



## UvA-DARE (Digital Academic Repository)

### Extensions of the Virasoro algebra constructed from Kac-Moody algebras using higher order Casimir invariants

Schoutens, K.; Bais, F.A.; Bouwknecht, P.; Surridge, M.

**DOI**

[10.1016/0550-3213\(88\)90631-1](https://doi.org/10.1016/0550-3213(88)90631-1)

**Publication date**

1988

**Published in**

Nuclear Physics B

[Link to publication](#)

**Citation for published version (APA):**

Schoutens, K., Bais, F. A., Bouwknecht, P., & Surridge, M. (1988). Extensions of the Virasoro algebra constructed from Kac-Moody algebras using higher order Casimir invariants. *Nuclear Physics B*, 304, 348-370. [https://doi.org/10.1016/0550-3213\(88\)90631-1](https://doi.org/10.1016/0550-3213(88)90631-1)

**General rights**

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

**Disclaimer/Complaints regulations**

If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: <https://uba.uva.nl/en/contact>, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.

## COSET CONSTRUCTION FOR EXTENDED VIRASORO ALGEBRAS

F.A. BAIS, P. BOUWKNEGT and M. SURRIDGE

*Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65,  
1018 XE Amsterdam, The Netherlands*

K. SCHOUTENS

*Institute for Theoretical Physics, University of Utrecht, Princetonplein 5, P.O. Box 80 006,  
3508 TA Utrecht, The Netherlands*

Received 5 October 1987

We discuss extensions of the Virasoro algebra obtained by generalizing the Sugawara construction to the higher order Casimir invariants of a Lie algebra  $\bar{g}$ . We generalize the GKO coset construction to the dimension-3 operator for  $\bar{g} = A_{N-1}$  and recover results of Fateev and Zamolodchikov if  $N = 3$ . Branching rules and generalizations to all simple, simply-laced  $\bar{g}$  are discussed.

### 1. Introduction

In a previous paper [1], henceforth referred to as I, we discussed extensions of the Virasoro algebra which we constructed from Kac–Moody algebras  $g$  by generalizing the Sugawara construction to higher order Casimir invariants of the underlying finite dimensional Lie algebra  $\bar{g}$ . This was done in an attempt to understand the occurrence of larger symmetries in  $d = 2$  conformal field theories and their implications for the physical spectra.

The starting point in I was a conformal field  $J(z)$  taking values in a Lie algebra  $\bar{g}$ . Its components  $J^a(z)$ , defined with respect to an antihermitian basis  $\{T^a, a = 1, 2, \dots, \dim(\bar{g}), \text{Tr}(T^a T^b) = -\delta^{ab}\}$ , satisfy the operator product expansion (OPE)

$$J^a(z)J^b(w) = \frac{-k \delta^{ab}}{(z-w)^2} + f^{abc} \frac{J^c(w)}{(z-w)} + \dots \quad (1.1)$$

The Fourier modes  $J_n^a$  satisfy the commutation relations of an untwisted affine

Kac–Moody algebra  $g$ . We proposed to consider the operators

$$T^{(\lambda_i)}(z) = \mathcal{N}^{(\lambda_i)}(g, k) d^{abc\dots} \left( J^a \left( J^b \left( J^c (\dots)(z) \right) \right) \right), \tag{1.2}$$

where  $\mathcal{N}^{(\lambda_i)}(g, k)$  is some normalization constant and  $d^{abc\dots}$  is the completely symmetric invariant tensor of order  $\lambda_i$  ( $i = 1, 2, \dots, l = \text{rank}(\bar{g})$ ) of  $\bar{g}$ , so that  $T^{(\lambda_i)} = d^{abc\dots} T^a T^b T^c \dots$  is the  $\lambda_i$ th order Casimir of the underlying Lie algebra  $\bar{g}$ .

In particular, the operator  $T(z) = T^{(2)}(z)$  is the usual Sugawara stress–energy tensor satisfying the OPE

$$T(z)T(w) = \frac{\frac{1}{2}c(g, k)}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{(z - w)} + \dots \tag{1.3}$$

with central charge

$$c(g, k) = \frac{k \dim(\bar{g})}{k + g}, \tag{1.4}$$

when the normalization is chosen as

$$\mathcal{N}^{(2)}(g, k) = \frac{-1}{2(k + g)}. \tag{1.5}$$

In these formulas  $g$  is the dual Coxeter number of  $g$  [2]. It is not very hard to show that the other fields  $T^{(\lambda_i)}$ ,  $i = 2, 3, \dots, l$ , are primary fields w.r.t.  $T(z)$  if the  $d$ -symbols are chosen to be traceless (the  $d$ -symbols can be chosen such that they are mutually orthogonal).

The currents  $T^{(\lambda_i)}(z)$  generate a set of local currents through OPE’s: two currents  $A(z)$  and  $B(z)$  generate  $(AB)_r(z)$ ,  $r \in \mathbf{Z}_{\geq 0}$ , where

$$A(z)B(w) = \sum_{r \geq 0} \frac{\{AB\}_r}{(z - w)^r} + O(z - w). \tag{1.6}$$

Let us denote by  $S$  the set of currents generated this way. The following is obvious

1.  $T^{(\lambda_i)}(z) \in S$ ,
2.  $1 \in S$ ,
3.  $A(z) \in S \Rightarrow \partial A(z) \in S$ ,
4.  $A(z), B(z) \in S \Rightarrow (AB)(z) \in S$ .

A case of special interest is when  $S$  is the *minimal* set of currents obeying these requirements, i.e. when all the fields generated in the singular part of OPE’s can be written as normal ordered products of the  $T^{(\lambda_i)}(z)$ ,  $i = 1, 2, \dots, l$ , and their derivatives.

The set  $S$  together with the products  $\{ \}_r: (A, B) \rightarrow \{AB\}_r, r \in \mathbb{Z}_{\geq 0}$ , can be viewed as an abstract operator algebra, which we will denote as  $\text{Vir}[g, k]$ . The operator algebra  $\text{Vir}[g, k]$  is equivalent to the commutator algebra of the Fourier modes  $A_n = \oint (dz/2\pi i) A(z) z^{n+\Delta_A-1}$ . However, due to the presence of composite expressions this commutator algebra is not a Lie algebra in general (an exception is the case  $\text{Vir}[A_1^{(1)}, k]$ ). It is therefore not very natural to pass to a formulation in terms of components  $A_n$  in this situation; we will instead do the analysis directly on the level of currents  $A(z)$  and products  $\{ \}_r, r \in \mathbb{Z}_{\geq 0}$ . In this context we define a representation of  $\text{Vir}[g, k]$  to be a linear map  $\rho$  from  $\text{Vir}[g, k]$  to a set  $S'$  of currents acting on some Hilbert space  $V$  such that

- (i)  $\rho(1) = 1_V,$
- (ii)  $\rho(\partial A)(z) = \partial(\rho(A))(z), \quad A \in \text{Vir}[g, k],$
- (iii)  $\rho(\{AB\}_r)(z) = \{\rho(A)\rho(B)\}_r, \quad r \geq 0, A, B \in \text{Vir}[g, k].$

In I we explicitly investigated the operator product algebra of the Sugawara stress-energy tensor

$$T(z) = \frac{-1}{2(k+g)} \delta^{ab}(J^a J^b)(z), \tag{1.7}$$

and the third-order Casimir operator

$$T^{(3)}(z) = \mathcal{N}^{(3)}(A_{N-1}^{(1)}, k) d^{abc}(J^a(J^b J^c))(z),$$

$$\mathcal{N}^{(3)}(A_{N-1}^{(1)}, k) = \frac{i}{3(N+k)} \sqrt{\frac{N}{2(N+2k)(N^2-4)}}, \tag{1.8}$$

for  $\bar{g} = A_{N-1}$  (for conventions and method of calculation we refer to I). Apart from the result that  $T(z)$  satisfies the Virasoro OPE and that  $T^{(3)}(z)$  is a primary field of dimension 3 we found that the OPE of  $T^{(3)}(z)$  and  $T^{(3)}(w)$  is given by

$$T^{(3)}(z)T^{(3)}(w) = \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3}$$

$$+ \frac{1}{(z-w)^2} [2b^2\Lambda(w) + \frac{3}{10} \partial^2 T(w) + R^{(4)}(w)]$$

$$+ \frac{1}{(z-w)} [b^2 \partial \Lambda(w) + \frac{1}{15} \partial^3 T(w) + \frac{1}{2} \partial R^{(4)}(w)]$$

$$+ \dots, \tag{1.9}$$

where

$$\Lambda(w) = (TT)(w) - \frac{3}{10} \partial^2 T(w),$$

$$c = c(A_{N-1}^{(1)}, k) = \frac{(N^2 - 1)k}{N + k}, \quad b^2 = \frac{16}{22 + 5c}. \quad (1.10)$$

