Dynamic models of research and development

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Chapter 4

Competition and Innovation

4.1 Introduction

Contemporary markets are flooded with imitations – it is hard to find a business model, a good, or service that is not a variation or an adaptation of some earlier version. Recently, even Samsung's lawyers could not tell the difference between Samsung's Galaxy Tab and Apple's iPad in court.[1] Imitators often even outperform innovators in business results (e.g., both Visa and Mastercard enjoy larger market shares than the first credit card issuer Diners Club, and currently Samsung's lead over Apple in smartphone market share has been widened further[2]). Similar observations led Levitt to claim already back in 1970s that "Imitation is not only more abundant than innovation, but actually a much more prevalent road to business growth and profits" (Levitt, 1966, p. 63). A business strategy specialist Oded Shenkar in his recent book, titled "Copycats: how smart companies use imitation to gain a strategic edge", even talks about an "imovation challenge" – companies that want to succeed need to fuse innovation and imitation as in the future it will not be possible anymore "to rely on innovation or imitation alone to drive competitive advantage" (Shenkar, 2010, p. 169).

Hardly any business idea is immune to imitation. In the words of Arrow (1962, p. 615): "No amount of legal protection can make a thoroughly appropriable commodity of something

[1] http://gizmodo.com/5849803/even-samsung-cant-tell-the-difference-between-its-tablet-and-ipad (accessed August 2, 2012)

[2] http://www.totaltele.com/view.aspx?ID=475297 (accessed August 2, 2012)

so intangible as information. The very use of the information in any productive way is bound to reveal it, at least in part. Mobility of personnel among firms provides a way of spreading information. Legally imposed property rights can provide only a partial barrier, since there are obviously enormous difficulties in defining in any sharp way an item of information and differentiating it from similar sounding items."

Indeed, Mansfield (1985) finds that rivals have information about new products or processes in 12 months or less. Similarly, Cabellero and Jaffe (1993) in their analysis of patent citations conclude that diffusion of information about innovations is so rapid that it can be regarded as being instantaneous. Shenkar (2010) concludes that the pace of imitation is nowadays increasing with the increased codification of knowledge and the advance of globalization: "In 1982 generics constituted a mere 2 percent of the U.S. prescription drug market, but by 2007 they made up 63 percent. In the early 1990s Cardizem lost 80 percent of the market to generic substitutes within five years; a decade later, Cardura lost a similar share in nine months; and Prozac, an Eli Lilly blockbuster drug, lost the same market share in only two months" (p. 6). On the other hand, Vonortas (1994), for instance, claims that the features of technological knowledge itself obstruct others from copying it. As he puts it, "technological knowledge involves a combination of poorly-defined and often incomplete know-how and a set of highly codified information which is hard to acquire and utilize effectively" (p. 415). In practice, the extent to which R&D (Research and Development) information leaks or spills over to competitors will most likely depend on specific characteristics of a particular industry and product in question.

The spillovers are prevalent in business practice, yet our understanding of their role in innovation activities of firms is still incomplete. On one hand, spillovers could improve market performance as imitators make market more competitive. On the other hand, innovators facing the danger of having their returns to research investments reduced by imitators could be less willing to innovate in the first place. Furthermore, the effect of spillovers on the innovation of asymmetric firms remains ambiguous as well. Empirical studies show that the follower often strives to catch up with the leader (see, e.g., Lerner, 1997). If imitating the leader is easier, the follower is able to progress faster and the industry is less likely to become monopolized.

However, the final result depends also on the way the leader responds to spillovers. The leader might increase his innovation efforts in order to widen the lead and hopefully drive the follower out of the market, or he might decrease his innovation efforts as the follower is free-riding on them.

The analysis of spillovers in the existing literature is typically limited to their effects on the innovation activities already in place. That is, the existence of the market and R&D process is already assumed and the question left then is how spillovers affect the (size of) R&D efforts. A distinguishing feature of our approach is that we do not limit ourselves to the question of how much to invest on a given market but recognize that any such a question is preceded by the question of whether to invest at all.[3] Specifically, we pay special attention to the determination of indifference points in a firm's investment function. At these points, a firm is indifferent between developing a given technology further and opting out. Consequently, we are able to analyze not only how spillovers affect the investments on existing markets, but also how they influence the likelihood that a new market will be formed, and if so, how does its likely structure (monopoly or oligopoly) relate to the level of spillovers.

