Meta-analytic structural equation modelling with missing correlations

Cheung and Chan (2005) proposed a two-stage method to conduct meta-analytic structural equation modelling (MASEM). MASEM refers to the technique of fitting structural equation models to pooled correlation or covariance matrices from several studies. Unfortunately, researchers do not always report all correlations between the variables of interest. In this paper, we propose a method to deal with missing correlations in the two-stage approach. We illustrate the proposed model with a meta-analysis of teacher-child relationships variables from 99 studies. In addition, using simulated data, we show that our method leads to more precise parameter estimates than the existing approach.


Received: 14 August 2012; Accepted: 20 November 2012

Keywords: Meta-analytic structural equation modelling, two-stage approach, missing correlations

Authors: Suzanne Jak*, Frans J. Oort*, Debora L. Roorda** and Helma M. Y. Koomen*
variables of interest will be collected from several studies. The analysis has two stages.

Stage 1: Pooling correlation matrices from several studies

Let \( R \) be the \( p \times p \) sample correlation matrix and \( p \) be the number of observed variables in the \( g \)th study. Some observed correlation matrices may have missing correlations. Moreover, not all studies may include all variables. The correlation matrices for the first three studies may be:

\[
R_1 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix},
\quad R_2 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix},
\quad R_3 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}.
\]

Here, Study 1 has all variables and correlations, Study 2 misses a variable, and Study 3 has all variables but misses a correlation. The OC approach accounts for missing variables, but not for missing correlations. In the OC approach, we account for missing correlations by adding a new matrix (matrix \( C \) in Equation 1) to the model. We substitute an arbitrary value (e.g. zero) in the observed matrix \( R_1 \) for a missing correlation. We obtain an estimate of the population correlation matrix \( R_{pop} \) of all \( p \) variables by fitting a multigroup SEM model, in which the model for each group (study) is:

\[
\Sigma_g = D_g (M_g R_{pop} M_g^\top) D_g^\top + C_g.
\]  

(1)

In this model, \( R_{pop} \) is the \( p \times p \) population correlation matrix with \( \text{diag}(R_{pop}) = I \), matrix \( M_g \) is a \( p \times p \) selection matrix that filters out the missing variables in study \( g \). Matrix \( M_g \) is constructed by taking a \( p \times p \) identity matrix and removing the rows corresponding to the missing variables in study \( g \). \( D_g \) is a \( p \times p \) diagonal matrix that accounts for differences in variances across the \( g \) studies. New in the OC approach is the addition of Matrix \( C_g \), which is used to account for missing correlations. Matrix \( C_g \) is a symmetric \( p \times p \) correction matrix, with fixed zeros for all present correlations and a free parameter for the missing correlations in study \( g \).

With the identification constraint \( \text{diag}(R_{pop}) = I \) and a free \( D_g \) matrix, the hypothesis that is being tested is equality of covariances (not of correlations) as the variances in each study do not necessarily equal unity. The homogeneity of covariance matrices (covariances and variances) can be tested by constraining the elements of the diagonal matrix \( D_g \) to be equal across studies, so that \( D_g = D \) for all \( g \) (Cheung & Chan, 2005). The unity of variances can be tested by constraining the elements of the diagonal matrix \( D_g \) to unity, \( D_g = I \) for all \( g \).

The model in Equation 1 is identical to the model in Stage 1 of Cheung and Chan’s two-stage approach, except for the correction matrices \( C_g \). In matrix \( C_g \), the free parameter for each missing correlation will take on a value that minimises the difference between the arbitrary chosen value for the missing correlation in the observed matrix \( R \) (e.g., zero), and the estimate in \( R_{pop} \) for the corresponding correlation. A chi-square measure of fit for the model in Equation 1 is obtained by comparing its -2 log likelihood with the -2 log likelihood of the saturated model. The saturated model is given by:

\[
\Sigma_s = D_s R_s D_s^\top.
\]

(2)

The difference between the -2 log likelihoods follows a chi-square distribution with degrees of freedom equal to the difference in numbers of parameters between the two models. When the chi-square test turns out significant, then the hypothesis of homogeneity of covariances is rejected. The chi-square statistic can also be used to calculate approximate fit indices such as the Root Mean Square Error of Approximation (RMSEA, Steiger & Lind, 1980).

