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Periodic, almost periodic and chaotic behaviour in Hicks' non-linear trade cycle model

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Abstract

Hicks' non-linear trade cycle model is an unstable multiplier–accelerator model together with an 'income ceiling' and an 'investment floor'. We show that in the simplest, two-dimensional version of the model the income time paths are always regular: either periodic or almost periodic (quasi-periodic) behaviour occurs. When consumption and/or investment is distributed over several time periods, higher dimensional versions of the model are obtained. We show that in the three-dimensional Hicks model irregular, chaotic income time paths may occur.

1. Introduction

In the last decade in economics there has been a rapidly growing interest in non-linear dynamic models exhibiting chaotic dynamical behaviour. The use of non-linear dynamic economic models, however, is not new and dates back to Kaldor (1940), Hicks (1950) and Goodwin (1951). In the fifties economists had not yet learned about chaos and the 'classical' non-linear economic models focused on regular, periodic behaviour rather than irregularity and chaos. Recently Lorenz (1989), Goodwin (1990) and Medio (1992) have shown the occurrence of chaos in a number of classical business cycle models, including a discrete time version of the Kaldor model, Metzler's inventory-cycle model with non-linear savings and investment functions and Goodwin's non-linear accelerator model with periodic investment outlays. In this paper, we show that the original 'classical' Hicks model can also generate chaotic dynamics.

Hicks' non-linear trade cycle model, as introduced by Hicks (1950), is an unstable linear multiplier–accelerator model together with a 'ceiling' (full employment upper bound) and a 'floor' (investment lower bound). The Hicks model is a piecewise linear dynamic model generating cycles. The simplest version of the Hicks model is a two-dimensional model. Extra time lags in the consumption and/or the investment equation of the model lead to higher dimensional versions. We will address the following question: *Does each time path in Hicks' non-linear trade cycle model converge to a periodic time path?*

2. The elementary Hicks model

Hicks' non-linear trade cycle model is perhaps the simplest non-linear model of the business cycle, and can be found in many textbooks on economic dynamics; see, for example, Allen (1965) or Blatt (1983). The simplest version of the model (*the elementary Hicks model*) is given by the following four equations:

$$\textit{consumption}: \quad C_t = mY_{t-1}\,, \tag{1}$$

$$\textit{total investment}: \quad I_t = I_t^{\text{ind}} + I^{\text{aut}}\,, \tag{2}$$

$$\textit{induced investment}: \quad I_t^{\text{ind}} = \max\{a(Y_{t-1} - Y_{t-2}), -I^{\text{f}}\}\,, \tag{3}$$

$$\textit{income}: \quad Y_t = \min\{C_t + I_t, Y^{\text{c}}\}\,. \tag{4}$$

According to (1) current consumption, C_t, is proportional to previous income, Y_{t-1}, with m the marginal propensity to consume, $0<m<1$. Equation (2) states that total investment, I_t, equals the constant autonomous investment, I^{aut}, plus induced investment, I_t^{ind}. Equations (3) and (4) are the two non-linear, or more precisely piecewise linear equations of the model. According to (3) induced investment is proportional to the growth in national income, as long as it is larger than the *investment floor*, $-I^{\text{f}}$. The negative net investment, $-I^{\text{f}}$, corresponds to zero gross investment. Based on economic data Hicks observed that the accelerator $a>1$. Finally, according to (4) income Y_t equals consumption plus investment, as long as it is smaller than the (full employment) *income ceiling*, Y^{c}. Substituting (1)–(3) into (4) a piecewise linear second-order difference equation, describing the dynamics of income is obtained:

$$Y_t = \min\{mY_{t-1} + \max\{a(Y_{t-1} - Y_{t-2}), -I^{\text{f}}\} + I^{\text{aut}}, Y^{\text{c}}\}\,. \tag{5}$$

