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Traveling Waves in the Complex Ginzburg-Landau Equation

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Summary. In this paper we consider a modulation (or amplitude) equation that appears in the nonlinear stability analysis of reversible or nearly reversible systems. This equation is the complex Ginzburg-Landau equation with coefficients with small imaginary parts. We regard this equation as a perturbation of the real Ginzburg-Landau equation and study the persistence of the properties of the stationary solutions of the real equation under this perturbation. First we show that it is necessary to consider a two-parameter family of traveling solutions with wave speed ν and (temporal) frequency ω ; these solutions are the natural continuations of the stationary solutions of the real equation. We show that there exists a two-parameter family of traveling quasi-periodic solutions that can be regarded as a direct continuation of the two-parameter family of spatially quasi-periodic solutions of the integrable stationary real Ginzburg-Landau equation. We explicitly determine a region in the (wave speed ν , frequency ω)-parameter space in which the weakly complex Ginzburg-Landau equation has traveling quasi-periodic solutions. There are two different one-parameter families of heteroclinic solutions in the weakly complex case. One of them consists of slowly varying plane waves; the other is directly related to the analytical solutions due to Bekki & Nozaki [3]. These solutions correspond to traveling localized structures that connect two different periodic patterns. The connections correspond to a one-parameter family of heteroclinic cycles in an o.d.e. reduction. This family of cycles is obtained by determining the limit behaviour of the traveling quasi-periodic solutions as the period of the amplitude goes to ∞ . Therefore, the heteroclinic cycles merge into the stationary homoclinic solution of the real Ginzburg-Landau equation in the limit in which the imaginary terms disappear.

Key words. pattern formation, modulation equations, traveling waves, perturbed integrable systems, elliptic functions

1. Introduction

The Ginzburg-Landau equation is a generic modulation equation describing the nonlinear evolution of patterns at near-critical conditions. The equation has been derived for many, mainly hydrodynamical, nonlinear stability problems, such as (binary) fluid convection [30], [34], Poiseuille flow [35], chemical reactions [26], and so on. In DiPrima et al. [11] and Newell [29] it has been shown that the Ginzburg-Landau equation generically appears as a modulation, or amplitude/envelope, equation for general systems at the onset of pattern formation. Recently, some important steps have been made toward a rigorous proof of its mathematical validity as a modulation equation ([23], [10], [19]).

The one-dimensional Ginzburg-Landau equation is, in rescaled form, given by

$$\frac{\partial u}{\partial t} = u + (1 + i\alpha) \frac{\partial^2 u}{\partial x^2} - (1 + i\beta)|u|^2 u, \quad (1.1)$$

where $u = u(x, t)$ is a complex-valued function; the real parameters α and β can be computed for a given underlying physical system (see [29]). We have chosen the (minus) sign in front of the nonlinearity in (1.1) (i.e., a supercritical bifurcation). This situation appears most frequently in the applications ([7], [26], [30], [34]) and is mathematically the most interesting, since there is some balance between the linear and the nonlinear terms. If $\alpha = \beta = 0$, equation (1.1) describes the onset of stationary periodic solutions in nonlinear stability problems. It is then called the real Ginzburg-Landau equation. This situation appears in systems that have a (spatial) reversibility symmetry (i.e., there is a reflection symmetry in the unbounded variable). Convection experiments are a good example of this [30]. (It should be noted that $(\alpha, \beta) \neq (0, 0)$ is also possible in reversible systems, see [24], [34], and the following paragraphs.) Equation (1.1) is called the complex Ginzburg-Landau equation if $(\alpha, \beta) \neq (0, 0)$.

Equation (1.1) is of significant importance in many kinds of nonlinear stability problems in physics, chemistry, and so on. Due to its generic character it has been the subject of many mathematical studies. These studies can, more or less, be divided into two groups. The first group of studies considers (1.1), and two- or three-dimensional generalizations of (1.1), on a bounded interval (mostly with periodic boundary conditions). We mention here: Doering et al. [14] (dimension of the attractor), Bartuccelli et al. [2] (estimates on the solutions in several norms), and Promislow [32] ((approximate) inertial manifolds). The second group considers (1.1) on an unbounded domain. Since (1.1) can only be derived for unbounded domains, this assumption is directly related to the derivation process. This paper belongs to the second group. The main aim of papers studying (1.1) on the real axis has been an exploration of the “universe of solutions” of (1.1) by some kind of reduction to a simpler system (Holmes [20] (using a center manifold reduction), Holmes [21], Doelman [12] (both near an integrable limit), and Bernoff [5] (asymptotic analysis of slowly varying solutions)).

In this paper we study the Ginzburg-Landau equation with coefficients with small imaginary parts, that is, we set $\alpha = \epsilon a$ and $\beta = \epsilon b$ in (1.1), for some $0 < \epsilon \ll 1$. This situation appears frequently in physical problems. One class of problems in which a “weakly complex” Ginzburg-Landau equation appears is that of reversible systems in which the nonlinear stability problem is described by a real Ginzburg-Landau equa-

tion ((binary fluid) convection [24], [34], electrically forced nematic liquid crystals [15], Taylor-Couette systems [7] (although here the equation has to be in two spatial dimensions), etc.). A small, nonreversible, that is, symmetry-breaking, extra effect will produce small imaginary terms in the modulation equation associated with the nonlinear stability problem. This effect might be a weak external force field (for instance, the coriolis force in convection) or a small change in the geometry of the experimental set-up that perturbs the symmetry (for instance, convection in a large annulus [4]). The small parameter ϵ measures, in these cases, the magnitude of the symmetry-breaking effects. Another class of problems in which the weakly complex Ginzburg-Landau equation appears is that of reversible systems in which there is an interaction between stationary and Hopf bifurcations at the onset of instability (binary fluid convection [24], [34], nematic crystals [15]). By varying a second control parameter (the separation ratio in convection, the intensity of a magnetic field in nematics) there is a transition from a real Ginzburg-Landau equation (stationary bifurcations) to a complex one (Hopf bifurcations: the periodic solutions have a nonzero temporal frequency). Near the transition, or codimension 2, point the temporal frequency becomes small [24]. Consequently, the associated modulation equation will be a Ginzburg-Landau equation with coefficients with small imaginary parts. The parameter ϵ is, in these cases, related to the temporal frequency of the periodic solutions bifurcating at the onset of instability. It should be noted that the situation is more complicated at the codimension 2 point, due to the interactions between the patterns [39] and a shielding effect [24].

It has been observed in many experiments that systems at near-critical conditions exhibit traveling patterns of nonperiodic type. For instance, the convection experiments performed by Bensimon et al. [4] exhibit patterns that consist of traveling patches of periodic motion surrounded by fluid at rest. Mathematically, these patterns can be interpreted as heteroclinic connections between the zero solution of the modulation equation (fluid at rest) and a traveling periodic solution. In this paper we study traveling wave solutions to the complex Ginzburg-Landau equation of the type

$$u(x, t) = \hat{u}(x + \nu t)e^{-i\omega t}, \quad (1.2)$$

where $\nu = \epsilon v$ and $\omega = \epsilon \omega$ in the case of small imaginary parts. In Section 2 we give this natural “Ansatz” a mathematical foundation. It should be noted that we consider here a supercritical modulation equation with only a cubic nonlinearity: in [37], [8], and [33] (see also the references cited in that paper) traveling waves of type (1.2) have been studied in a subcritical modulation equation with an additional fifth-order nonlinearity. Since equations of this type do not appear generically from the derivation process (see [13]), we do not consider these extended systems here. However, most of the methods developed in this paper can be applied to these equations.

Mathematically, the advantage of studying the weakly complex Ginzburg-Landau equation is that the p.d.e. (1.1) can be reduced, using (1.2), to a 3-D dynamical system that is near an integrable limit: $\epsilon = 0$, which corresponds to the real Ginzburg-Landau equation. The stationary problem associated with the real equation is integrable (Section 3) and has large families of periodic, quasi-periodic, and homoclinic solutions. The homoclinic solutions correspond to localized structures in the underlying physical system, that is, the homoclinic solutions represent patterns that tend to a periodic pattern at $x \rightarrow \pm\infty$. Kramer & Zimmerman [25] have shown the relevance of

the quasi-periodic and homoclinic solutions in the creation or annihilation process of periodic rolls in reversible systems. The main result of this paper is that both these families of stationary solutions, the quasi-periodic and the homoclinic, can be continued into modified, but very similar, families of traveling solutions of the weakly complex Ginzburg-Landau equation. In other words, the integrable structure of the real equation persists (in some sense) under the nonsymmetric perturbations. The quasi-periodic solutions have a well-determined, nonzero wave speed ν and frequency ω in the weakly complex case. We determine a precise relationship between the wave speed and frequency of these solutions and the stationary quasi-periodic solutions in the real limit. This yields a bounded region F in (ν, ω) -space for which quasi-periodic solutions to (1.1) exist (Section 4). One of the boundaries of this region is related to the homoclinic solutions. We show that to any point (i.e., any (ν, ω) combination) on this curve there corresponds a reduced o.d.e. that has a heteroclinic cycle. Therefore, we find a family of heteroclinic cycles (in any weakly complex Ginzburg-Landau equation) that is a continuation of the family of stationary homoclinic solutions in the real case. Any heteroclinic cycle consists of two different connections between the same two periodic points (which are on an $O(\epsilon)$ distance from each other). Physically this means that to any stationary homoclinic structure in the real case there correspond two different traveling localized structures, a “fast” one and a “slow” one, in the weakly complex case. Both these structures are front solutions to the modulation equation: they connect two different periodic patterns (from $x \rightarrow -\infty$ to $x \rightarrow \infty$). The fast connection remains close to the homoclinic solution; that is, like the homoclinic solution, it makes an $O(1)$ detour in phase space by connecting the two periodic points. The slow connection remains close to the family of periodic solutions of (1.1); that is, it is a slowly varying plane wave. In Figure 7 we have plotted typical examples of a slow (7a) and a fast (7b) connection, as solutions of the p.d.e.

It should be noted that both types of connections have already appeared separately in the literature. The slow connection is a slowly varying plane wave; that is, the solution is locally $O(\epsilon)$ near a plane wave, which is a periodic solution of the type $R \exp[i(Lx + Wt)]$ (see Section 2); R , L , and W vary on long ($O(1/\epsilon)$) length and time scales. These types of solutions have already been studied by Bernoff [5]. In that paper the techniques developed by Howard & Kopell [22] were applied to the Ginzburg-Landau equation. In [5] and [22] the existence of slowly varying plane wave connections was shown without any assumptions on the magnitudes of α and β in (1.1). It has not been necessary to apply the techniques of [5] and [22] in the present paper since $\alpha = \epsilon a$ and $\beta = \epsilon b$ in (1.1). The existence of the slow connections follows quite directly from the work of Fenichel ([17], see Section 4). The fast connections, which are certainly not slowly varying plane waves, are given by the explicit solutions of (1.1) due to Bekki & Nozaki [3]. In [3] a family of front solutions to (1.1) was constructed using algebraic techniques (see Section 4). This method works for any α and β in (1.1). Due to the very different natures of the techniques applied in [5] and [22] on the one hand and in [3] on the other hand, it has so far not been possible to relate these two types of heteroclinic connections. In this paper we show that two connections, one of the plane wave type, one of the Bekki & Nozaki type, together make up one heteroclinic cycle in the phase space of the reduced o.d.e. associated with (1.1) (see Fig. 6b). Furthermore, we show that

these heteroclinic cycles can be considered as a limit of the family of traveling quasi-periodic solutions (as the period of $|u| = |\hat{u}(x + \nu t)|$ goes to ∞). This situation is exactly like the real case (the limit $\epsilon = 0$) in which the family of quasi-periodic solutions, which corresponds to a family of periodic solutions in the o.d.e. reduction, is bounded by the family of homoclinic orbits (and, this boundary is obtained by taking the limit of the period of $|u| = |\hat{u}(x)|$ to ∞ , see Section 3, Fig. 1). The homoclinic solution corresponds to the fast Bekki & Nozaki solution for $\epsilon = 0$ (see Section 4). The slow plane wave connection disappears in the limit $\epsilon = 0$.

In the last part of this introduction we present a sketch of our methods and a more detailed discussion of our results by describing the contents of each of the following sections. Furthermore, we discuss briefly the relation between this paper and earlier work on the weakly complex Ginzburg-Landau equation [12].

In a preparatory Section (2), we present a mathematical foundation to (physically plausible) Ansatz (1.2). By a center manifold reduction we show that there exist, locally, near the unstable periodic solutions of (1.1), quasi-periodic solutions to the Ginzburg-Landau equation of type (1.2), independent of the magnitudes of α or β in (1.1). We find that, if $\alpha = \epsilon a$ and $\beta = \epsilon b$, the wave speed and the frequency are also $O(\epsilon)$: $\nu = \epsilon \nu$ and $\omega = \epsilon \omega$ and, thus, that solutions of the form (1.2) correspond directly to the stationary solutions of the real Ginzburg-Landau equation ($\epsilon = 0$).

In Section 3 we extend this local result to a global result by studying the o.d.e. reduction of (1.1), which is obtained by substitution of (1.2) into (1.1). We show that the global family of quasi-periodic solutions of the real Ginzburg-Landau equation persists under the perturbation, or, more precisely, we prove that to any quasi-periodic solution of the real Ginzburg-Landau equation there corresponds a unique wave speed–frequency pair $(\epsilon \nu_*, \epsilon \omega_*)$ such that the complex Ginzburg-Landau equation ($\alpha = \epsilon a$ and $\beta = \epsilon b$) has a traveling quasi-periodic solution of type (1.2) with wave speed $\epsilon \nu_*$ and frequency $\epsilon \omega_*$, which is a continuation of the original quasi-periodic solution. Our methods are based on an $O(\epsilon)$ approximation of the Poincaré map associated with the perturbed integrable o.d.e. ([21], [12]). The proof of this result boils down to a detailed analysis of the approximation of the Poincaré map P . We express this approximation in a linear combination of two types of complete elliptic expressions (Section 3.3). We note here that the ideas used to analyse and to control this map in this special system can be generalized directly to more general perturbed integrable o.d.e.'s (see also Remark 3.13). Due to this precise control of the Poincaré map (and therefore of the flow generated by the perturbed system) we are able to determine explicitly a bounded region in parameter space that prescribes the wave speed–frequency pairs (ν, ω) for which equation (1.1) has traveling quasi-periodic solutions. (The Poincaré map has no fixed points for parameter combinations outside this region.) This region, F , does not depend on the coefficients $\alpha = \epsilon a$ and $\beta = \epsilon b$ of Equation (1.1) if we consider it in (A, B) -space instead of (ν, ω) -space, with $A = (w - a)/(b - a)$ and $B = \nu/(b - a)$, see Fig. 5 (there is no degeneration near $b = a$, see Section 3).

We remark here that this result is a significant extension of the work done in [12]: in that paper it was shown that none of the quasi-periodic solutions of the integrable limit, except for a family of degenerate periodic solutions, survive the perturbation. However, in [12] we considered $\nu = 0$. It is an immediate consequence of the analysis

in this paper that the quasi-periodic solutions of (1.1) necessarily have a nonzero wave speed (for $(\alpha, \beta) \neq (0, 0)$, see Remark 4.2). Another important addition to [12] is the fact that we show in this paper the existence of a unique pair (ν, ω) for which a given quasi-periodic, or degenerate periodic, solution persists, whereas in [12] we could not derive such a uniqueness result.

In Section 4 we study the influence of the perturbation on the family of homoclinic solutions of the integrable system. Here, the structure of the unperturbed system is significantly changed by the perturbation: we show that this type of homoclinic solution cannot persist. However, we determine an $O(\epsilon^2)$ narrow strip in (A, B) -space near the boundary of region F in which one can expect traveling waves of soliton type (localized structures) to (1.1). These structures correspond to heteroclinic or homoclinic connections in the o.d.e. reduction. The family of solitonlike solutions constructed by Bekki and Nozaki [3] defines a wave speed–frequency curve inside this strip. We interpret this family geometrically as the fast part of a heteroclinic cycle that connects two nearby periodic solutions of (1.1). The existence of the slow part of the cycle follows quite directly from the general theory on perturbations of hyperbolic structures developed by Fenichel (see [17] and the references cited there). This family of heteroclinic cycles is shown to merge into the family of homoclinic solutions of the integrable system in the limit $\epsilon \downarrow 0$. We conclude the paper with a short section in which we discuss our results.