When the model fit is not acceptable, then the hypothesis of homogeneity of covariances is not tenable, so that the estimation of \( R_{pop} \) is not valid. Researchers may then create clusters of more similar studies, and construct separate pooled correlation matrices for all clusters of studies. Alternatively, a random effects model could be used, which estimates variances (and covariances) between the pooled correlation coefficients across studies. In this paper we do not consider random effects models.

The model from Equation 1 can be fitted using maximum likelihood estimation with any structural equation modelling program. However, writing the syntax can be very laborious in some programs. OpenMx (Boker et al., 2011) is a very flexible R-package, allowing the use of all R functions (R Development Core Team, 2011).

Stage 2: Fitting structural equation models

At Stage 2, the pooled correlation matrix from Stage 1 is used as the input matrix in an SEM analysis. Cheung and Chan (2005) propose using weighted least squares estimation at this stage. Weighted least squares estimation takes the asymptotic covariance matrix of the correlation coefficients from Stage 1 as the weight matrix in the fit function. Some correlation coefficients at Stage 1 are estimated using information from more studies than other correlation coefficients. As a result, coefficients that are based on more studies will have smaller variance in the asymptotic covariance matrix, and thus get more weight in the estimation process than coefficients that are based on less studies.

SEM generally requires the use of covariance matrices, however, the input matrix at Stage 2 is a
correlation matrix. Treating the correlation matrix as a covariance matrix leads to incorrect results when estimating confidence intervals or when testing specific hypotheses (Cudeck, 1989). To obtain correct results at Stage 2, we add a so-called estimation constraint. This constraint enforces the diagonal of the model implied correlation matrix to identity.

Illustrative example

Data

Roorda, Koomen, Spilt and Oort (2011) collected 99 studies that reported correlations between positive teacher-student relations and negative teacher-student relations on the one hand and student engagement and student achievement on the other hand. Correlations between positive teacher-student relations and negative teacher-student relations were collected afterwards for the present paper. Of these studies, 63 were conducted at primary schools and 36 at secondary schools. In total, there where 129,184 respondents (sample sizes ranging from 42 to 39,553). Based on leading theories about teacher-student relations (Connell & Wellborn, 1991; Pianta, 1999), teacher-student relations were considered as exogenous variables and engagement and achievement as endogenous variables. Out of the 99 studies, 20 studies missed a correlation between two variables, and 90 studies did not include one or more of the four variables.

Results

Table 1a gives the fit results of the several models we fitted to the 99 correlation matrices, using OpenMx (Boker et al., 2011). Model 1 is a saturated model, meaning that a correlation matrix is estimated for each study, without equality restrictions across studies. This model is used as a baseline model, to obtain fit indices for Models 2 to 4. Model 2 is a model in which we restricted all covariances to be equal, without restrictions on the variances across studies (equal covariances). Model 3 is a model in which we restricted all covariances and variances to be equal across studies (equal variances and covariances). Model 4 is a model in which we additionally restricted all variances to be unity (equal variances across studies).

All three models at Stage 1 had significant chi-square values, indicating that the models do not fit