Equation (5) has a unique equilibrium, $Y^{\text{e}} = I^{\text{aut}}/(1-m)$, and by assumption $Y^{\text{e}} < Y^{\text{c}}$. A simple computation shows that the equilibrium, Y^{e}, is unstable, since the accelerator $a>1$. Writing $x_t = Y_t$ and $y_t = Y_{t-1}$, (5) is transformed into

$$\begin{aligned} x_{t+1} &= \min\{mx_t + \max\{a(x_t - y_t), -I^{\text{f}}\} + I^{\text{aut}}, Y^{\text{c}}\}\,, \\ y_{t+1} &= x_t\,. \end{aligned} \tag{6}$$

Consequently, the elementary Hicks model is given by a difference equation $(x_{t+1}, y_{t+1}) = H(x_t, y_t)$, where H is the two-dimensional (2-D) piecewise linear map

$$H(x, y) = (\min\{mx + \max\{a(x-y), -I^{\text{f}}\} + I^{\text{aut}}, Y^{\text{c}}\}, x)\,. \tag{7}$$

The map H has an unstable equilibrium, $E = (Y^{\text{e}}, Y^{\text{e}})$, and one can easily show that all time paths are bounded. We investigate the following question: *Does a time path in the elementary Hicks model always converge to a periodic time path?* Although the Hicks model can be found in many textbooks on economic dynamics, it seems that this question has not yet been answered.

The next result completely describes the dynamics of the 2-D Hicks model:

Theorem. For $a>1$ the map H in (7) has an attracting set K, which is a piecewise linear closed curve with the unstable equilibrium E lying inside. All time paths, except for the equilibrium, are attracted to the attracting set K, and rotate with a unique rotation number $\rho(H)$ (i.e. a unique average rotation) around the unstable equilibrium. Two possibilities can occur:

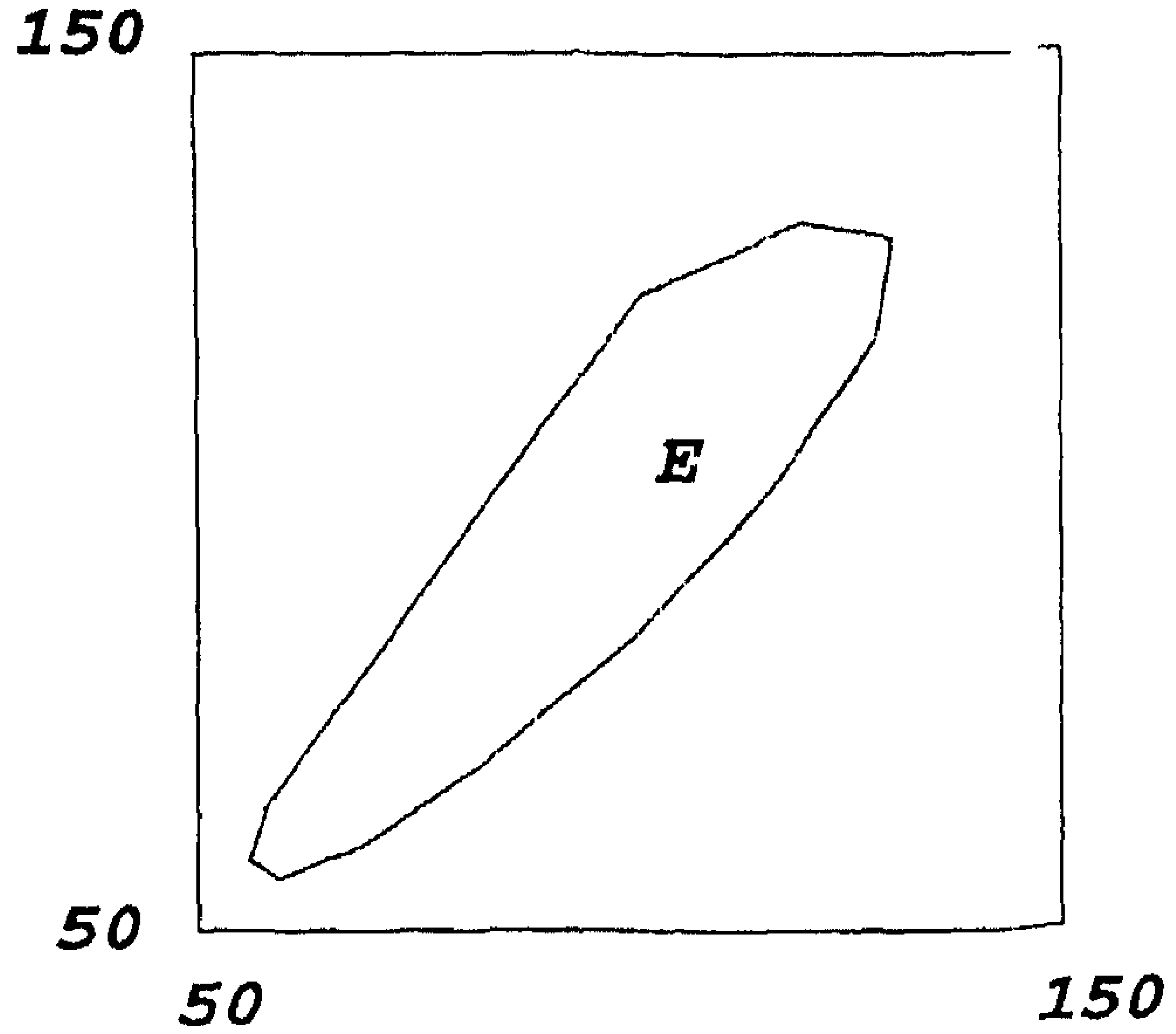

Fig. 1. A quasi-periodic attractor. The parameters are: $Y^c = 150$, $I^f = 10$, $I^{aut} = 20$, $m = 0.75$ and $a = 1.25$. The figure shows 5000 points of the orbit of $(x_0, y_0) = (Y^c, Y^c)$ after a transient of 10 time periods.

(A) *The rotation number, $\rho(H) = p/q$, is rational* [*with* $gcd(p, q) = 1$ *and* $0 < p/q < 1$]. *In that case all time paths, except for the unstable equilibrium, converge to a (stable) periodic orbit with period q.*

(B) *The rotation number, $\rho(H) = \alpha$, is irrational (with $0 < \alpha < 1$). In that case all time paths, except for the unstable equilibrium, are aperiodic, that is they are not periodic and they do not converge to a periodic time path.*

A proof of the theorem can be found in Hommes (1991). Case (B) is often referred to as *quasi-periodic* dynamics. An example of a *quasi-periodic attractor* (or a periodic attractor with a very long period) is shown in Fig. 1. The corresponding income time paths, although not exactly periodic, are roughly periodic. All orbits, except for the equilibrium, are dense in the attractor. In fact, the dynamics is equivalent to an irrational rotation on a circle.

We emphasize that, although aperiodic time paths can occur, the dynamics of the elementary Hicks model is always regular. The time paths are either periodic or almost periodic (quasi-periodic) and all time paths rotate with the same average rotation around the unstable equilibrium. There is no sensitive dependence on initial states. Chaotic, irregular income fluctuations do not occur in the simplest version of Hicks' nonlinear trade cycle model.

3. The three-dimensional Hicks model

In the elementary Hicks model all consumption lags one period behind income, while all induced investment outlays are concentrated in the time period immediately following the originating change in income. Hicks (1950) already posed the following two questions: (1) *What difference does it make when some consumption is lagged more than one period behind income?* (2) *What difference does it make if induced investment is postponed or distributed over a number of* *periods*? With distributed consumption and/or investment, higher dimensional versions of the Hicks model are obtained. With consumption distributed over the three time periods following income and induced investment distributed over the two time periods following the change in income, the Hicks model is a three-dimensional (3-D) piecewise linear model. The 3-D Hicks model is in fact the simplest version of the model, after the elementary Hicks model. We investigate whether the regularity in the dynamics of the elementary Hicks model also holds for the 3-D version. In particular we investigate the following fundamental problem: *Can Hicks' non-linear trade cycle model with distributed consumption lags and investment lags generate chaotic time paths*?