2. Bifurcation into Traveling Waves

In this section we will show the existence of, and determine the structure of, a family of traveling quasi-periodic solutions to the Ginzburg-Landau equation. The method is very classical and has already been applied (in more details than we will present here) to equations of Ginzburg-Landau type by Holmes [20]: we study the bifurcations of unstable periodic solutions of (1.1) of the form

$$u(x, t) = R \exp[i(Lx + Wt)], \quad (2.1)$$

with R, L , and W (real) parameters. We will not perform the necessary center manifold analysis in great detail; we will not do any explicit computations (for these we refer the reader to [20]). We are primarily interested in showing the existence of traveling quasi-periodic solutions and in deriving their structure (this was not done in [20]). Substitution of (2.1) into (1.1) yields

$$R^2 + L^2 = 1, \quad (2.2)$$

$$W = -(\alpha L^2 + \beta R^2); \quad (2.3)$$

that is, for any value of α and β there exists a semicircle of periodic solutions in the (L, R) -plane. Note that these solutions are stationary if $\alpha = \beta = 0$ in (1.1), that is, in the case of the real Ginzburg-Landau equation. To study the stability of a periodic solution of type (2.1) we fix an $L = L_0$, $R = R_0$, and $W = W_0$ such that (2.2) and (2.3) hold and perturb this solution: we set

$$u(x, t) = (R_0 + U(x, t)) \exp[i(L_0 x + W_0 t)]. \quad (2.4)$$

Although it is natural to consider perturbations $U(x, t)$ that decay fast enough for $x \rightarrow \infty$ so that we can set

$$U(x, t) = \int_{-\infty}^{\infty} \hat{u}(n, t) e^{inx} dn, \quad (2.5)$$

we restrict ourselves to the class of periodic perturbations with wavenumber l and use l as a parameter (like in [20]):

$$U(x, t) = \sum_{n=-\infty}^{\infty} \hat{U}_n(t) e^{inlx}. \quad (2.6)$$

Note that if one is only interested in the linear stability of the solutions of type (2.1), it does not make any difference whether one considers perturbations of type (2.5) or of type (2.6): in both situations the problem reduces in linear approximation to studying a family of linear 2×2 -systems in, respectively, $\hat{u}(n, t)$ and $\hat{u}^*(-n, t)$ for (2.5) and $\hat{U}_n(t)$ and $\hat{U}_{-n}^*(t)$ for (2.6); both types of perturbations will give exactly the same stability criteria. This phenomenon of decoupling of corresponding “modes” is called side-band interaction (see [36]). However, if one is interested in nonlinear phenomena, for instance bifurcations, it is more convenient to use (2.6) instead of (2.5): substituting (2.4) with (2.6) into U_n (1.1) yields a (countably) infinite-dimensional dynamical system:

$$\begin{aligned} \frac{d}{dt} \hat{U}_n = & [1 - iW_0 - (1 + i\alpha)(nl + L_0)^2 - 2(1 + i\beta)R_0^2] \hat{U}_n - (1 + i\beta)R_0^2 \hat{U}_{-n}^* \\ & - (1 + i\beta)R_0 \sum_{k_1+k_2=n} (\hat{U}_{k_1} \hat{U}_{k_2} + 2\hat{U}_{k_1} \hat{U}_{-k_2}^*) - (1 + i\beta) \sum_{k_1+k_2+k_3=n} \hat{U}_{k_1} \hat{U}_{k_2} \hat{U}_{-k_3}^*, \end{aligned} \quad (2.7)$$

for $n = 0, \pm 1, \pm 2, \dots$. Observe the decoupling into 2×2 -systems in $\hat{U}_n(t)$ and $\hat{U}_{-n}^*(t)$ ($n = 0, 1, \dots$) in the linear approximation. Since we will use the result in Section 2, we will give here a very short sketch of the linear stability analysis in the special case $\alpha = \beta = 0$. The linear system is now

$$\begin{pmatrix} (d/dt) \hat{U}_n \\ (d/dt) \hat{U}_{-n}^* \end{pmatrix} = \begin{pmatrix} -R_0^2 - 2nlL_0 - n^2l^2 & -R_0^2 \\ -R_0^2 & -R_0^2 + 2nlL_0 - n^2l^2 \end{pmatrix} \begin{pmatrix} \hat{U}_n \\ \hat{U}_{-n}^* \end{pmatrix}, \quad (2.8)$$

for $n = 1, 2, \dots$ (we have used (2.2) and (2.3) to simplify (2.8)). Note that for $n = 0$, (2.8) can be considered as a 2×2 real system, that has one eigenvalue that is identically zero (due to the fact that we are linearizing along a periodic solution) and one that is always negative. The eigenvalue problem associated with (2.8) reads

$$\lambda^2 + 2\lambda[R_0^2 + n^2l^2] + n^2l^2[n^2l^2 - 4L_0^2 + 2R_0^2] = 0. \quad (2.9)$$

Applying the condition that both eigenvalues have to have a negative real part yields one inequality: $2R_0^2 - 4L_0^2 > -n^2l^2$, or, by using (2.2),

$$6R_0^2 - 4 > -n^2l^2. \quad (2.10)$$

Hence, the first mode to become unstable is the $n = 1$ mode, which becomes unstable at $l = l_c = (\pm) \sqrt{4 - 6R_0^2}$. Note that although solutions with

$$R_0^2 > \frac{2}{3} \quad (2.11)$$

are always stable, solutions with $R_0^2 < \frac{2}{3}$ are unstable as solutions of (1.1), since (2.10) has to hold for all l . This result is originally due to Eckhaus [16]. This type of analysis can also be done in the general case $(\alpha, \beta) \neq (0, 0)$; however, the analysis becomes more involved since it is no longer possible to obtain an easy expression like (2.9). Yet, one can observe directly that the $n = 0$ mode is always stable (with one eigenvalue identically zero), and that if $R_0 \exp[i(L_0 x + W_0 t)]$ is unstable then the $n = 1$ mode is the first to become unstable at a certain $l = l_c(\alpha, \beta, R_0)$; all modes are stable if $l > l_c$. Furthermore, it is easy to derive that the boundaries of the interval of stability, that is, $L_0 = \pm \sqrt{\frac{1}{3}}$, are only changed by an $O(\epsilon)$ amount in the weakly complex case ($\alpha = \epsilon a$, $\beta = \epsilon b$ in (1.1)). For more details on the linear stability analysis we refer the reader to [36] and [28].

We now start the bifurcation analysis near $l = l_c$ (i.e., we assume that $R_0 \exp[i(L_0 x + W_0 t)]$ is unstable): we set $l = l_c + \delta$, with δ , $|\delta| \ll 1$, the new bifurcation parameter. Before we perform this bifurcation analysis, we present its outcome:

Theorem 2.1. *If $R_0 \exp[i(L_0 x + W_0 t)]$, with R_0 , L_0 , and W_0 satisfying (2.2) and (2.3), is unstable against perturbations of type (2.6) as solution of (1.1), then system (2.7) undergoes a Hopf bifurcation at a certain $l = l_c(\alpha, \beta, R_0)$, which yields the (local) existence of a family of traveling quasi-periodic solutions of the Ginzburg-Landau equation having the structure*

$$u(x, t) = \hat{u}(x + vt)e^{-i\omega t}. \quad (2.12)$$

This result is not very surprising; however, we mention it here to give a mathematical background to Ansatz (2.12). By Theorem 2.1 we know that we have to consider solutions of type (2.12) to be able to find quasi-periodic solutions. In order to apply the center manifold theorem to system (2.7) (near $l = l_c$) we have to bring it into standard form. For more details on the center manifold theorem we refer to [9], [18]. We construct a new basis $\{V_+(t), V_-(t)\}$ for the linear space spanned by $\{\hat{U}_1(t), \hat{U}_{-1}^*(t)\}$ such that, in the linear approximation,

$$\frac{d}{dt}V_{\pm} = \lambda_{\pm}V_{\pm},$$

with $\operatorname{Re}\lambda_+ = -c_1^r\delta + O(\delta^2)$, $c_1^r \gg |\delta|$, and $\operatorname{Re}\lambda_- \ll -|\delta|$ ($\lambda_{\pm} = \lambda_{\pm}(\alpha, \beta, R_0, \delta)$). We will not present the computational details. We then know that there are functions $h_-(V_+)$ and $h_n(V_+)$ ($n = 0, \pm 2, \pm 3, \dots$) such that the center manifold is given by

$$V_- = h_-(V_+), \quad \hat{U}_n = h_n(V_+). \quad (2.13)$$

Due to the structure of the nonlinearities of (2.7) we may conclude

$$h_0(V_+) = O((V_+)^2), \quad h_n(V_+) = O((V_+)^{|n|}) \quad (2.14)$$

(for $n = \pm 2, \pm 3, \dots$). On the center manifold the flow is governed by

$$\frac{d}{dt}V_+ = \lambda_+ V_+ - c_3 |V_+|^2 V_+ + \text{h.o.t.}, \quad (2.15)$$

with complex coefficient $c_3 = c_3(\alpha, \beta, R_0, \delta)$, $\text{Rec}_3 \gg |\delta|$. Hence, since $d/d\delta \lambda_+ \neq 0$ and $\text{Rec}_3 = O(1)$, we can conclude (see, for instance, [18]) that there is a Hopf bifurcation at $l = l_c$ (i.e., $\delta = 0$): there is a unique periodic solution of the nonlinear system (2.7) for $-1 \ll \delta < 0$, that is, $l < l_c$, which, up to higher-order terms, coincides with the periodic solution of (2.15):

$$|V_+|^2 = \frac{\lambda_+}{\text{Rec}_3}.$$

That is, $V_+(t) = \hat{v}_+ \exp(i\hat{w}_+ t) + \text{h.o.t.}$, for some constants $\hat{v}_+$ and $\hat{w}_+$, and, by (2.14):

$$\hat{U}_n(t) = \hat{u}_n \exp(i\hat{w}_+ n t) + \text{h.o.t.}, \quad (2.16)$$

for $n = 0, \pm 1, \pm 2, \dots$ and some constants $\hat{u}_n$. Note that the higher-order corrections consist of products with terms like $\exp(i\hat{w}_+ m t)$, $m = 0, \pm 1, \pm 2, \dots$; hence, a priori $\hat{U}_n(t)$ is only periodic with (an $O(\delta)$ correction of) period $2\pi/\hat{w}_+$ and is not an n th harmonic of $\hat{U}_1(t)$. However, by looking at (2.7) carefully we observe that there exist solutions of (2.7) exactly of the form

$$\hat{U}_n(t) = u_n e^{iwn t} \quad (2.17)$$

for $n = 0, \pm 1, \pm 2, \dots$ and some constants u_n and w . Hence, since the Hopf bifurcation theorem describes the appearance of a unique periodic solution, we may conclude that the periodic solution bifurcating at $l = l_c$ is exactly of the form (2.17).

Substituting (2.17) into (2.6) yields

$$U(x, t) = \sum_{n=-\infty}^{\infty} u_n \exp[in(lx + wt)],$$

that is, $U(x, t) = f(lx + wt)$, a periodical function in $(lx + wt)$ and, by (2.4),

$$\begin{aligned} u(x, t) &= \left(R_0 + f\left(l\left[x + \frac{w}{l}t\right]\right) \right) \exp\left\{i\left[L_0\left(x + \frac{w}{l}t\right) + \left(W_0 - \frac{wL_0}{l}\right)t\right]\right\}, \\ &= \hat{u}(x + \nu t)e^{-i\omega t}, \end{aligned}$$

with the obvious definitions of ν and ω . Note that $\hat{u}$ is (in general) a quasi-periodic function in the variable $\xi = x + \nu t$. This concludes the proof of Theorem 2.1.

This theorem gives a mathematical background to considering solutions of (1.1) of the structure (2.12). The structure becomes even more simple in the case of the real Ginzburg-Landau equation.

Corollary 2.2. *If $\alpha = \beta = 0$ in (1.1) then $\nu = \omega = 0$ in (2.12), that is, the bifurcating quasi-periodic solutions are stationary; $\nu = \epsilon\nu$, $\omega = \epsilon\omega$ if $\alpha = \epsilon a$ and $\beta = \epsilon b$ for some $0 < \epsilon \ll 1$.*

The proof is rather straightforward.

Proof. Since equation (2.7) has only real coefficients if $\alpha = \beta = 0$, it follows directly (after checking that λ_+ and λ_- are also real) that all coefficients in (2.14) and (2.15) are real. Thus, $\hat{w}_+ = 0$ in (2.16) and $w = 0$ in (2.17). Combining this with the fact that $W_0 = 0$ yields the corollary for $\alpha = \beta = 0$. The same type of argument gives the result for $\alpha = \epsilon a$ and $\beta = \epsilon b$.

In the next section we will see that the stationary problem associated with the real Ginzburg-Landau equation, that is, a complex Duffing equation

$$0 = u_{xx} + u - |u|^2 u, \quad (2.18)$$

is integrable (this is a well-known result, see for instance [30], [25]). Hence, we can explicitly find the quasi-periodic solutions “predicted” by Theorem 2.1. A simple analysis of this integrable system shows that the unstable periodic solution corresponds to a critical point, of center type, of system (2.18) on a level set of an integral Ω (see Section 3). The bifurcating quasi-periodic solutions correspond to periodic solutions close to this critical point (“close” since Theorem 2.1 is a local result; see Fig. 1). However, this local family of quasi-periodic solutions extends directly to a “global” family of quasi-periodic solutions that merges into a homoclinic solution. This structure cannot be explained by Theorem 2.1. In Sections 3 and 4 we shall show that a similar structure can be found in the case $\alpha = \epsilon a$ and $\beta = \epsilon b$ for some $0 < \epsilon \ll 1$.

Fig. 1. The phase-portrait of the unperturbed flow on a $\Omega = \Omega_0$ level set. The periodic solutions near centerpoint $(\rho_c, 0)$ correspond to the quasi-periodic solutions described by Theorem 2.1.