In the expansion (1.9), apart from the identity operator and its descendants, an extra primary field  $R^{(4)}(z)$  and a corresponding descendant field  $\partial R^{(4)}(z)$  are present in the singular terms. For general  $N$  the field  $R^{(4)}(z)$  contains a term proportional to the 4th order Casimir of  $A_{N-1}$ . For  $N = 3$ , however, no independent 4th order Casimir exists and one therefore expects some simplifications. In I we have shown that in the vertex operator realization of the level 1 representation of  $A_2^{(1)}$  the field  $R^{(4)}(z)$  vanishes identically. This shows that for  $k = 1$  the field  $R^{(4)}(z)$  plus the fields generated in OPE's with  $R^{(4)}(z)$  form an ideal in the complete operator algebra and can consistently be put equal to zero. Thus the operator algebra  $\text{Vir}[A_2^{(1)}, 1]$  is minimal in the sense described above; we read off from (1.9) that it is actually equivalent to the algebra given by Zamolodchikov in [3] with  $c = 2$ . It was shown in I that in the vertex operator realization for  $A_2^{(1)}$ ,  $k = 1$ , the expression for  $T^{(3)}(z)$  reduces to the free field realization of this operator as given by Fateev and Zamolodchikov [4].

Our main goal in this paper is to present an extension of the Goddard, Kent, Olive (GKO) coset construction of a so-called coset Virasoro algebra to the dimension-3 field  $T^{(3)}(z)$ . This will allow us to make contact with the results for  $c < 2$  in [4].

In fact we expect that a similar extended coset construction can be given more generally whenever we have a Kac–Moody subalgebra  $g' \subset g$ . The operators resulting from this construction constitute a representation of an extended Virasoro algebra which we will denote by  $\text{Vir}[g, g', k]$ . In this paper we focus on  $\text{Vir}[g \oplus g, g, (k, 1)]$  where  $g \subset g \oplus g$  is the diagonal embedding.

This paper is organized as follows. In sect. 2 we explicitly give the construction of the coset dimension-3 operator  $\tilde{T}^{(3)}(z)$  of  $\text{Vir}[A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)}, (k_1, k_2)]$ . For  $N = 3$ ,  $k_2 = 1$ , the operator product algebra is shown to reduce to Zamolodchikov's spin-3 algebra. In section 3 we present some results on the representation theory of  $\text{Vir}[g \oplus g, g, (k, 1)]$  for simple, simply-laced  $\bar{g}$ . We do not have a complete proof of these results but we give a number of non-trivial consistency checks which strongly support our formulas. In particular we will point out the relation of  $\text{Vir}[A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)}, (1, 1)]$  with the parafermion algebras [7]. Sect. 4 deals with the branching rules for irreducible HWM's of  $g \oplus g$  into HWM's of  $g_{\text{diag}} \oplus \text{Vir}[g \oplus g, g, (k, 1)]$ . Also here we give no complete proof but we go through a number of convincing consistency checks, one of which is the construction of some

modular invariants. In sect. 5 we end with some conclusions and remarks. An appendix is added to explain some of our notations.

## 2. Coset construction

In this section we discuss how to extend the GKO coset construction [5, 6] to the Casimir operators of higher conformal dimension. We make this explicit for the dimension-3 operator associated to  $A_{N-1}^{(1)}$  ( $N \geq 3$ ) For  $N = 3$  we compare the resulting operator product algebra with Zamolodchikov's spin-3 algebra.

Before we address the construction of the extended coset Virasoro algebras let us first review the ordinary GKO coset construction.

The construction starts from a set of currents (with central charge  $k$ ) corresponding to an untwisted affine Kac–Moody algebra  $g$ . The Sugawara construction (1.2) gives a field  $T(z)$  satisfying a Virasoro algebra with central charge  $c(g, k)$  as in (1.4). Let  $g' \subset g$  be a Kac–Moody subalgebra of  $g$ . Restricting the Sugawara construction to the currents corresponding to  $g'$  we obtain the Virasoro current  $T'(z)$  with central charge  $c(g', k')$ . The value  $k'$  is determined by the embedding  $g' \subset g$  through  $k' = jk$  where  $j$  is the Dynkin index of this embedding [9]. Now it is easily shown that the difference

$$\tilde{T}(z) = T(z) - T'(z) \tag{2.1}$$

also generates a Virasoro algebra  $\widetilde{\text{Vir}}$  with central charge  $c(g, g', k)$  given by

$$c(g, g', k) = c(g, k) - c(g', k'). \tag{2.2}$$

An important property of the coset Virasoro algebra  $\widetilde{\text{Vir}}$  is that in the commutator algebra it commutes with the subalgebra  $g'$ . This allows us to write every HWM  $L(\Lambda)$  of  $g$  as a sum over irreducible HWM's of the direct sum  $g' \oplus \widetilde{\text{Vir}}$ . It can be shown that an irreducible integrable HWM  $L(\Lambda)$  is finitely reducible into irreducible HWM's of  $g' \oplus \widetilde{\text{Vir}}$  if and only if  $c(g, g', k) < 1$  [10–16]. This property is very powerful. For the diagonal embedding  $A_1^{(1)} \subset A_1^{(1)} \oplus A_1^{(1)}$  at level  $(k, 1)$  it has been shown that most (if not all) properties of the Virasoro algebra can be derived from those of  $A_1^{(1)}$  by exploiting this embedding.

For  $c(g, g', k) \geq 1$  we have no finite reducibility in terms of  $g' \oplus \widetilde{\text{Vir}}$  alone. This indicates that for those cases we should look for an extension of the coset Virasoro algebra which includes more generators than  $\tilde{T}(z)$  alone. It is the aim of this section to provide such an extension which will allow us to extend the finite reducibility theorem beyond  $c(g, g', k) < 1$ .

The following is a tentative definition of what we mean by an extended coset Virasoro operator algebra.

For a given untwisted affine Kac–Moody algebra  $g$  let  $\text{Vir}[g, k]$  be the operator algebra generated by the Casimir operators  $T^{(\lambda_i)}(z)$  at level  $k$ . Let  $g'$  be a Kac–Moody subalgebra of  $g$ . The extended coset Virasoro operator algebra  $\text{Vir}[g, g', k]$  should have the following properties

(i) the generators of  $\text{Vir}[g, g', k]$  are local currents constructed by taking normal ordered products of the currents for  $g$  at level  $k$  and their derivatives.

(ii)  $\text{Vir}[g, g', k]$  contains the Virasoro generators  $\tilde{T}(z)$  as in (2.1) and coset analogues  $\tilde{T}^{(\lambda_i)}(z)$ ,  $i = 2, 3, \dots, l$ , of  $T^{(\lambda_i)}(z)$  which transform under  $\tilde{T}(z)$  as primary fields of dimension  $\lambda_i$ .

(iii) Mutual operator products of  $\text{Vir}[g, g', k]$  with the currents of  $g'$  are regular, i.e. in the commutator algebra  $\text{Vir}[g, g', k]$  and  $g'$  commute.

