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Feichtinger, G.; Herold, W.; Hommes, C.H.

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Chaos in a Simple Deterministic Queueing System

GUSTAV FEICHTINGER

Institute for Econometrics, Operations Research and Systems Theory, University of Technology, Vienna, Argentinierstraße 8, 1040 Vienna, Austria

CARS H. HOMMES

Department of Economic Statistics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands

WOLFGANG HEROLD

Institute for Econometrics, Operations Research and Systems Theory, University of Technology, Vienna, Argentinierstraße 8, 1040 Vienna, Austria

Abstract: We present a simple discrete-time deterministic queueing model, with one server and two queueing lines. The input rates of both queues are constant and their sum equals the server-capacity. In each time period the server has to decide how much time to spend on each of the two queues. The servers decision rule is a nonlinear, but increasing function of the difference between the two queue-lengths. We investigate how the dynamical behaviour of the queue-lengths and the service process depend on the 'steepness' of the decision function and the ratio of the input rates of the two queues. We show that if the decision function is steep, then for many input-ratios chaotic dynamics occurs.

1 Introduction

Much of queueing theory deals with *probabilistic* models. Both the number of customers and the unfinished work as functions of time are stochastic processes with *discontinuous* jumps, e.g., at instants of customer arrivals to the system. In attacking practical problems, however, engineers have developed crude *deterministic* models to estimate the gross behaviour of queueing systems. This point of view assumes that the discrete and random arrivals can be approximated by a non-random continuum as if it were a fluid flowing into a reservoir. Approximating the departure process in a similar way queueing systems are treated as *continuous fluid* rather than as discrete customer flows. Despite the fact that such fluid approximations are so coarse as to neglect all things dealt with in stochastic queueing theory, they enable us to study some important practical problems in a deterministic manner; see, e.g., Newell (1971) and Kleinrock (1976).

Usually, fluid approximations are dealt with in a continuous time framework. There are, however, exceptions in the literature. Erramilli and Forys (1991)

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motivate the discrete scale in their model by switching times. In the model we are going to discuss a discrete time scale is taken as basis for the periodic (say daily) decision how much time to allocate to the service of each of two queueing lines.

In this paper we present a simple discrete-time deterministic queueing model, where in each period an individual has to decide how much time to spend on each of two different activities. We assume that for each of the two activities the input rate is constant, and that the capacity of the individual equals the sum of the two input rates. The decision how much time to spend on each of the two activities in each period depends in an adaptive way upon the difference in the queue length of the two activities. The decision function is assumed to be a nonlinear, but increasing function of the difference between the queue-lengths. We investigate how the queue-lengths and the activities of the individual evolve over time. In particular, we investigate how the dynamical behaviour depends on the 'steepness' of the adaptive decision function and the ratio of the two input rates. Although our queueing model is strikingly simple, it turns out that complex dynamical behaviour can occur. We will show that when the decision function is 'steep', for many input-ratios chaotic dynamics occurs.

Chaotic dynamics is characterized by erratic, seemingly unpredictable time paths. If one would see such chaotic time paths without knowing how they were generated, one might wrongly think that the time paths came from a stochastic system. The erratic and unpredictable nature of the time paths however, is caused by nonlinear deterministic laws. In the present model a simple, fixed adaptive decision rule, which intuitively makes sense, is responsible for the irregular dynamical behaviour of the queue lengths.

The 'classical' example of a 1-dimensional difference equation exhibiting chaotic dynamics is the extensively studied quadratic difference equation $x_{n+1} = \lambda x_n(1 - x_n)$. The dynamical behaviour of the quadratic difference equation changes from a stable equilibrium for say $\lambda = 2.5$, via the so-called period doubling bifurcation route, into chaotic behaviour for λ between say 3.6 and 4. The 1-dimensional difference equation $x_{n+1} = f(x_n)$ describing the dynamics in our queueing-model however, is quite different. In particular the map f has two critical points (i.e. two points where f has a local extremum), while the quadratic map only has one critical point. The map f depends on two parameters, namely the 'steepness' of the decision function and the ratio of the two input rates. The bifurcation scenario, that is the qualitative changes in the dynamics when a parameter changes, turns out to be much more complicated than the bifurcation scenario w.r.t. the parameter λ for the quadratic map. We will see that, if the steepness of the decision function is high, then as the input-ratio changes both period doubling and period halving bifurcations occur and regular (periodic) and irregular (chaotic) behaviour alternate several times.