3. Existence and Uniqueness of Quasi-Periodic Solutions

3.1. The O.D.E. Reduction

In Section 2 we proved the existence of (traveling) quasi-periodic solutions of the Ginzburg-Landau equation near unstable periodic solutions of the type $R \exp[i(Lx + Wt)]$. We showed that these quasi-periodic solutions have the following structure

$$u(x, t) = \hat{u}(x + \nu t) e^{-i\omega t},$$

(the signs in front of the ν and the ω are chosen for convenience). Substituting this expression into (1.1) yields a 2-D complex o.d.e. for $\hat{u}(\xi)$, with $\xi = x + \nu t$. This o.d.e. has four parameters: α , β , ν , and ω . Since α and β are given by the equation, we will consider them to be fixed. The quantities ν and ω will be considered as free parameters that can be varied at will (it will turn out, as the outcome of some transformations, that this distinction is not important). We study this system near the integrable limit $\alpha = \beta = \nu = \omega = 0$ (see [25], [12], and the following paragraphs), hence we set $\alpha = \epsilon a$, $\beta = \epsilon b$, $\nu = \epsilon \nu$, and $\omega = \epsilon \omega$ for some $0 < \epsilon \ll 1$ (Corollary 2.2). Since there is a “phase-invariance” in the Ginzburg-Landau equation, we introduce polar coordinates:

$$u(x, t) = \hat{u}(x + \epsilon \nu t) e^{-i\epsilon \omega t} = \rho(\xi) \exp\{i[\theta(\xi) - \epsilon \omega t]\}.$$

This yields, by substitution, a 3-D real o.d.e. Hence, solutions of this three-dimensional system correspond to families of solutions of the p.d.e. because one can always shift the phase of a solution of this o.d.e. by a constant. We will discuss the correspondence between solutions of this o.d.e. and the p.d.e. without mentioning this (see also Remark 3.1). By defining $V(\xi) = \rho_\xi(\xi)$ and $\Omega(\xi) = \rho^2(\xi)\theta_\xi(\xi)$ one can write down this 3-D o.d.e.:

$$\begin{cases} \rho_\xi &= V, \\ V_\xi &= -\rho + \rho^3 + \frac{\Omega^2}{\rho^3} + \frac{\epsilon}{\Delta} \nu V - \frac{a\epsilon^2}{\Delta} \rho[(\omega - a) - \rho^2(b - a)] + \frac{a\epsilon^2}{\Delta} \nu \frac{\Omega}{\rho}, \\ \Omega_\xi &= -\frac{\epsilon}{\Delta} \rho^2[(\omega - a) - \rho^2(b - a)] + \frac{\epsilon}{\Delta} \nu \Omega - \frac{a\epsilon^2}{\Delta} \nu \rho V, \end{cases} \quad (3.1)$$

with $\Delta = 1 + \epsilon^2 a^2$. Note that $\Omega(\xi)$ is a conserved quantity in the integrable limit $\epsilon \downarrow 0$ (i.e., $\Omega_\xi(\xi) \equiv 0$). System (3.1) reduces for $\epsilon = 0$ to

$$\rho_{xx} + \rho - \rho^3 - \frac{\Omega_0^2}{\rho^3} = 0, \quad (3.2)$$

on a $\Omega = \Omega_0$ level set (with $\xi = x$ since $\nu = 0$). This planar system has a second integral

$$K = V^2 + \rho^2 - \frac{1}{2}\rho^4 + \frac{\Omega_0^2}{\rho^2}. \quad (3.3)$$

One can check that the unperturbed planar system has no bounded solutions for $|\Omega_0| > \sqrt{\frac{4}{27}}$. Note that the unbounded solutions of (3.1) or (3.2) correspond to

nonphysical solutions of the full p.d.e. We consider here only the part of the flow generated by (3.2) in the $\rho > 0$ half-plane, system (3.2) has two critical points in this half-plane for $|\Omega_0| \in (0, \sqrt{4/27})$: $\rho(\xi) = \rho_c$, a center with $\rho_c = \rho_c(\Omega_0) \in (0, \sqrt{2/3})$ and $\rho(\xi) = \rho_s > \rho_c$, a saddle with $\rho_s = \rho_s(\Omega_0) \in (\sqrt{2/3}, 1)$ (see Fig. 1). These critical points correspond to periodic solutions of the p.d.e.:

$$u(x, t) = u(x) = \rho_{c,s} \exp \{ i [(\Omega_0 / \rho_{c,s}^2)x + \phi_0] \},$$

with ϕ_0 an arbitrary phase. Hence, these critical points correspond to the periodic solutions considered in Section 2. Note that the saddles are stable as solutions of (1.1), see (2.11): $\rho_s^2 = R_0^2 > \frac{2}{3}$, but the centers are unstable. The quasi-periodic solutions of Theorem 2.1 are stationary (Corollary 2.2: $\epsilon = 0$, hence $\alpha = \beta = 0$) and, as was already remarked in Section 2, correspond to the periodic solutions of planar system (3.2) close to centerpoint ρ_c (see Fig. 1).

Define the value of (second) integral K , see (3.3), at $\rho(\xi) \equiv \rho_c$, $\Omega = \Omega_0$: $K_c = K_c(\Omega_0)$; analogously $K_s = K_s(\Omega_0) = K(\rho(\xi) \equiv \rho_s, \Omega = \Omega_0)$. We observe that there are periodic solutions to (3.2) for $K \in (K_c(\Omega_0), K_s(\Omega_0))$; hence, the family of locally bifurcating quasi-periodic solutions at an unstable periodic solution of type (2.1) of (1.1) can be extended globally. The “global” quasi-periodic solutions merge into a homoclinic solution for $K \uparrow K_s$. Note that this homoclinic solution corresponds to a nonmoving solitonlike “localized structure,” connecting to the periodic solution $\rho_s e^{il_s x}$, with $l_s = \Omega_0 / \rho_s^2$ as $x \rightarrow \infty$ and $x \rightarrow -\infty$.

In Section 2 we have shown that there exist quasi-periodic solutions close to the unstable periodic solutions of (1.1) (of type (2.1)) for any α and β . The main topic of the subsequent analysis is, Can this local family of quasi-periodic solutions be “embedded” in some kind of global family of quasi-periodic solutions, like it is the case for $\alpha = \beta = 0$? Also, do these global solutions again merge into some kind of solitonlike homoclinic, or perhaps heteroclinic, structure? Note that this structure probably will move with a certain wavespeed and a certain period in time. Another way to phrase these questions is, What happens to the integrable structure of (3.2), with $\Omega_0 \in [-\sqrt{4/27}, \sqrt{4/27}]$, for $\epsilon \neq 0$? Can we show the existence of periodic solutions to the perturbed system (3.1) by adjusting v and w ? What is the influence of the perturbation on the family of homoclinic solutions?

In this section we will study the existence of periodic solutions of (3.1). In Section 4 we will consider the last question and study (3.1) near the homoclinic connections. Before we can begin the analysis of (3.1), we need to gather some information on the integrable system. We define a region E in (K, Ω) -space:

$$E = \left\{ (K, \Omega) : \Omega \in \left[-\sqrt{\frac{4}{27}}, \sqrt{\frac{4}{27}} \right], K \in [K_c(\Omega), K_s(\Omega)] \right\}. \quad (3.4)$$

Thus, if $(K_0, \Omega_0) \in E - \partial E$ then (K_0, Ω_0) corresponds to a periodic solution of the integrable system. The boundary ∂E can be divided into $\partial E_c \cup \partial E_s \cup (2/3, \pm \sqrt{4/27})$: a point (K_0, Ω_0) on ∂E_c corresponds to a centerpoint of (3.2), (K_0, Ω_0) on ∂E_s corresponds to the homoclinic solution to the saddle ρ_s together with ρ_s itself, and the (degenerate) points $(2/3, \pm \sqrt{4/27})$ are related to the (saddle-node) bifurcation at which ρ_c and ρ_s merge. Region E is related to a volume S in (ρ, V, Ω) -space: ∂S is the surface of homoclinic loops of the integrable system; the interior of S consists of

Fig. 2. The region E in the (K, Ω) -plane.

all periodic solutions of the integrable system. Region E is plotted in Fig. 2; one can check that the boundary of E can be parametrized by

$$\partial E = \{(K, \Omega) : \Omega^2 = Y^2 - Y^3, K = 2Y - \frac{3}{2}Y^2, Y \in [0, 1]\}, \quad (3.5)$$

in which $Y = \rho_c^2$ for $Y \in [0, \frac{2}{3})$, $Y = \rho_s^2$ for $Y \in (\frac{2}{3}, 1]$. Region E , or, equivalently, volume S , will play an important role in the analysis of the perturbed system: system (3.1) can only have bounded solutions inside or $O(\epsilon)$ near to region E or S , since solutions of (3.1) will remain, on a $O(1)$ timescale, $O(\epsilon)$ close to solutions of the integrable system, which will all become unbounded outside E/S . Hence, the only physically relevant solutions of (3.1) are those that remain $O(\epsilon)$ close to E/S for all ξ .

Remark 3.1. We have introduced polar coordinates to eliminate the phase-invariance in the solutions of the Ginzburg-Landau equation (thus, we reduced a four-dimensional o.d.e. to a three-dimensional one). However, using polar coordinates can be dangerous, since it is not directly clear from (3.1) or (3.2) that the radius $\rho(\xi)$ cannot become zero. The flow generated by the integrable system will not pass through $\rho = 0$ for

$\Omega_0 \neq 0$; however, for $\Omega_0 = 0$ all noncritical solutions of (3.2) change sign. These solutions correspond to standing-wave solutions of (1.1) that periodically cross the origin of the complex plane. Hence, one has to be careful near $\Omega(\xi) = 0$ when analysing the flow of (3.1). We will find that we do not need to study this situation as long as $\nu \neq 0$ (see Remark 4.2); hence, this degeneration is not very important in this paper. In [12] we have studied (3.1) with $\nu = 0$; consequently in this system the $\Omega = 0$ plane and the probability of solutions crossing to the $\rho < 0$ half-space are very important. For more details on these degenerations, we refer to [12]. In Remark 4.9 we will again mention briefly the effect of the $\Omega = 0$ plane degeneration on our analysis.

3.2. Quasi-Periodic Solutions: The Main Theorem

The central question of this subsection is, For a given pair $(K_*, \Omega_*) \in E - \partial E$ corresponding to a periodic solution of the unperturbed system, can we determine a certain wave speed ν_* and frequency w_* such that this periodic solution “survives” the perturbation of (3.1)? This question will be answered (affirmatively) in Theorem 3.3. Before we can formulate this theorem we have to define a suitable Poincaré map, which will also be our main tool to analyze the flow generated by system (3.1). First, we rewrite system (3.1) into a somewhat more convenient form:

$$\begin{cases} \rho_\xi = V, \\ V_\xi = -\rho + \rho^3 + \frac{\Omega^2}{\rho^3} + \hat{\epsilon}BV - \hat{\epsilon}\tilde{\epsilon}\frac{1}{\rho}[A\rho^2 - \rho^4 - B\Omega], \\ \Omega_\xi = -\hat{\epsilon}[A\rho^2 - \rho^4 - B\Omega] - \hat{\epsilon}\tilde{\epsilon}B\rho V, \end{cases} \quad (3.6)$$

with $\hat{\epsilon} = (b - a)/\Delta\epsilon$ ($\Delta = 1 + \epsilon^2 a^2$), $\tilde{\epsilon} = a\epsilon = a\Delta/(b - a)\hat{\epsilon}$, and

$$A = \frac{w - a}{b - a}, \quad B = \frac{\nu}{b - a}. \quad (3.7)$$

In the sequel we will again drop the “hats” on the ϵ s (however, see Section 4.2). Note that $A = A(w)$ and $B = B(\nu)$ are the only two parameters left in the problem: we have reduced the four-dimensional parameter space to a two-dimensional one; consequently there is no difference between fixed parameters (a and b) and free parameters (ν and w), both A and B can be varied at will. Note that $a\Delta/(b - a)$ may, a priori, still have a direct influence by the $\tilde{\epsilon}$ terms; however, since these terms only appear at an $O(\epsilon^2)$ level, it will turn out that this influence is negligible. Although we assume throughout the subsequent analysis that $a \neq b$, this is no restriction: if $b = a$, one can make all perturbation terms disappear in (3.1) by choosing $w = b = a$, $\nu = 0$; hence, the system can be considered to be integrable if $b = a$.

Before we focus our attention on the question of the existence of periodic solutions of this system we make some basic observations. First we note that the critical points of (3.6) are also critical points of the integrable system (3.2), since in both the perturbed and the unperturbed case equations $V = 0$ and $\Omega^2 = \rho^4 - \rho^6$ have to be satisfied. This fact will be of crucial importance in the analysis of (3.6) near the perturbed homoclinic

Fig. 3. The number and character of the critical points of (3.6), as function of A and B , in the (A, B) -plane (Ce. = perturbed center point, Sa. = perturbed saddle point).

orbits of the integrable system. The additional equation $A\rho^2 - \rho^4 - B\Omega = 0$ has to be satisfied when $\epsilon \neq 0$; solving these three equations yields

$$\rho_{\pm}^2 = \frac{1}{2} \left[-(B^2 - 2A) \pm \sqrt{B^2(B^2 - 4A + 4)} \right]. \quad (3.8)$$

Hence, system (3.6) has either no critical points, two perturbed centers ($0 < \rho^2 < \frac{2}{3}$), two perturbed saddles ($\frac{2}{3} < \rho^2 < 1$), four (perturbed) centers, four (perturbed) saddles, two centers and two saddles, or is in a transition between two (or more) of these possibilities. In Fig. 3 we plot the various regions in the (A, B) -plane.

The divergence of the flow is nonzero, unlike the unperturbed case; hence, system (3.6) is nonconservative. Note that this system is conservative for $\nu = 0$, that is, $B = 0$. In that case there are also some additional symmetries: $\xi \rightarrow -\xi$, $\rho \rightarrow \rho$, $V \rightarrow -V$, $\Omega \rightarrow -\Omega$, and $\rho \rightarrow -\rho$, $V \rightarrow -V$ (this one is important near $\Omega_0 = 0$,

see [12] and Remark 3.1). The first symmetry is conserved as a symmetry in parameter space:

$$B \rightarrow -B, \quad \xi \rightarrow -\xi, \quad \rho \rightarrow \rho, \quad V \rightarrow -V, \quad \Omega \rightarrow -\Omega. \quad (3.9)$$

Hence, because there is a symmetry between the waves in the Ginzburg-Landau equation that travel to the right and the waves that travel to the left, one can assume $B \geq 0$.

As in Holmes [21] and Doelman [12] we analyse the flow governed by system (3.6) using a Poincaré map P ; the map P is defined only in an $O(\epsilon)$ neighborhood of region E , E_ϵ . Consider a solution $\Gamma_\epsilon(\xi) = (\rho_\epsilon(\xi), V_\epsilon(\xi), \Omega_\epsilon(\xi))$ of (3.6) with $\Gamma_\epsilon(0) = (\rho_0, 0, \Omega_0)$ and $(dV_\epsilon/d\xi)(0) < 0$; K_0 is the value of the second integral K at the point $(\rho_0, 0, \Omega_0)$: $K_0 = \rho_0^2 - \frac{1}{2}\rho_0^4 + (\Omega_0^2/\rho_0^2)$, see (3.3), $(K_0, \Omega_0) \in E_\epsilon$. Let $(\rho_1, 0, \Omega_1)$ be the next intersection of $\Gamma_\epsilon(\xi)$ with the $V = 0$ plane such that $(dV_\epsilon/d\xi) < 0$ (note that this point $(\rho_1, 0, \Omega_1)$ does not necessarily exist). The Poincaré map P is a map from E_ϵ to itself defined by

$$P(K_0, \Omega_0) = (K_1, \Omega_1),$$

with $K_1 = K(\rho = \rho_1, V = 0, \Omega = \Omega_1) = \rho_1^2 - \frac{1}{2}\rho_1^4 + (\Omega_1^2/\rho_1^2)$; $(K_1, \Omega_1) \in E_\epsilon$, due to the character of the flow induced by the unperturbed system outside S (the (ρ, V, Ω) -space equivalent of E). Define now $\Delta K(K_0, \Omega_0)$ and $\Delta\Omega(K_0, \Omega_0)$ by

$$P(K_0, \Omega_0) = (K_1, \Omega_1) = (K_0 + \Delta K(K_0, \Omega_0), \Omega_0 + \Delta\Omega(K_0, \Omega_0)).$$

The expressions ΔK and $\Delta\Omega$ are of magnitude $O(\epsilon)$ and can be calculated explicitly up to $O(\epsilon^2)$:

$$\Delta K(K_0, \Omega_0) = \int_0^{T_\epsilon(K_0, \Omega_0)} K_\xi(\Gamma_\epsilon(\xi)) d\xi,$$

where $T_\epsilon(K_0, \Omega_0)$ is the return-“time” of Γ_ϵ . By (3.3) and (3.6) we have

$$K_\xi = 2\epsilon \left[\frac{1}{2}B\rho^4 + (\Omega - B)\rho^2 + (BK - A\Omega) \right] + O(\epsilon^2).$$