Like in previous chapters, our focus here is on process innovation. That is, the firms increase their production efficiency by exerting R&D efforts. This higher production efficiency in turn makes them stronger competitors on the Cournot product market. We allow for initial unit production costs of firms (representing initial technology levels) being above the choke price (the lowest price at which the quantity sold is zero) and we explicitly take firms' product market participation constraints into account. In consequence, firms' R&D process and production do not need to coexist at all times and firms can enter or exit the product market and initiate or cease their R&D processes at different times. Here our work builds upon the previous chapters, where we observed that the existing literature on strategic process innovation holds on to the assumption of "low enough" initial production costs, and thereby implicity imposes the coexistence of production and R&D at all times, without a proper

[3]Elmer Bolton, a scientist-manager at the DuPont company, one of the most innovative corporations in American business history, was famous for saying to company's chemists who in his opinion lacked the awareness that the success of the company depends on its products being commercially exploitable: "This is very interesting chemistry, but somehow I don't hear the tinkle of the cash register" (Hounshell and Smith, 1988).

justification. Obviously, this assumption is in contradiction with the real life observation that for great many new technologies, research starts long before the first prototype sees the light. In our critical re-assessment of competition policies on R&D cooperatives in Chapter 3, we showed how incomplete and misleading conclusions based on this restrictive assumption can be. Furthermore, our model allows for asymmetric positions of firms, such that we are also able to study investment decisions of the leader in relation to the follower at different levels of spillovers and firms' relative positions.

The rigor of our approach has its price - indifference points in firms' policy functions bring about a possibility that the latter exhibit discontinuities, which in turn implies a possible existence of multiple regions of non-differentiability in the value functions. This poses significant problems to numerical schemes. We progress by first exploiting the fact that adding random noise to the R&D process smooths up the policy functions and value functions, making numerical schemes easier to implement. We then consider a solution to the related stochastic optimization problem, interesting in its own right, as an approximating solution to the deterministic game when the noise level tends to zero. That is, we obtain a solution to the deterministic game as a vanishing viscosity solution (see Başar and Olsder, 1995, Ch. 5.7). We solve for a feedback Nash equilibrium of the differential game, characterized by a system of highly nonlinear implicit partial differential equations, by a variant of the numerical method of lines (Schiesser, 1991): we transform the system of partial differential equations into the system of ordinary differential equations and consider the solution to the latter as an initial boundary value problem. Solution methods for the latter require the values of the solution to the game at the boundaries of the state space over which we seek a solution before they can proceed. The problem is that true values at all boundaries are *ex ante* not known to us. We solve this problem by exploiting the fact that the characteristics of the associated first-order Hamilton-Jacobi-Bellman partial differential equations leave the state space at the boundaries. This implies that the solution in the interior of the state space is unaffected by the precise specification of the boundaries, possibly excepting a small strip along the boundaries. This enables us to obtain an accurate approximating solution over an interior region of interest.

We show that in general duopoly results in the product market only if initial asymmetries between the firms are not too large.[4] The duopoly on the product market is characterized by regression toward the mean phenomenon: asymmetries between the firms tend to vanish over time.

Our results qualify the indication in the literature that larger spillovers might prevent the monopolization of an industry (Petit and Tolwinski, 1999). We show that this is true only when initial production costs of the leader are high and so also his incentives to exert R&D efforts are high. At relatively lower unit costs of the leader, when additional R&D efforts benefit the leader progressively less and the follower progressively more, the incentives of the leader to exert R&D efforts can be rather low. This makes it harder for the follower to catch up with the leader. Notably, the ability to copy is not worth much when there is little to copy. Consequently, *lower* cost asymmetries can suffice to induce the monopolization of the industry at *larger* spillovers.

We show that through increasing complementarities in R&D, larger spillovers always increase the chance that an expensive technology that calls for investments in advance of production will be brought to production. Though, the level to which such a technology is developed can be lower due to lower R&D investments of firms that try to free-ride on each other along the way. In this sense, spillovers increase production efficiency only up to a point.

