Relativistic warm cylindrically symmetric MHD wind

The equations for a relativistic hot flow were written down by Vlahakis & Königl (2003a). The assumptions remained the same, time-independence, ideal MHD, axial-symmetry, and self-similarity, with the addition of a zero azimuthal electric field ($E_\phi = 0$), which is generated due to relativistic effects. The velocity is given by:

$$ V = \frac{\Psi_A}{4\pi\gamma\rho_0} B + \sigma \Omega \hat{\phi}, \quad \frac{V_p}{B_p} = \frac{\Psi_A}{4\pi\gamma\rho_0}, \quad (2.26) $$

where $\rho_0$ is the baryon rest-mass density. The five parameters constant along a field line are the field angular velocity:

$$ \Omega = \frac{V_\phi}{\sigma} - \frac{\Psi_A}{4\pi\gamma\rho_0} \frac{B}{\sigma}, \quad (2.27) $$

the mass-to-magnetic flux ratio:

$$ \Psi_A = \frac{4\pi\gamma\rho_0 V_p}{B_p}, \quad (2.28) $$

the total (kinetic plus magnetic) specific angular momentum:

$$ L = \xi \gamma \sigma V_\phi - \frac{\sigma B_\phi}{\Psi_A}, \quad (2.29) $$

where $\xi c^2$ is the specific (per baryon mass) relativistic enthalpy, the total energy-to-mass flux ratio $\mu c^2$ with:

$$ \mu = \xi \gamma - \frac{\sigma \Omega B_\phi}{\Psi_A c^2}, \quad (2.30) $$

and the specific entropy:

$$ Q = \frac{P}{\rho_0^\Gamma}. \quad (2.31) $$

Since gravity had again to be ignored to allow self-similar relativistic flow, the modified slow point was not solved for. For the far-field solution it was deemed an asymptotically cylindrical flow was the only physically acceptable solution, with the MFP at infinite height. So in effect only the Alfvén point was crossed.

The free parameters in this model are $F$, which controls the current distribution and is given by:

$$ F = 1 + \frac{d \log (I)}{d \log (r)}, \quad (2.32) $$

where $I$ is the current, $\Gamma$, the adiabatic index, $\theta_A$, the poloidal spherical angle of the Alfvén point, $\phi_A$, the angle of the field line at the Alfvén point, $x_A$, the cylindrical
2.2 Extending the model

As can be seen in Table 2.1, we extend the previous work by developing a model that not only allows both the temperature and the velocity of the flow to become relativistic, but also crosses all three singular points.

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<th>Gravity</th>
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Table 2.1: Overview of radially self-similar wind models, indicating whether the flow has thermal energy and whether the flow can attain relativistic velocities. Also indicated is which singular points are crossed, named the slow magnetosonic separatrix surface (SMSS), or modified slow point (MSP), the Alfvén surface (AS), and the fast magnetosonic separatrix surface (FMSS), or modified fast point (MFP).

radius of the Alfvén point in units of the light cylinder radius, $\sigma_M$, the Michel magnetisation parameter, $q$, denoting the relative entropy content of the adiabatic plasma, and $B_0\sigma_M^{2-F}$, which sets the overall physical scaling.

As can be seen in Table 2.1, we extend the previous work by developing a model that not only allows both the temperature and the velocity of the flow to become relativistic, but also crosses all three singular points.

2.2 Extending the model

We would like to derive a model that allows us to cross all three critical points in a relativistic flow, so it is possible to relate the height of the MFP to properties close to the black hole (around the MSP). To this end we want to compare the relativistic wind equation of Vlahakis & Königl (2003a, hereafter VK) without gravity, with the wind equation given in Vlahakis et al. (2000, hereafter VTST) which does include gravity, in order to obtain a gravity term, which includes only the kinetic inertia, and neglects the thermal and electromagnetic inertias (Polko et al. 2013a). We will call this term the kinetic gravity term. We need to derive the VK wind equation, since this equation is not given within the paper. We also want to derive a gravity term that includes all inertias, called the full gravity term, and compare it with the kinetic gravity term. For this we need to rederive new forms of the relativistic energy and transfield equations. We will also derive the non-relativistic transfield equation for completeness.
2 Background and Methodology

2.2.1 Non-relativistic wind model with gravity

One way to obtain the wind equation from the energy and transfield equation is by using the determinant method. When we write both these equations in the form:

\[ A \frac{dM^2}{d\theta} + B \frac{d\psi}{d\theta} = C, \quad (2.33) \]

using subscripts 1 for the energy equation and subscripts 2 for the transfield equation, we have two equations and two unknowns, leading to the solutions:

\[ \frac{dM^2}{d\theta} = \frac{C_1 B_2 - C_2 B_1}{A_1 B_2 - A_2 B_1} \quad (2.34) \]

and:

\[ \frac{d\psi}{d\theta} = \frac{C_2 A_1 - C_1 A_2}{A_1 B_2 - A_2 B_1}. \quad (2.35) \]

The first equation is the wind equation. Since every product in both the numerator and the denominator of the formula is composed of a single term from the energy equation and a single term from the transfield equation, if we multiply all terms from one equation with the same factor, this factor will cancel out in the final equation. We will make use of this to simplify the calculations. To show this method works, we test it on the VTST equations, since that wind equation is already given. We write down the terms obtained from the derivative of the energy equation, derive the transfield equation and find its terms, and then calculate the wind equation.

Terms from the energy equation

Here we give the expressions for the determinant method obtained from the derivative of the energy equation with respect to \( \theta \):

\[ A_1 = \left[ \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \right] \times \left[ -2 \lambda_{VTST}^2 \frac{M^2 (1 - G^2)^2 \cos^2(\psi + \theta)}{G^2 (1 - M^2)^3 \sin^2(\theta)} \right. \]
\[ + \left. \Gamma \mu_{VTST} \frac{\cos^2(\psi + \theta)}{M^2} \frac{2 M^2}{G^4} \right], \quad (2.36) \]

\[ B_1 = \left[ \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \right] \left[ -2 M^4 \frac{\tan(\psi + \theta)}{G^4} \right], \quad (2.37) \]

\[ C_1 = \left[ \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \right] \left[ \frac{2 \kappa_{VTST}^2 \sin(\theta)}{G} \frac{\sin(\psi + \theta) \cos(\psi + \theta)}{\sin^2(\theta)} \right. \]
\[ - \left. 2 \frac{M^4 \cos(\psi)}{G^4 \sin(\theta) \cos(\psi + \theta)} + 2 \lambda_{VTST}^2 \frac{(2 M^2 - 1) G^4 - M^4 \cos(\psi) \cos(\psi + \theta)}{G^2 (1 - M^2)^2 \sin^3(\theta)} \right]. \quad (2.38) \]

The common factors have been extracted to make the eventual multiplication and comparison easier.
2.2 Extending the model

Derivation of the transfield equation

The transfield equation is the inner product of the sum of forces acting on a field line and the unit vector normal to the field line. The forces are the kinetic force, the thermal pressure force, the electromagnetic force and the gravitational force:

\[- \rho (V \cdot \nabla) V - \nabla P + \frac{(\nabla \times B) \times B}{4\pi} + \rho \nabla \frac{GM}{r} = 0. \quad (2.39)\]

Taking the inner product, the kinetic force becomes:

\[-\rho (V \cdot \nabla) V \cdot \hat{n} = \left[ -M^2 \sin^2(\theta) \frac{\partial \psi}{\partial \theta} - \lambda_{\text{VTST}}^2 \frac{G^2(G^2 - M^2)^2}{M^2(1 - M^2)^2} \sin(\psi) \frac{\cos(\psi + \theta)}{\sin(\theta)} \right] \times \left[ \frac{B_0^2 \alpha F^{-2}}{4\pi \sigma G^4 \cos(\psi + \theta)} \right]. \quad (2.40)\]