Solution $\Gamma_\epsilon(\xi)$ of (3.6) can be approximated by a solution $\Gamma_0(\xi)$ of the unperturbed system: $\Gamma_0(\xi) = (\rho_0(\xi), V_0(\xi), \Omega_0)$ with $(\rho_0(\xi), V_0(\xi))$ satisfying (3.2); $\Gamma_0(0) = (\rho_0, 0, \Omega_0) = \Gamma_\epsilon(0)$. Note that if $(K_0, \Omega_0) \in E_\epsilon - E$ one has to adapt the initial value of $\Gamma_0(\xi)$ a little ($= O(\epsilon)$) such that $(\tilde{K}_0, \tilde{\Omega}_0) \in E$, this has only an $O(\epsilon^2)$ influence on $\Delta K(K_0, \Omega_0)$. Since $|\Gamma_\epsilon(\xi) - \Gamma_0(\xi)| = O(\epsilon)$ on a $O(1)$ “time”-scale, we may write

$$\Delta K(K_0, \Omega_0) = 2\epsilon \int_0^{T_0(K_0, \Omega_0)} \left[\frac{1}{2}B\rho_0^4 + (\Omega_0 - B)\rho_0^2 + (BK_0 - A\Omega_0) \right] d\xi + O(\epsilon^2),$$

with $T_0(K_0, \Omega_0)$ the period of periodic solution $(\rho_0(\xi), V_0(\xi))$ of (3.2). By solving for $\rho_\xi = V$ in (3.3) we can change variables:

$$\Delta K(K_0, \Omega_0) = 4\epsilon \int_{\check{\rho}(K_0, \Omega_0)}^{\hat{\rho}(K_0, \Omega_0)} \frac{[\frac{1}{2}B\rho^4 + (\Omega_0 - B)\rho^2 + (BK_0 - A\Omega_0)]}{\sqrt{\frac{1}{2}\rho^4 - \rho^2 - (\Omega_0^2/\rho^2) + K_0}} d\rho + O(\epsilon^2),$$

where $\check{\rho}(K_0, \Omega_0)$ and $\hat{\rho}(K_0, \Omega_0)$ are defined as the intersections of $(\rho_0(\xi), V_0(\xi))$ with the ρ -axis (see Fig. 1). Setting $R = \rho^2$ (and $\check{R} = \check{\rho}^2$, $\hat{R} = \hat{\rho}^2$) we find

$$\begin{aligned} \Delta K(K_0, \Omega_0) &= 2\epsilon \int_{\hat{R}(K_0, \Omega_0)}^{\check{R}(K_0, \Omega_0)} \frac{[\frac{1}{2}BR^2 + (\Omega_0 - B)R + (BK_0 - A\Omega_0)]}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}} dR + O(\epsilon^2), \\ &= \epsilon \oint \frac{[\frac{1}{2}BR^2 + (\Omega_0 - B)R + (BK_0 - A\Omega_0)]}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}} dR + O(\epsilon^2), \end{aligned} \quad (3.10)$$

a contour integral in the complex plane around the interval $[\check{R}(K_0, \Omega_0), \hat{R}(K_0, \Omega_0)]$ on the real line. Since

$$\oint \frac{(d/dR)(\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2)}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}} dR = 0,$$

we have

$$\oint \frac{R^2 dR}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}} = -\frac{2}{3} \oint \frac{(-2R + K_0) dR}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}}.$$

Using this we simplify (3.10) into

$$\Delta K(K_0, \Omega_0) = 2\epsilon \int_{\hat{R}(K_0, \Omega_0)}^{\check{R}(K_0, \Omega_0)} \frac{(\Omega_0 - \frac{1}{3}B)R + (\frac{2}{3}BK_0 - A\Omega_0)}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}} dR + O(\epsilon^2). \quad (3.11)$$

By introducing

$$T_0(K_0, \Omega_0) = \oint \frac{dR}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}}, \quad (3.12)$$

(note that $T_0(K_0, \Omega_0)$ is (again) the period of periodic solution $(\rho_0(\xi), V_0(\xi))$ of (3.2)) and

$$T_1(K_0, \Omega_0) = \oint \frac{R dR}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}}, \quad (3.13)$$

we finally arrive at

$$\Delta K(K_0, \Omega_0) = \epsilon \left[\left(\frac{2}{3}BK_0 - A\Omega_0 \right) T_0(K_0, \Omega_0) + \left(\Omega_0 - \frac{1}{3}B \right) T_1(K_0, \Omega_0) \right] + O(\epsilon^2). \quad (3.14)$$

Exactly along the same lines one derives

$$\Delta\Omega(K_0, \Omega_0) = -\epsilon \int_{\hat{R}(K_0, \Omega_0)}^{\tilde{R}(K_0, \Omega_0)} \frac{(A - \frac{4}{3})R + (\frac{2}{3}K_0 - B\Omega_0)}{\sqrt{\frac{1}{2}R^3 - R^2 + K_0R - \Omega_0^2}} dR + O(\epsilon^2), \quad (3.15)$$

as in (3.11), or

$$\Delta\Omega(K_0, \Omega_0) = -\frac{1}{2}\epsilon \left[\left(\frac{2}{3}K_0 - B\Omega_0 \right) T_0(K_0, \Omega_0) + \left(A - \frac{4}{3} \right) T_1(K_0, \Omega_0) \right] + O(\epsilon^2). \quad (3.16)$$

Remark 3.2. These expressions are a priori only valid for K_0 not close to $K_c(\Omega_0)$ or $K_s(\Omega_0)$. However, approximations $\Delta K(K_0, \Omega_0)$ and $\Delta\Omega(K_0, \Omega_0)$ of the Poincaré map P are valid for $K \uparrow K_s(\Omega_0)$ and are even valid at $K = K_s(\Omega_0)$ if ΔK and $\Delta\Omega$ are finite at $(K_s(\Omega_0), \Omega_0)$. The quantities ΔK and $\Delta\Omega$ are Melnikov functions in this case and measure the splitting-distance between the (fast) stable and (fast) unstable manifolds of the (perturbed) saddle point of (3.6). The quantities $\Delta K(K_s(\Omega_0), \Omega_0)$ and $\Delta\Omega(K_s(\Omega_0), \Omega_0)$ can be finite only if the point $(\rho_s(\Omega_0), 0, \Omega_0)$ is (also) a critical point of the perturbed system (see also Section 4 in which we discuss the significance of the $O(\epsilon)$ slow manifolds). For more details (and proofs) we refer to [18] and [38]. However, as we shall see in Section 4, one should remain careful in using (3.14) and (3.16) near $K_s(\Omega_0)$. Furthermore, there can also be “problems” near $K_c(\Omega_0)$: there Ω_ξ is (also) not small compared to ρ_ξ and V_ξ ; hence one cannot approximate the solutions of (3.6) by the solutions of the unperturbed system (for more details on this, see [12]). Since the bifurcation analysis of Section 2 is valid in this region, it is not necessary to consider this situation in this paper.

We now return to our main question: can a periodic solution of the unperturbed system, corresponding to a point $(K_0, \Omega_0) \in E - \partial E$ “survive” the perturbation? We can now reformulate this question in terms of the Poincaré map P : can we choose parameters A and B such that the Poincaré map associated with the perturbed system (3.6) has a fixed point at a given point $(K_0, \Omega_0) \in E - \partial E$? The return-map P has a fixed point at (K_0, Ω_0) if $\Delta K(K_0, \Omega_0) = \Delta\Omega(K_0, \Omega_0) = 0$; that is, by (3.14) and (3.16), P has a fixed point at (K_0, Ω_0) if

$$\begin{pmatrix} -\Omega_0 T_0 & -\frac{1}{3}T_1 + \frac{2}{3}K_0 T_0 \\ T_1 & -\Omega_0 T_0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\Omega_0 T_1 \\ \frac{4}{3}T_1 - \frac{2}{3}K_0 T_0 \end{pmatrix} + O(\epsilon), \quad (3.17)$$

where $T_i = T_i(K_0, \Omega_0)$, $i = 1, 2$. This is a linear 2×2 system in unknowns A and B . Note that we may drop the $O(\epsilon)$ term in (3.17) from now on: if one can find a solution $(\tilde{A}_0, \tilde{B}_0)$ to (3.17) for $\epsilon = 0$ then the implicit function theorem guarantees the existence of a solution (A_0, B_0) to the $\epsilon \neq 0$ system $O(\epsilon)$ close to $(\tilde{A}_0, \tilde{B}_0)$ (if some nondegeneracy conditions are satisfied, see the following paragraphs below). Hence, one expects that for any given (K_0, Ω_0) there will be a unique solution (A_0, B_0) for linear system (3.17), unless the determinant of this system is zero. We can now formulate the main result of this section.

Theorem 3.3. *For any $(K_0, \Omega_0) \in E - \partial E$ there exists a unique pair $(A_0(K_0, \Omega_0), B_0(K_0, \Omega_0))$ such that the Poincaré map P , associated with system (3.6) in which $A = A_0$ and $B = B_0$, has a fixed point at (K_0, Ω_0) .*

This result yields that to any stationary quasi-periodic solution of the real Ginzburg-Landau equation there corresponds a unique slowly moving quasi-periodic solution to a complex Ginzburg-Landau equation with coefficients with small imaginary parts ϵa and ϵb . These solutions have a well-defined (slow) wave speed ϵv and a (slow) frequency $\epsilon \omega$. Hence we have extended the local existence of traveling quasi-periodic solutions, which was a result of the bifurcation analysis of Section 2, to a global existence (and uniqueness) result. In Section 4 we shall determine a region F in the (A, B) -plane that determines for which values of A and B and thus, by (3.7), for which wave speeds v and frequencies ω , the complex Ginzburg-Landau equation with coefficients with small imaginary parts can have traveling quasi-periodic solutions.

The only thing needed for a proof of Theorem 3.3 is a result that controls the determinant of the matrix in (3.17): we have to show that

$$-T_1(K, \Omega) \left[\frac{1}{3} T_1(K, \Omega) - \frac{2}{3} K T_0(K, \Omega) \right] - \Omega^2 T_0^2(K, \Omega) \neq 0, \quad (3.18)$$

for all $(K, \Omega) \in E - \partial E$. Note that if this determinant is not equal to zero, one can invert the matrix in (3.17), and thus one can uniquely solve (3.17) for $\epsilon \neq 0$. We will present the proof of Theorem 3.3 in the following subsection.

3.3. The Proof of Theorem 3.3: Analysing T_0 and T_1

This subsection is almost completely devoted to the proof of Theorem 3.3. This proof is based on a detailed analysis of the elliptic expressions $T_0(K, \Omega)$ and $T_1(K, \Omega)$ (see (3.12) and (3.13)). The results of this analysis will be presented in a number of lemmas, and the proof of Theorem 3.3 will be a consequence of the implications of these lemmas. The lemmas are also important as isolated results: some of them will be used again in Section 4. It should be noted that although the results obtained in this subsection are based completely on the properties of the unperturbed, integrable system, only by applying the outcome of the analysis to the proof of Theorem 3.3 can we make the connection to the perturbed system. Furthermore, the methods developed in this subsection can be applied to more general integrable systems, yielding similar conclusions (see Remark (3.13)).

Theorem 3.3 is true unless

$$\frac{1}{3} \left(\frac{T_1}{T_0} \right) - \frac{2}{3} K \left(\frac{T_1}{T_0} \right) + \Omega^2 = 0 \quad (3.19)$$

somewhere inside E . Note that $T_0(K, \Omega)$ is always (strictly) positive since T_0 is the period of a periodic solution of the unperturbed system for $(K, \Omega) \in E$. To avoid an overly cumbersome notation, we introduce

$$\chi(K, \Omega) = \frac{T_1(K, \Omega)}{T_0(K, \Omega)}. \quad (3.20)$$

By solving (3.19) we observe that we need to show that

$$\chi(K, \Omega) \neq K \pm \sqrt{K^2 - 3\Omega^2}. \quad (3.21)$$

Using (3.5) one can check that $K^2 - 3\Omega^2 \geq 0$ in E . Equation (3.21) defines two functions $F^+(K, \Omega)$ and $F^-(K, \Omega)$,

$$F^\pm(K, \Omega) = K \pm \sqrt{K^2 - 3\Omega^2}, \quad (3.22)$$

which are two separated parts of a branch of a hyperbola if Ω is kept fixed at a certain value Ω_0 (see Fig. 4). By consulting (3.12) and (3.13) we find

$$\chi(K_c(\Omega_0), \Omega_0) = \rho_c^2, \quad \chi(K_s(\Omega_0), \Omega_0) = \rho_s^2 \quad (3.23)$$

(note that both T_0 and T_1 diverge if $K \uparrow K_s(\Omega_0)$; however, $\chi(K_s(\Omega_0), \Omega_0)$ is well defined and equal to $R_s = \rho_s^2$ since T_0 and T_1 are completely determined by their (blow-up) behavior near R_s in the limit $K \uparrow K_s(\Omega_0)$). Using (3.5) one also observes

$$\begin{aligned} F^\pm(K_{c,s}(\Omega_0), \Omega_0) &= 2Y - \frac{3}{2}Y^2 \pm \sqrt{\frac{9}{4}Y^4 - 3Y^3 + Y^2}, \\ &= 2Y - \frac{3}{2}Y^2 \pm Y|(1 - \frac{3}{2}Y)|. \end{aligned}$$

Fig. 4. Functions $F^+(K, \Omega_0)$ and $F^-(K, \Omega_0)$ on the interval $[K_c(\Omega_0), K_s(\Omega_0)]$. The dotted line represents the curve of $\chi(K, \Omega_0) = T_1(K, \Omega_0)/T_0(K, \Omega_0)$ (see Lemmas 3.4–3.12).

Hence, since $Y = \rho_c^2$ for $0 < Y < \frac{2}{3}$ and $Y = \rho_s^2$ for $\frac{2}{3} < Y < 1$, we have

$$\begin{aligned} F^+(K_2(\Omega_0), \Omega_0) &= Y_s = \rho_s^2(\Omega_0), \\ F^-(K_c(\Omega_0), \Omega_0) &= Y_c = \rho_c^2(\Omega_0). \end{aligned}$$

Thus, determinant (3.18) of system (3.17) $\equiv 0$ on ∂E , or

$$\begin{aligned} F^+(K_s(\Omega_0), \Omega_0) &= \chi(K_s(\Omega_0), \Omega_0), \\ F^-(K_c(\Omega_0), \Omega_0) &= \chi(K_c(\Omega_0), \Omega_0). \end{aligned}$$

We will prove Theorem 3.3 by showing that

$$F^+(K, \Omega_0) > \chi(K, \Omega_0) \quad \text{for } K < K_s(\Omega_0) \quad (3.24)$$

and

$$F^-(K, \Omega_0) < \chi(K, \Omega_0) \quad \text{for } K > K_c(\Omega_0) \quad (3.25)$$

(see Fig. 4). These two inequalities will follow from the subsequent lemmas. First, we study the behavior of $\chi(K, \Omega)$ as function of K .

Lemma 3.4. *For $(K, \Omega) \in E - \partial E$ we have*

$$\frac{\partial}{\partial K} \chi = \frac{1}{2D} \left[(2K^2 - 8\Omega^2 + 3K\Omega^2) - 4(K^2 - 3\Omega^2)\chi + \left(K - \frac{9}{2}\Omega^2 \right) \chi^2 \right] \quad (3.26)$$

with

$$D = D(K, \Omega) = 2K^2 - 4K^3 + 18K\Omega^2 - 8\Omega^2 - \frac{27}{2}\Omega^4. \quad (3.27)$$

Proof. This differential equality can be obtained like a similar result in [1]. Define

$$J_i(K, \Omega) = \oint \frac{R^i}{\left(\sqrt{1/2R^3 - R^2 + KR - \Omega^2} \right)^3} dR$$

($i = 0, 1, 2, \dots$). Then, since

$$T_i(K, \Omega) = \oint \frac{R^i (1/2R^3 - R^2 + KR - \Omega^2)}{\left(\sqrt{1/2R^3 - R^2 + KR - \Omega^2} \right)^3} dR,$$

we find

$$\begin{aligned} T_0 &= \frac{1}{2}J_3 - J_2 + KJ_1 - \Omega^2J_0, \\ T_1 &= \frac{1}{2}J_4 - J_3 + KJ_2 - \Omega^2J_1. \end{aligned}$$

Furthermore,

$$\oint \frac{d}{dR} \left(\frac{R^i}{\sqrt{1/2R^3 - R^2 + KR - \Omega^2}} \right) dR = 0,$$

which yields

$$\frac{3}{2}J_2 - 2J_1 + KJ_0 = 0,$$

$$\frac{3}{2}J_3 - 2J_2 + KJ_1 = 2T_0,$$

$$\frac{3}{2}J_4 - 2J_3 + KJ_2 = 4T_1.$$

All in all, we now have five linear equations in five unknowns $J_0, J_1, \dots, J_4$. Solving these equations results in

$$J_1(K, \Omega) = \frac{-1}{D(K, \Omega)} [2(K^2 - 3\Omega^2)T_0(K, \Omega) - (K - \frac{9}{2}\Omega^2)T_1(K, \Omega)], \quad (3.28)$$

$$J_2(K, \Omega) = \frac{-1}{D(K, \Omega)} [(2K^2 - 8\Omega^2 + 3K\Omega^2)T_0(K, \Omega) - 2(K^2 - 3\Omega^2)T_1(K, \Omega)], \quad (3.29)$$

with $D(K, \Omega)$ as defined in (3.27). Similar expressions can be obtained for J_0, J_3 , and J_4 . Note that

$$\frac{\partial}{\partial K} T_i(K, \Omega) = -\frac{1}{2} J_{i+1}(K, \Omega), \quad (3.30)$$

for $i = 0, 1, 2, \dots$, and thus

$$\frac{\partial}{\partial K} \chi = \partial \frac{\partial}{\partial K} \left(\frac{T_1}{T_0} \right) = -\frac{T_0 J_2 - T_1 J_1}{2T_0^2}.$$

Substitution of (3.28) and (3.29) into this expression yields Lemma 3.4.