The paper is organized as follows. In section 2 we describe the queueing-model. Section 3 presents a definition of chaotic dynamics. In section 4 we analyze the bifurcation scenario w.r.t. the two parameters of the model. Section 5 contains some concluding remarks.

2 The Model

The following model deals with a very simple system, which can be described in queueing terms as follows: We consider two parallel $D/D/1/\infty$ systems in continuous time where the overall constant service capacity is distributed to the servers at times $1, 2, 3, \ldots$ in an adaptive way depending on the differences of the queue lengths. The system's total utilisation is $\rho = 1$. It is well known that for $\rho = 1$ only deterministic queues are positive recurrent, but here the adaptive control makes the things more complex. It will turn out that, depending on the control policy and the ratio of the input rates of the stations "stable" or "chaotic" behaviour occurs.

The queueing model describes one server and two queueing lines. As an example we consider an individual facing two different activities X and Y, but one may think of many other applications, e.g. a firm which has to decide how much to spend on research and development. In each time period, say at the morning of each day, the individual has to decide how much time to spend on each of the two different activities. We assume that, in each period, the input rate of new activities is constant, say α for activity X and β for activity Y. In each time period the individual will spend Φ_X on activity X and Φ_Y on activity Y. The sum of the two input rates in each period is assumed to be equal to the total service-capacity of the individual. By scaling to 1 we therefore have

$$\alpha + \beta = 1 \quad \text{and} \quad \Phi_X + \Phi_Y = 1 \tag{1}$$

where α, β, Φ_X and Φ_Y are all nonnegative. The ratio of the two input rates is given by $\alpha/\beta = \alpha/(1-\alpha)$. Let x_t and y_t, $t \in \mathbb{N}$, denote the queue lengths of X and Y, that is x_t and y_t denote the work which remains to be done at the end of period t for each of the two activities X and Y. The amount of time Φ_X and Φ_Y that will be spent on activities X and Y in period $t+1$ are determined by an adaptive feedback rule depending on the difference of the queue lengths x_t and y_t. The new queue lengths x_{t+1} and y_{t+1} are then given by the old queue length plus the total input minus the time spent on that activity, that is

$$x_{t+1} = x_t + \alpha - \Phi_X(x_t - y_t) \tag{2}$$

$$y_{t+1} = y_t + \beta - \Phi_Y(x_t - y_t) \ . \tag{3}$$

Adding (2) and (3) yields

$$x_{t+1} + y_{t+1} = x_t + y_t + \alpha + \beta - (\Phi_x + \Phi_y) \ . \tag{4}$$

Using (1) we get

$$x_{t+1} + y_{t+1} = x_t + y_t = L \ . \tag{5}$$

Equation (5) simply states that the sum of the queue lengths is constant over time. We assume that L is large enough, so that the queues are never empty. We choose $L = 2 > 1$, and note that other L-values ≥ 2 will lead to the same results. Solving (5) for y_t and substituting into (2) we get the following 1-dimensional difference equation:

$$x_{t+1} = x_t + \alpha - \Phi(2x_t - 2) \ , \tag{6}$$

where we have written Φ instead of Φ_X. We now have reduced the model to a 1-dimensional dynamical system, describing how the queue length of activity X evolves over time. The model is now complete, except for the adaptive feedback rule describing the decision how much time to spend on each of the two activities. We call the function Φ the decision rule or policy function. We will consider two possibilities, which are both intuitively plausible. They say that longer queues are served with higher priority.

1. *All-or-nothing decision:* the individual decides to spend all of his/her time on the activity corresponding to the longer queue. The decision function Φ is then given by the Heaviside function Φ_1:

$$\Phi_1(x - y) = \begin{cases} 1 & \text{if } x \geq y \\ 0 & \text{if } x < y \ . \end{cases} \tag{7}$$

2. *Mixed solution:* the individual decides to spend most of his/her time to the activity corresponding to the longer queue. For the decision function Φ we choose the S-shaped, logistic function Φ_2 given by

$$\Phi_2(x - y) = \frac{1}{[1 + e^{k(y-x)}]} \ . \tag{8}$$

We note that what follows also holds for other S-shaped decision functions Φ. The parameter k tunes the 'steepness' of the S-shape. Observe that for $k \to \infty$ the function Φ_2 approaches the Heaviside function Φ_1, so that case 1 occurs as a limiting case of case 2.