The *3-D Hicks model* is given by the following four equations:

$$C_t = m_1 Y_{t-1} + m_2 Y_{t-2} + m_3 Y_{t-3} , \tag{8}$$

$$I_t = I_t^{\text{ind}} + I^{\text{aut}} , \tag{9}$$

$$I_t^{\text{ind}} = \max\{a_1(Y_{t-1} - Y_{t-2}) + a_2(Y_{t-2} - Y_{t-3}), -I^{\text{f}}\} , \tag{10}$$

$$Y_t = \min\{C_t + I_t, Y^{\text{c}}\} . \tag{11}$$

The parameters m_1, m_2 and m_3 are called the *partial consumption coefficients*, while a_1 and a_2 are called the *partial investment coefficients*. Substituting (8)–(10) into (11) and writing $x_t = Y_t$, $y_t = Y_{t-1}$ and $z_t = Y_{t-2}$, we get:

$$\begin{aligned} x_{t+1} &= \min\{m_1 x_t + m_2 y_t + m_3 z_t + \max\{a_1(x_t - y_t) + a_2(y_t - z_t), -I^{\text{f}}\} + I^{\text{aut}}, Y^{\text{c}}\} , \\ y_{t+1} &= x_t , \\ z_{t+1} &= y_t . \end{aligned} \tag{12}$$

Hence, the 3-D Hicks model is given by a difference equation $(x_{t+1}, y_{t+1}, z_{t+1}) = H(x_t, y_t, z_t)$, where H is the 3-D piecewise linear map given by

$$\begin{aligned} H(x, y, z) = (&\min\{m_1 x + m_2 y + m_3 z + \max\{a_1(x - y) + a_2(y - z), -I^{\text{f}}\} \\ &+ I^{\text{aut}}, Y^{\text{c}}\}, x, y) . \end{aligned} \tag{13}$$

Let $m = m_1 + m_2 + m_3$, $0 < m < 1$, be the overall marginal propensity to consume. We write $Y^{\text{e}} = I^{\text{aut}}/(1 - m)$ and we assume that $Y^{\text{e}} < Y^{\text{c}}$. The map H has a unique equilibrium given by $E = (Y^{\text{e}}, Y^{\text{e}}, Y^{\text{e}})$. Define the income floor-level $Y^{\text{f}} = (I^{\text{aut}} - I^{\text{f}})/(1 - m)$. Note that $Y^{\text{f}} < Y^{\text{l}} < Y^{\text{c}}$. Let D be the set $D = \{(x, y, z) | Y^{\text{f}} \leq x, y, z \leq Y^{\text{c}}\}$. A simple computation shows that H maps D into itself. Therefore, all time paths in the 3-D Hicks model are bounded. *What can be said about the dynamics of the* 3-*D Hicks model when the equilibrium is unstable*?

In Figs. 2 and 3 we present pictures of the (x, y)-projections of some time paths in the 3-D Hicks model, after a short transient time (i.e. the first say 10–20 points of the orbits are omitted). The corresponding values of the parameters are given in the captions of the figures. Figures 2(a) and 3(a) show two examples of so-called *strange*, *chaotic attractors*. Here the word 'strange' refers to the complicated geometric structure of the attractors, while the word 'chaotic' refers to the erratic dynamical behaviour on the attractor. The pictures suggest a Cantor-like structure [see also the enlargements in figs. 2(c), 2(d), 3(b), 3(c) and 3(d)] and the corresponding largest Lyapunov exponents are positive, indicating sensitivity to initial states. The Lyapunov exponents have been computed by using the DYNAMICS-program [Yorke, (1992)]. For mathematical details con-