Based on this lemma we can show

Lemma 3.5.

$$\frac{\partial}{\partial K} \chi(K, \Omega) > 0 \quad \text{for } (K, \Omega) \in E - \partial E.$$

Proof. By substitution of parametrization (3.5) of ∂E into (3.27) we obtain: $D(K, \Omega) \equiv 0$ on ∂E . Analyzing D in the interior of E yields $D(K, \Omega) > 0$ for $(K, \Omega) \in E - \partial E$. We define the quadratic polynomial $P_{K,\Omega}(X)$:

$$P_{K,\Omega}(X) = (2K^2 - 8\Omega^2 + 3K\Omega^2) - 4(K^2 - 3\Omega^2)X + (K - \frac{9}{2}\Omega^2)X^2; \quad (3.31)$$

that is, $P_{K,\Omega}(X = \chi(K, \Omega))$ is the numerator of (3.26). It is easy to check that the discriminant of polynomial $P_{K,\Omega}(X)$ is equal to $-KD(K, \Omega)$, which is negative definite inside E . Hence, since the coefficient $(K - \frac{9}{2}\Omega^2)$ in front of the X^2 term in $P_{K,\Omega}(X)$ is positive inside E , we may conclude that $P_{K,\Omega}(\chi(K, \Omega)) > 0$ in $E - \partial E$, and thus, by (3.26), we may conclude that $(\partial/\partial K)\chi > 0$ inside E .

Since $F_-(K, \Omega_0)$ is a decreasing function of K (we fixed $\Omega = \Omega_0$, see Fig. 4), we can conclude that inequality (3.25) is true. Inequality (3.24) needs some more work; we have to determine an upper bound for $\chi(K, \Omega_0)$. Note that both Lemma 3.4 and Lemma 3.5 are only valid in the interior of E . The behavior (and the existence) of $(\partial/\partial K)\chi$ near ∂E is considered in Lemmas 3.9, 3.10, and 3.12. Observe that it is a priori not possible to extend (3.26) to $(K, \Omega) \in \partial E$ since $D(K, \Omega) = 0$ on ∂E .

Define $X_{min}(K, \Omega)$ to be the value of X at which the polynomial $P_{K,\Omega}(X)$ is minimal:

$$X_{min}(K, \Omega) = \frac{2(K^2 - 3\Omega^2)}{K - 9/2\Omega^2}. \quad (3.32)$$

Note that the denominator of (3.32) is positive inside E . Since $\chi(K, \Omega_0)$ is an increasing function of K and we know by (3.23) that $\chi(K_{c,s}(\Omega_0), \Omega_0) = \rho_{c,s}^2(\Omega_0) = Y_{c,s}(\Omega_0)$ (with $0 \leq Y_c(\Omega_0) \leq \frac{2}{3} \leq Y_s(\Omega_0) \leq 1$), we are only interested in $X \in [Y_c(\Omega_0), Y_s(\Omega_0)]$ for $P_{K,\Omega_0}(X)$. Simple calculations yield the following:

Lemma 3.6.

$$X_{min}(K_c(\Omega_0), \Omega_0) = Y_c = \chi(K_c(\Omega_0), \Omega_0),$$

$$X_{min}(K_s(\Omega_0), \Omega_0) = Y_s = \chi(K_s(\Omega_0), \Omega_0),$$

and

$$\frac{\partial}{\partial K} X_{min}(K, \Omega_0) > 0 \quad \text{in } E - \partial E.$$

Hence $Y_c(\Omega_0) \leq X_{min}(K, \Omega_0) \leq Y_s(\Omega_0)$ for $K \in [K_c(\Omega_0), K_s(\Omega_0)]$.

Inequality (3.24) will be a consequence of the result $Y_c \leq \chi(K, \Omega_0) \leq X_{min}(K, \Omega_0)$ (Lemma 3.11). However, before we can show this, we need some preliminary results. The next lemma is based on classical results, see for instance Lawden [27].

Lemma 3.7.

$$T_0(K, \Omega) = \frac{2\sqrt{2}}{\sqrt{R_3 - \hat{R}}} \mathcal{H}\left(\sqrt{\frac{\check{R} - \hat{R}}{R_3 - \hat{R}}}\right), \quad (3.33)$$

in which $\mathcal{H}(k) = \int_0^{\pi/2} (1/\sqrt{1 - k^2 \sin^2 \theta}) d\theta$, the complete elliptic integral of the first kind. The quantities $\hat{R}$ and $\check{R}$ are already used in (3.10); R_3 , $\check{R}$, and $\hat{R}$ are the three ze-

ros of the polynomial $\frac{1}{2}R^3 - R^2 + KR - \Omega^2$ with $R_3 = R_3(K, \Omega) \geq \check{R}(K, \Omega) \geq \hat{R}(K, \Omega) \geq 0$. Furthermore, for $0 < \delta \ll 1$,

$$T_0(K_c(\Omega_0) + \delta, \Omega_0) = \frac{\pi \sqrt{2}}{\sqrt{2 - 3Y_c}} + O(\delta). \quad (3.34)$$

Proof. Equality (3.33) is based on straightforward computations. Due to the term $\sqrt{(\check{R} - \hat{R})/(R_3 - \hat{R})}$ in (3.33) approximation (3.34) is not immediately clear. One way to show (3.34) is by using yet another description of T_0 :

$$T_0(K, \Omega) = \frac{\pi \sqrt{2}}{\mathcal{M}(\sqrt{R_3 - \check{R}}, \sqrt{R_3 - \hat{R}})}, \quad (3.35)$$

with $\mathcal{M}(a, b)$ the algebraic-geometric mean of a and b , defined by $0 < a = a_0 \leq b = b_0$, $a_{n+1} = \sqrt{a_n b_n}$, $b_{n+1} = \frac{1}{2}(a_n + b_n)$, $\mathcal{M}(a, b) = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$. This equality is based on $\mathcal{H}(k) = [\pi/2\mathcal{M}(\sqrt{1 - k^2}, 1)]$ (see again [27]). Observe that $T_0(K, \Omega)$ is completely under control now by this explicit formulation (and $T_0(K, \Omega)$ is now easy to compute numerically, see [6]). Approximation (3.34) follows from the (easily derivable) fact: $\mathcal{M}(a + s\hat{\delta}, a + t\hat{\delta}) = a + \frac{1}{2}(s + t)\hat{\delta} + O(\hat{\delta}^2)$ (for any $0 < \hat{\delta} \ll 1$), combined with the observations $\hat{R}(K_c(\Omega_0) + \delta, \Omega_0) = Y_c - y_c \sqrt{\delta} + O(\delta)$ and $\check{R}(K_c(\Omega_0) + \delta, \Omega_0) = Y_c + y_c \sqrt{\delta} + O(\delta)$ for some $y_c > 0$. Note that the quantity $(\pi \sqrt{2} / \sqrt{2 - 3Y_c})$ is equal to the period of the linearized system of (3.2) around center $(\rho_c(\Omega_0), 0)$.

Remark 3.8. By substitution of $K = K_c + \delta$ and the approximations of $\hat{R}$ and $\check{R}$ into integral expression (3.12) one obtains by a direct perturbation expansion only a “deviation” of $O(\sqrt{\delta})$ in (3.34) instead of $O(\delta)$. This is not enough, since we need a result on the differentiability of $T_0(K, \Omega_0)$ at $K = K_c$.

Lemma 3.9. $T_0(K, \Omega_0)$ and $T_1(K, \Omega_0)$ are (continuously) differentiable with respect to K in $K = K_c(\Omega_0)$.

Proof. The result for $T_0(K, \Omega_0)$ follows immediately from (3.34): $\lim_{\delta \downarrow 0} (1/\delta)(T_0(K_c + \delta, \Omega) - T_0(K_c, \Omega))$ exists. Hence, by (3.30), $J_1(K_c, \Omega_0)$ is well defined. Substituting $K = K_c + \delta$ into (3.28) and expanding (using (3.5)) $D(K_c + \delta, \Omega_0)$ in powers of δ , we find that $D(K_c + \delta, \Omega_0) = 3\delta(2 - 3Y_c)^2 + O(\delta^2)$, which yields that $T_1(K_c + \delta, \Omega_0) = \text{some constant} + O(\delta)$. Hence, $(\partial/\partial K)T_1(K_c(\Omega_0), \Omega_0)$ exists.

Lemma 3.10.

$$\frac{\partial}{\partial K} \chi(K_c(\Omega_0), \Omega_0) = \frac{4 - 3Y_c}{2(2 - 3Y_c)^2}. \quad (3.36)$$

Proof. By Lemma 3.9 we know that $\chi(K_c(\Omega_0) + \delta, \Omega_0) = Y_c + s\delta + \text{h.o.t.}$ Substituting this and the expansion $K = K_c + \delta$ into (3.26) yields (using (3.5))

$$\frac{\partial}{\partial K} \chi(K_c(\Omega_0) + \delta, \Omega_0) = \frac{\delta(4 - 3Y_c) + O(\delta^2)}{2\delta(2 - 3Y_c)^2 + O(\delta^2)}.$$

Hence, the result follows.

A priori one might remark that the analysis in the previous lemmas is somewhat overdetailed; however, we shall see in Lemma 3.12 that this type of reasoning is necessary to obtain the correct results on $(\partial/\partial K)\chi$ near $K = K_s$. Using the differentiability of χ as function of K at $K = K_c$ we can derive the following essential result:

Lemma 3.11. $Y_c(\Omega_0) \leq \chi(K, \Omega_0) \leq X_{min}(K, \Omega_0)$ for all $K \in [K_c(\Omega_0), K_s(\Omega_0)]$ (where X_{min} is defined in (3.32)).

Proof. First we note that both χ and X_{min} are continuous functions on $[K_c, K_s]$ connecting Y_c and Y_s . By (3.32) we find

$$\frac{\partial}{\partial K} X_{min}(K, \Omega_0) = 2 \frac{K^2 - 9K\Omega_0^2 + 3\Omega_0^2}{(K - 9/2\Omega_0^2)^2}, \quad (3.37)$$

and, by (3.5),

$$\frac{\partial}{\partial K} X_{min}(K_c(\Omega_0), \Omega_0) = 2 \frac{7 - 6Y_c}{(2 - 3Y_c)^2}.$$

Comparing this to (3.36) in Lemma 3.10 yields $(\partial/\partial K)X_{min}(K_c(\Omega_0), \Omega_0) > (\partial/\partial K)\chi(K_c(\Omega_0), \Omega_0)$ for all $|\Omega_0| < \sqrt{4/27}$ (note that $Y_c = Y_s = 2/3$ for $|\Omega_0| = \sqrt{4/27}$). Hence $\chi(K, \Omega) < X_{min}(K, \Omega)$ for K slightly larger than K_c . If χ passes X_{min} , then $\chi(K_*, \Omega_0) = X_{min}(K_*, \Omega_0)$ at a certain K_* ; $(\partial/\partial K)\chi$ can be computed explicitly at this point:

$$\frac{\partial}{\partial K} \chi(K_*, \Omega_0) = \frac{P_{K_*, \Omega_0}(X_{min})}{2D(K_*, \Omega_0)} = \frac{K_*}{2[K_* - (9/2)\Omega_0^2]} \quad (3.38)$$

(using (3.26), (3.27), and (3.32)). However, direct computations (i.e., comparing (3.37) and (3.38)) show that $(\partial/\partial K)X_{min}(K_*, \Omega_0) > (\partial/\partial K)\chi(K_*, \Omega_0)$ for any $K_* > K_c(\Omega_0)$. Thus, χ cannot pass X_{min} , and Lemma 3.11 has been proven.

Now we can finish the proof of Theorem 3.3 by showing that inequality (3.24) holds. Due to Lemma 3.11 this can be achieved by showing that

$$F^+(K, \Omega_0) \geq X_{min}(K, \Omega_0) \quad \text{for } K < K_s(\Omega_0), \quad (3.39)$$

which is a straightforward computation since we have determined explicit formulas for F^+ and X_{min} ((3.22) and (3.32)). Inequality (3.24) can, for instance, be justified by observing that $(\partial^2/\partial K^2)F^+(K, \Omega_0) \leq 0$ (see Fig. 4) and $(\partial^2/\partial K^2)X_{min}(K, \Omega_0) \geq 0$ (note that $F^+(K, 0) = 2K = X_{min}(K, 0) > \chi(K, 0)$ on $(0, \frac{1}{2})$).

In the next section we will need to have some knowledge of the behavior of $\chi(K, \Omega)$ near $K = K_s(\Omega)$. Observe that one can expect a somewhat singular

behavior since $\chi(K, \Omega_0) < X_{\min}(K, \Omega_0)$ in $(K_c(\Omega_0, K_s(\Omega_0)))$ and $\chi(K_s(\Omega_0), \Omega_0) = X_{\min}(K_s(\Omega_0), \Omega_0)$, while $(\partial/\partial K)\chi(K, \Omega_0) < (\partial/\partial K)X_{\min}(K, \Omega_0)$ if $\chi(K, \Omega_0) = X_{\min}(K, \Omega_0)$ for any $K \in [K_c(\Omega_0, K_s(\Omega_0))]$: it seems quite difficult for $\chi(K, \Omega_0)$ to “catch up” with $X_{\min}(K, \Omega_0)$.

Lemma 3.12. *For $0 < \delta \ll 1$ we have*

$$\chi(K_s(\Omega_0) - \delta, \Omega_0) = Y_s(\Omega_0) - \frac{1}{4(3Y_s - 2)|\log \delta|} + \text{h.o.t.} \quad (3.40)$$

(thus: $(\partial/\partial K)\chi(K, \Omega_0) \rightarrow \infty$ for $K \uparrow K_s(\Omega_0)$).