We investigate the following problem. *What can be said about the dynamics of this queueing-model, when the decision function Φ is given by* (7) *or* (8)? First we concentrate on the case where Φ is the logistic function in (8). The dynamics of the model is determined by a 1-dimensional difference equation $x_{n+1} = f(x_n)$, where the map f is given by

$$f(x) = x + \alpha - \Phi(2x - 2) \tag{9}$$

with Φ the S-shaped, logistic function. A *fixed point* or *equilibrium point* is a point x_{eq} satisfying $f(x_{eq}) = x_{eq}$. Obviously, a fixed point of the map f is determined by the equation $\Phi(2x - 2) = \alpha$, that is a fixed point x is the x-coordinate of the intersection point of the horizontal line $y = \alpha$ and the graph of $\Phi(2x - 2)$. Since Φ is strictly increasing and its image is the interval $(0, 1)$, and since $0 < \alpha < 1$ it follows that the map f has a unique fixed point x_{eq}.

The derivative f' of the map f is given by

$$f'(x) = 1 - 2\Phi'(2x - 2) \tag{10}$$

with

$$\Phi'(2x - 2) = \frac{ke^{k(2-2x)}}{[1 + e^{k(2-2x)}]^2} \,. \tag{11}$$

Recall that the parameter k tunes the 'steepness' of the S-shape of the logistic function Φ. The derivative of the map $\Phi(2x - 2)$ assumes its maximum value at $x = 1$ and this maximum value equals $k/4$. Consequently $f'(x)$ assumes its minimum at $x = 1$ and $f'(1) = 1 - k/2$. Hence, for $0 < k \leq 2$ we have $f'(x) \geq 0$ for all x, so that f is increasing and has a globally stable equilibrium for all $0 < \alpha < 1$. For $k > 2$ the map f has two *critical points*, that is two points where f has a local extremum. For $0 < k < 4$ and for all $0 < \alpha < 1$, the map f has a stable equilibrium, since $-1 < 1 - k/2 < f'(x_{eq}) < 1$. From a graphical analysis it follows that in this case the equilibrium is globally stable. What happens when $k > 4$? For $\alpha = 1/2$ the equilibrium x_{eq} is unstable, since $f'(x_{eq}) = 1 - k/2 < -1$, when $k > 4$. *What can be said about the dynamics of the model when the equilibrium is unstable?*

3 Chaotic Dynamics

The answer to the question above depends on the two parameter values α and k. In particular, we will see that if the 'steepness' of the logistic function Φ is sufficiently large, that is, if the parameter k is large enough, then for suitable choices of the input rate α chaotic dynamical behaviour occurs. For an introduction to chaotic dynamics see e.g. Devaney (1989), Guckenheimer and Holmes (1986) or Rasband (1990). There exist several definitions of chaos in the literature. In this section we present the definition we will use.

Let $f: \mathbb{R} \to \mathbb{R}$ be an 1-dimensional map generating a dynamical system $x_{n+1} = f(x_n)$. In order to define chaos we need several other notions first. The *orbit* or *trajectory* of a point x is the set $\{x, f(x), f^2(x), f^3(x), \dots\}$, where for any integer k, f^k denotes the map f iterated k times; the point x is called the *initial state* of the orbit. A point x is called a *periodic point* with period p, if $f^p(x) = x$ and $f^k(x) \neq x$, for $0 < k < p$. A point x is called an *aperiodic point* if (1) x is not a periodic point, (2) the orbit of x does not converge to a periodic orbit, and (3) the orbit of x is bounded.

An important feature of chaotic dynamics is the so-called sensitive dependence on initial conditions. We say that the map f has *sensitive dependence on initial conditions* w.r.t. a set S, if there exists a positive constant D such that for each point x in S and for each neighbourhood U of x, there exists a point y in U, and a positive integer N, such that the distance $|f^N(x) - f^N(y)| > D$. The definition says that arbitrarily close to each point x in S there exists a point y, such that the orbits of x and y are not close to each other forever. In a system exhibiting sensitive dependence on initial conditions, it is difficult to make long term predictions.

Now we present a definition of chaos. The dynamical system $x_{n+1} = f(x_n)$ is called *chaotic* if the following three properties are satisfied: (1) there exists a set P of infinitely many periodic points with different period, (2) there exists an uncoutable set A of aperiodic points, and (3) the map f has sensitive dependence on initial conditions w.r.t. the set $\Lambda = P \cup A$.