The thermal pressure force becomes:

\[-\nabla P \cdot \hat{n} = -\frac{\mu_{\text{VTST}}}{M^2} G^4 \left[ (F - 2) + \sin(\psi + \theta) \cos(\psi + \theta) \frac{\Gamma}{2M^2} \frac{\partial M^2}{\partial \theta} \right] \times \left[ \frac{B_0^2 \alpha F^{-2}}{4\pi \sigma G^4 \cos(\psi + \theta)} \right]. \quad (2.41)\]

The electromagnetic force becomes:

\[
\frac{(\nabla \times B) \times B}{4\pi} \cdot \hat{n} = \left\{ \begin{array}{l}
\frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ -(F - 2) - \frac{\cos(\psi) \sin(\psi + \theta)}{\sin(\theta)} + \frac{\partial \psi}{\partial \theta} \right] \\
+ \lambda_{\text{VTST}}^2 \frac{G^2(1 - G^2)^2}{(1 - M^2)^2} \left[ -(F - 1) - \frac{2G^2}{1 - G^2} \frac{\cos(\psi) \sin(\psi + \theta)}{\sin(\theta)} \right] \\
+ \frac{\sin(\psi + \theta) \cos(\psi + \theta)}{1 - M^2} \frac{\partial M^2}{\partial \theta} \right\} \left[ \frac{B_0^2 \alpha F^{-2}}{4\pi \sigma G^4 \cos(\psi + \theta)} \right]. \quad (2.42)\]

And the gravitational force becomes:

\[\rho \nabla \frac{GM}{r} \cdot \hat{n} = -\frac{\kappa_{\text{VTST}}^2}{G} \frac{\sin(\theta)}{M^2} G^4 \cos^2(\psi + \theta) \left[ \frac{B_0^2 \alpha F^{-2}}{4\pi \sigma G^4 \cos(\psi + \theta)} \right]. \quad (2.43)\]
2 Background and Methodology

The full transfield equation is then given by:

\[
\begin{aligned}
&\left\{-M^2 \sin^2(\theta) \frac{\partial \psi}{\partial \theta} - \lambda^2_{VTST} \frac{G^2(G^2 - M^2)^2}{M^2(1 - M^2)^3} \sin(\psi) \cos(\psi + \theta) \right. \\
&\quad - \frac{\mu_{VTST}}{M^2} G^4 \left[ (F - 2) + \sin(\psi + \theta) \cos(\psi + \theta) \frac{\Gamma}{2M^2} \frac{\partial M^2}{\partial \theta} \right] \\
&\quad + \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ -(F - 2) - \frac{\cos(\psi) \sin(\psi + \theta)}{\sin(\theta)} + \frac{\partial \psi}{\partial \theta} \right] \\
&\quad + \lambda^2_{VTST} \frac{G^2(1 - G^2)^2}{(1 - M^2)^2} \\
&\quad \times \left[ -(F - 1) - \frac{2G^2}{1 - G^2} \frac{\sin(\psi) \sin(\psi + \theta)}{\sin(\theta)} + \frac{\sin(\psi + \theta) \cos(\psi + \theta) \frac{\partial M^2}{\partial \theta}}{1 - M^2} \right] \\
&\quad \left. - \frac{\kappa^2_{VTST}}{G} \frac{G^4}{M^2} \cos^2(\psi + \theta) \right\} \frac{B_0^2 \theta^{F-2}}{4\pi \nu G^4 \cos(\psi + \theta)} \frac{\sin(\theta)}{\sin(\psi + \theta)} = 0.
\end{aligned}
\]

(2.44)

And the individual components used for the determinant method:

\[
A_2 = \left[ \frac{B_0^2 \theta^{F-2}}{4\pi \nu G^4 \cos(\psi + \theta)} \right] \left[ \frac{\lambda^2_{VTST}}{G^4} \right] \frac{G^2(1 - G^2)^2}{(1 - M^2)^3} \sin(\psi + \theta) \cos(\psi + \theta) \\
\quad - \frac{\mu_{VTST} \theta^{2\Gamma}}{M^2 G^4} \sin(\psi + \theta) \cos(\psi + \theta),
\]

(2.45)

\[
B_2 = \left[ \frac{B_0^2 \theta^{F-2}}{4\pi \nu G^4 \cos(\psi + \theta)} \right] \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ \frac{1}{\cos^2(\psi + \theta)} - M^2 \right],
\]

(2.46)

\[
C_2 = \left[ \frac{B_0^2 \theta^{F-2}}{4\pi \nu G^4 \cos(\psi + \theta)} \right] \left\{ \frac{\lambda^2_{VTST}}{M^2(1 - M^2)^2} \frac{G^2(G^2 - M^2)^2}{\sin(\theta)} \sin(\psi) \cos(\psi + \theta) \\
\quad + \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ (F - 2) + \frac{\cos(\psi) \sin(\psi + \theta)}{\sin(\theta)} \right] \\
\quad + \lambda^2_{VTST} \frac{G^2(1 - G^2)^2}{(1 - M^2)^2} \left[ (F - 1) + \frac{2G^2}{1 - G^2} \frac{\cos(\psi) \sin(\psi + \theta)}{\sin(\theta)} \right] \\
\quad \frac{\mu_{VTST}}{M^2 \Gamma} (F - 2)^G + \frac{\kappa^2_{VTST} \sin(\theta)}{G} \frac{G^4}{M^2} \cos^2(\psi + \theta) \right\}.
\]

(2.47)
The wind equation is then given by \((C_1B_2 - C_2B_1) / (A_1B_2 - A_2B_1)\):

\[
\frac{dM^2}{d\theta} = -2\tan(\psi + \theta) \left[ -\frac{\kappa_{\text{VTST}}^2 \sin(\theta)}{G} - \mu_{\text{VTST}}(F - 2)M^{1-2\Gamma} \\
+ \frac{M^4}{G^4}(1 - M^2)\frac{\cos(\psi) \sin(\theta)}{\sin(\psi + \theta)} - \frac{M^4}{G^4}(F - 2)\frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \\
- \lambda_{\text{VTST}}^2 \frac{M^4}{G^2}(F - 2) \left( \frac{1 - G^2}{1 - M^2} \right)^2 + \lambda_{\text{VTST}}^2 M^2 \frac{G^4 - M^2}{1 - M^2} \\
- \lambda_{\text{VTST}}^2 \frac{\cos(\psi)}{\sin(\theta) \sin(\psi + \theta)} \frac{(2M^2 - 1)G^4 - M^4}{G^2(1 - M^2)} \\
\times \left\{ \frac{\Gamma}{2} \frac{\mu_{\text{VTST}}}{M^{2\Gamma}} (1 - M^2) - \lambda_{\text{VTST}}^2 \frac{M^2}{G^2} \left( \frac{1 - G^2}{1 - M^2} \right)^2 \\
+ \frac{M^4 \sin^2(\theta)}{G^4} - \frac{M^2 \sin^2(\theta)}{G^4 \cos^2(\psi + \theta)} \right\}^{-1},
\]  

(2.48)

which is, indeed, the same as given in VTST.