Proof. Set $K = K_s(\Omega_0) - \delta$; since $\chi(K, \Omega_0)$ is a continuous function of K , we can set $\chi(K_s(\Omega_0) - \delta, \Omega_0) = Y_s(\Omega_0) - \sigma(\delta)$ for some function $\sigma(\delta)$ with $\sigma(\delta) \downarrow 0$ if $\delta \downarrow 0$. Substituting this into (3.26) (and using (3.5)) yields

$$\frac{\partial}{\partial K}\chi(K_s(\Omega_0) - \delta, \Omega_0) = \frac{-\delta(4/3 - Y_s) - 1/2\sigma^2(Y_s - 2/3) + O(\delta^2, \sigma\delta)}{-6\delta(Y_s - 2/3)^2 + O(\delta^2)}. \quad (3.41)$$

If we assume that $\sigma(\delta) = s\delta + \text{h.o.t.}$ for some number s , then $(\partial/\partial K)\chi(K_s(\Omega_0), \Omega_0)$ exists and is equal to s (as in Lemma 3.10). Thus, $(\partial/\partial K)\chi(K_s(\Omega_0), \Omega_0)$ can be computed and compared to $(\partial/\partial K)X_{\min}(K_s(\Omega_0), \Omega_0)$. This comparison yields that $(\partial/\partial K)\chi(K_s(\Omega_0), \Omega_0) < (\partial/\partial K)X_{\min}(K_s(\Omega_0), \Omega_0)$; hence, $\chi(K, \Omega_0)$ cannot arrive at the same point, Y_s , as $X_{\min}(K, \Omega_0)$ for $K \uparrow K_s(\Omega_0)$ (Lemma 3.11), which contradicts Lemma 3.6. So, $\sigma(\delta) \neq O(\delta)$, $(\sigma(\delta)/\delta) \rightarrow \infty$ if $\delta \downarrow 0$. Using this observation in (3.41) yields

$$\frac{\partial}{\partial K}\chi(K_s(\Omega_0) - \delta, \Omega_0) = \frac{\sigma^2}{4\delta(3Y_s - 2)} + \text{h.o.t.}$$

Since $\chi(K_s(\Omega_0) - \delta, \Omega_0) = Y_s(\Omega_0) - \sigma(\delta)$, we have $(\partial/\partial K)\chi = (d/d\delta)\sigma$, that is,

$$\frac{d}{d\delta}\sigma = \frac{\sigma^2}{4\delta(3Y_s - 2)} + \text{h.o.t.}$$

(note that $Y_s > \frac{2}{3}$). Solving this equation results in (3.40).

Remark 3.13. As we already noted in the introduction, the control on the approximation of the Poincaré map obtained from the properties of T_0 and T_1 that we have derived in the lemmas of this subsection can also be obtained for more general integrable systems. Furthermore, the preceding lemmas can be used (for this or for more general systems) to construct a converging sequence of upper and lower bounds on $\chi = (T_1/T_0)$. The upper and lower bounds of Lemma 3.11 can be used to construct two sharper bounds by substituting them into (3.26). Lower bound $Y_c(\Omega) = L_1(K, \Omega)$ yields an upper bound $U_2(K, \Omega)$ by integrating, and upper bound $X_{\min}(K, \Omega) = U_1(K, \Omega)$ gives a lower bound $L_2(K, \Omega)$. Analogously, these bounds lead, by substituting in

(3.26) and integrating, to two yet sharper bounds $L_3(K, \Omega)$ and $U_3(K, \Omega)$, and so on. Although one has to be aware of some subtleties (the behavior of the U_n for $n > 1$ near $K_s(\Omega)$), one can show that this “squeezing” sequence converges to $\chi(K, \Omega)$.

4. Heteroclinic Cycles

4.1. Taking the Limit: $(K, \Omega) \rightarrow \partial E$

In this section we study the influence of the perturbation on the family of homoclinic solutions to the integrable system. Since Theorem 3.3 is valid everywhere in $E - \partial E$, we can study the limit behavior of $A_0(K_0, \Omega_0)$ and $B_0(K_0, \Omega_0)$ (as defined in this theorem) for $K_0 \rightarrow K_s(\Omega_0)$, that is, by considering the limit in which the periodic solutions of (3.2) merge into the homoclinic solution. Thus, before we can set up a detailed analysis of system (3.6) near the saddle points, we have to find these critical values A^* and B^* of A_0 and B_0 by studying the behavior of the solutions $(A_0(K_0, \Omega_0), B_0(K_0, \Omega_0))$ for $K_0 \rightarrow K_s(\Omega_0)$. Doing so, we are determining one of the boundaries of a region F in the (A, B) -plane, which is defined by the following: if $(A_0, B_0) \in F$ then there is a $(K_0, \Omega_0) \in E$ such that the quasi-periodic solution of (3.2) related to the point (K_0, Ω_0) is a fixed point of the Poincaré map P associated to system (3.6) with $A = A_0$ and $B = B_0$. In other words, points inside F correspond, by solving (3.7), to wave speed–frequency pairs (v, w) at which the Ginzburg-Landau equation has quasi-periodic traveling wave solutions (of type (2.12)). There are no quasi-periodic traveling waves for (v, w) -combinations outside F . Hence, region F is in itself an important object, and it can be determined by solving linear system (3.17):

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{-\frac{1}{3}T_1^2 + \frac{2}{3}KT_1T_0 - \Omega^2T_0^2} \begin{pmatrix} -\frac{4}{9}T_1^2 + (\frac{10}{9}K - \Omega^2)T_1T_0 - \frac{4}{9}K^2T_0^2 \\ \Omega(-T_1^2 + \frac{4}{3}T_1T_0 - \frac{2}{3}KT_0^2) \end{pmatrix},$$

(we have ignored the $O(\epsilon)$ term). By introducing $\chi = \chi(K, \Omega)$, as in (3.20), we find

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{-\frac{1}{3}\chi^2 + \frac{2}{3}K\chi - \Omega^2} \begin{pmatrix} -\frac{4}{9}\chi^2 + (\frac{10}{9}K - \Omega^2)\chi - \frac{4}{9}K^2 \\ \Omega(-\chi^2 + \frac{4}{3}\chi - \frac{2}{3}K) \end{pmatrix}. \quad (4.1)$$

As noted above, the boundaries of F can be determined by considering the limits $K \rightarrow K_c(\Omega_0)$ and $K \rightarrow K_s(\Omega_0)$ for $\Omega_0 \in (-\sqrt{4/27}, \sqrt{4/27})$ in (4.1). The following theorem gives a precise description of F and ∂F :

Theorem 4.1. *F is a bounded region, and its boundary can be divided into three parts: $\partial F = \partial F_c \cup (\frac{4}{3}, \pm \frac{2}{3}\sqrt{3}) \cup \partial F_s$ with (up to $O(\epsilon)$),*

$$\partial F_c = \left\{ (A, B) : A(Y) = \frac{Y(6 - 7Y)}{4 - 5Y}, B(Y) = \pm \frac{2Y\sqrt{1 - Y}}{4 - 5Y}, Y = Y_c \in \left[0, \frac{2}{3}\right] \right\}, \quad (4.2)$$

Fig. 5. The region F in the (A, B) -plane.

$$\partial F_s = \left\{ (A, B) : A(Y) = 2 - Y, B(Y) = \pm 2\sqrt{1 - Y}, Y = Y_s \in \left(\frac{2}{3}, 1\right] \right\}. \quad (4.3)$$

Observe that the “ $\pm$ ” in (4.2) and (4.3) are due to the fact that $\Omega = \pm Y\sqrt{1 - Y}$ at ∂E (see (3.5)). Region F is plotted in Fig. 5. Note that it is very similar to region E (Fig. 2). Since we considered waves traveling to the left (we chose a “+” in front of the wave speed ν in (2.12)), the upper part of E is mapped onto the upper part of F .

Proof. The proof is entirely based on (4.1) and Lemmas 3.10 and 3.12. As in the proof of Lemma 3.10 we (first) set $K = K_c(\Omega_0) + \delta$ for some $0 < \delta \ll 1$. We then know that $\chi(K_c(\Omega_0) + \delta, \Omega_0) = Y_c + \delta[(4 - 3Y_c)/2(2 - 3Y_c)^2] + \text{h.o.t.}$ Substituting this into (4.1) and using (3.5) again yields $(B(K_c(\Omega_0) + \delta, \Omega_0), B(K_c(\Omega_0) + \delta, \Omega_0)) = (A(Y_c), B(Y_c)) + O(\delta)$ with $A(Y_c)$ and $B(Y_c)$ as in (4.2). Analogously, by substitution of (3.40) (Lemma 3.12) in (4.1) we find $(A(K_s(\Omega_0) - \delta, \Omega_0), B(K_s(\Omega_0) -$

$\delta, \Omega_0)) = (A(Y_s), B(Y_s)) + O([1/|\log \delta|])$ with $A(Y_s)$ and $B(Y_s)$ as in (4.3). The points $(\frac{4}{3}, \pm \frac{2}{3}\sqrt{3})$ correspond directly to saddle-node bifurcation cusps $(2/3, \pm \sqrt{4/27})$ of E . It is clear from approximations (3.14) and (3.16) that F should be inside ∂F (for instance, assume that F is outside ∂F , then the Poincaré map P has to have fixed points for $B = 0$, $A > \frac{4}{3}$; however, by (3.16) we observe that $\Delta\Omega(K, \Omega)$ is always negative in this case).

Remark 4.2. From (4.1) it follows immediately that $B = 0$ implies $\Omega = 0$ (and vice versa). Hence, for $B = 0$ the periodic solutions on the $\Omega = 0$ level set survive the perturbation. However, as was noted in Remark 3.1, the $\Omega = 0$ plane is degenerated in the integrable case. This situation is analyzed in detail in [12] where system (3.1) is studied for $\nu = 0$ (i.e., $B = 0$ in (3.6)). It has been shown that the degenerated solutions in the $\Omega = 0$ plane survive the perturbation for w (i.e., A) in a certain range. These solutions are also degenerated. The quantity $\rho(\xi)$ is periodically positive and negative, but, as in the integrable case, this can be controlled (by the extra symmetries in the $\nu = 0$ system, see subsection 3.2). Hence, by [12] we may conclude that Theorem 3.3 is also true for $\Omega_0 = 0$.

Note that ∂F_s can also be written in the form

$$\partial F_s = \left\{ B^2 - 4A + 4 = 0, B \in \left(-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3} \right) \right\}; \quad (4.4)$$

that is, (by (3.8)) system (3.6) has two pairs of coinciding critical points for $(A, B) \in \partial F_s$. Consulting (3.8), or Fig. 3, yields that these points are perturbed and degenerated saddles. Furthermore, there are no critical points to (3.6) if $B^2 - 4A + 4 < 0$; thus there cannot exist any localized structures (i.e., connections between periodic points) if $B^2 - 4A + 4 < 0$. It is a priori not clear that if we pick a point $(A_0, B_0) \in \partial F$, corresponding to a pair $(K_s(\Omega_0), \Omega_0) \in \partial E$, that one of the two degenerated saddles is equal to the saddle $(\rho_s(\Omega_0), 0, \Omega_0)$ of the corresponding unperturbed system. However, it is not difficult to check that if one fixes $\Omega = \Omega_0$, (i.e., Y_c and Y_s have been chosen) and consider the corresponding point $(A_0, B_0) \in \partial F_s$ that system (3.6) has a critical point (which is one of the two described above) that is equal to $(\rho_s(\Omega_0), 0, \Omega_0)$, thus $(\rho_s(\Omega_0), 0, \Omega_0)$ is a perturbed and degenerated saddle. Hence, $\Delta K(K, \Omega)$ and $\Delta\Omega(K, \Omega)$ are defined (and bounded, see Remark 3.2) at $(K_s(\Omega_0), \Omega_0)$. One has to use formulas (3.11) and (3.15) to compute ΔK and $\Delta\Omega$ in these points, since T_0 and T_1 are unbounded there, hence (3.14) and (3.16) cannot be used. For this same reason one cannot use system (3.17) to conclude that $\Delta K(K, \Omega) = \Delta\Omega(K, \Omega) = 0$ is possible at $K = K_s(\Omega)$: system (3.17) reduces (by substitution of (3.5) and (4.3)) to a singular system (the determinant = 0): $T_1 - Y_s T_0 = 0$, which is not defined, since both T_0 and T_1 approach ∞ if $K \uparrow K_s$. Note that one gets a well-defined equality by dividing by T_0 : $\chi - Y_s = 0$, which is satisfied per construction at $K = K_s(\Omega_0)$. Hence, on the curve (which turns out to be a straight line) in (A, B) -space at which $(\rho_s(\Omega_0), 0, \Omega_0)$ is a critical point of (3.6) (see (3.8)), there is only one point (or an $O(\epsilon)$ neighbourhood of this point) at which the homoclinic orbit of this saddle in the unperturbed system possibly survives: the point of this curve on ∂F , since this is the only point for which $\chi - Y_s = 0$, which is a necessary (but not sufficient) condition for having a homoclinic solution to (3.6).

Substitution of (3.5) and (4.3) into (3.11) yields a measure for the splitting distance on the K -axis between the (fast) stable and unstable manifolds of $(\rho_s(\Omega_0), 0, \Omega_0)$ (see Remark 3.2):

$$\begin{aligned}\Delta K(K_s, \Omega_0) &= -2\epsilon \left(\pm \sqrt{1 - Y_s} \right) \left(Y_s - \frac{2}{3} \right) \int_{\hat{R}}^{Y_s} \frac{Y_s - R}{\sqrt{1/2(R - \hat{R})(Y_s - R)^2}} dR + O(\epsilon^2), \\ &= -2\epsilon \left(\pm \sqrt{1 - Y_s} \right) \left(Y_s - \frac{2}{3} \right) \int_{\hat{R}}^{Y_s} \frac{1}{\sqrt{1/2(R - \hat{R})}} + O(\epsilon^2),\end{aligned}\quad (4.5)$$

in which the “ $\pm$ ” are again due to the fact that $\Omega_0 = \pm Y_s \sqrt{1 - Y_s}$. We derive a similar expression for $\Delta\Omega(K_s, \Omega_0)$:

$$\Delta\Omega(K_s, \Omega_0) = -\epsilon \left(Y - \frac{2}{3} \right) \int_{\hat{R}}^{Y_s} \frac{1}{\sqrt{1/2(R - \hat{R})}} + O(\epsilon^2). \quad (4.6)$$

One can check that in this situation $\hat{R} = 2(1 - Y_s)$. Hence, (4.5) and (4.6) can be evaluated explicitly:

$$\Delta K(K_s, \Omega_0) = -4\sqrt{6}\epsilon \left(\pm \sqrt{1 - Y_s} \right) \left(Y_s - \frac{2}{3} \right) \sqrt{Y_s - \frac{2}{3}} + O(\epsilon^2), \quad (4.7)$$

$$\Delta\Omega(K_s, \Omega_0) = -2\sqrt{6}\epsilon \left(Y_s - \frac{2}{3} \right) \sqrt{Y_s - \frac{2}{3}} + O(\epsilon^2). \quad (4.8)$$

Hence, since $Y_s \in (\frac{2}{3}, 1]$ for $\Omega_0 \in (-\sqrt{4/27}, \sqrt{4/27})$,

$$\Delta K(K_s, \Omega_0) \neq 0, \quad \Delta\Omega(K_s, \Omega_0) \neq 0. \quad (4.9)$$

One has to be very careful in interpreting (4.9): it is not yet justified to conclude that there are no homoclinic orbits to a (perturbed) saddle possible. However, if such a homoclinic orbit exists it will have to make some “detour” in phase space (see Claim 4.4 for a more mathematical description of this interpretation of (4.9)). This can be explained by studying the local flow governed by (3.6) near the curve of saddle points $(\rho_s(\Omega), 0, \Omega)$ of the unperturbed system in somewhat more detail. Note that this curve can be represented by

$$\left\{ (\rho, V, \Omega) = (\sqrt{Y_s}, 0, \pm Y_s \sqrt{1 - Y_s}, Y_s \in (\frac{2}{3}, 1]) \right\}. \quad (4.10)$$

The linearized flow around a saddle $(\rho_s(\Omega), 0, \Omega)$ of (3.6) has two fast ($= O(1)$) directions, corresponding to the stable and unstable manifolds, respectively $W_+^0(\Omega_0)$ and $W_-^0(\Omega_0)$, of the saddle $(\rho_s(\Omega), 0)$ on the $\Omega = \Omega_0$ level set of (3.2) and a slow ($= O(\epsilon)$) direction corresponding to a slow flow $O(\epsilon)$ close to the curve of unperturbed saddles (4.10) (see Fig. 6a). Since curve (4.10) is locally an intersection of two (hyperbolic) surfaces $(\{W_+^0(\Omega), \Omega \in (-\sqrt{4/27}, \sqrt{4/27})\}$ and $\{W_-^0(\Omega), \Omega \in (-\sqrt{4/27}, \sqrt{4/27})\})$, the following result holds.

Fig. 6a. The critical points $(\rho_s^+, 0, \Omega_s^+)$ and $(\rho_s^-, 0, \Omega_s^-)$ and their local character in the (ρ, V, Ω) phase space. We have assumed that (4.13) is satisfied, that is, the one-dimensional unstable manifold of $(\rho_s^-, 0, \Omega_s^-)$ comes $O(\epsilon^2)$ close to the one-dimensional stable manifold of $(\rho_s^+, 0, \Omega_s^+)$. The slow connection predicted by Theorem 4.3 is also shown. Note that the distance between the two critical points is $O(\epsilon)$, while the excursion made by the unstable manifold of $(\rho_s^-, 0, \Omega_s^-)$ is $O(1)$.