This definition is often referred to as *topological chaos*. We point out that in a topologically chaotic system the set Λ in the third property above may have Lebesgue measure zero. In that case we say that the system exhibits *transient chaos*; for most (in the sense of Lebesgue-measure) orbits only the initial part of the time path is erratic and most orbits eventually converge to a stable periodic orbit. For practical purposes however transient chaos may be considered as erratic behaviour, since the initial erratic phase of many time paths may be very long and/or the stable periodic orbit may have a very high period.

4 Analysis of the Dynamics

Recall from section 3 that for $0 < k < 4$ and for all $0 < \alpha < 1$, the map f in (9) has a globally stable equilibrium. We now will investigate the dynamics for $k > 4$. In that case we already know from section 2 that for $\alpha = 0.5$ the equilibrium is unstable. First we investigate the dynamics by computer simulations. In particular we investigate which bifurcations (i.e. qualitative changes) occur, when the parameter α is varied.

In figure 1 we present bifurcation diagrams w.r.t. the parameter α, for different choices of the parameter k. The bifurcation diagrams have been constructed as follows. For $\alpha = 0$ compute say the first 100 points of the orbit with initial state $x_0 = 0$, and thereafter plot the next say 200 points of this orbit. Next increase the parameter α a little bit by say 0.001, compute the first 100 points of the orbit of $x_0 = 0$ and plot the next 200 points again. This procedure is repeated until the parameter α reaches its maximum value 1. The result is a picture showing the long term behaviour of the model as a (multi-valued) function of the parameter α.

Figure 1 shows that when the parameter k is not too large (e.g. $k = 5$ in figure 1a and $k = 7$ in figure 1b), the bifurcation diagrams with respect to α are simple. Chaotic dynamics does not occur and only finitely many bifurcations occur. Notice however, that both period doubling and period halving bifurcations occur. For larger values of k (e.g. $k = 7.3$ in figure 1c and $k = 8$ in figure 1d) chaotic behaviour arises after infinitely many period doubling bifurcations, as α is increased from 0 to 0.3. However, when α is further increased from 0.3 to 0.5 chaos disappears, after infinitely many period halving bifurcations. For $0.5 < \alpha < 1$ the bifurcation scenario is qualitatively the same as for $0 < \alpha < 0.5$, since the system is symmetric w.r.t. $\alpha = 0.5$ and $x = 1$. The following theorem explains the dynamical behaviour observed in the bifurcation diagrams of figure 1.

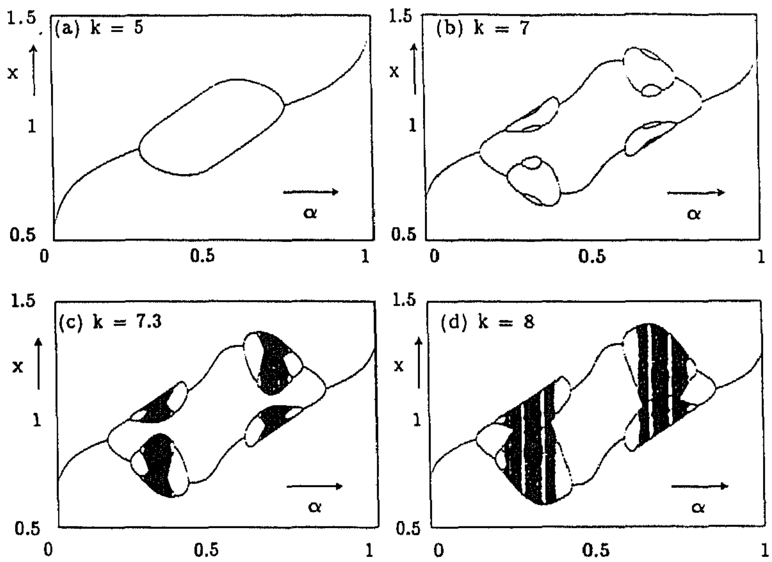

Fig. 1. *Bifurcation diagrams w.r.t. the parameter* α (input rate), *for different values of the parameter* k (steepness of the adaptive decision rule)

Theorem. When the parameter k is sufficiently large then there exist parameter values α_1 and α_2, $0 < \alpha_1 < 0.5 < \alpha_2 < 1$, such that for the dynamics of $x_{n+1} = f(x_n)$ the following properties hold:

1) *for α sufficiently close to 0 there exists a globally stable equilibrium.*
2) *for an interval of α-values containing α_1, the dynamics is chaotic.*
3) *for α sufficiently close to 0.5 there exists a stable period 2 orbit, and the map f has no periodic points with period different from 1 or 2.*
4) *for an interval of α-values containing α_2, the dynamics is chaotic.*
5) *for α sufficiently close to 1 there exists a globally stable equilibrium.*

The proof of the theorem is essentially the same as in Hommes (1991, ch. 1, and 1994) where a similar theorem is proven, for a difference equation $x_{n+1} = g(x_n)$ describing the price behaviour in a simple economic model (the cobweb model with adaptive expectations). The basic idea of the proof of properties (2) and (4) in the theorem is to show that when k is sufficiently large, then for a suitable choice of α the map f has a period 3 orbit. The properties then follow from the well known "*Period* 3 *implies chaos*" result of Li and Yorke (1975). As a corollary of the theorem it follows that, for k sufficiently large, infinitely many period doubling bifurcations occur in the parameter intervals $(0, \alpha_1)$ and $(0.5, \alpha_2)$, while infinitely many period halving bifurcations occur in the parameter intervals $(\alpha_1, 0.5)$ and $(\alpha_2, 1)$. In this sense the theorem explains the pictures in figure 1.

Let us briefly describe the implications of the theorem for the dynamical behaviour of the activities of the individual and the queue lengths in the model. Assume that the decision function Φ is steep, i.e. that the parameter k is large. Recall that α is the input rate of activity X, while $1 - \alpha$ is the input rate for Y. For α close to 0 (and $1 - \alpha$ close to 1) there is a stable equilibrium, meaning that in the long run, in each time period the individual spends a fixed proportion of his time to each of the two activities, and he/she spends most of the time to the activity Y with the highest input rate. For α close to 1 (and $1 - \alpha$ close to 0) we have the same behaviour, with the activities X and Y interchanged. For α close to 0.5, that is when the input rates of the two activities are almost equal, the equilibrium is unstable, and there is a stable period 2 orbit. This means that in one period most of the time is spend to activity X, in the next period most of the time to Y, in the next time period most of the time to X again, etc. Chaotic behaviour occurs when α is somewhere between 0 and 0.5 or between 0.5 and 1, e.g. for $\alpha = 1/3$ and $\alpha = 2/3$. In that case the ratio of the input rates $(\alpha/1 - \alpha)$ is 1/2 and 2/1 respectively. Hence, a steep decision function together with a situation where the input rate of one activity is twice the input rate of the other activity leads to irregular queue lengths.

In figure 2 we present some other bifurcation diagrams w.r.t. α, for still larger values of k, namely for $k = 11$ (figure 2a and an enlargement in figure 2b) and for $k = 100$ (figure 2c and an enlargement in figure 2d). In figure 1 the maximum number of period doubling/period halving reversals seems to be 3. In figure 2a some period doubling bifurcations have been indicated by a D, and some period

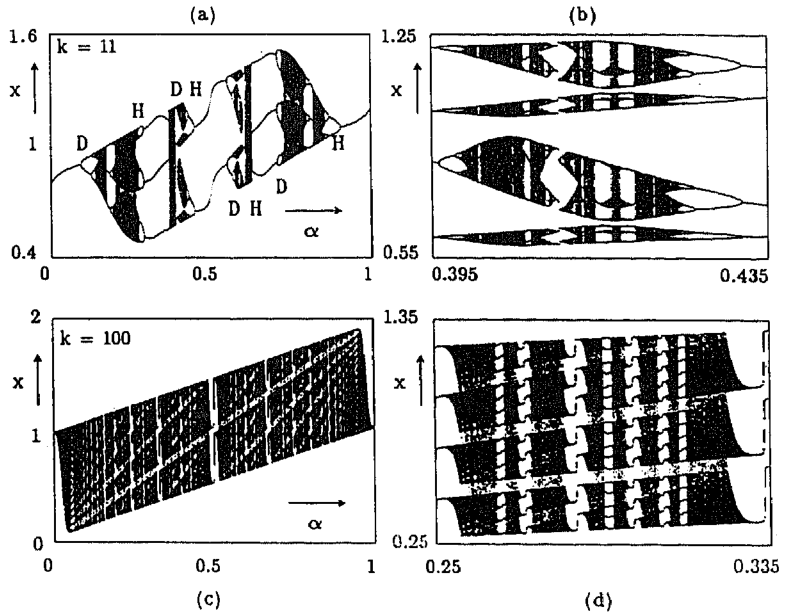

Fig. 2. *Bifurcation diagrams w.r.t. the parameter α, for $k = 11$ (a and an enlargement in b) and for $k = 100$ (c and an enlargement in d)*