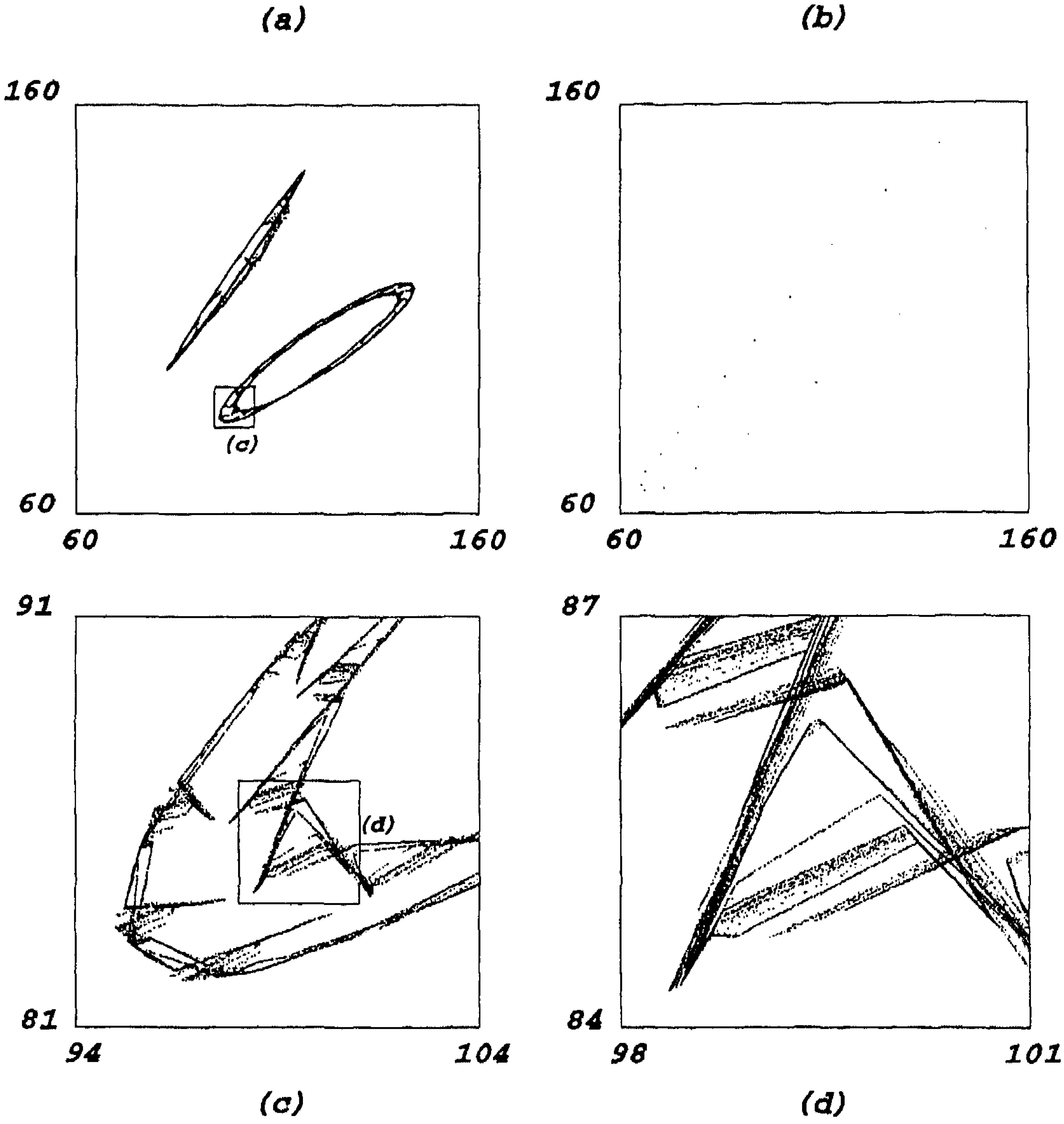

Fig. 2. Coexistence of a strange chaotic attractor and a stable period-20 orbit. The parameters are: $Y^c = 150$, $I^f = 10$, $I^{aut} = 20$, $m_1 \approx 0.54$, $m_2 = 0.25$, $m_3 = 0$, $a_1 = 0.6$ and $a_2 = 1.4$. (a) The orbit of $(x_0, y_0, z_0) = (120, 100, 150)$ converges to a strange, chaotic attractor. The corresponding largest Lyapunov exponent is $\lambda \approx 0.013$. (b) The orbit of $(x_0, y_0, z_0) = (Y^c, Y^c, Y^c)$ converges to a stable period-20 orbit. Only 19 points are visible, since two points of the period-20 orbit are projected to the point $(x, y) = (Y^c, Y^c)$. (c)–(d) Enlargements of the strange chaotic attractor in (a).

cerning chaos, strange attractors and Lyapunov exponents, see, for example, Guckenheimer and Holmes (1986) and Eckmann and Ruelle (1985).

In the first example (Fig. 2), in addition to the strange, chaotic attractor, a stable period–20 orbit occurs, see Fig. 2(b). Hence, we have *coexistence* of (at least) two different attractors. It depends on the initial state whether regular (periodic) or irregular (chaotic) behaviour occurs. In the second example (Fig. 3) it seems that the model has a unique strange, chaotic attractor; the