<table>
<thead>
<tr>
<th>Model</th>
<th>-2 log likelihood</th>
<th>df</th>
<th>( \chi^2 )</th>
<th>RMSEA + 95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>STAGE 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Saturated</td>
<td>271981.9</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>2. Equal covariances (</td>
<td>D_g</td>
<td>= \text{free} )</td>
<td>277400.2</td>
<td>214</td>
</tr>
<tr>
<td>3. Equal variances and covariances (</td>
<td>D_g</td>
<td>= D )</td>
<td>277957.2</td>
<td>474</td>
</tr>
<tr>
<td>4. Variances equal to unity (</td>
<td>D_g</td>
<td>= I )</td>
<td>277957.4</td>
<td>478</td>
</tr>
<tr>
<td><strong>STAGE 2</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Mediation model (based on Model 2)</td>
<td>144.73</td>
<td>2</td>
<td>144.73</td>
<td>0.0235 [0.0197 ; 0.0274]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model</th>
<th>-2 log likelihood</th>
<th>df</th>
<th>( \chi^2 )</th>
<th>RMSEA + 95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>STAGE 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Saturated</td>
<td>249821.0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>2. Equal covariances (</td>
<td>D_g</td>
<td>= \text{free} )</td>
<td>254872.7</td>
<td>193</td>
</tr>
<tr>
<td>3. Equal variances and covariances (</td>
<td>D_g</td>
<td>= D )</td>
<td>255420.2</td>
<td>434</td>
</tr>
<tr>
<td>4. Variances equal to unity (</td>
<td>D_g</td>
<td>= I )</td>
<td>255420.3</td>
<td>438</td>
</tr>
<tr>
<td><strong>STAGE 2</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Mediation model (based on Model 2)</td>
<td>178.48</td>
<td>2</td>
<td>78.48</td>
<td>0.0261 [0.0223 ; 0.0301]</td>
</tr>
</tbody>
</table>
the data exactly. The RMSEAs were all below .05, indicating close approximate fit (Browne & Cudeck, 1992). As we do not have any hypothesis on the variances being equal across studies, we take the result of Model 2 as the estimate of the population correlation matrix. This correlation matrix is given in Table 2, and is used as the input for the Stage 2 analysis.

The Stage 2 model is based on social-motivational theory (Connell & Wellborn, 1991), in which it has been hypothesised that student engagement acts as a mediator in the association between teacher-student relations and student achievement. Empirical studies have provided some support for the mediating role of engagement (e.g., Hughes, Luo, Kwok, & Loyd, 2008). Therefore, the path model we fitted was a mediation model, in which the influence of positive and negative teacher-child relationships on student achievement was mediated by student engagement. This model fitted the population correlation matrix from Stage 1 closely according to the RMSEA.

Figure 1 provides a graphical representation of the model, with standardised parameter estimates and 95% confidence intervals. Positive teacher-student relations had a positive effect on Student engagement ($\beta = .296, p < .05$). Negative teacher-student relations had a negative influence of about the same size ($\beta = -.255, p < .05$). Student engagement had a medium sized positive effect on Student achievement ($\beta = .322, p < .05$). The indirect effect of Positive teacher-student relations on Student achievement via Student engagement was small and positive ($\beta = .095, p < .05$). Negative teacher-student relations had a similar small-sized negative indirect effect on Student achievement ($\beta = -.082, p < .05$). The model explained 20.8% of the variance in Student engagement, and 10.0% of the variance in Student achievement.

Results when deleting variables with missing correlations
We compared our results with the results obtained with the OV approach. This involved the deletion of a variable in 20 of the 99 studies. As can be seen in Table 1b, this approach leads to a loss of degrees of freedom and slightly worse model fit. However, the models still fit closely according to the RMSEA. Some parameter estimates are different from the previous analysis, and the likelihood based confidence intervals are somewhat wider. Figure 2 shows a graphical comparison of the parameter estimates and confidence intervals from the two analyses. Each graph pictures the parameter estimate (the dot) with its 95% confidence interval (the line) for the analysis with the OC approach (upper part) and for the analysis with the OV approach (lower part).

Simulation study
In order to investigate the effect of accounting for the missing correlations (OC approach), compared with deleting variables associated with missing correlations (OV approach), we performed an analysis of simulated data. We generated complete data for 100 studies. The pooled correlation matrix was estimated based on the full data, and based on data with missing correlations using the two approaches. The data were simulated under extreme conditions, so that differences between the two analysis methods became more apparent than in the illustration. Our expectation is that the OC approach leads to models with more power, better parameter estimates and smaller confidence intervals than the OV approach.

Data generation
We chose values of the population correlations based on the data from Roorda et al. (see Table 2). For each of the 500 replications, complete raw data were

<table>
<thead>
<tr>
<th>Table 2: Pooled correlation matrix across 99 studies, N = 129,184</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Positive teacher-student relations</td>
</tr>
<tr>
<td>2. Negative teacher-student relations</td>
</tr>
<tr>
<td>3. Student engagement</td>
</tr>
<tr>
<td>4. Student achievement</td>
</tr>
</tbody>
</table>

Figure 1 Path model with parameter estimates and their 95% confidence intervals
Figure 2 Parameter estimates (dots) with 95% confidence intervals (lines) of all parameters for the analysis with the OC approach (upper dot + line) and the OV approach (lower dot + line).