We show that larger random shocks to firms' production costs are favorable to the likelihood that a technology will be developed further as firms are stimulated by the chance of a favorable shock to their production costs in the future more than they are destimulated by the equal chance of an unfavorable shock. Stochasticity also increases the likelihood that the product market will be competitive as the chance of a larger favorable shock in the future increases the endurance of the follower.

We find comparably large investments of firms at low spillovers and high initial unit costs of both firms. There, a small cost advantage of one firm leads to a behavior that can be considered predatory: the leader exerts high R&D efforts which are profitable in that they induce the follower to give up. When firms start from a symmetric situation, their behavior,

[4] This conclusion is similar to Doraszelski (2003) who finds action-reaction behavior in a patent race model with history-dependent R&D stocks: the follower catches up with the leader provided his initial stock of knowledge is of sufficient size and gives up otherwise.

however, resembles a preemption race: each firm invests a lot trying to win the race in which a small lead suffices for gaining a monopoly position.

4.2 Model

The dynamic game is defined in continuous time and over an infinite horizon: $t \in [0, \infty)$. There are two firms which potentially both compete in a market for a homogenous good with demand given by

$$p(t) = \max\left\{A - q_i(t) - q_j(t), 0\right\}, \tag{4.1}$$

where $p(t)$ is the market price, $q_i(t)$ is the quantity produced by firm $i = \{1, 2\}$, $q_j(t)$ is the quantity produced by its rival $(i \neq j)$, and A is the choke price (the lowest price at which the quantity sold is zero). At the outset of the game, each firm obtains an exogenous technology $c_i(0)$. For simplicity, we assume that firms may differ in their production cost, but they are identical in every other aspect. While both firms produce with constant returns to scale, each firm can reduce its unit cost $c_i(t) > 0$ by investing in R&D. This process is subject to spillovers. Firm i exerts R&D effort $k_i(t) \geq 0$ and as a consequence of these investments, its unit cost (state variable) evolves over time according to

$$\frac{dc_i}{dt} \equiv \dot{c}_i(t) = c_i(t)(-k_i(t) - \beta k_j(t) + \delta), \tag{4.2}$$

where $k_j(t)$ is the R&D effort exerted by its rival and where $\beta \in [0, 1]$ is a degree of spillovers. Notice that equation (4.2) is not linear and that consequently the game is not linear-quadratic. Low values of β correspond to strong intellectual property protection and the ability of firms to prevent involuntary leaks of information. The reverse is true for high values of β. We treat the value of β as given for firms.[5] Observe in (4.2) that the smaller the c_i, the smaller the effect of particular k_i on $\dot{c}_i$. Further innovations require increasingly more R&D efforts. The

[5] In general, β may be one of a firm's strategic variables. See Katsoulacos and Ulph (1998) and Amir, Evstigneev and Wooders (2003) for an attempt to endogenize the degree of spillovers. Von Hippel (1988) provides empirical evidence for firms being consensually involved in information sharing. See also Shenkar (2010). Amir, Amir and Jin (2000) allow for spillovers to differ between firms.

parameter $\delta > 0$ is the constant rate of efficiency reduction due to the ageing of technology and organizational forgetting. Exerting R&D effort is costly. This cost is per unit of time given by

$$\Gamma_i(k_i(t)) = b(k_i(t))^2, \tag{4.3}$$

where $b > 0$ is inversely related to the cost-efficiency of the R&D process. In assuming decreasing returns to R&D, we follow the bulk of the literature (see Chapter 2 for a discussion of model's assumptions). Both firms discount the future with the same constant rate $\rho > 0$. The instantaneous profit of firm i is:

$$\pi_i(t) = \begin{cases} (A - q_i(t) - q_j(t) - c_i(t))\, q_i(t) - bk_i(t)^2 & \text{if} \quad p(t) > 0, \\ -c_i(t)q_i(t) - bk_i(t)^2 & \text{if} \quad p(t) = 0, \end{cases} \tag{4.4}$$

yielding its total discounted profits over time:

$$\Pi_i = \int_0^\infty \pi_i(t)e^{-\rho t}dt. \tag{4.5}$$

4.2.1 Rescaling

Our model depends on five parameters: $A, b, \delta, \beta,$ and ρ. Some of these can be set to 1 by choosing the measurement scale of units appropriately. The five-dimensional parameter space then reduces to a three-dimensional one.