### 2.2.2 Relativistic wind model without gravity

The right-hand side of the wind equation consists of a numerator and a denominator. Only the denominator is given in VK, so we will first derive the denominator using the determinant method and compare it with the one given, showing this method works. After a successful comparison, we will derive the numerator to obtain the full wind equation. As above, we will label the terms from the energy equation with the subscript 1 and those from the transfield equation with subscript 2.
2 Background and Methodology

Terms from the energy equation

The terms from the derivative of the energy equation with respect to $\theta$, with the common factor $\left[-2F^2\sigma^2_M G^6(1 - M^2 - x^2)^2 \sin^2(\theta) \tan(\psi + \theta)\right]$, are:

\[
A_1 = \frac{\mu^2}{F^2\sigma^2_M} \frac{M^2}{G^2} \frac{(1 - G^2)^2}{(1 - M^2 - x^2)^3} \sin^3(\theta) \sin(\psi + \theta) \frac{M^2 \cos(\psi + \theta)}{G^4 \sin(\psi + \theta)} + \frac{\mu^2}{F^2\sigma^2_M} \frac{M^2}{G^2} \frac{(1 - G^2)^2}{(1 - M^2 - x^2)^3} \sin^3(\theta) \sin(\psi + \theta) \frac{M^2 \cos(\psi + \theta)}{G^4 \sin(\psi + \theta)}
\]

\[
B_1 = \frac{M^4}{G^4},
\]

\[
C_1 = \frac{\xi^2 x_A^4}{F^2\sigma^2_M} \cos(\psi) \cos^2(\psi + \theta) \frac{\mu^2}{G^4} \frac{M^2}{G^4} \frac{(1 - M^2 - x^2)^2}{(1 - M^2 - x^2)^2} - 1
\]

\[
+ \frac{2x^2}{1 - M^2 - x^2} \frac{\xi^2}{G^4(1 - M^2 - x^2)^2} \cos^2(\psi) \cos^2(\psi + \theta) \frac{M^2}{G^4(1 - M^2 - x^2)^2}
\]

\[
- \frac{\mu^2}{\xi^2} \frac{G^2(2 - M^2 - x^2)(1 - M^2 - x^2)(1 - x_A^2)}{G^4(1 - M^2 - x^2)^2}
\]

Terms from the transfield equation

The terms from the transfield equation are:

\[
A_2 = \left[ B_0^2 \alpha F^{-2} \right] \frac{\sin(\theta)}{4\pi \sigma G^4} \cos(\psi + \theta) \frac{\xi^2 x_A^4}{F^2\sigma^2_M} \sin(\psi + \theta) \cos(\psi + \theta)
\]

\[
\times \left[ \frac{\mu^2}{\xi^2} \frac{x_A^2(1 - G^2)^2}{G^4(1 - M^2 - x^2)^3} - \frac{(\Gamma - 1)(\xi - 1)}{G^4} \frac{M^2}{G^4} \right]
\]

\[
B_2 = \left[ B_0^2 \alpha F^{-2} \right] \frac{\sin(\theta)}{4\pi \sigma G^4} \cos(\psi + \theta) \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \frac{(1 - x^2)^2}{M^2 - x^2}
\]

\[
C_2 = \left[ B_0^2 \alpha F^{-2} \right] \frac{\sin(\theta)}{4\pi \sigma G^4} \cos(\psi + \theta) \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \frac{F - 2F x^2 + x^2 + (1 + x^2) \cos(\psi) \sin(\psi + \theta)}{\sin(\theta)}
\]

\[
+ \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ F - 2F x^2 + x^2 + (1 + x^2) \cos(\psi) \sin(\psi + \theta)\right]
\]

\[
+ \frac{x_A^2 M^2}{F^2\sigma^2_M} \left( \frac{1 - G^2}{1 - M^2 - x^2} \right)^2 \left[ F - 1 \right] + \frac{2G^2}{1 - G^2} \frac{1 - M^2 - x^2}{1 - M^2 - x^2} \frac{\sin(\theta)}{\sin(\theta)}
\]

\[
+ 2 \frac{M^2}{\Gamma} \frac{F - 2 \xi(\xi - 1)x^4}{M^2}
\]
2.2 Extending the model

The denominator of the wind equation

The denominator from the determinant method is:

\[
D_{\text{DM}} = \left[ 2F^2 \sigma_M^2 G^6 (1 - M^2 - x^2)^2 \sin^2(\theta) \right] \times \left[ \frac{x_A^4}{F^2 \sigma_M^2} - \frac{M^2 \sin^2(\theta)}{G^4 \cos^2(\psi + \theta)} \right] \left[ \frac{1}{M^2} - \frac{1}{(1 - G^2)^2} \right] + \frac{M^4 \sin^2(\theta)}{G^4} - \frac{M^2 \sin^2(\theta)}{G^4 \cos^2(\psi + \theta)} \left( 1 - x^2 \right). \tag{2.55}
\]

We want to compare the denominator we obtain from applying the determinant method (with subscript DM) to the denominator given by equation (26) in VK (with subscript VK):

\[
D_{\text{VK}} = \left[ \frac{x_A^4}{F^2 \sigma_M^2} - \frac{M^2 \sin^2(\theta)}{G^4 \cos^2(\psi + \theta)} \right] \left[ \frac{1}{M^2} - \frac{1}{(1 - G^2)^2} \right] + \frac{M^4 \sin^2(\theta)}{G^4} - \frac{M^2 \sin^2(\theta)}{G^4 \cos^2(\psi + \theta)} \left( 1 - x^2 \right). \tag{2.56}
\]

Apart from the overall scaling, which means the numerators would also have a different scaling, the two denominators are the same. By comparing with the denominator of the VTST wind equation in (2.48), we can see that we have extracted the right scaling.

The numerator of the wind equation

Based on the successful comparison of the denominators, we will now try to match the numerators as well as possible, by writing both the VTST and VK wind equations in the form that makes them most similar in structure. In order to find the correct normalisation (as terms can be arbitrarily put in the numerator or denominator), we look at the extracted factors to match the denominators. In formulas:

\[
\frac{\mathcal{N}_{\text{DM}}}{\mathcal{D}_{\text{DM}}} = \frac{A_1 B_2 - C_2 B_1}{A_1 B_2 - B_1 A_2} = \frac{A \cdot \text{CN}}{B \cdot \text{CD}} = \frac{C \cdot \text{CN}}{D \cdot \text{CD}} = \frac{\mathcal{N}_{\text{VTST}}}{\mathcal{D}_{\text{VTST}}} : \text{VTST}, \tag{2.57}
\]

where DM stands for determinant method and CN and CD stand for common numerator and common denominator respectively. A is what we want to determine, B is
given by equation (2.55), and C and D by equation (2.48):

\[
B = 2F^2 \sigma_M^2 G^6 (1 - M^2 - x^2)^2 \sin^2(\theta),
\]

\[
C = -\frac{2 \sin(\psi + \theta)}{\cos(\psi + \theta)} = -2 \tan(\psi + \theta),
\]

\[
D = 2,
\]

\[
A = \frac{CB}{D} = -2F^2 \sigma_M^2 G^6 (1 - M^2 - x^2)^2 \sin^2(\theta) \tan(\psi + \theta).
\]

So, equation (2.61) is the factor that should be extracted from the numerator of the VK wind equation for it to be compared to the VTST numerator. Writing out the numerator of the VK wind equation \((C_1B_2 - C_2B_1)\):

\[
N_{DM} = -2 \frac{\Gamma - 1}{\Gamma} (F - 2) \frac{x_A^4}{F^2 \sigma_M^2} \xi (\xi - 1) M^2 + \frac{M^4}{G^4} (1 - M^2 - x^2) \frac{\cos(\psi) \sin(\theta)}{\sin(\psi + \theta)}
\]

\[
- \frac{M^4}{G^4} (F - 2 - Fx^2 + x^2) \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} - 2x^2 \frac{M^4 \cos(\psi) \sin(\theta) \sin(\psi + \theta)}{\cos^2(\psi + \theta)}
\]

\[
+ \frac{\mu^2 \chi_A^2}{F^2 \sigma_M^2 G^2} \left[ \frac{G^2 - M^2 - x^2}{1 - M^2 - x^2} \right]^2 - (F - 1) M^2 \left[ \begin{array}{c} 1 - G^2 \\ G^2(1 - M^2 - x^2) \end{array} \right]^2
\]

\[
+ \frac{\mu^2 \chi_A^2}{F^2 \sigma_M^2 G^2} \cos(\psi) \left[ \frac{(G^2 - M^2 - x^2)^2 + 2G^2 M^2 (1 - G^2)}{G^2(1 - M^2 - x^2)} \right].
\]