Here we construct explicitly the dimension-3 operator contained in  $\text{Vir}[A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)}, (k_1, k_2)]$  where  $A_{N-1}^{(1)} \subset A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}$  is the diagonal embedding. Denoting the generators of  $A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}$  as  $J_{(1)}^a(z)$  and  $J_{(2)}^a(z)$  and those of the diagonal  $A_{N-1}^{(1)}$  as  $J'^a(z)$  we have the relation

$$J'^a(z) = J_{(1)}^a(z) + J_{(2)}^a(z), \quad k' = k_1 + k_2. \tag{2.3}$$

The Virasoro generator for  $\text{Vir}[A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)}, (k_1, k_2)]$  is given by

$$\tilde{T}(z) = T_{(1)}(z) + T_{(2)}(z) - T'(z), \tag{2.4}$$

with  $T_{(1)}(z)$  and  $T_{(2)}(z)$  as in (1.6). The coset central charge is

$$\begin{aligned} \tilde{c} &= c(A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)}, (k_1, k_2)) \\ &= (N^2 - 1) \left[ \frac{k_1}{k_1 + N} + \frac{k_2}{k_2 + N} - \frac{k_1 + k_2}{k_1 + k_2 + N} \right]. \end{aligned} \tag{2.5}$$

In order to write down the coset analogue  $\tilde{T}^{(3)}(z)$  of (1.7) we inspect the requirements (i), (ii) and (iii). From (i) and (ii) we learn that  $\tilde{T}^{(3)}(z)$  is a normal ordered product cubic in the Kac–Moody currents  $J_{(1)}^a(z)$  and  $J_{(2)}^a(z)$  (it is easily seen that bilinear terms involving a derivative are no good). The requirement (iii) implies that  $\tilde{T}^{(3)}(z)$  is a singlet under the underlying  $A_{N-1}$  subalgebra of  $g'$ . This restricts us further to the terms

$$(J_{(1)}^a Q_{(1)}^a)(z), \quad (J_{(1)}^a Q_{(2)}^a)(z), \quad (J_{(2)}^a Q_{(1)}^a)(z), \quad (J_{(2)}^a Q_{(2)}^a)(z) \tag{2.6}$$

where

$$Q^a(z) = d^{abc}(J^b J^c)(z). \tag{2.7}$$

Notice that the combination  $T_{(1)}^{(3)}(z) + T_{(2)}^{(3)}(z) - T'^{(3)}(z)$ , which is the natural

analogue of  $\tilde{T}(z)$  in (2.4), does not satisfy the requirement (ii). We therefore allow a general linear combination of the terms appearing in (2.6) and proceed by imposing the requirements (ii) and (iii). Extensively using the results of I (in particular (2.11), (2.12) and the techniques of appendix A) we have found that the requirement that  $\tilde{T}^{(3)}(z)$  is a primary field of dimension 3 under  $\tilde{T}(z)$  uniquely fixes  $\tilde{T}^{(3)}(z)$  up to a normalization factor  $B_N(k_1, k_2)$ :

$$\begin{aligned} \tilde{T}^{(3)}(z) = B_N(k_1, k_2) \{ & k_2(N + k_2)(N + 2k_2)(J_{(1)}^a Q_{(1)}^a)(z) \\ & - 3(N + k_1)(N + k_2)(N + 2k_2)(J_{(2)}^a Q_{(1)}^a)(z) \\ & + 3(N + k_1)(N + k_2)(N + 2k_1)(J_{(1)}^a Q_{(2)}^a)(z) \\ & - k_1(N + k_1)(N + 2k_1)(J_{(2)}^a Q_{(2)}^a)(z) \}. \end{aligned} \tag{2.8}$$

It turns out that the remaining requirement (iii), expressed as

$$\tilde{T}^{(3)}(z) J^a(w) = \text{regular}, \tag{2.9}$$

is now already satisfied.

The next step in our construction is to determine the structure of the operator algebra  $\text{Vir}[A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)}(k_1, k_2)]$ , i.e. to calculate the OPE  $\tilde{T}^{(3)}(z)\tilde{T}^{(3)}(w)$ . After much arithmetic one finds

$$\begin{aligned} \tilde{T}^{(3)}(z)\tilde{T}^{(3)}(w) = & \frac{\tilde{c}/3}{(z-w)^6} + \frac{2\tilde{T}(w)}{(z-w)^4} + \frac{\partial\tilde{T}(w)}{(z-w)^3} \\ & + \frac{1}{(z-w)^2} [2b^2\tilde{\Lambda}(w) + \frac{3}{10}\partial^2\tilde{T}(w) + \tilde{R}^{(4)}(w)] \\ & + \frac{1}{(z-w)} [b^2\partial\tilde{\Lambda}(w) + \frac{1}{15}\partial^3\tilde{T}(w) + \frac{1}{2}\partial\tilde{R}^{(4)}(w)] \\ & + \dots, \end{aligned} \tag{2.10}$$

where

$$\tilde{\Lambda}(w) = (\tilde{T}\tilde{T})(w) - \frac{3}{10}\partial^2\tilde{T}(w), \quad b^2 = \frac{16}{22 + 5\tilde{c}} \tag{2.11}$$



provided the normalization is fixed as

$$B_N(k_1, k_2) = \frac{i}{3(N+k_1)(N+k_2)(N+k_1+k_2)} \times \sqrt{\frac{N}{2(N+2k_1)(N+2k_2)(3N+2k_1+2k_2)(N^2-4)}} \quad (2.12)$$

The field  $\tilde{R}^{(4)}(z)$  is a primary field of dimension 4 under  $\tilde{T}(z)$ . Notice that the structure of the OPE (2.10) in  $\text{Vir}[A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)}(k_1, k_2)]$  is similar to that of the dimension-3 operator in  $\text{Vir}[A_{N-1}^{(1)}, k]$ , eq. (1.9).

For general  $N$  the field  $\tilde{R}^{(4)}(z)$  will contain the coset version of the 4th order Casimir of  $A_{N-1}$ . One therefore expects that for  $N=3$  the expression for  $\tilde{R}^{(4)}(z)$  simplifies. For general  $(k_1, k_2)$  this is not the case but we found that the field  $\tilde{R}^{(4)}(z)$  decouples from the theory if one of the levels  $(k_1, k_2)$  is equal to one!

We derived this remarkable result by showing that if we put  $k_2 = 1$  and we explicitly insert the vertex operator realization for the level 1 currents  $J_{(2)}^a(z)$  then the field  $\tilde{R}^{(4)}(z)$  vanishes identically. Let us briefly discuss some of the details of this calculation. First of all, for  $k_2 = 1$  many of the terms in  $\tilde{R}^{(4)}(z)$  vanish simply because they contain a multiplicative factor  $(k_2 - 1)$ . The remaining terms can be written as

$$\begin{aligned} \tilde{R}^{(4)}(z) = & C_1(k_1) \left( R_{(1)}^{(2)ab} R_{(2)}^{(2)ab} \right)(z) + C_2(k_1) \left( J_{(1)}^a R_{(2)}^{(3)a} \right)(z) \\ & + C_3(k_1) \left( J_{(2)}^a R_{(2)}^{(3)a} \right)(z), \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} R^{(3)a}(z) = & 3(J^a T)(z) + f^{abc}(J^b \partial J^c)(z) - 2 \partial^2 J^a(z), \\ R^{(2)ab}(z) = & (3 \delta^{ab} \delta^{cd} + 24 \delta^{ac} \delta^{bd} - 8 f^{acm} f^{bdm})(J^c J^d)(z), \end{aligned} \quad (2.14)$$

and  $C_i(k_1)$  are some constants with the property  $C_i(k_1 = 1) = 0$  whose precise form is irrelevant for the following discussion. The fields  $R^{(2)ab}(z)$  and  $R^{(3)a}(z)$  are primary fields of dimension 2 and 3 respectively (we remark that  $(J^a R^{(3)a})(z)$  is proportional to the field  $R^{(4)}(z)$  occurring in the expansion (1.9)). Furthermore, by inspecting the OPE's of  $J^a(z)$  with  $R^{(2)ab}(w)$  and  $R^{(3)a}(w)$  one finds that  $R^{(2)ab}$  and  $R^{(3)a}$  transform respectively as the 27 and the 8 representations of the  $SU(3)$  algebra generated by the  $J_0^a$ . By inserting the vertex operator realization of the currents  $J_{(2)}^a(z)$  (remember  $k_2 = 1$ ) and using the techniques described in I one easily shows that both  $R^{(2)ab}(z)$  and  $R^{(3)a}(z)$  vanish.