Theorem 4.3. *There are $O(\epsilon)$ slow solutions of (3.6) that are $O(\epsilon)$ close to (4.10) as long as $\Omega(\xi) \in (-\sqrt{4/27}, \sqrt{4/27})$. If system (3.6) has two or more critical points on (4.10) (i.e., these are perturbed saddles) then these points are connected to each other by $O(\epsilon)$ slow solutions that are $O(\epsilon)$ close to the curve (4.10).*

See Fig. 6a for a sketch of a slow connection between two perturbed saddles of (3.6). This theorem is a consequence of more general results obtained by Fenichel (see

Fig. 6b. A typical member of the family of heteroclinic cycles in the (ρ, V, Ω) phase space. The existence of this family is predicted in Remark 4.7 and is established in Theorems 4.3 and 4.8. The fast part is given by a Bekki & Nozaki solution.

[17] and the references given there). Details of the proof can be found in Howard and Koppell [22] in which a similar result has been proved for $\lambda - \omega$ systems. These systems describe a mathematical reaction-diffusion model for the Belousov-Zhabotinskii reaction and are mathematically very closely related to the Ginzburg-Landau equation. The evolution of $K(\xi)$ and $\Omega(\xi)$ of solutions of (3.6) of the type described by Theorem 4.3 (or of solutions that remain for an $O(1)$ time $O(\epsilon)$ close to these solutions and evolve in the same (up to $O(\epsilon)$) direction as these solutions) cannot be measured by the approximations of ΔK and $\Delta \Omega$ constructed in Section 3. Hence, (4.9) is not necessarily true for solutions like those described above. However, from (4.9) and Theorem 4.3 one may conclude the following.

Claim 4.4. *If $\Gamma_{\Omega_0}^\epsilon(\xi)$ is a solution of (3.6) corresponding to a fixed point of Poincaré map P and $\Gamma_{\Omega_0}^\epsilon(\xi) \rightarrow (\rho_s(\Omega_0), 0, \Omega_0)$ for $\xi \rightarrow \pm\infty$, and if $\Gamma_{\Omega_0}^0(\xi)$ is the homoclinic orbit of integrable system (3.2) at the level set Ω_0 , then*

$$\lim_{\xi \rightarrow \pm\infty} \frac{d}{d\xi} \Gamma_{\Omega_0}^\epsilon(\xi) - \lim_{\xi \rightarrow \pm\infty} \frac{d}{d\xi} \Gamma_{\Omega_0}^0(\xi) \neq O(\epsilon),$$

either for $\xi \rightarrow -\infty$ or for $\xi \rightarrow \infty$: $\Gamma_{\Omega_0}^\epsilon(\xi)$ either approaches or leaves $(\rho_s(\Omega_0), 0, \Omega_0)$ in a direction that differs $O(1)$ from the direction of $\Gamma_{\Omega_0}^0(\xi)$.

Thus, a homoclinic orbit of (3.6) has to either approach or leave $(\rho_s(\Omega_0), 0, \Omega_0)$ along the direction of a slow manifold (described by Theorem 4.3). The following result shows the importance of the slow solutions described by this theorem.

Claim 4.5. *Let $\Omega = C_s(K)$ be the image of curve (4.10) in (K, Ω) -space (i.e., $\{(K, \pm C_s(K)) : K \in [\frac{1}{2}, \frac{2}{3}]\} = \partial E_s$, see (3.5) and Fig. 2) then*

$$\frac{d}{dK} C_s(K = K_s(\Omega_0)) = \left| \frac{\Delta \Omega(K_s(\Omega_0), \Omega_0)}{\Delta K(K_s(\Omega_0), \Omega_0)} \right| + O(\epsilon),$$

with $\Delta K(K_s(\Omega_0), \Omega_0)$ and $\Delta \Omega(K_s(\Omega_0), \Omega_0)$ as in (4.7) and (4.8).

Proof. The saddle boundary of ∂E is given by $(K_s(Y_s), \Omega_s(Y_s)) = (2Y_s - \frac{3}{2}Y_s^2, \pm Y_s \sqrt{1 - Y_s})$, $Y_s \in (\frac{2}{3}, 1]$, (3.5); hence,

$$\frac{d}{dK} C_s(K = K_s(\Omega_0)) = \frac{(d/dY_s)\Omega_s}{(d/dY_s)K_s} = \frac{1}{2\sqrt{1 - Y_s}},$$

which is up to $O(\epsilon)$ equal to the quotient of (4.7) and (4.8).

Remark 4.6. In this claim we implicitly assumed $(A, B) \in \partial F_s$; however, because we did not choose a special value of Ω_0 , the result is valid anywhere on ∂F_s . This will also be the case for the subsequent analysis of this section: the position of (A, B) on ∂F_s will not be prescribed (i.e., Ω_0 , or equivalently Y_s , will be considered “free”).

Thus, by Claim 4.5 we know that the fast unstable manifold of a perturbed saddle of (3.6) returns $O(\epsilon^2)$ close to the curve of unperturbed saddles (4.10) and

hence, since $\Delta K(K_s, \Omega_0)$ and $\Delta \Omega(K_s, \Omega_0)$ are $O(\epsilon)$, $O(\epsilon^2)$ close to the slow flow along (the perturbation of this) curve. It is not difficult to check that if $(A, B) \in \partial F_s$, that is, if $(\rho_s(\Omega_0), 0, \Omega_0)$ is a degenerate (perturbed) saddle and (4.4) holds exactly, $(\rho_s(\Omega_0), 0, \Omega_0)$ will be mapped by the Poincaré map P (see (4.7) and (4.8)) to a part of the slow manifold which is unstable, that is, which is moving away from $(\rho_s(\Omega_0), 0, \Omega_0)$. Hence, the (fast) unstable manifold of $(\rho_s(\Omega_0), 0, \Omega_0)$ cannot return to $(\rho_s(\Omega_0), 0, \Omega_0)$. Note that writing “ P maps $(\rho_s(\Omega_0), 0, \Omega_0)$ to” is abuse of language since the Poincaré map is not defined on the critical points of (3.6) (these points can be interpreted as degenerate fixed points of P). Hence, instead of writing this, we should write “ P maps the fast unstable manifold of $(\rho_s(\Omega_0), 0, \Omega_0)$ to.” For simplicity we will keep on using this terminology.

Note that we determined ∂F_s by the highest-order parts of the approximations (3.11) and (3.15) of the Poincaré map P ; hence it is not natural to restrict ourselves to points (A, B) exactly on ∂F_s as defined in (4.3). Therefore, we introduce two new parameters $\hat{A}$ and $\hat{B}$ by

$$A = 2 - Y_s + \hat{A}\epsilon^2, \quad B = 2\sqrt{1 - Y_s} + \hat{B}\epsilon^2 \quad (4.11)$$

(for $Y_s \in (\frac{2}{3}, 1]$). We consider here only $B \geq 0$ and $\Omega \geq 0$. The case $B \leq 0$, $\Omega \leq 0$ is similar by the symmetry (3.9). Substituting (4.11) into (3.8) yields two symmetrical pairs of critical points of (3.6) of (perturbed) saddle type: $(\rho_s^\pm, 0, \pm\Omega_s^\pm)$ with

$$\begin{aligned} (\rho_s^\pm)^2 &= Y_s \pm 2\epsilon\sqrt{1 - Y_s}\sqrt{\hat{B}\sqrt{1 - Y_s} - \hat{A}} + O(\epsilon^2), \\ (\Omega_s^\pm)^2 &= Y_s^2 - Y_s^3 \mp 2\epsilon Y_s(3Y_s - 2)\sqrt{1 - Y_s}\sqrt{\hat{B}\sqrt{1 - Y_s} - \hat{A}} + O(\epsilon^2) \end{aligned} \quad (4.12)$$

(thus, we have to impose the condition $\hat{B}\sqrt{1 - Y_s} - \hat{A} > 0$ on $(\hat{A}, \hat{B})$, which means that (A, B) has to be left of the curve (4.4) in Fig. 3). The critical points $(\rho_s^+, 0, \pm\Omega_s^+)$ and $(\rho_s^-, 0, \pm\Omega_s^-)$ are $O(\epsilon)$ apart, since we adapted A and B $O(\epsilon^2)$ in (4.11). Again, one can check that on the slow connection between these two saddles (Theorem 4.3) the flow is moving from $(\rho_s^+, 0, \Omega_s^+)$ to $(\rho_s^-, 0, \Omega_s^-)$ (i.e., $(\rho_s^+, 0, \Omega_s^+)$ has a one-dimensional (fast) stable manifold and a two-dimensional unstable one, consisting of a fast and a slow part; $(\rho_s^-, 0, \Omega_s^-)$ has a one-dimensional (fast) unstable manifold and a two-dimensional stable manifold; see Fig. 6a). The $O(\epsilon^2)$ adjustments of A and B have no $O(\epsilon)$ influence on ΔK and $\Delta \Omega$ (given in (4.7) and (4.8)); thus, we observe that both $(\rho_s^+, 0, \Omega_s^+)$ and $(\rho_s^-, 0, \Omega_s^-)$ are mapped by the Poincaré map $O(\epsilon^2)$ close to a position on (4.10) with an Ω -coordinate that is $O(\epsilon)$ smaller than, respectively, Ω_s^+ or Ω_s^- . By local analysis one can check that the image of $(\rho_s^+, 0, \Omega_s^+)$ cannot flow back, or be mapped back, to $(\rho_s^+, 0, \Omega_s^+)$, as in the $\hat{A} = \hat{B} = 0$ case. However, by varying $\hat{A}$ and $\hat{B}$ (which has no influence on the $O(\epsilon)$ part of $P(\rho_s^-, 0, \Omega_s^-)$) we adjust the relative position of $(\rho_s^\pm, 0, \Omega_s^\pm)$ on (4.10). By choosing

$$\hat{B}\sqrt{1 - Y_s} - \hat{A} = \frac{2}{3}(Y_s - \frac{2}{3}) + O(\epsilon) \quad (4.13)$$

(see (4.12) and (4.7), (4.8)) the map P takes $(\rho_s^-, 0, \Omega_s^-)$ $O(\epsilon^2)$ close to $(\rho_s^+, 0, \Omega_s^+)$ (in this case we have $\Delta \Omega(K_s(\Omega_s), \Omega_s) = \Omega_s^- - \Omega_s^+ + O(\epsilon^2)$ and $\Delta K(K_s(\Omega_s), \Omega_s) =$

$K_s^- - K_s^+ + O(\epsilon^2)$ with $K_s^\pm$ defined by $(\rho_s^\pm, 0, \Omega_s^\pm)$ and (3.3)). Hence, in this case the one-dimensional unstable manifold of $(\rho_s^-, 0, \Omega_s^-)$ comes $O(\epsilon^2)$ close to $(\rho_s^+, 0, \Omega_s^+)$ and thus $O(\epsilon^2)$ close to the one-dimensional stable manifold of $(\rho_s^+, 0, \Omega_s^+)$. Both manifolds are $O(\epsilon)$ close to the homoclinic loop of the unperturbed system (3.2) at $\Omega_0^2 = Y_s^2 - Y_s^3$ (for at least a part of their orbit). A sketch of this situation is presented in Fig. 6a. Note that $\hat{A}$ and $\hat{B}$ only have to satisfy (4.13) in order to create the situation described above, and there is still a one-dimensional freedom of choice left in $(\hat{A}, \hat{B})$.

An important question, which will be answered affirmatively in the next subsection, is, of course, Is it possible to connect the unstable manifold of $(\rho_s^-, 0, \Omega_s^-)$ to the stable manifold of $(\rho_s^+, 0, \Omega_s^+)$ by choosing $(\hat{A}, \hat{B})$ correctly? Note that this will be a connection in a three-dimensional space between two one-dimensional orbits. If such a connection exists, then there is a heteroclinic cycle in perturbed system (3.6) connecting $(\rho_s^+, 0, \Omega_s^+)$ and $(\rho_s^-, 0, \Omega_s^-)$, consisting of a slow part (Theorem 4.3) and a fast part (involving a large excursion, like the homoclinic orbits of the integrable system, see Fig. 6b). This cycle corresponds to two different “localized structures” in (1.1), both moving with the same well-defined wave speed and temporal frequency (computable from (3.7)), both connecting the same two different, but $O(\epsilon)$ close, periodic patterns at ξ or $x \rightarrow \pm\infty$; the slow connection is like a continuous modulation of the first periodic pattern into the second pattern, the fast connection has a completely different local structure, see Figs. 7a and 7b.

Remark 4.7. In general one would not expect a connection between two one-dimensional orbits in a three-dimensional space. However, there is a (heuristic) geometrical argument that suggests the existence of these connections. The interval $\{\Omega = \Omega_0, K \in (K_c(\Omega_0), K_s(\Omega_0))\}$ in region E is mapped by (4.1) into a curve C_{Ω_0} in F : a point on C_{Ω_0} corresponds to a system (3.6) with a periodic solution associated to some (K, Ω_0) . By taking the limit toward ∂E_s in E the periodic solution of the unperturbed system merges into the homoclinic orbit. One would expect that something similar has to happen in (3.6) at the ∂F_s limit of C_{Ω_0} ; however, due to Claim 4.4, an “ordinary” homoclinic connection is not possible. The heteroclinic cycle described above (see also Fig. 6b) is a plausible substitute for the homoclinic connection: the periodic solution can merge into the heteroclinic cycle if $C_{\Omega_0} \rightarrow \partial F_s$. In this sense the heteroclinic cycle is a natural continuation of the homoclinic orbit of the integrable case. Note that this argument is independent of Y_s ; that is, one should expect a family $(Y_s \in (\frac{2}{3}, 1])$ of these heteroclinic cycles. In Remark 4.10 we will indicate that the heteroclinic cycle is not the only possible natural continuation of the homoclinic orbit.

The existence of a family of heteroclinic solutions to (3.6) can also be predicted by a counting argument: for any given value of “parameter” Y_s (between $\frac{2}{3}$ and 1) we have constructed a band of $O(\epsilon^2)$ length (see (4.11)) and $O(\epsilon^3)$ width (see (4.13)) in which the unstable manifold of $(\rho_s^-, 0, \Omega_s^-)$ comes $O(\epsilon^2)$ close to the stable manifold of $(\rho_s^+, 0, \Omega_s^+)$ (see Fig. 6a), while varying the parameters in this band has an $O(\epsilon^2)$ influence on the Poincaré map. Hence, there is a two-dimensional freedom in parameter space: it is natural to expect a connection between two one-dimensional manifolds in a three-dimensional space for a special “isolated” parameter combination. Since this argument does not depend on Y_s , one can expect a family of heteroclinic cycles.

Fig. 7a. A sketch of the imaginary part of the slow heteroclinic connection between saddles $(\rho_s^-, 0, \Omega_s^-)$ and $(\rho_s^+, 0, \Omega_s^+)$ as a solution of equation (3.6) for a certain time t .

Fig. 7b. A sketch of the imaginary part of the fast heteroclinic connection between saddles $(\rho_s^-, 0, \Omega_s^-)$ and $(\rho_s^+, 0, \Omega_s^+)$ as a solution of equation (3.6) for a certain time t . This curve is given by the Bekki & Nozaki solution that is described in Subsection 4.2.