Note: v1 = Positive teacher-student relations, v2 = Negative teacher-student relations, v3 = Student engagement, v4 = Student achievement
drawn from the multivariate normal distribution, with means equal to zero, and variance covariance matrix equal to the population correlations. We chose 100 as the number of studies, with each study having a sample size of 100. So, each dataset contained 100 x 100 = 10000 scores on 4 variables. For each study, the correlation matrix was calculated and included in the meta-analysis.

Results with complete data
Fitting the model from Equation 1 (with \( D_g = I \)) to the complete data in 500 samples led to the average correlations in Table 3a. Percentages of estimation bias in all parameters are calculated as 100 \( \times (\text{mean estimated value} - \text{population value}) / \text{population value} \). According to Muthén, Kaplan and Hollis (1987), estimation bias less than 10% can be considered negligible. With complete data, estimation bias was below 1% for all parameters. The model had 994 degrees of freedom, the average chi-square value was 294.48 (SD = 41.28).

Results with missing correlations
The correlation between variables 1 and 3 was deleted randomly for 80% of the studies. Also for 80% of the studies we randomly deleted the correlation between variables 2 and 3. In this way, about 64% of the studies missed both correlations, and about 32% studies missed one of them, while only about 4% of the studies had complete data.

Using the OC approach, we obtained the pooled correlation matrix shown in Table 3b. The model had 834 degrees of freedom, the average chi-square value was 119.29 (SD = 39.60). The estimated correlations based on the data with missing correlations were close to the population correlations. The largest difference was found for the correlation between variables 1 and 4, which deviated 1.27% from the population correlation. The results in Table 3c were obtained by removing one variable for each missing correlation (the OV approach). This model had 618 degrees of freedom, the average chi-square value was 62.85 (SD = 29.98). The results do not differ very much from the results in Table 3b. The largest difference was found for the correlation between variables 1 and 3, which deviated -0.84 % from the population value.

The bias in parameter estimates is not very different across the three models (complete data vs. OC approach vs. OV approach). However, a structural

<table>
<thead>
<tr>
<th>Variable 1</th>
<th>Variable 2</th>
<th>Variable 3</th>
<th>Variable 4</th>
<th>Variable 1</th>
<th>Variable 2</th>
<th>Variable 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable 1</td>
<td>1</td>
<td>-0.371</td>
<td>0.356</td>
<td>0.156</td>
<td>-0.54%</td>
<td>-0.56%</td>
</tr>
<tr>
<td>Variable 2</td>
<td>-0.371</td>
<td>1</td>
<td>-0.343</td>
<td>1</td>
<td>-0.56%</td>
<td>-0.66%</td>
</tr>
<tr>
<td>Variable 3</td>
<td>-0.343</td>
<td>1</td>
<td>1</td>
<td>0.283</td>
<td>-0.64%</td>
<td>-0.66%</td>
</tr>
<tr>
<td>Variable 4</td>
<td>0.156</td>
<td>-0.151</td>
<td>0.283</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variable 1</th>
<th>Variable 2</th>
<th>Variable 3</th>
<th>Variable 4</th>
<th>Variable 1</th>
<th>Variable 2</th>
<th>Variable 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable 1</td>
<td>1</td>
<td>-0.373</td>
<td>0.359</td>
<td>0.159</td>
<td>0.00%</td>
<td>0.28%</td>
</tr>
<tr>
<td>Variable 2</td>
<td>-0.373</td>
<td>1</td>
<td>-0.349</td>
<td>-0.153</td>
<td>1.27%</td>
<td>0.66%</td>
</tr>
<tr>
<td>Variable 3</td>
<td>-0.349</td>
<td>-0.349</td>
<td>1</td>
<td>-0.27%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variable 4</td>
<td>0.159</td>
<td>-0.153</td>
<td>0.287</td>
<td>1</td>
<td>-0.64 %</td>
<td>-0.66 %</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variable 1</th>
<th>Variable 2</th>
<th>Variable 3</th>
<th>Variable 4</th>
<th>Variable 1</th>
<th>Variable 2</th>
<th>Variable 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable 1</td>
<td>1</td>
<td>-0.371</td>
<td>0.355</td>
<td>0.156</td>
<td>-0.54%</td>
<td>-0.84 %</td>
</tr>
<tr>
<td>Variable 2</td>
<td>-0.371</td>
<td>1</td>
<td>0.345</td>
<td>-1.01 %</td>
<td>-0.64 %</td>
<td>-0.66 %</td>
</tr>
<tr>
<td>Variable 3</td>
<td>-0.345</td>
<td>-1.01 %</td>
<td>1</td>
<td>-1.27 %</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variable 4</td>
<td>0.156</td>
<td>-0.151</td>
<td>0.283</td>
<td>1</td>
<td>-0.64 %</td>
<td>-0.66 %</td>
</tr>
</tbody>
</table>
difference can be seen in width of the likelihood based confidence intervals. These are smaller with the use of the OC approach, compared with the OV approach. Figure 3 gives a graphical representation of the parameter estimates and the associated 95% likelihood based confidence intervals. The upper dot and line denote the average parameter estimate and confidence interval of the analysis with the OC approach, while the lower dot and line denote the average parameter estimate and confidence interval of the analysis with the OV approach. The dotted vertical line shows the population value of the parameter. As expected, omitting information leads to larger confidence intervals. The difference is most clearly seen in the correlations between variables 3 and 4 in the lower right corner of the figure. It is not surprising that the difference is so apparent for the correlation between variables 3 and 4. As we deleted the correlation between variables 1 and 3 and between variables 2 and 3, both methods use equal amounts of information about these correlations. However, where our method still uses information about the correlation between variables 3 and 4, this information is often deleted in the other approach, leading to less precise parameter estimates.