We do not fully understand why the terms occurring in (2.13) turn out to have nice transformation properties under both  $\tilde{T}(z)$  and  $J_0^a$  but we believe that this observation is essential if one tries to generalize the present construction to the higher order Casimirs.

The operator product algebra  $\text{Vir}[A_2^{(1)} \oplus A_2^{(1)}, A_2^{(1)}, (m-3, 1)]$ , ( $m = 4, 5, \dots$ ) has thus been proven to be equal to Zamolodchikov's spin-3 algebra  $\mathscr{W}$  with central charge given by (2.5)

$$c(m) = 2 \left( 1 - \frac{12}{m(m+1)} \right), \quad m = 4, 5, \dots \tag{2.15}$$

In the remainder of this section we summarize some results on the representation theory of this algebra [4], which form a starting point for the generalizations we present in sect. 3–5. These results were obtained by Fateev and Zamolodchikov, who constructed a free field representation in the spirit of Feign and Fuks [17], and Dotsenko and Fateev [18]. They considered highest weight modules  $L(h^{(i)})$  of  $\mathscr{W}$  which contain a highest weight vector  $|h^{(i)}\rangle$  obeying

$$\begin{aligned} L_0^{(i)} |h^{(i)}\rangle &= h^{(i)} |h^{(i)}\rangle, \\ L_n^{(i)} |h^{(i)}\rangle &= 0, \quad n > 0, \quad i = 2, 3 \end{aligned} \tag{2.16}$$

where  $L_n^{(i)} = \oint (dz/2\pi i) z^{n+i-1} T^{(i)}(z)$ . The main result of [4] is that precisely for the  $c$ -values given in (2.15) there exist (completely) degenerate representations  $L \begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix}$  of  $\mathscr{W}$  with

$$h^{(2)} \begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix} = \frac{[(m+1)r - ms] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} [(m+1)r - ms] - 6}{6m(m+1)}, \tag{2.17}$$

$$\begin{aligned} h^{(3)} \begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix} &= \sqrt{\frac{2}{3}} [(m+1)(r_2 - r_1) - m(s_2 - s_1)] \\ &\times \frac{[(m+1)(2r_1 + r_2) - m(2s_1 + s_2)][(m+1)(2r_2 + r_1) - m(2s_2 + s_1)]}{9m(m+1)\sqrt{(2m+5)(2m-3)}}, \end{aligned} \tag{2.18}$$

such that the finite set of primary fields

$$\bigoplus_{\Sigma r_i \leq m-1} \bigoplus_{\Sigma s_i \leq m} \left[ \Phi \begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix} \right] \tag{2.19}$$

generate a closed operator algebra, with OPE coefficients that resemble the  $A_2$  Clebsch–Gordan coefficients. The construction we have given above clearly demonstrates why it is that properties of  $A_2$  come into the game.

In sect. 3 we will discuss how these representation theoretical results can be extended to  $\text{Vir}[g \oplus g, g, (k, 1)]$  for all simple simply-laced Lie algebras  $\bar{g}$ . In particular we will give the analogues of the expressions above.

### 3. Generalizations

It is rather cumbersome to find explicit expressions for the other extended Virasoro algebras  $\text{Vir}[g, k]$  and the coset algebras  $\text{Vir}[g, g', k]$  by the methods we have explored for  $A_2^{(1)}$ .

In I we showed that the expression for  $T^{(3)}(z)$  in the algebra  $\text{Vir}[A_2^{(1)}, k = 1]$  can be reduced to an expression depending only on the Cartan subalgebra generators, thus giving a free field representation which is determined by the  $d$ -symbols in the Cartan subalgebra only. Assuming that a similar reduction can also be done in more general cases we can try to obtain information on the extended Virasoro algebras by generalizing the free field construction as given by Fateev and Zamolodchikov [4].

Though we have not yet completed our analysis, the structure given below is seen to emerge. Independent arguments for this generalization were presented in sect. 5 of I. The main result is an expression for the conformal dimensions of primary fields  $\Phi(p, q)$  which generate highest weight states for irreducible representations of the extended coset Virasoro algebras  $\text{Vir}[g \oplus g, g, (k, 1)]$ . This information is then used in the following section to postulate branching rules for certain products of Kac–Moody representations.

In the following we make frequent use of the notation of Kac and Peterson [2, 19] concerning Kac–Moody algebra theory. The appendix contains a summary of notation and definitions.

Let  $\bar{g}$  be a finite dimensional, simple, simply-laced Lie algebra. Consider  $l = \text{rank}(\bar{g})$  free scalar fields  $\varphi^a(z)$  with two-point function

$$\langle \varphi^a(z) \varphi^b(w) \rangle = -2 \delta^{ab} \log(z - w) \tag{3.1}$$

coupled to a background charge  $\gamma^a$ . The stress–energy tensor  $T(z) = T^{(2)}(z)$  reads

$$T(z) = -\frac{1}{4} \delta^{ab} (\partial \varphi^a \partial \varphi^b)(z) + i \gamma^a \partial^2 \varphi^a(z). \tag{3.2}$$

The central charge for its corresponding Virasoro algebra is given by

$$c = l - 24 |\gamma|^2. \tag{3.3}$$

Let  $T^{(\lambda_i)}(z)$ ,  $i = 2, \dots, l$ , be the expression for a higher order Casimir in the free field realization.

A set of highest weight states in the spectrum is provided by the vertex operators  $V_\beta(z) = \exp(i\beta\varphi)(z)$  which are primary fields of conformal dimension

$$h^{(2)}(\beta) = h^{(2)}(2\gamma - \beta) = (\beta, \beta - 2\gamma). \tag{3.4}$$

It is easy to see that the state  $V_\beta(0)|0\rangle$  is also a highest weight state w.r.t.

$$\left\{ L_n^{(\lambda_i)} = \oint \frac{dz}{2\pi i} z^{n+\lambda_i-1} T^{(\lambda_i)}(z) \right\}$$

of  $L_0^{(\lambda_i)}$  eigenvalue, say,  $h^{(\lambda_i)}(\beta)$ .

In order to construct null-states following Fateev and Zamolodchikov [4] we have to find solutions to

$$h^{(2)}(\beta) = 1. \tag{3.5}$$

In principle we could take any background charge  $\gamma$ , but the most natural generalization of the construction in [4] seems to be to take  $\gamma = \alpha_0 \bar{\rho}$ , where  $\bar{\rho}$  is the defining vector of the principal  $A_1$  embedding in  $\bar{g}$  defined in the appendix. In this case (3.5) allows for  $2l$  solutions

$$\beta = \alpha_\pm \bar{\alpha}_i, \quad i = 1, 2, \dots, l, \tag{3.6}$$

where

$$\begin{aligned} \alpha_\pm &= \frac{1}{2}(\alpha_0 \pm \sqrt{\alpha_0^2 + 2}), \\ \alpha_+ + \alpha_- &= \alpha_0, \quad \alpha_+ \alpha_- = -\frac{1}{2}, \end{aligned} \tag{3.7}$$

and  $\bar{\alpha}_i$ ,  $i = 1, 2, \dots, l$ , are the simple roots of  $\bar{g}$ .