halving bifurcations have been indicated by a H. Figure 2a shows that at least 7 reversals occur. Apparently the bifurcation scenario becomes more and more complicated as the steepness of the decision function Φ (i.e. the parameter k) increases. Recall from section 2 that as $k \to \infty$, the decision function Φ converges to the Heaviside function Φ_1. With $\Phi = \Phi_1$ the map f in (9) is given by

$$f(x) = \begin{cases} x + \alpha - 1 \ , & x \geq 1 \\ x + \alpha \ , & x < 1 \ . \end{cases} \tag{12}$$

Note that f maps the interval $J = [\alpha, \alpha + 1)$ into itself and all orbits are attracted to J. By identifying the endpoints, the interval J can be identified with a circle, and the map f can be identified with a rotation on the circle over an angle $2\pi\alpha$. Hence the dynamical behaviour of the queueing model in the case $\Phi = \Phi_1$ is equivalent to a rigid rotation on a circle. Using the properties of rotations on the circle (see e.g. Devaney (1989)) it follows that for rational values $\alpha = p/q$ of the input rate, every point x is periodic with period q. In that case, of every q time periods p time periods are completely spend on the first activity, while the remaining $q - p$ time periods are spend on the other activity. The order in which the two activities alternate is determined by the 'rotation number' p/q. On the other hand, when the parameter α is irrational, say $\alpha = \vartheta$, then every point x is aperiodic, and every orbit is dense in the interval $[\alpha, \alpha + 1]$. In fact, for $\alpha = \vartheta$

irrational, the dynamical behaviour is quasi-periodic, i.e. topologically equivalent to an irrational rotation over an angle $2\vartheta\pi$. In the case when the decision function is the logistic function, i.e. $\Phi = \Phi_2$, then for k large the dynamics approaches this periodic/quasi-periodic dynamical behaviour corresponding to the Heaviside function. For illustration see figures 2c and 2d.

5 Discussion and Concluding Remarks

We have analyzed the dynamical behaviour of a simple, discrete time, deterministic queueing model, with one server and two queueing lines. We have shown that, if the 'steepness' of the servers decision function is high, and the ratio of the two input rates is e.g. 2, then chaotic behaviour in the queue lengths can occur. We emphasize that the erratic behaviour of the queue lengths is caused by the fixed *nonlinear* adaptive control rule of the server, and not by e.g. stochastically fluctuating input rates. The results presented here depend mainly on a theorem proved by one of the authors (Hommes (1991, p. 9, and 1994)) and are illustrated by simulations.

Our simple model seems to be one of the first examples of queueing models exhibiting chaotic dynamics. Another example of chaos in a more complex discrete time queueing model was presented by Erramilli and Forys (1991). Our model is very simple, and the model could be made more realistic in several ways. For example one might allow for varying, e.g. state-dependent, or stochastic input rates of the two queues. One might also allow for more than two queueing lines and more than one server. Obviously, when chaos is already possible in a very simple framework, one can expect that it also occurs in more realistic and more complicated queueing-models.

In the present model we used a *first-order* approximation for queues in which we replaced the arrival and service process by their mean values, thereby creating a deterministic continuous fluid approximation to queues. In reality, these processes are often random in nature, and the approximation can be improved by including the variances for the arrival and the departure processes, respectively. This leads to a *second-order* approximation to queueing systems denoted as diffusion approximation. The mathematical tool of these approximation is the one-dimensional Fokker-Planck equation (see Kleinrock, 1976, sect. 2.8). An interesting question would be whether this continuous approach might be carried over to extend our time-discrete system to include second-order terms.

We would like to emphasize that the chaotic behaviour does not only occur in discrete time models, but can also occur in continuous time models of dimension 3 or larger. Continuous time in our simple framework would rule out the chaotic behaviour, since our model is 1-dimensional. Our simple example shows however, that it may be worthwhile to investigate the effect of similar nonlinearities in continuous queueing models of dimension 3 or larger. In our view this would be an important topic for further research.

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