Now we identify all parts separately with the VTST parts, taking the non-relativistic \((V \ll c)\), and the non-force-free MHD \((x = 0)\) limits:

\[
-2 \frac{\Gamma - 1}{\Gamma} \frac{x_A^4}{F^2 \sigma_M^2} \xi (\xi - 1) (F - 2) M^2 = -\mu_{VTST} (F - 2) M^{4 - 2\xi},
\]

\[
\frac{M^4}{G^4} (1 - M^2 - x^2)^0 \frac{\cos(\psi) \sin(\theta)}{\sin(\psi + \theta)} = \frac{M^4}{G^4} (1 - M^2) \frac{\cos(\psi) \sin(\theta)}{\sin(\psi + \theta)},
\]

\[
- \frac{M^4}{G^4} \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ F - 2 - Fx^2 (F - 1) \right] = -\frac{M^4}{G^4} (F - 2) \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)},
\]

\[
\frac{\mu^2 \chi_A^2}{F^2 \sigma_M^2 G^2} \left[ \frac{(G^2 - M^2 - x^2)^0 - (F - 1) M^2 (1 - G^2)^2}{(1 - M^2 - x^2)^2} \right]
\]

\[
= -\frac{\mu^2 \chi_A^2}{F^2 \sigma_M^2 G^2} M^4 (F - 2) \left[ 1 - G^2 \right]^2 + \frac{\mu^2 \chi_A^2}{F^2 \sigma_M^2 G^2} M^4 - M^2.
\]
\[
\left[ \frac{(G^2 - M^2 - \lambda^2)^0}{G^2(1 - M^2 - \lambda^2)} \right] = -\frac{(2M^2 - 1)G^4 - M^4}{G^2(1 - M^2)}. \tag{2.67}
\]

For comparison we give the full numerator of the VTST wind equation:

\[
\mathcal{N}_{VTST} = -2\frac{\sin(\psi + \theta)}{\cos(\psi + \theta)} \left\{ -\frac{\kappa_{VTST}^2 \sin(\theta)}{G} - \mu_{VTST}(F - 2) M^{4-2\Gamma} + \frac{M^4}{G^4(1 - M^2)} \frac{\cos(\psi) \sin(\theta)}{\sin(\psi + \theta)} \frac{M^4}{G^4(F - 2)} \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} - \frac{\mu^2 x_A^6}{F^2 \sigma_M^2} \frac{M^4}{G^2} \frac{1}{1 - M^2} \right\}, \tag{2.68}
\]

The only term not accounted for is the gravity term, \(-\kappa_{VTST}^2 \sin(\theta) / G\). In order to use this term in the VK framework, we need a prescription for \(\kappa_{VTST}\) in terms of the VK variables. By comparing the equations for the velocity, magnetic field, density and pressure, we can obtain expressions in VK notation for \(V, B, \lambda_{VTST}, \rho, P\) from the parts between the square brackets:

\[
V_{VTST} = -\left[ V, \alpha^{-1/4} \right] \frac{M^2 \sin(\theta)}{G^2 \cos(\psi + \theta)} \hat{b} + V, \alpha^{-1/4} \lambda_{VTST} \frac{G^2 - M^2}{G(1 - M^2)} \hat{\phi}, \tag{2.69a}
\]

\[
V_{VK} = -\left[ \frac{F \sigma_M}{\gamma \xi x_A^2} \right] \frac{M^2 \sin(\psi + \theta)}{G^2 \cos(\psi + \theta)} \hat{b} + \frac{\mu x_A c}{\gamma \xi} \frac{G^2 - M^2 - \lambda^2}{G(1 - M^2 - \lambda^2)} \hat{\phi}, \tag{2.69b}
\]

\[
B_{VTST} = -\left[ B_x \right] \alpha^{\lambda^2} \frac{\sin(\theta)}{G^2 \cos(\psi + \theta)} \hat{b} - B_x \alpha^{\lambda^2} \left[ \lambda_{VTST} \right] \frac{1 - G^2}{G(1 - M^2)} \hat{\phi}, \tag{2.70a}
\]

\[
B_{VK} = -\left[ B_0 \right] \alpha^{\lambda^2} \frac{\sin(\theta)}{G^2 \cos(\psi + \theta)} \hat{b} - B_0 \alpha^{\lambda^2} \left[ \frac{\mu x_A^4}{F \sigma_M} \right] \frac{1 - G^2}{G(1 - M^2 - \lambda^2)} \hat{\phi}, \tag{2.70b}
\]

\[
\rho_{VTST} = \frac{1}{M^2}, \tag{2.71a}
\]

\[
\rho_0 = \frac{B_0^2}{4\pi F^2 \sigma_M^2} \left[ \frac{\lambda^4}{c^2} \right] \frac{1}{M^2}. \tag{2.71b}
\]
2 Background and Methodology

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<td>Pressure at the Alfvén radius along the reference field line</td>
</tr>
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Table 2.2: Corresponding non-relativistic and relativistic terms and their meaning.

\[
P_{VTST} = a^{F - 2 - \Gamma (F - 3/2)} P_* \left( \frac{\rho}{P_*} \right)^\Gamma = \left[ \frac{P_*}{M^{2\Gamma}} \right] a^{F - 2}. \quad (2.72a)
\]

\[
P_{VK} = \left[ \frac{B_0^2 \Gamma - 1}{4 \pi} \frac{x_A^2}{\Gamma} \frac{\xi (\xi - 1)}{F^2 \sigma^2 M_c} \frac{M^2}{M^2} \right] a^{F - 2}. \quad (2.72b)
\]

Table 2.2 lists all VTST terms and their corresponding VK terms.

**Definition of the constants**

With these conversions we can now also list the four constants of the non-relativistic model and express them in their corresponding relativistic form. They are $\kappa_{VTST}$, the gravity, or mass loss parameter, which we can now fill in; $\lambda_{VTST}$, the specific angular momentum of the flow in units of $V_0 \sigma_*$, from equation (2.70b); $\mu_{VTST}$, the gas entropy parameter, by comparing the denominators in equations (2.48) and (2.55); and...
2.2 Extending the model

\( \epsilon_{VTST} \), the Bernoulli constant, from the energy equation:

\[
\begin{align*}
\kappa_{VTST} &= \sqrt{\frac{GM}{\sigma_0 V_z^2}} = \sqrt{\frac{GM}{c^2 \sigma_A F^2 \sigma_M^2} \frac{\mu^2 x_A^4 (1 - M^2 - \chi_A^2)^2}{(1 - M^2 - \chi^2)^2}}, \\
\lambda_{VTST} &= \frac{\mu^2 x_A^6}{F^2 \sigma_M^2}, \\
\mu_{VTST} &= 2 \frac{\Gamma - 1}{\Gamma} \frac{x_A^4}{F^2 \sigma_M^2} \frac{\xi^2 (\xi - 1)}{\xi - (\Gamma - 1) (\xi - 1)} \frac{M^2}{M^2}, \\
\mu_{VTST} &= 2 \frac{\Gamma - 1}{\Gamma} \frac{x_A^4}{F^2 \sigma_M^2} \frac{\xi (\xi - 1)}{\xi - (\Gamma - 1) (\xi - 1)} \frac{M^2}{M^2} \quad (\xi - 1 \ll 1), \\
\nu_{VTST} &= \frac{1}{2} \frac{M^4}{G^4} \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} + \frac{1}{2} \frac{\mu^2 x_A^6}{F^2 \sigma_M^2} \frac{G^2 (G^2 - M^2)}{G^2 (1 - M^2)^2} \\
&\quad + \frac{x_A^4}{F^2 \sigma_M^2} \frac{\xi^2 (\xi - 1)}{\xi - (\Gamma - 1) (\xi - 1)} - \frac{GM}{\sigma_0 V_0^2} \frac{\sin(\theta)}{G} + \frac{\mu^2 x_A^6}{F^2 \sigma_M^2} \frac{1 - G^2}{1 - M^2}.
\end{align*}
\]