The fact that the vertex operators

$$V_i^{(\pm)}(z) = (e^{i\alpha_\pm \bar{\alpha}_i \varphi})(z) \tag{3.8}$$

have conformal dimension 1 assures that

$$T(z)V_i^{(\pm)}(w) = \partial_w \left( \frac{1}{z-w} V_i^{(\pm)}(w) \right) + \dots \tag{3.9}$$

Certainly (3.6) does not give all solutions to (3.5), but there is reason to expect that they are precisely the solutions such that

$$T^{(\lambda_j)}(z)V_i^{(\pm)}(w) = \partial_w \left[ \frac{h^{(\lambda_j)}(\alpha_\pm \bar{\alpha}_i)}{\lambda_j - 1} \frac{1}{(z-w)^{\lambda_j-1}} V_i^{(\pm)}(w) + \dots \right] + \dots \tag{3.10}$$

This implies that the states

$$\oint \frac{dz_1}{2\pi i} V_i^{(+)}(z_1) \dots \oint \frac{dz_n}{2\pi i} V_i^{(+)}(z_n) V_{2\gamma - \beta - n\alpha_+ \bar{\alpha}_i}(0) |0\rangle \tag{3.11}$$

occurring in the Verma module with highest weight vector  $V_\beta(0)|0\rangle$  will be null states. The condition for these states to be non-vanishing is

$$2\alpha_+^2(n-1) + 2\alpha_+(\bar{\alpha}_i, 2\gamma - \beta - n\alpha_+ \bar{\alpha}_i) = -m - 1, \quad m, n = 1, 2, \dots \tag{3.12}$$

This equation can be reduced to

$$(\bar{\alpha}_i, \beta) = \alpha_+(1-n) + \alpha_-(1-m). \tag{3.13}$$

In terms of the fundamental weights  $\bar{\Lambda}_i, i = 1, 2, \dots, l$ , of  $\bar{g}$  the solution of (3.13) reads

$$\beta(p_i, q_i) = -(p_i \alpha_+ + q_i \alpha_-) \bar{\Lambda}_i, \quad i = 1, 2, \dots, l, \quad p_i, q_i = 0, 1, 2, \dots \tag{3.14}$$

Writing  $\bar{p} = \sum_{i=1}^l p_i \bar{\Lambda}_i$  we can interpret the  $p_i, i = 1, 2, \dots, l$ , as the Dynkin labelling of a finite dimensional representation of  $\bar{g}$ . It is convenient to add one more positive integer  $p_0$  to the set  $\{p_i\}$  and associate to it a dominant integral Kac-Moody weight  $p = \sum_{i=0}^l p_i \Lambda_i \in P_+^k$  of level  $k = \sum_{i=0}^l a_i^v p_i$ .

If one substitutes (3.14) into (3.4) the result is that degenerate representations  $L(p, q)$  of the free field algebra  $\{T^{(\lambda)}(z)\}$  exist for

$$h^{(2)}(p, q) = (\bar{p}\alpha_+ + \bar{q}\alpha_- + \alpha_0 \bar{\rho}, \bar{p}\alpha_+ + \bar{q}\alpha_- + \alpha_0 \bar{\rho}) - \frac{l-c}{24}. \tag{3.15}$$

where

$$c = l - 24\alpha_0^2 |\bar{\rho}|^2 = l - 2g\alpha_0^2 (\dim \bar{g}), \tag{3.16}$$

$$\alpha_\pm = \sqrt{\frac{l-c}{8g(\dim \bar{g})}} \left[ 1 \pm \sqrt{1 + \frac{4g(\dim \bar{g})}{l-c}} \right]. \tag{3.17}$$

In general the operator product algebra of the fields  $\Phi(p, q)$  corresponding to degenerate representations  $L(p, q)$  will involve infinitely many of the fields  $\Phi(p, q)$ . Taking however  $2\alpha_+^2 = (m+1)/m$  (or equivalently  $\alpha_0^2 = 1/2m(m+1)$ ) where  $m = g+1, g+2, \dots$  we find the so-called main sequence of minimal models [20] with

$$c(m) = l \left( 1 - \frac{g(g+1)}{m(m+1)} \right), \quad m = g+1, g+2, \dots \tag{3.18}$$

Since the coset charge  $c(g \oplus g, g, (m - g, 1))$  of the diagonal embedding  $g \subset g \oplus g$  equals  $c(m)$  (use (A.10)) we see that this main sequence is the one relevant for the coset algebra  $\text{Vir}[g \oplus g, g, (m - g, 1)]$ . It should be possible to prove positivity by using this correspondence.

Comparing the above with the results for  $g = A_2^{(1)}$  leads us to the following claim.

Let  $\bar{g}$  be a finite dimensional, simple, simply-laced Lie algebra,  $g = \bar{g}^{(1)}$ . The operator algebra  $\text{Vir}[g, k = 1]$  is minimal in the sense described in sect. 1. The Cartan subalgebra reduction of the vertex operator representation of  $\text{Vir}[g, 1]$ , which has  $c = l$ , results in a free field representation of this algebra; It corresponds to the above construction with background charge 0, i.e.  $m \rightarrow \infty$ . The coset operators  $\tilde{T}^{(\lambda)}(z)$ , defined for the diagonal embedding  $g \subset g \oplus g$  at level  $(m - g, 1)$ ,  $m = g + 1, \dots$ , generate the operator algebra  $\text{Vir}[g, g \oplus g, (m - g, 1)]$  which differs from  $\text{Vir}[g, 1]$  only through the value of the central charge which is now  $c = c(m) < l$  as in (3.18). These  $c$ -values, which correspond to the free field construction in the presence of a background charge with strength  $\alpha_0^2 = 1/2m(m + 1)$ , are precisely such that unitary HWM's  $L(p, q)$  of the algebra exist. The unitary HWM's for  $c = c(m)$  are parametrized by dominant integral  $g$ -weights  $p$  and  $q$  of level  $m - g$  and  $m - g + 1$  respectively. The fields

$$\bigoplus_{p \in P_+^{m-g}} \bigoplus_{q \in P_+^{m-g+1}} [\Phi(p, q)] \tag{3.19}$$

form a closed operator algebra. The conformal dimensions of the fields  $\Phi(p, q)$ , found by combining (3.15), (3.17) and (3.18), read

$$h_{p,q}^{(2)(m)} = \frac{((m + 1)\bar{p} - m\bar{q}, (m + 1)\bar{p} - m\bar{q} + 2\bar{\rho})}{2m(m + 1)}, \tag{3.20}$$

$p \in P_+^{m-g}, q \in P_+^{m-g+1}.$

Precise information on how the HWM's  $L(p, q)$  are contained in level  $(m - g, 1)$  tensor product representations of  $g \oplus g$  is provided by a branching rule which we present in sect. 4.

If we write (3.20) in terms of the labels  $r_i = p_i + 1$  and  $s_i = q_i + 1$ ,  $i = 1, 2, \dots, l$ , and the inverse Cartan matrix  $G$  of  $\bar{g}$  the expression becomes

$$h_{r,s}^{(2)(m)} = \frac{((m + 1)r_i - ms_i)G_{ij}((m + 1)r_i - ms_i) - (g/12) \dim \bar{g}}{2m(m + 1)},$$

$$r_i, s_i = 1, 2, \dots, \quad \sum_{i=1}^l a_i^v r_i \leq m - 1, \quad \sum_{i=1}^l a_i^v s_i \leq m. \tag{3.21}$$

In this form it can more easily be recognized as a generalization of the dimension formula for the Virasoro algebra [21] and of (2.17) for  $g = A_2^{(1)}$ .

We stress that the validity of (3.20) and (3.21) in particular hinges on the assumptions that  $\text{Vir}[(g \oplus g, g, (k, 1)]$  exists and that eq. (3.10) holds. There is little doubt that the proposed formula is correct because it passes the following tests.

(i) For all the  $c$ -values (3.18) in the region  $c < 1$  [15] one can check that eq. (3.21) produces  $h^{(2)}$  eigenvalues which are allowed by unitarity. Especially interesting is the case  $c(E_8^{(1)} \oplus E_8^{(1)}, (2, 1)) = 21/22$  where the  $h^{(2)}$  values given by (3.21) give the subset of all Virasoro  $h$ -values at  $c = 21/22$  contained in the non-standard modular invariant combination  $(A_{10}, E_6)$  [22].