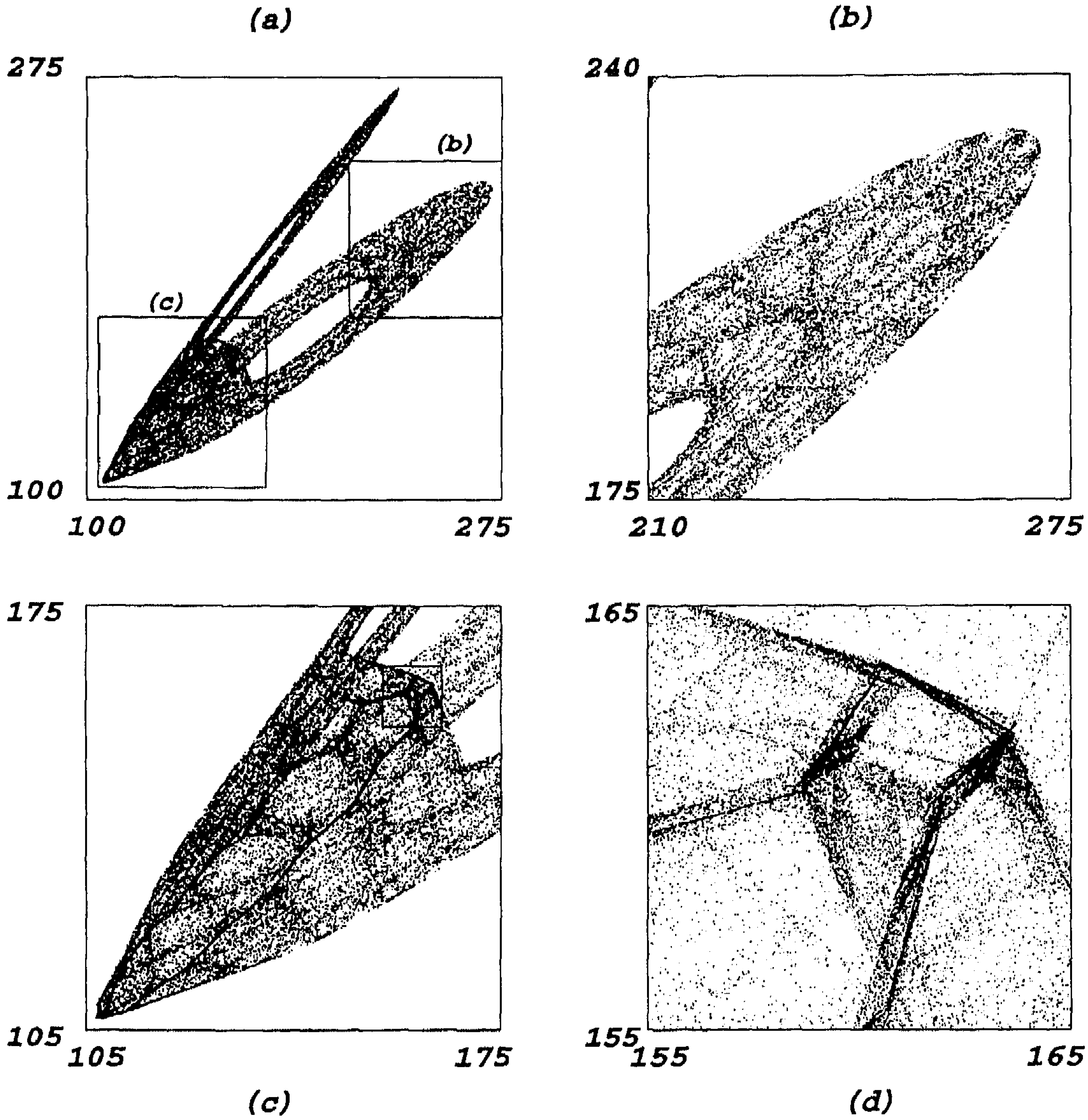

Fig. 3. A strange, chaotic attractor with corresponding largest Lyapunov exponent $\lambda \approx 0.022$. The parameters are: $Y^c = 275$, $I^f = 10$, $I^{aut} = 20$, $m_1 = 0.55$, $m_2 = 0.2$, $m_3 = 0.1$, $a_1 = 0.25$ and $a_2 = 1.27$. Figures 3(b)–(d) show enlargements of Fig. 3(a). Almost all time paths seem to converge to the strange chaotic attractor.

orbits of almost all (in the sense of Lebesgue measure) initial states seem to converge to this strange, chaotic attractor.

Notice that in both examples we have $m_1 > m_2 > m_3 \geq 0$, so that the highest fraction of income is consumed with a delay of one period, while the lowest fraction of income is consumed with a delay of three periods. Furthermore, in both examples $a_2 > a_1$ so that most of the induced investment takes place in the second time period following the originating change in income. As for the example pointed out by Allen (1965), such an investment pattern may be the rule rather than the exception. Usually, the time needed for investment decisions and investment outlays is longer than the shortest consumption lag. Therefore, in the 3-D Hicks model irregular, chaotic income fluctuations may occur for reasonably realistic parameter values.

4. Concluding remarks

We have investigated the dynamics of Hicks' non-linear trade cycle model. In the simplest version of the model the income fluctuations are always regular: either periodic or almost periodic behaviour occurs. Irregular, chaotic income fluctuations do not occur in the elementary Hicks model.

However, extra time lags in the consumption or the investment equation may change the nature of the income fluctuations quite dramatically. Our numerical results indicate that even in the one but simplest version of the model, the 3-D Hicks model, irregular, chaotic income fluctuations occur. This simple non-linear business cycle model illustrates the fact that the introduction of extra time lags (which is often more realistic) may cause a considerable complication of the dynamical behaviour of a model.

The Hicks model is a prototype of a model with very simple type non-linearities: 'ceilings' (upper bounds) and 'floors' (lower bounds) imposed on an (unstable) linear system. In particular, for economics these types of non-linearities may be important. We have seen that these simple types of non-linearities may lead to erratic dynamical behaviour.

Non-linear economic models can provide an endogenous explanation of economic fluctuations, in contrast to attributing them to exogenous shocks. In recent years non-linearities have been detected in several economic time series. These results stress the importance of investigating non-linear models. Stated in the words of Brock (1988): 'The evidence adduced to date should make us suspicious of "Frischian" models. There is enough contrary evidence now that we should be investigating nonlinear models. It has taken almost 50 years for this particular idea of Hicks to catch on.'

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