2.2.3 The Alfvén Regularity Condition

Since many terms in the numerator and denominator of the wind equation are of the form 0/0 at the Alfvén point, we need to rewrite the wind equation to start off the integration. First we define the parameter \( p_A \) to have the value of \( dM^2/d\theta \) at the Alfvén point. The procedure is to fill in the Alfvén values for all singular terms in the wind equation. The resulting equation, which is called the Alfvén Regularity Condition (ARC), can be solved for \( p_A \), although this generally has to be done numerically. We will derive the ARC in VK. We will introduce a place holder gravity term (\( G \)) in the numerator of the wind equation to show how it gets manipulated.
2 Background and Methodology

Derivation of the relativistic Alfvén Regularity Condition

The full VK wind equation is given by:

\[
\frac{dM^2}{d\theta} = -2 \frac{\Gamma - 1}{\Gamma} (F - 2) \frac{x_A^4}{F^2 \sigma_M^2} \xi (\xi - 1) M^2 + \frac{M^4}{G^4} (1 - M^2 - x^2) \frac{\cos(\psi) \sin(\theta)}{\sin(\psi + \theta)}
\]

\[
- \frac{M^4}{G^4} \left( F - 2 - Fx^2 + x^2 \right) \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} - 2x^2 \frac{M^4 \cos(\psi) \sin(\theta) \sin(\psi + \theta)}{G^4} - \frac{x^2}{M^2} \left( G^2 - M^2 - x^2 \right)^2 \left( F - 1 \right) M^2 \left( \frac{1 - G^2}{1 - M^2 - x^2} \right)^2
\]

\[
+ \frac{\mu^2 x_A^4}{F^2 \sigma_M^2} \frac{\cos(\psi)}{G^2 \sin(\theta) \sin(\psi + \theta)} \left[ G^2 - M^2 - x^2 + 2G^2 M^2 \left( 1 - G^2 \right) \right]
\]

\[
\times \left[ \frac{x_A^4}{F^2 \sigma_M^2} \frac{(\Gamma - 1) \xi^2 (\xi - 1) \left( 1 - M^2 - x^2 \right)}{2 - \Gamma} + \frac{\mu^2 x_A^4 M^2}{F^2 \sigma_M^2} \frac{(1 - G^2)^2}{G^2 (1 - M^2 - x^2)^2}
\]

\[
+ \frac{M^4 \sin^2(\theta)}{G^4} - \frac{M^2 \sin^2(\theta)}{G^4 \cos^2(\psi + \theta)} \left( 1 - x^2 \right) \right]^{1}.
\]

To obtain the equation at the Alfvén point, we make the following substitutions in the wind equation:

\[
G = 1,
\]

\[
M = (1 - x_A^2),
\]

\[
\left( \frac{dM^2}{d\theta} \right)_A = p_A,
\]

\[
\sigma_A = \frac{2x_A^2 \cos(\psi_A)}{p_A \sin(\theta_A) \cos(\theta_A + \psi_A)},
\]

\[
\frac{1 - M^2 - x_A^2}{1 - M^2 - x^2}_A = \frac{1}{\sigma_A + 1},
\]

\[
\frac{1 - G^2}{1 - M^2 - x^2}_A = \frac{\sigma_A}{x_A^2 (\sigma_A + 1)},
\]

\[
\frac{G^2 - M^2 - x^2}{1 - M^2 - x^2}_A = \frac{x_A^2 - \sigma_A (1 - x_A^2)}{x_A^2 (\sigma_A + 1)},
\]
\[ 0 = \mathcal{G}_A + 2 \frac{\Gamma - 1}{\Gamma} \frac{F - 2}{F^2 \sigma_M^2} \xi_A (\xi_A - 1) (1 - x_A^2) x_A^4 A + \frac{\sin^2(\theta_A) (1 - x_A^2)^2}{\cos^2(\psi_A + \theta_A)} \left[ (F - 1)(1 - x_A^2) - 1 \right] A + \frac{\mu^2 x_A^2}{F^2 \sigma_M^2} (F - 1) \sigma_A^2 (\sigma_A + 1)^2 - \frac{\mu^2 x_A^2}{F^2 \sigma_M^2} \frac{1 - x_A^2}{(\sigma_A + 1)^2} \left[ x_A^2 - \sigma_A (1 - x_A^2) \right]^2 A + 2 \frac{\cos(\psi_A) \sin(\theta_A) \sin(\psi_A + \theta_A)}{\cos^2(\psi_A + \theta_A)} x_A^2 (1 - x_A^2) \frac{\sigma_A + 1}{\sigma_A}. \] (2.86)

The above equation is the form we will use to determine \( p_A \) through \( \sigma_A \), so the gravitational addition to the ARC is simply the value of the gravity term at the Alfvén point. If we want to use equation (B6) in VK instead, we would have to add the following term to its right hand side:

\[ - \mathcal{G}_A \frac{F^2 \sigma_M^2}{\mu^2 x_A^2} (\sigma_A + 1)^2. \] (2.87)

2.2.4 The kinetic gravity term

To include a gravity term into the equations, we need a prescription for it. A gravity term that only has the kinetic inertia, but neglects the thermal and electromagnetic inertias, is given in VTST. We will call it the kinetic gravity term, to contrast it with a gravity term that includes all these inertias. We can rewrite the gravity term found in VTST in a form compatible with VK and, making sure the scaling is correct, include it into the VK wind equation. We can also include a Paczyński-Wiita potential and will give the appropriate expressions as well.

Adding kinetic Newtonian gravity to the Alfvén Regularity Condition

By comparing the wind equations of VTST and VK, we can identify the gravity term in VTST. Using the definition of \( \kappa_{\text{VTST}} \) from VTST and the substitutions from table 2.2, the gravity term in VTST can now be rewritten in terms of VK:

\[ \kappa_{\text{VTST}} = \sqrt{\frac{GM}{c^2 \sigma_A V^2}} = \sqrt{\frac{GM}{c^2 \sigma_A} \frac{\mu^2 x_A^2 (1 - M^2 - x_A^2)^2}{F^2 \sigma_M^2 (1 - M^2 - x^2)^2}}. \] (2.88)
Writing out the whole term found in the numerator of the VTST wind equation, we obtain the full kinetic gravity term:

\[ G_{\text{kin}} = -\frac{\kappa_{\text{VTST}}^2 \sin(\theta)}{G} = -\frac{GM}{c^2} \frac{\mu^2 x^4}{F^2 \sigma^2_M (1 - M^2 - x^2)^2} \sin(\theta). \] (2.89)

Filling in the Alfvén values gives the gravity term addition to the ARC:

\[ G_{\text{kin, A}} = -\frac{GM}{c^2} \frac{\mu^2 x^4}{F^2 \sigma^2_M (\sigma + 1)^2} \frac{1}{\sigma_A} \sin(\theta_A). \] (2.90)