(ii) For  $\text{Vir}[A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)}, (1, 1)]$  the values

$$c(A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)}, (1, 1)) = \frac{2(N-1)}{N+2}, \quad N = 2, 3, \dots \quad (3.22)$$

exactly reproduce the set of  $c$ -values of the  $\mathbb{Z}_N$  parafermion algebras [7]. Moreover, in this case the  $h^{(2)}$  values (3.20) agree precisely with the set given by [7, 8]. By (3.20) we have

$$h_{\Lambda_k, \Lambda_l + \Lambda_m}^{(2)(N+1)} = \frac{1}{2N(N+1)(N+2)} \left\{ (N+1)(\xi(l, l) + \xi(m, m)) \right. \\ \left. + 2(N+1)(N+2)(\xi(k, k) - \xi(k, l) - \xi(k, m)) + 2(N+1)^2 \xi(l, m) \right\}, \\ k, l, m = 0, 1, \dots, N-1, \quad (3.23)$$

where

$$\xi(a, b) = \begin{cases} (N-a)b & a \geq b \\ (N-b)a & a \leq b. \end{cases}$$

So for instance in the region  $k \geq l \geq m$

$$h_{\Lambda_k, \Lambda_l + \Lambda_m}^{(2)(N+1)} = h_{\hat{l}} + \frac{(N - \hat{k})(\hat{k} - \hat{l})}{N}, \quad (3.24)$$

where

$$h_{\hat{l}} = \frac{\hat{l}(N - \hat{l})}{2N(N + 2)}, \quad \hat{l} = l - m, \quad \hat{k} = k - m, \\ \hat{l} = 0, 1, \dots, \hat{k}, \quad \hat{k} = 0, 1, \dots, N - m - 1.$$

Formula (3.24) is precisely the result of ref. [8]. The other regions can be identified similarly.

Let us conclude this section by making a remark about  $\text{Vir}[(g \oplus g, g, (1, k))]$ . From the expression (3.20) for  $h^{(2)}(p, q)$  it is clear that  $h^{(2)}$  is at least invariant under the action of the automorphism group of the weight lattice of  $\bar{g}$  which in particular contains the Weyl group  $W(\bar{g})$  of  $\bar{g}$ . In general, however, the eigenvalues  $h^{(\lambda)}(p, q)$  of the higher order Casimir operators will break this symmetry. Consider for example the case of  $g = A_2^{(1)}$ . By explicitly working out the action of the Weyl group one finds that  $h^{(2)}$  is invariant under the order-6 group  $S_3 \sim W(A_2)$  generated by two elements  $R$  and  $S$  with  $R^2 = S^3 = 1$  which act on the labels  $\begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix}$  as

$$\begin{aligned}
 R: \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} &\rightarrow \begin{bmatrix} p_2 & q_2 \\ p_1 & q_1 \end{bmatrix}, \\
 S: \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} &\rightarrow \begin{bmatrix} m-3-(p_1+p_2) & m-2-(q_1+q_2) \\ p_1 & q_1 \end{bmatrix}. \quad (3.25)
 \end{aligned}$$

It is however easy to check from (2.18) that  $h^{(3)}$  is in general only invariant under  $S$ . This breaks the symmetry to  $\mathbb{Z}_3$  [4]. This breaking also shows up if one computes the table of  $h^{(2)}$  values for all  $p \in P_+^{m-3}$ ,  $q \in P_+^{m-2}$ . One finds that the  $h^{(2)}$  values are either 3- or 6-fold degenerate. Those which are 3-fold degenerate have  $h^{(3)}$  eigenvalue 0 whereas the 6-fold degenerates split into two sets of 3 with opposite  $h^{(3)}$  eigenvalue.

We expect that in the case  $g = A_{N-1}^{(1)}$  one should be able to prove in the same way that the symmetry of the coset model is  $\mathbb{Z}_N$ , which we know to be the case for the first term, i.e.  $\text{Vir}[(A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)}, (1, 1)]$  where we have shown the correspondence with the  $\mathbb{Z}_N$ -parafermion models.

### 4. Branching rules

Let us from now on assume that for  $\bar{g}$  simple and simply-laced the coset algebra  $\mathscr{W} = \text{Vir}[g \oplus g, g, (m-g, 1)]$  can be constructed and that our formula (3.21) for the conformal dimensions of the fields is correct. As pointed out in sect. 2, one expects  $\mathscr{W}$  to provide the additional structure necessary to extend the finite reducibility theorem for the diagonal embedding  $g \subset g \oplus g$  with certain  $c$ -values larger than unity. In other words, we claim that the irreducible HWM's of  $g \oplus g$  at level  $(m-g, 1)$  ( $m = g+1, g+2, \dots$ ) are finitely reducible into irreducible HWM's of  $g_{\text{diag}} \oplus \mathscr{W}$  at level  $m-g+1$  for  $g_{\text{diag}}$  and coset charge  $c(m)$  given by (3.18).

Let  $\chi_{L(p,q)}^{\mathscr{W}}$  be the character of the irreducible HWM  $L(p, q)$  of  $\mathscr{W}$ , i.e.

$$\chi_{L(p,q)}^{\mathscr{W}} = q^{-c(m)/24} \text{Tr}_{L(p,q)} q^{\tilde{L}_0}, \quad q = e^{2\pi i \tau}, \quad (4.1)$$



where  $\tilde{L}_0$  is the zero component of the coset stress–energy tensor  $\tilde{T}(z)$ . The full character also contains information on the  $\tilde{L}_0^{(3)}$  eigenvalue of the states, but the specialization (4.1) is sufficient for our purposes.

We propose the following branching rule

$$\chi_{\tilde{L}(r)}^g \chi_{\tilde{L}(p)}^g = \sum_{\substack{q \in P_+^{m-g+1} \\ \bar{r} + \bar{p} - \bar{q} \in \Lambda_R(\bar{g})}} \chi_{\tilde{L}(q)}^g \chi_{\tilde{L}(p,q)}^{\mathscr{W}}, \tag{4.2}$$

for  $r \in P_+^1$ ,  $p \in P_+^{m-g}$ , where the sum runs over all irreducible integrable HWM's  $L(q)$  at level  $m - g + 1$  such that  $\bar{r} + \bar{p} - \bar{q}$  is an element of the root lattice  $\Lambda_R(\bar{g})$  of  $\bar{g}$ .  $\chi_{\tilde{L}(\Lambda)}^g$  denotes the character of the HWM  $L(\Lambda)$  of  $g$  in its homogeneous specialization [2, 19] (i.e. only  $L_0$  eigenvalues).

For  $g = A_1^{(1)}$  this branching rule is well-known [6]. Though the generalization (4.2) is presumably hard to prove we make four independent checks which provide strong evidence for its validity.

(i) In order to prove that both sides of the branching rule (4.2) have the same transformation property under  $\tau \rightarrow \tau + 1$  we have to check that the conformal dimensions on both sides of (4.2) match modulo positive integers. Let  $L(\Lambda)$  be an integrable HWM of  $g$  at level  $k = \sum_{i=0}^l a_i^v m_i$ , then

$$h_{L(\Lambda)}^{(2)g} = \frac{(\bar{\Lambda}, \bar{\Lambda} + 2\bar{\rho})}{2(k + g)}. \tag{4.3}$$

Therefore

$$\begin{aligned} & h_{\tilde{L}(\bar{r})}^{(2)g} + h_{\tilde{L}(\bar{p})}^{(2)g} - h_{\tilde{L}(\bar{q})}^{(2)g} - h_{\tilde{L}(p,q)}^{\mathscr{W}} \\ &= \frac{(m + 1)(\bar{p}, \bar{p} + 2\bar{\rho}) - m(\bar{q}, \bar{q} + 2\bar{\rho})}{2m(m + 1)} \\ & \quad - \frac{((m + 1)\bar{p} - m\bar{q}, (m + 1)\bar{p} - m\bar{q} + 2\bar{\rho})}{2m(m + 1)} + \frac{(\bar{r}, \bar{r} + 2\bar{\rho})}{2(g + 1)} \\ &= -\frac{1}{2}(\bar{p} - \bar{q}, \bar{p} - \bar{q}) + \frac{1}{2(g + 1)}(\bar{r}, \bar{r} + 2\bar{\rho}). \end{aligned} \tag{4.4}$$

Now use that  $\bar{p} - \bar{q} = -\bar{r} + \alpha$  where  $\alpha \in \Lambda_R(\bar{g})$ , and that  $n = \frac{1}{2}(\alpha, \alpha) + (\bar{r}, \alpha) \in \mathbb{Z}_{\geq 0}$  for simply-laced  $\bar{g}$ . The expression (4.4) reduces to

$$n - \frac{1}{2(g + 1)}(g(\bar{r}, \bar{r}) - 2(\bar{r}, \bar{\rho})).$$

It is not very hard to show explicitly that  $g(\bar{r}, \bar{r}) - 2(\bar{r}, \bar{\rho})$  vanishes identically for all level 1 integrable HWM's  $L(r)$  of simply-laced  $g$ . This proves our assertion.