Discussion

In this paper we have proposed a method to incorporate missing correlations in MASEM. The method was demonstrated with an example from teacher-child interactions. Using simulated data, the method was compared with the current practice, which is to delete one of the variables that is associated with the missing correlation. As the OC approach uses more information than the OV approach we expected that our method would lead to better parameter estimates and smaller confidence intervals. Results from the very small simulation study indicated that the OC approach leads to models with larger degrees of freedom and smaller confidence intervals. The parameter estimates were close to the population values for all methods, indicating that...
deleting one or two variables in 80% of the studies did not really influence parameter recovery.

A possible explanation of the similar results with respect to parameter bias between the methods is that in our study the missingness of the correlation coefficients were introduced randomly. Maximum likelihood estimation with data missing at random is known to lead to unbiased parameter estimates (e.g., Enders & Bandolash, 2001; Newman, 2003). In true meta-analysis, the missing correlations may not be missing at random, and it would be interesting to investigate the effect of not-random missingness. For example, if the correlations between variables 3 and 1 and variables 3 and 2 are mainly missing in studies where the correlation between variable 4 and 3 is high, then if we delete variable 3 in the OV approach, the information about the correlation between variable 4 and 3 is lost, and the parameter will be underestimated. Using the OC approach, all remaining information would be used and the parameter estimate is expected to be closer to the true value.

References


R Development Core Team (2011). R: A Language and Environment for Statistical Computing. [Computer software]. Vienna, Austria.


SUZANNE JAK
PhD candidate in Methods and Statistics, Research Institute of Child Development and Education, University of Amsterdam. Her research interests include structural equation modelling, measurement bias, multilevel structural equation modelling and meta-analytical structural equation modelling.

FRANS J. OORT
Full professor of Methods and Statistics and Director of the Research Institute of Child Development and Education, University of Amsterdam. His research interests include measurement and measurement bias, and statistical modelling, especially non-standard applications of structural equation modelling.

DEBORA L. ROORDA
Postdoc research fellow at the School Psychology and Child and Adolescent Development Department at the University of Leuven, Belgium. Her research interests focus on teacher-student interactions and their influence on children’s emotional well-being, school engagement, and achievement.

HELMA M. Y. KOOMEN
An assistant professor in special education at the Research Institute of Child Development and Education at the University of Amsterdam. Her research interests address student-teacher interaction and teachers’ pedagogical practices in relation to emotional well-being, social adjustment, task behaviours, and achievement of children with and without special need.