**Adding kinetic Paczyński-Wiita gravity to the Alfvén Regularity Condition**

Since:

\[ \frac{\sin(\theta)}{\sigma_A G} = \frac{1}{r}, \] (2.91)

and the Paczyński-Wiita potential is given by the substitution:

\[ \frac{1}{r} \rightarrow \frac{1}{r - r_S} = \left[ r - 2GM/c^2 \right]^{-1}, \] (2.92)

where \( r_S = 2GM/c^2 \) is the Schwarzschild radius, the gravity term including the Paczyński-Wiita potential is given by:

\[ G_{\text{kinPW}} = -\frac{GM}{c^2} \frac{\mu^2 x^4}{F^2 \sigma^2_M (1 - M^2 - x^2)^2} \left[ \frac{\sigma_A G}{\sin(\theta)} - 2GM \right]^{-1} \] (2.93)

\[ = -\frac{\mu^2 x^4}{F^2 \sigma^2_M (1 - M^2 - x^2)^2} \left[ \frac{c^2 \sigma_A G}{GM \sin(\theta)} - 2 \right]^{-1}. \] (2.94)

Filling in the Alfvén values gives the gravity term addition to the ARC:

\[ G_{\text{kinPW, A}} = -\frac{\mu^2 x^4}{F^2 \sigma^2_M (\sigma + 1)^2} \left[ \frac{c^2}{GM \sin(\theta_A)} - 2 \right]^{-1}. \] (2.95)

**2.2.5 The full gravity term**

Another approach is to start with a general relativistic equation for gravity and see how the VK equations would change if we keep gravity, while still making the same approximations (Polko et al. 2013b). Although retaining gravity violates the assumptions used, we can check to what extent this happens afterwards. We will give a list of the equations that are modified by the inclusion of gravity.
2.2 Extending the model

To obtain a full gravity term, including the kinetic, thermal, and electromagnetic inertias, we have to modify both the energy equation and the transfield equation. We modify the energy equation based on an equation for $\mu$ that does depend on gravity. The transfield equation has an extra component accounting for gravity. This component is in the form of equations (A8) in VK, so we need to derive a new transfield equation from these equations in order to get the correct scaling for this new term. We will start with the energy equation.

A new equation for $\mu$

In general relativistic MHD (GRMHD) in the presence of gravity it is not $\mu$ that stays constant, but rather (Meier 2012):

$$(\mu - 1)c^2 + \mu \Phi = \text{constant} \equiv \mu' c^2 - c^2.$$  \hspace{1cm} (2.96)

With

$$\Phi = -\frac{GM}{r}$$  \hspace{1cm} (2.97)

the gravitational potential, it follows that

$$\frac{\Phi}{c^2} = -\frac{r_g}{r} = -\frac{GM \sin(\theta)}{c^2 \sigma_A G},$$  \hspace{1cm} (2.98)

and therefore at every point we should compute

$$\mu' = \mu \left[ 1 - \frac{GM \sin(\theta)}{c^2 \sigma_A G} \right] = \mu_A \left[ 1 - \frac{GM \sin(\theta_A)}{c^2 \sigma_A} \right],$$  \hspace{1cm} (2.99)

or, if we include a Paczyński-Wiita potential:

$$\mu' = \mu \left\{ 1 - \frac{GM}{c^2} \left[ \frac{\sigma_A G}{\sin(\theta)} - \frac{2GM}{c^2} \right]^{-1} \right\} = \mu_A \left\{ 1 - \left[ \frac{\sigma_A}{GM \sin(\theta_A)} - 2 \right]^{-1} \right\}. \hspace{1cm} (2.100)$$

A new addition to $C_1$

Since the parameter $\mu$ is no longer constant, there is an additional term in the derivative of the modified energy equation. This term has no derivatives of $M^2$ or $\psi$ with respect to $\theta$, so it is added to the $C_1$ term:

$$C_1^+ = \left[ -2 \tan(\psi + \theta) G^6 F^2 \sigma_M^2 (1 - M^2 - x^2)^2 \sin^2(\theta) \right] - \left[ \frac{GM \sin(\theta)}{c^2 \sigma_A G} \right] \left[ 1 - \frac{GM \sin(\theta)}{c^2 \sigma_A} \right] \frac{\mu^2 x_A^2 \cos^2(\psi + \theta)}{F^2 \sigma_M^2 \sin^2(\theta)}. \hspace{1cm} (2.101)$$
2 Background and Methodology

The transfield equation

There also appears an additional gravitational force in the transfield equation. Since the transfield equation given in VK is not from the actual force equation, we need to derive a new transfield equation with the proper scaling. The centrifugal force term is given by

\[ f_{\perp} = \frac{B_0^2 \alpha F^{-2}}{4\pi \varpi G^4} \left[ \frac{x^4 \mu^2 x^2}{F^2 \sigma_M^2 M^2} \left( \frac{G^2 - M^2 - x^2}{F^2 - x^2} \right)^2 \sin(\psi) \right], \quad (2.102) \]

the inertial force term by

\[ f_I = \frac{\alpha F^{-2}}{4\pi \varpi G^4 \cos(\psi + \theta)} \left[ \frac{2M^2 \sin^2(\theta) - M^2 \sin(\theta) \cos(\theta)}{\cos(\psi + \theta)} \right. \\
\left. + M^2 \sin(\psi) \sin(\theta) - M^2 \sin^2(\theta) \cos^2(\psi + \theta) \frac{\partial}{\partial \theta} \tan(\psi + \theta) \right], \quad (2.103) \]

the pressure force term by

\[ f_p = \frac{B_0^2 \alpha F^{-2}}{4\pi \varpi G^4 \cos(\psi + \theta)} \frac{x^4}{F^2 \sigma_M^2 M^2} \left[ -2 \frac{\Gamma - 1}{\Gamma} (F - 2) \frac{\xi (\xi - 1)}{M^2} \right. \\
\left. - \frac{\Gamma - 1}{\xi - (\Gamma - 1) (\xi - 1)} \frac{\xi^2 (\xi - 1)}{M^4} \sin(\psi + \theta) \cos(\psi + \theta) \frac{dM^2}{d\theta} \right], \quad (2.104) \]

the electric force term by

\[ f_E = \frac{B_0^2 \alpha F^{-2}}{4\pi \varpi G^4 \cos(\psi + \theta)} \frac{x^4 \mu^2 x^2}{F^2 \sigma_M^2 M^2} \left\{ -x^2 \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} - x^2 \frac{\cos(\psi) \sin(\theta) \sin(\psi + \theta)}{\cos^2(\psi + \theta)} \right\}, \quad (2.105) \]

and the magnetic force term by

\[ f_B = \frac{B_0^2 \alpha F^{-2}}{4\pi \varpi G^2 \cos(\psi + \theta)} \frac{x^4 \mu^2 x^2}{F^2 \sigma_M^2 M^2} \left\{ \frac{1 - G^2}{1 - M^2 - x^2} \right\}^2 \left[ -(F - 1) \right. \\
\left. + \left[ \frac{1 - M^2 - x^2}{1 - G^2} \frac{dG^2}{d\theta} + \frac{dM^2}{d\theta} \right] \sin(\psi + \theta) \cos(\psi + \theta) \right\} \\
\left. + \left[ - \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} - e^{-1} \frac{\sin(\psi) \sin(\theta)}{\cos(\psi + \theta)} + \sin^2(\theta) \frac{\partial}{\partial \theta} \tan(\psi + \theta) \right) \right\}, \quad (2.106) \]
2.2 Extending the model

with the full transfield equation the sum of these parts:

\[
f_{C_\perp} + f_{I_\perp} + f_{P_\perp} + f_{E_\perp} + f_{B_\perp} = 0. \tag{2.107}
\]