(ii) In all the cases we checked (4.2) reproduces the well-known branching rules for tensor products of the finite dimensional representations of  $\bar{g}$  occurring at the highest grade of the Kac–Moody HWM. It should be remarked that if one succeeds in proving (4.2) one actually has a generating formula for branching rules of tensor products of representations.

(iii) One can use the branching rule (4.2) to construct modular invariant sesquilinear combinations of characters (string partition function). For details of this procedure of constructing modular invariants we refer to ref. [16]. For  $\mathscr{W} = \text{Vir}[A_2^{(1)} \oplus A_2^{(1)}, A_2^{(1)}, (m-3, 1)]$  the result is the following. First of all, the set  $\{X_{L(p,q)}^{\mathscr{W}} | p \in P_+^{m-3}, q \in P_+^{m-2}\}$  is stable under the modular group. Secondly, there exist modular invariants of  $\mathscr{W}$  labelled by three modular invariants  $H_{\Lambda\Lambda'}^{(k)}$  of  $A_2^{(1)}$  at level  $k = 1, m-3$  and  $m-2$ . If we use the explicit form of the two  $A_2^{(1)}$  invariants at level 1 [23]

$$\text{I: } (1, 1) + (3, 3) + (\bar{3}, \bar{3}), \quad \text{i.e. } H_{\Lambda\Lambda'}^{(1)} = \delta_{\Lambda\Lambda'}, \quad (4.5)$$

$$\text{II: } (1, 1) + (3, \bar{3}) + (\bar{3}, 3), \quad (4.6)$$

corresponding to the level 1 WZW models on the group manifolds of  $SU(3)$  and  $SU(3)/\mathbb{Z}_3$  respectively, the results are

$$\text{I: } H_{pq, p'q'}^{\mathscr{W}, c(m)} = H_{pp'}^{(m-3)} H_{qq'}^{(m-2)} \delta((\bar{p} - \bar{p}') - (\bar{q} - \bar{q}')), \quad (4.7)$$

$$\text{II: } H_{pq, p'q'}^{\mathscr{W}, c(m)} = H_{pp'}^{(m-3)} H_{qq'}^{(m-2)} \delta((\bar{p} + \bar{p}') - (\bar{q} + \bar{q}')), \quad (4.8)$$

where  $\delta(\lambda)$  is defined by

$$\delta(\lambda) = \begin{cases} 1 & \lambda \in \Lambda_R(A_2) \\ 0 & \text{otherwise.} \end{cases} \quad (4.9)$$

If we use in particular  $H_{\Lambda\Lambda'}^{(k)} = \delta_{\Lambda\Lambda'}$  for both  $k = m-3$  and  $m-2$  we obtain from I

$$H_{pq, p'q'}^{\mathscr{W}, c(m)} = \delta_{pp'} \delta_{qq'}.$$

Taking into account the redundancy in the labelling with  $p$  and  $q$  we can rewrite this as

$$H_{pq, p', q'}^{\mathscr{W}, c(m)} \sim 3 \delta_{h^{(2)}, h^{(2)}} \delta_{h^{(3)}, h^{(3)}} \quad (4.10)$$

which is precisely the “unitary” combination we would have expected to occur anyhow. As a second example, if one takes  $m = 4$  ( $c = 4/5$ ) and uses the known

SU(3) and SU(3)/ $\mathbb{Z}_3$  invariants at level 1 and 2, the eqs. (4.7) and (4.8) give a total of 8 possibilities which can all be identified as either (4.10) or

$$H_{pq, p'q'}^{\mathscr{W}, c(m)} \sim \delta_{h^{(2)}, h^{(2)}} \delta_{h^{(3)}, -h^{(3)}}. \tag{4.11}$$

However, if we take into account only the  $\tau$ -dependence of the characters (as in (4.1)) both (4.10) and (4.11) reduce to the partition function of the 3-states Potts model (eq. (1.7) in I).

From the analogy with the  $A_1^{(1)}$  case one might expect that (4.7) and (4.8) give all modular invariants for  $\mathscr{W}$ .

On the other hand, by using the result (4.10) one can for instance inductively construct the modular invariant partition function at level  $k$  for the SU(3)/ $\mathbb{Z}_3$  WZW model by starting with the level-1 invariant (4.6). One finds successively

$$\begin{aligned} k = 2: & \quad (1, 1) + (8, 8) + (3, \bar{3}) + (\bar{3}, 3) + (6, \bar{6}) + (\bar{6}, 6), \\ k = 3: & \quad (1 + 10 + \bar{10}, 1 + 10 + \bar{10}) + 3(8, 8), \\ k = 4: & \quad (1, 1) + (8, 8) + (27, 27) + (10, 10) + (\bar{10}, \bar{10}) \\ & \quad + (6, \bar{6}) + (\bar{6}, 6) + (3, 24) + (\bar{3}, \bar{24}) + (24, 3) + (\bar{24}, \bar{3}) \\ & \quad + (15_1, \bar{15}_1) + (\bar{15}_1, 15_1) + (15_2, \bar{15}_2) + (\bar{15}_2, 15_2). \end{aligned}$$

The  $k = 2$  and  $k = 3$  results were already given by Gepner and Witten [23].

It is clear that the above construction can be extended to all simply laced  $g$  [24]. It may even be possible to achieve a complete classification of all modular invariants this way.

(iv) For the special case of  $g = A_2^{(1)}$  we have been able to show, by explicitly acting with the operator  $\tilde{L}_0^{(3)}$  on the highest weight vector occurring on the l.h.s. of (4.2), that the  $h^{(3)}$  eigenvalues given in (2.18) are also in agreement with the branching rule (4.2)\*.

### 5. Discussion

In this paper we have shown that a coset construction for the 3rd order Casimir operator  $T^{(3)}(z)$  of  $\text{Vir}[A_{N-1}^{(1)}, k]$  exists and we determined the corresponding operator algebra. In particular this determines the complete coset algebra  $\text{Vir}[A_2^{(1)} \oplus$

\* In this way we discovered that eq. (5.6) in ref. [4] is wrong by a factor of 2.

$A_2^{(1)}, A_2^{(1)}, (k_1, k_2)$ ]. For  $k_2 = 1$  this algebra has been shown to reduce miraculously to Zamolodchikov's spin-3 algebra.

We conjectured the existence of coset algebras  $\text{Vir}[g, g', k]$  for every Kac–Moody subalgebra  $g'$  of  $g$  and gave expressions for the conformal dimensions  $h^{(2)}(p, q)$  of a main sequence  $c = c(m)$  of minimal models corresponding to the diagonal coset algebras  $\text{Vir}[g \oplus g, g, (m - g, 1)]$  for simple, simply-laced  $\bar{g}$ . We also postulated branching rules for HWM's of  $g \oplus g$  in terms of HWM's of  $g_{\text{diag}} \oplus \text{Vir}[g \oplus g, g, (m - g, 1)]$ .

It is clear that a lot of work remains to be done in proving these conjectures (though a number of consistency checks have made them more than plausible) and extending them to all coset algebras  $\text{Vir}[g, g', k]$  or even to super Kac–Moody algebras. We believe that a complete understanding of  $\text{Vir}[g, g', k]$  might very well lead to a classification of all (rational) 2D conformal field theories, which would be of great interest in string theory and statistical mechanics.