4.2. A Geometrical Interpretation of the Bekki & Nozaki Solutions

In this subsection we show that the family of solutions of the Ginzburg-Landau equation constructed by Bekki and Nozaki [3] can be interpreted as the fast parts of the heteroclinic cycles described above (see also Fig. 6b). In [3], solutions to (1.1) satisfying the following Ansatz are studied:

$$u_{BN}(x, t) = e^{(1/2)iq\xi} G(\xi) F(\xi)^{-(1+i\lambda)} e^{-i\omega t}, \quad (4.14)$$

with $\xi = x + \nu t$, $G(\xi) = \mu(e^{k\xi} + ze^{-k\xi})$, and $F(\xi) = e^{k\xi} + e^{-k\xi}$; ω , ν , q , λ , μ , k , and $z = z_r + iz_i$ are parameters. Substituting (4.14) into (1.1) yields eight coupled nonlinear algebraic equations (see also [33] for more details). “Miraculously,” one of these equations can be eliminated, so Ansatz (4.14) describes a one-parameter family of solutions of (1.1). In general one needs computer assistance to determine these solutions ([33]). However, in the near-real case (i.e., $\alpha = \epsilon a$, $\beta = \epsilon b$) it is possible to expand the parameters in series in ϵ and perform a perturbation analysis. We will present here only the outcome of these (straightforward) computations; for more details we refer the reader to [3] and [33]. Define

$$Q_+ = \frac{1}{2}(q + 2\lambda k), \quad Q_- = \frac{1}{2}(q - 2\lambda k). \quad (4.15)$$

We find

$$\lambda = \epsilon \ell = \frac{2}{3}(b - a)\epsilon + O(\epsilon^3),$$

and therefore define Q by

$$Q_+ = Q_- + \epsilon Q, \quad (4.16)$$

and derive a relation between Q_- and Q :

$$Q^2 + \frac{8}{3}(b - a)^2 Q_-^2 = \frac{8}{9}(b - a)^2 + O(\epsilon),$$

which again indicates that we will find a family of solutions. For the moment we forget about the $O(\epsilon)$ corrections and set

$$Q_- = \frac{1}{3}\sqrt{3} \sin \phi, \quad (4.17)$$

$$Q = \frac{2}{3}(b - a) \cos \phi, \quad (4.18)$$

for $\phi \in [0, 2\pi]$ (a priori). It turns out that the wave speed ν and frequency ω are both $O(\epsilon)$ (hence the Bekki & Nozaki solutions fit in the framework of this paper):

$$\begin{aligned} \nu &= \epsilon \nu = 2\epsilon(b - a)Q_- + O(\epsilon^2), \\ \omega &= \epsilon \omega = \epsilon[b + (b - a)Q_-] + O(\epsilon^2), \end{aligned}$$

which gives

$$A = 1 + Q_-^2 + O(\epsilon), \quad B = 2Q_- + O(\epsilon) \quad (4.19)$$

where A and B are as defined in (3.7). Thus, $B^2 - 4A + 4 = O(\epsilon)$; that is, (A, B) is $O(\epsilon)$ close to ∂F_s , so we are in a situation like the one described in the previous subsection. The quantity Q_- is a parameter and plays a role similar to Y_s . There is a direct relation between Q_- and Y_s : we derive

$$\mu^2 = 1 - Q_-^2$$

(thus, by (4.17), $\mu^2 \in [\frac{2}{3}, 1]$) and

$$\mu^2 |z|^2 = 1 - Q_+^2 = 1 - Q_-^2 + O(\epsilon)$$

(by (4.16)). Hence $|z|^2 = 1 + O(\epsilon)$. Observe that for $\xi \rightarrow \pm\infty$,

$$|u_{BN}|^2 \rightarrow \mu^2, \mu^2 |z|^2;$$

that is, u_{BN} connects two periodic solutions of the type $R_{1,2} \exp[i(Kx + Wt)]$ with $R_1 = \mu$ and $R_2 = \mu|z|$; that is, the periodic solutions are $O(\epsilon)$ close to each other. Since $Y_s = \rho_s^2 = R^2$ (in this situation), we can set

$$Y_s = \mu^2 = 1 - Q_-^2, \quad (4.20)$$

which gives the direct relation between Q_- and Y_s . A more detailed analysis yields

$$\begin{aligned} |z|^2 &= 1 - 2\epsilon \frac{QQ_-}{1 - Q_-^2} + O(\epsilon^2), \\ &= 1 - \frac{4}{3} \sqrt{6(b-a)} \epsilon \frac{\sin 2\phi}{5 + \cos 2\phi} + O(\epsilon^2), \end{aligned} \quad (4.21)$$

(by (4.17) and (4.18)). Thus, the difference, $\Delta R(\phi)$, between the squares of the absolute values of the periodic solutions at $\xi = \pm\infty$ is

$$\begin{aligned} \Delta R(\phi) &= \mu^2 - \mu^2 |z|^2 = 2\epsilon QQ_-, \\ &= \frac{2}{9} \sqrt{6(b-a)} \epsilon \sin 2\phi + O(\epsilon^2). \end{aligned} \quad (4.22)$$

This is exactly the critical distance such that the Poincaré map P maps the unstable manifold of the saddle $(\rho_s^-, 0, \Omega_s^-)$ onto (or: $O(\epsilon^2)$ near to) saddle $(\rho_s^+, 0, \Omega_s^+)$, which was determined in the previous subsection. This can be checked by combining (4.12) and (4.13):

$$\Delta R(Y_s) = 4\hat{\epsilon} \sqrt{1 - Y_s} \left(\frac{2}{3} \left(Y_s - \frac{2}{3} \right) \right)^{(1/2)} + O(\hat{\epsilon}^2),$$

in which we used $\hat{\epsilon}$ instead of ϵ (see (3.6)); the u_{BN} computations are based on ϵ whereas the Poincaré map computations of the previous section are based on $\hat{\epsilon} = (b-a)\epsilon + O(\epsilon^3)$ ((3.6)). Substitution of (4.20) into the expression $\Delta R(Y_s)$ and use of (4.17) yields that $\Delta R(Y_s) = \Delta R(\phi)$ for $\phi \in [0, \frac{1}{2}\pi]$. Note that considering only $\phi \in [0, \frac{1}{2}\pi]$ is no restriction: due to the symmetry (3.9) we may consider $B > 0$, that is, $\nu > 0$, and thus $\phi \in [0, \pi]$. Moreover, there is an additional symmetry in Ansatz (4.14): it is possible to restrict oneself to situations in which $|z| < 1$ (i.e., $\phi \in [0, \frac{1}{2}\pi] \cup [\pi, \frac{3}{2}\pi]$

by (4.22)): $|z| > 1$ can be transformed to $|z| < 1$ in (4.14) by: $z \rightarrow 1/z$, $k \rightarrow -k$, and $\mu \rightarrow \mu|z|$ (this transformation creates an additional phase $e^{i\text{Arg}(z)}$; however, this is not important due to the phase symmetry in (1.1)). Furthermore,

$$k = \frac{1}{2} \sqrt{2} \cos \phi.$$

Hence the symmetry in u_{BN} is a direct reflection of the fact that $(A(\phi), B(\phi))$, as in (4.19), covers ∂F_s two times as a function of ϕ for $\phi \in [0, 2\pi]$.

It is now clear that u_{BN} is a heteroclinic solution in system (3.6) connecting $(\rho_s^-, 0, \Omega_s^-)$ and $(\rho_s^+, 0, \Omega_s^+)$. However, it is not yet shown that u_{BN} is exactly the orbit in (ρ, V, Ω) -space that is a continuation of the homoclinic orbit of the integrable system as described in Remark 4.7. To show this one has to study the behavior of $u_{BN}(\epsilon)$ for $\epsilon \downarrow 0$. In the above computations there is not yet a result on the behavior of the Ω -coordinate of u_{BN} . Observe that

$$\Omega(\xi) = \frac{1}{2i} \left[\left(\frac{d}{d\xi} u_{BN} \right) u_{BN}^* - \left(\frac{d}{d\xi} u_{BN}^* \right) u_{BN} \right].$$

Evaluating this expression yields

$$\Omega(\xi) = (1 - Q_-^2) Q_- + O(\epsilon);$$

hence $\Omega(\xi) \rightarrow \Omega_0$, a constant, for $\epsilon \downarrow 0$ (the main ingredient in this computation is the observation that $i q(z + z^*) - 2k(z - z^*) = 2i q + O(\epsilon)$). By a similar analysis for the ρ coordinate of u_{BN} , that is, $|u_{BN}|$, it can be checked that the family of heteroclinic u_{BN} solutions merges into the family of homoclinic solutions of the integrable system if $\epsilon \downarrow 0$ (one, for instance, has to show that u_{BN} approaches and leaves the saddles along the direction of the fast manifold). Thus we can conclude this analysis with the following theorem:

Theorem 4.8. *The family of Bekki & Nozaki solutions $u_{BN}(x, t; \phi, \epsilon)$ is a family of heteroclinic connections between the fast one-dimensional unstable manifold and the fast one-dimensional stable manifold of two nearby saddles in (3.6) that merges into the family of homoclinic solutions of the unperturbed system if $\epsilon \downarrow 0$.*

By performing the perturbation analysis in more detail we are now able to compute $\hat{A}$ and $\hat{B}$ (which are defined in (4.11), see also (4.19)) up to any accuracy. Combining Theorem 4.3 with Theorem 4.8 yields the existence of a family of heteroclinic cycles in (3.6) for a computable curve in (A, B) -space (which is $O(\epsilon^2)$ close to ∂F_s as defined in (4.3)), as was “predicted” in Remark 4.7. In Fig. 6b we have sketched a typical member of the family of heteroclinic cycles in phase space. In Figs. 7a and 7b we show (respectively) the slow part and the fast part of a heteroclinic cycle as solutions of the Ginzburg-Landau equation.

Remark 4.9. All of the above analysis is a priori only valid $O(\epsilon)$ away from the endpoints (or critical points) of ∂E or ∂F . One has to be careful if one considers Y_s close to $\frac{2}{3}$ or 1. At $Y_s = \frac{2}{3}$ the saddle and center of the integrable system merge (in a saddle-node bifurcation), and the perturbation may have a significant influence on the

character of the critical points $O(\epsilon)$ close to $Y_s = \frac{2}{3}$. Near $Y_s = 1$ one has to be aware of the degenerations of the unperturbed flow in the $\Omega_0 = 0$ level set (see Remark 3.1, Remark 4.2, and [12]). It is possible to carry through the above analysis in this case. One then finds a degenerate cycle between four (perturbed) saddles consisting of two slow and two fast parts connecting two saddles with $\rho_{\pm} > 0$ (and $O(\epsilon)$ close to 1) with two saddles with $\rho_{\pm} < 0$; hence the radius of $u(\xi, t)$ has to be zero for some values of ξ . The complete heteroclinic structure is $O(\epsilon)$ close to the $\Omega = 0$ plane. This situation can be controlled by the symmetries of system (3.1) with $v = 0$ (see Section 3.2) exactly like in [12]. Note that in this case u_{BN} reduces to a more classical solution (a straightforward computation yields that $z = -1$ in this case; that is, $|G(\xi)/F(\xi)|$ is a tanh function) that is exactly of the type described by Nozaki and Bekki in an earlier paper [31].

Remark 4.10. The heteroclinic cycles described by the Bekki & Nozaki solutions will not necessarily be the only “localized structures” described by system (3.6). On the contrary, it is very plausible that there are many more (families of) solutions of heteroclinic or homoclinic type possible in (3.6). These localized structures will appear for wave speed-frequency combinations such that A and B are in the $O(\epsilon^2)$ narrow strip near ∂F_s described by (4.11). The Bekki & Nozaki solutions are connections between two one-dimensional manifolds, it is also very natural to expect intersections between the two dimensional unstable and stable manifolds of saddles $(\rho_s^{\pm}, 0, \Omega_s^{\pm})$, which will give another family of heteroclinic connections that differ from the u_{BN} family in the sense that these solutions will approach and leave the saddles along the slow manifolds (and that this family will generically exist for an open region in (A, B) -space, instead of a curve). Furthermore, by playing with $\hat{A}$ and $\hat{B}$, one might expect homoclinic solutions to $(\rho_s^-, 0, \Omega_s^-)$ that satisfy the conditions described in Claim 4.4: solutions that are close to u_{BN} but “miss” $(\rho_s^+, 0, \Omega_s^+)$ and enter $(\rho_s^-, 0, \Omega_s^-)$ along the slow manifold (or heteroclinic cycles that first miss $(\rho_s^+, 0, \Omega_s^+)$ then move to $(\rho_s^-, 0, \Omega_s^-)$ along the slow connection but miss again, make another loop close to the unperturbed homoclinic orbit and enter $(\rho_s^+, 0, \Omega_s^+)$ along the fast direction, etc.). And, it is natural to expect periodic solutions of (3.6) that correspond to periodic points of the Poincaré map P , that is, orbits which circle around two or more times. We do not consider these types of solutions further in this paper.

5. Conclusions

In this paper we have studied the weakly complex Ginzburg-Landau equation. This type of modulation equation appears naturally in nonlinear stability problems that have a reversibility symmetry and that are weakly perturbed by an extra asymmetrical effect (an external force, a slightly transformed geometry), or in reversible systems in which there is an interaction between a stationary bifurcation and a Hopf bifurcation at the onset of instability. We refer the reader to [4], [7], [15], [24], [34], [39] for examples of systems that exhibit these features. We have shown that the essence of the integrable structure of the stationary solutions of the real Ginzburg-Landau equation continues into the (nonintegrable) weakly complex situation. The real Ginzburg-Landau equation describes the onset of stationary bifurcations in reversible systems. The associated

stationary problem is integrable and has a two-parameter family of quasi-periodic solutions and a one-parameter family of homoclinic localized, solitonlike solutions (see Fig. 1). In [25] the importance of these solutions to the roll-generating or -annihilating process in reversible systems has been discussed. We have determined in this paper the existence of a two-parameter family of traveling quasi-periodic solutions and two one-parameter families of traveling localized solitonlike solutions in the weakly complex Ginzburg-Landau equation, which can be considered as continuations of the families of stationary solutions, quasi-periodic and homoclinic, which appear in the integrable structure of the real Ginzburg-Landau equation. We have shown that it is necessary to study solutions of the type $u(x, t) = \hat{u}(x + \nu t)e^{-i\omega t}$ in order to be able to extend the integrable structure of the real case to the case of complex coefficients (see Section 2). We have determined explicitly (see Fig. 5 and Theorem 4.1) a bounded region F in the (wave speed ν , frequency ω)-space in which traveling quasi-periodic solutions exist. Any quasi-periodic solution corresponding to a point inside this region merges into a stationary quasi-periodic solution if one considers the limit towards the case of real coefficients. The boundary of F can be divided into two parts, a part ∂F_c where the period of the radius $|u(\xi)|$ of u (with $\xi = x + \nu t$) tends towards zero and a part ∂F_s where this period tends to ∞ . At the ∂F_c part of the boundary the quasi-periodic solutions merge into the periodic solutions (of the type $u(x, t) = R \exp[i(Lx + Wt)]$) of the Ginzburg-Landau equation that are unstable. By taking the limit towards ∂F_s in F we find that the corresponding quasi-periodic solutions merge into a family of heteroclinic cycles (see Fig. 6b). These heteroclinic cycles consist of two parts, a fast part and a slow part. Both parts correspond to a traveling solitonlike solution of the Ginzburg-Landau equation that connects two different stable periodic patterns as $\xi \rightarrow \infty$ and $\xi \rightarrow -\infty$. Note that exactly the same happens in the integrable, real, case: the stationary quasi-periodic solutions merge into the family of unstable stationary periodic solutions when the period of the amplitude tends to zero; when this period tends to ∞ , the quasi-periodic solutions merge into the family of homoclinic solutions, and these homoclinic solutions correspond to stationary localized structures connecting a stable periodic pattern at $x = -\infty$ with the same pattern at $x = +\infty$. We show that the family of fast parts of the heteroclinic cycles is given by the explicit analytical solutions due to Bekki and Nozaki [3]. The slow connections can be regarded as slowly varying waves and are therefore related to the solutions found by Howard & Kopell [22] and Bernoff [5]. In Figs. 7a and 7b we have plotted two typical examples of such solitonlike solutions of the complex Ginzburg-Landau equation: 7a shows a slow connection and 7b a fast connection. Note that the difference between these patterns is very localized, in this relatively short distance the fast solution makes its excursion through phase space (see Fig. 6b.). In the limit towards noncomplex coefficients this family of heteroclinic cycles merges into the family of stationary homoclinic solutions of the real Ginzburg-Landau equation.

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