Casting the transfield equation above in the following form:

\[
A_2 \frac{dM^2}{d\theta} + B_2 \frac{d\psi}{d\theta} = C_2, \tag{2.108}
\]
yields the following determinant parts:

\[
A_2 = \frac{B_0^2 \alpha^{F-2}}{4\pi \Sigma G^4} \begin{bmatrix}
\xi^2 \lambda^4 & \sin(\psi + \theta) \cos(\psi + \theta) \\
\frac{\mu^2}{\xi^2} \left(1 - M^2 - x^2\right)^2 & \left(\Gamma - 1\right) \left(\xi - 1\right) G^4
\end{bmatrix}, \tag{2.109}
\]

\[
B_2 = \frac{B_0^2 \alpha^{F-2}}{4\pi \Sigma G^4} \begin{bmatrix}
\sin(\theta) \\
\sin^2(\theta) \cos^2(\psi + \theta) - M^2
\end{bmatrix}, \tag{2.110}
\]

\[
C_2 = \frac{B_0^2 \alpha^{F-2}}{4\pi \Sigma G^4} \left\{ \begin{array}{l}
\frac{x^4 \mu^2 x^2}{F^2 \sigma_M^2 M^2} \left( G^2 - M^2 - x^2 \right)^2 \sin^2(\psi + \theta) \\
+ \frac{1}{\Gamma} \frac{1 - G^2}{F^2 \sigma_M^2 M^2} \left( 2G^2 - F - 1 \right) \sin^2(\theta)
\end{array} \right\}. \tag{2.111}
\]

The gravity term in the transfield equation

The gravitational force in the transfield equation is given by (Meier 2012):

\[
f_G = \left( \gamma \rho + \frac{\xi}{c^2} \right) \left( \nabla \Phi \cdot \hat{n} \right) = \frac{B_0^2 \alpha^{F-2}}{4\pi \Sigma G^4} \begin{bmatrix}
\sin(\theta) \\
\cos^2(\psi + \theta)
\end{bmatrix} \times \left\{ \begin{array}{l}
\frac{x^4}{F^2 \sigma_M^2 M^2} \left( \frac{\mu^2}{M^2} \left(1 - M^2 - x^2\right)^2 + \frac{\lambda^4}{2G^4(1 - M^2 - x^2)^2} \right) \\
- \frac{\Gamma - 1}{\Gamma} \frac{\xi (\xi - 1) x^4}{M^2}
\end{array} \right\} \left[ \frac{G M \sin(\theta)}{c^2 \sigma_A G} \cos^2(\psi + \theta) \right]. \tag{2.112}
\]
2 Background and Methodology

The term above has the scaling of the physical transfield equation. To get it in the form of equation (B2e) we divide by \( \frac{R_n^{2F-2}}{4\pi\sigma G^4} \frac{\sin(\theta)}{\cos(\psi+\theta)} \) to obtain:

\[
C_2^+ = \left\{ \frac{x^4}{2F^2\sigma_M^2} \left[ \frac{\mu^2 (1 - M^2 - x^2_\Lambda)^2}{M^2 (1 - M^2 - x^2)} + \frac{\mu^2 x^2 (1 - G^2)^2}{2G^4 (1 - M^2 - x^2)^2} \frac{\Gamma - 1}{\Gamma M^2} \right] + \frac{1}{2} \frac{(1 + x^2) \sin^2(\theta)}{\cos^2(\psi+\theta)} \right\} \left[ \frac{GM \sin(\theta)}{c^2} \sigma_A G^2 \right] \cos^2(\psi + \theta),
\]

which, taking all the plusses and minuses into account, should also be the addition to \( C_2 \).

The full gravity term with a Newtonian potential

Due to the distributivity of summation, it is possible to calculate the gravity term separately from the rest of the numerator. This term is given by \( C_1^+ B_2 - C_2^+ B_1 \), which,

\[
B_1 = \left[ -2 \tan(\psi + \theta) G^6 F^2 \sigma_M^2 (1 - M^2 - x^2)^2 \sin^2(\theta) \right] \frac{M^4}{G^4},
\]

and:

\[
B_2 = \left[ \frac{R_n^{2F-2}}{4\pi\sigma G^4} \frac{\sin(\theta)}{\cos(\psi+\theta)} \right] \left[ (1 - M^2 - x^2) + M^2 \sin^2(\psi + \theta) \right] \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)},
\]

turns into (neglecting the denominator of \( C_1^+ \) and the common factor of \( B_1 \) and \( C_1^+ \) and \( B_2 \) and \( C_2^+ \)):

\[
G_{\text{full}} = - \frac{G M \sin(\theta)}{c^2} \sigma_A G \left\{ \frac{\mu^2 x^4}{F^2\sigma_M^2} \frac{(1 - M^2 - x^2_\Lambda)^2}{(1 - M^2 - x^2)} (1 - x^2) \right.
- \frac{\mu^2 x^4_\Lambda}{F^2\sigma_M^2} \frac{x^2 (G^2 - M^2 - x^2)^2}{G^4(1 - M^2 - x^2)^2} (1 - x^2)
+ \frac{\mu^2 x^4_\Lambda}{F^2\sigma_M^2} \frac{M^2 x^2 (G^2 - M^2 - x^2)^2}{G^4 (1 - M^2 - x^2)^2} \cos^2(\psi + \theta)
+ \frac{1}{2} \frac{\mu^2 x^4_\Lambda}{F^2\sigma_M^2} \frac{M^4 x^2}{G^4 (1 - G^2)^2} \cos^2(\psi + \theta)
- \frac{x^4_\Lambda}{F^2\sigma_M^2} \frac{\Gamma - 1}{\Gamma} \frac{GM \sin(\theta)}{\cos^2(\psi + \theta)} \left[ (1 - M^2 - x^2)^2 \right]
- \frac{1}{2} \frac{M^4}{G^4} (1 + x^2) \sin^2(\theta) \right\}.
\]
If we neglect the thermal, magnetic and electric contributions, we are left with:

\[
G_{\text{kin,rel}} = \frac{GM}{c^2} \frac{\mu^2 x_A^4}{\sigma_A G} \frac{(1 - M^2 - x_A^2)^2}{(1 - M^2 - x^2)^2} \left[ (1 - M^2 - x^2) + M^2 \sin^2(\psi + \theta) \right]
\]

This differs from the kinetic gravity term in equation (2.89) only in the addition of \((1 - x^2)\), which goes to 1 as \(x^2\) is small in the non-relativistic limit.

### Adding full Newtonian gravity to the Alfvén Regularity Condition

Since our wind equation has an additional gravitational term, this term evaluated at the Alfvén point should be added to the ARC as well. We set all parameters to their Alfvén values and substitute in equations (2.79 – 2.86) wherever a 0/0 occurs:

\[
G_{\text{full, A}} = -\frac{GM \sin(\theta_A)}{c^2} \frac{x_A^4}{\sigma_A} \left\{ \frac{\mu_A^2}{(\sigma_A + 1)^2} \left[ 1 - \left( \frac{x_A^2 - (1 - x_A^2) \sigma_A}{x_A^2} \right)^2 \right] \right\}
\]

\[
+ \frac{\mu_A^2}{x_A^2 (\sigma_A + 1)^2} \left[ (1 - x_A^2) + \frac{\mu_A^2 \sigma_A^2 (1 - x_A^2)^2}{2x_A^2 (\sigma_A + 1)^2} \right]
\]

\[
- \frac{\Gamma - 1}{\Gamma} \varepsilon_A (\xi_A - 1) (1 - x_A^2)
\]

\[
+ \frac{1}{2} \frac{F^2 \sigma_M^2 (1 - x_A^2) \sin^2(\theta_A)}{x_A^4 \cos^2(\psi + \theta_A)} \left[ 1 + x_A^2 \right] \cos^2(\theta_A + \psi_A) \right\}. \quad (2.118)
\]
The full gravity term with a Paczyński-Wiita potential