In all the work on 2D conformal field theory which has been done so far there seems to be an intriguing relationship among a triplet

- (i) a (rational) 2D conformal field theory.
- (ii) a coset pair  $(g, g')$ ,
- (iii) an integrable lattice model.

In this paper we have tried to work out the relationship between (i) and (ii). Concerning (iii) we can remark that recently Jimbo et al. [25, 26] constructed a family of exactly solvable two-dimensional lattice models by extending the construction of “restricted solid on solid” (RSOS) models by Andrews et al. [27]. These models are based on the weight space of  $A_{N-1}^{(1)}$ . It is conjectured that the local state probabilities of these models in the continuum limit are related to the irreducible decomposition of characters for the coset pair  $(A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}, A_{N-1}^{(1)})$  as is known for  $N = 2$  [28]. It therefore seems likely that the coset Casimir models we have constructed are the continuum limits of the RSOS models of Jimbo et al. It would be very interesting to unravel this correspondence and, in particular, to make explicit the higher symmetries in these lattice models.

#### Note added

After completion of this paper we received a preprint by Fateev and Lykhanov [29] containing results for  $A_{N-1}^{(1)}$  agreeing with ours in sect. 3. We also received a thesis by Hayashi [30] who studied higher order Casimir operators.

We would like to thank M. Jimbo for providing refs. [29, 30] and O. Foda for interesting comments. P.B. and K.S. are financially supported by the “Stichting voor Fundamenteel Onderzoek der Materie” (FOM) and M.S. is supported by a SERC/NATO postdoctoral fellowship.

**Appendix**

In this appendix we give some conventions for our notation. We will follow mostly the conventions of Kac and Peterson [2, 19].

An untwisted affine Kac–Moody algebra will be denoted by  $g$ , its underlying finite dimensional Lie algebra by  $\bar{g}$ . A bar on a root or weight of  $g$  will always denote the corresponding root or weight in  $\bar{g}$ .

$h$  and  $h^*$  are respectively the Cartan subalgebra of  $g$  and its dual. The simple roots of  $g$  are  $\{\alpha_i, i = 0, 1, \dots, l = \text{rank } \bar{g}\} \subset h^*$ . Their duals  $\{\alpha_i^v, i = 0, 1, \dots, l\} \subset h$  are such that the Cartan matrix  $a_{ij}$  is given by

$$a_{ij} = \langle \alpha_j, \alpha_i^v \rangle. \tag{A.1}$$

The element  $\rho \in h^*$  is defined by

$$\langle \rho, \alpha_i^v \rangle = 1, \quad i = 0, 1, \dots, l, \tag{A.2}$$

and the central element  $c$  of  $g$  is given by

$$c = \sum_{i=0}^l a_i^v \alpha_i^v, \tag{A.3}$$

where the dual coxeter labels  $a_i^v$  constitute a left zero eigenvector of the Cartan matrix  $a_{ij}$ . The dual Coxeter number  $g$  is defined by

$$g = \langle \rho, c \rangle = \sum_{i=0}^l a_i^v. \tag{A.4}$$

The irreducible integrable highest weight modules (HWM's)  $L(\Lambda)$  are labelled by the integral dominant weights  $\Lambda$ :

$$\Lambda \in P_+ = \{ \lambda \in h^* \mid \langle \lambda, \alpha_i^v \rangle \in \mathbb{Z}_{\geq 0}, \quad i = 0, 1, \dots, l \}. \tag{A.5}$$

Every  $\Lambda \in P_+$  can be expressed as a sum  $\Lambda = \sum_{i=0}^l m_i \Lambda_i$ , where  $\{\Lambda_i\}$  is a set of fundamental weights

$$\langle \Lambda_i, \alpha_j^v \rangle = \delta_{ij}, \quad i, j = 0, 1, \dots, l, \tag{A.6}$$

and the positive integers  $m_i = \langle \Lambda, \alpha_i^v \rangle$  are the so-called Dynkin labels of the HWM. The level  $k$  of a  $\Lambda \in P_+$  is the positive integer

$$k = \langle \Lambda, c \rangle = \sum_{i=0}^l a_i^v m_i; \tag{A.7}$$

we denote

$$P_+^k = \{ \Lambda \in P_+, \langle \Lambda, c \rangle = k \}. \quad (\text{A.8})$$

Finally, we often used the Freudenthal strange formula

$$\frac{|\bar{\rho}|^2}{2g} = \frac{\dim \bar{g}}{24} \quad (\text{A.9})$$

and a formula only valid for simple, simply-laced algebras  $\bar{g}$

$$\dim \bar{g} = l(g + 1). \quad (\text{A.10})$$

### References

- [1] F.A. Bais, P. Bouwknegt, K. Schoutens and M. Surridge, Nucl. Phys. B304 (1988) 348
- [2] V.G. Kac, Infinite dimensional Lie algebras (Cambridge University Press, 1985)
- [3] A.B. Zamolodchikov, Theor. Math. Phys. 65 (1986) 1205
- [4] V.A. Fateev and A.B. Zamolodchikov, Nucl. Phys. B280 [FS18] (1987) 644
- [5] P. Goddard, A. Kent and D.I. Olive, Phys. Lett. B152 (1985) 88
- [6] P. Goddard, A. Kent and D.I. Olive, Comm. Math. Phys. 103 (1986) 105
- [7] A.B. Zamolodchikov and V.A. Fateev, Sov. Phys. JETP 62 (1985) 215
- [8] D. Gepner and Z. Qiu, Nucl. Phys. B285 [FS19] (1987) 423
- [9] F.A. Bais, F. Englert, A. Taormina and P. Zizzi, Nucl. Phys. B279 (1987) 529
- [10] V.G. Kac and D.H. Peterson, Proc. Natl. Acad. Sci. USA 78 (1981) 3308
- [11] P. Goddard, W. Nahm and D.I. Olive, Phys. Lett. B160 (1985) 111
- [12] P. Goddard and D.I. Olive, Nucl. Phys. B257 [FS14] (1985) 226
- [13] P. Goddard and D.I. Olive, Int. J. Mod. Phys. A1 (1986) 303
- [14] P.G. Bouwknegt and W. Nahm, Phys. Lett. B184 (1987) 359
- [15] P. Bowcock and P. Goddard, Nucl. Phys. B285 [FS19] (1987) 651
- [16] P. Bouwknegt, Nucl. Phys. B290 [FS20] (1987) 507
- [17] B.L. Feigin and D.B. Fuks, Funct. Anal. Appl. 16 (1982) 114
- [18] V.I. Dotsenko and V.A. Fateev, Nucl. Phys. B240 [FS12] (1984) 312
- [19] V.G. Kac and D.H. Peterson, Adv. Math. 53 (1984) 125
- [20] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333
- [21] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett 52 (1984) 1575; *in* Vertex operators in mathematics and physics, eds. J. Lepowski et al., p. 419 (Springer, 1985); Comm. Math. Phys. 107 (1986) 535
- [22] A. Cappelli, C. Itzykson and J.B. Zuber, Nucl. Phys. B280 [FS18] (1987) 445; Comm. Math. Phys. 113 (1987) 1
- [23] D. Gepner and E. Witten, Nucl. Phys. B278 (1986) 493
- [24] Work in progress
- [25] M. Jimbo, T. Miwa and M. Okado, Mod. Phys. Lett. B1 (1987) 73
- [26] M. Jimbo, T. Miwa and M. Okado, Kyoto preprint RIMS-572 (April 1987) Lett. Math. Phys., to be published
- [27] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193
- [28] E. Date, M. Jimbo, T. Miwa and M. Okado, Kyoto preprint RIMS-548 (Sept. 1986)
- [29] V.A. Fateev and S.L. Lykhanov, The models of two dimensional conformal quantum field theory with  $Z_N$  symmetry, Landau preprint
- [30] T. Hayashi, Sugawara operators and the Kac-Kazhdan conjecture, Thesis, Dept. of Mathematics, Nagoya University