The gravity term with a Paczyński-Wiita potential is given by:

\[
G_{\text{fullPW}} = -\frac{\mu^2 x_A^4}{F^2 C^2_M} \left(1 - M^2 - x_A^2\right)^2 \left(1 - x^2\right)
- \frac{\mu^2 x_A^4}{F^2 C^2_M} \frac{x^2(G^2 - M^2 - x^2)^2}{G^4(1 - M^2 - x^2)\cos^2(\psi + \theta)}
+ \frac{\mu^2 x_A^4}{F^2 C^2_M} \frac{M^2 G^4}{1 - M^2 - x^2} \cos^2(\psi + \theta)
+ \frac{\mu^2 x_A^4}{F^2 C^2_M} \frac{M^4 G^4}{1 - M^2 - x^2} \cos^2(\psi + \theta)
- \frac{x_A^4}{F^2 C^2_M} \frac{\Gamma - 1}{\Gamma} \xi(\xi - 1) M^2 \cos^2(\psi + \theta)
+ \frac{1}{2} \frac{M^4}{G^4} (1 + x^2) \sin^2(\theta) \left[\frac{c^2}{G M \sin(\theta)} - 2\right]^{-1}. \tag{2.119}
\]

Adding full Paczyński-Wiita gravity to the Alfvén Regularity Condition

The addition to the ARC of a gravity term with a Paczyński-Wiita potential is then given by:

\[
G_{\text{fullPW,A}} = -\left[\frac{c^2}{G M \sin(\theta_A)} - 2\right]^{-1} \frac{x_A^4}{F^2 C^2_M} \left[\frac{\mu_A^2}{(\sigma_A + 1)^2} \left[1 - \frac{[x_A^2 - (1 - x_A^2)\sigma_A]^2}{x_A^2}\right] \right]
+ \mu_A^2 \frac{\sigma_A}{x_A^2(\sigma_A + 1)^2} \left(1 - x_A^2\right) + \mu_A^2 \sigma_A^2 \frac{1}{2x_A^2(\sigma_A + 1)^2}
- \frac{\Gamma - 1}{\Gamma} \xi\left(\xi_A - 1\right) \left(1 - x_A^2\right)
+ \frac{1}{2} \frac{F^2 C^2_M}{x_A^2 \cos^2(\theta_A + \theta_A)} \left(1 + x_A^2\right) \cos^2(\theta_A + \psi_A). \tag{2.120}
\]

2.3 Numerical method for finding solutions

In this section we will describe the numerical method we have employed to find solutions to the equations, from the calculations required to start the integration, via the integration steps, to the iteration towards a solution that crosses both the MSP and MFP.
2.3 Numerical method for finding solutions

2.3.1 Initial setup

We start by specifying values for the parameters \( F, \Gamma, \theta_A, \psi_A, x_A, \sigma_M, q \) explained in section 2.1.5, and the mass of the compact object \( M \), and the physical radius of the Alfvén point \( \sigma_A \). Since these parameters are completely degenerate, we combine them by expressing \( \sigma_A \) in gravitational radii.

Using the equation for \( M \) at the Alfvén point \( (M_A^2 = 1 - x_A^2) \), we can calculate the specific enthalpy at the Alfvén point:

\[
M^2 = q \frac{\xi}{(\xi - 1)^{1/(\Gamma - 1)}}, \tag{2.121}
\]

using a cubic root solver. This approach works when \( \Gamma \) has values \( 4/3 \) or \( 5/3 \), but not in the general case. Next we apply the appropriate expression of the Alfvén regularity condition to calculate \( p_A \) using the Newton-Raphson technique. With this value we can calculate:

\[
\sigma_A = \frac{2x_A^2 \cos(\psi_A)}{p_A \sin(\theta_A) \cos(\theta_A + \psi_A)}, \tag{2.122}
\]

and consequently:

\[
\mu^2 = \frac{(\sigma_A + 1)^2}{x_A^2 - \sigma_A (1 - x_A^2)^2} \left[ x_A^2 \xi_A^2 + \frac{F^2 \sigma_M^2 (1 - x_A^2)^2 \sin^2(\theta_A)}{x_A^2 \cos^2(\theta_A + \psi_A)} \right], \tag{2.123}
\]

to finally obtain \( \mu' \) using either equation (2.99) or (2.100). Now we have all the required values to start off the integration.

2.3.2 Integration step

We use the Runge–Kutta method with Cash–Karp coefficients, modified to continue until the integration step becomes zero, to solve simultaneously the differential equation for the cylindrical radius of the field line:

\[
\frac{dG^2}{d\theta} = \frac{2G^2 \cos(\psi)}{\sin(\theta) \cos(\psi + \theta)}, \tag{2.124}
\]

and the wind equation with the appropriate gravity term, using \( \theta \) as our independent variable. From the new values of \( \theta, G, \) and \( M \) we can calculate:

\[
x = x_A G, \tag{2.125}
\]
\[ \mu \text{ from equation (2.99) or (2.100), } \xi \text{ from equation (2.121), and } \psi \text{ from the energy equation:} \]

\[
\frac{\mu^2 G^4 \left(1 - M^2 - x^2_\Lambda\right)^2 - x^2 \left(G^2 - M^2 - x^2\right)^2}{\xi^2 G^4 \left(1 - M^2 - x^2\right)^2} = 1 + \frac{F^2 \sigma^2_M M^4 \sin^2(\theta)}{\xi^2 x^0 \cos^2(\psi + \theta)}. \tag{2.126}
\]

With these values we can initiate the next integration step until the integration fails due to zero step size.

### 2.3.3 Iteration towards a solution

Once we have a full integration of a field line, there are in general four possible results. At both the MSP and MFP either the denominator of the wind equation crossed (or tends to \(2\)) zero before the numerator, or the numerator crossed zero before the denominator. Based on these results we change the fitting parameters \(x^2_\Lambda\) (for the MFP) and \(q\) (for the MSP) until the other crossed first, signifying a double crossing, or a smooth transition of the corresponding singular point in between those two values of the fitting parameter (see figure 2.3). We have found it convenient to change \(x^2_\Lambda\) until an MFP is found, then change \(q\) until an MSP is found, and then repeat. After two such steps it is possible to interpolate from these four points in order to get a better estimate of the parameter values giving a solution. It is also possible to use a linear combination of \(x^2_\Lambda\) and \(q\) to follow the MFP and MSP line and converge to a solution faster, though in cases where these lines are almost parallel, this approach may be less stable. The diagnostic plot shown in figure 2.3 can be used to quickly sample a large range in \(x^2_\Lambda\) and \(q\), and visually estimate the correct parameter values.

---

\(^2\)If the value of the denominator becomes small while the numerator is still large, the integration reaches zero step size due to the resulting large acceleration, so in general the denominator does not cross zero.
2.3 Numerical method for finding solutions

Figure 2.3: Diagnostic plot to help locate solutions. The four colours correspond to the four possible results of an integration. Yellow: the numerator crosses zero first at both the MFP and MSP. Red: the denominator crosses first at the MFP and the numerator at the MSP. Green: the denominator crosses first at the MFP and the numerator at the MSP. Blue: the denominator crosses first at the MFP and MSP. The line between the yellow/green and red/blue area denotes parameter values which smoothly cross the MFP; the line between the yellow/red and green/blue area denotes parameter values which smoothly cross the MSP. Where these two lines cross lies a solution with both an MFP and MSP. The big block size is due to finite sampling of the $x_N$-$q$-plane.