Lemma 10. *By choosing the units of* t*,* q_i*,* q_j*,* c_i*,* c_j*,* k_i*, and* k_j *appropriately, we can assume* $A = 1$*,* $b = 1$*, and* $\delta = 1$*. The state equation changes to*

$$\dot{c}_i(t) = c_i(t)(1 - (k_i(t) + \beta k_j(t))\phi) \tag{4.6}$$

with $\phi = A/\delta\sqrt{b}$*.*

Proof. See Appendix 4.A. □

The new parameter $\phi = A/\delta\sqrt{b}$ captures the profit potential of a technology: a higher A

implies higher potential sales revenue, a higher b implies more costly R&D efforts, whereas a higher δ implies that each unit of R&D effort reduces the marginal cost by less. Hence, a higher (lower) ϕ corresponds to a higher (lower) profit potential of a technology.

4.2.2 Equilibrium

In our two-firm differential game, each firm tries to maximize its total discounted profits by selecting a strategy which specifies a quantity produced and an R&D effort exerted at each point in time.

When selecting their strategies, firms have a lot of possibilities. In case firms use open-loop strategies, they precommit themselves at the outset of the game to a fixed schedule of actions over the entire planning horizon. That is, they specify an entire time path of quantities produced and R&D efforts exerted at the beginning of the game and commit themselves to stick to these preannounced plans over the entire horizon no matter what. Alternatively, firms can use feedback strategies. In this case, strategies of firms are functions of time and the *current* values of state variables (see Başar and Olsder, 1995). Firms are not required to precommit. If the unit cost of a competing firm is reduced at some point in time, the opponent reacts by choosing a quantity and an R&D effort level that take this change in unit costs into account. As players condition their actions on the current state of the system and react immediately to any changes in state variables, feedback strategies capture the essence of strategic interactions. Consequently, while harder to derive, feedback Nash equilibrium is a more satisfactory solution concept than open-loop equilibrium. We therefore seek a solution to our dynamic game in the class of feedback Nash equilibria. The latter are derived by a dynamic programming approach and are by construction strongly time consistent or subgame perfect (players have no incentives to deviate unilaterally at any stage of the game).[6]

In our case, a feedback strategy for firm i specifies its quantity produced and R&D effort exerted (control variables) for every possible combination of the two firms' unit costs (state

[6] In the literature, strategies that depend on the current value of state variables (and on the time variable) only are also called Markov strategies. Equilibria in which all players use Markov strategies are called Markov equilibria. If such equilibria are also subgame perfect, they are called Markov-perfect equilibria (see Maskin and Tirole, 2001). Feedback Nash equilibria are by definition Markov perfect.

variables).

Product market and equilibrium output levels

We assume that firms compete in a product market by strategically setting their output levels. The analysis of the product market is simplified by the fact that quantity variables, unlike R&D efforts, do not appear in the equations for the state variables. Hence, production feedback strategies of the dynamic game, associating output levels with unit costs, $q_i = \psi_i(c_1, c_2)$, are static Cournot-Nash strategies of each corresponding instantaneous game.

Proposition 3. *A strategy profile $\psi^*(c_1, c_2) = (\psi_1^*(c_1, c_2), \psi_2^*(c_1, c_2))$, where*

$$\textit{i)} \quad \psi_i^*(c_1, c_2) = \frac{1 - 2c_i + c_j}{3} \quad \textit{if} \quad 2c_1 - c_2 < 1, \quad 2c_2 - c_1 < 1 \tag{4.7}$$

$$\textit{ii)} \quad \psi_i^*(c_1, c_2) = 0, \quad \psi_j^*(c_1, c_2) = \frac{1 - c_j}{2} \quad \textit{if} \quad 2c_i - c_j \geq 1, \quad c_j < 1 \tag{4.8}$$

$$\textit{iii)} \quad \psi_1^*(c_1, c_2) = 0, \quad \psi_2^*(c_1, c_2) = 0 \quad \textit{if} \quad c_1 \geq 1, \quad c_2 \geq 1 \tag{4.9}$$

is a feedback Nash equilibrium of a quantity setting duopoly in the product market $(i, j = \{1, 2\}, i \neq j)$.

Equilibria on the product market are illustrated in Figure 4.1. Both firms produce positive amounts only for combinations of unit costs corresponding to the shaded region. There, the market price is higher than the unit cost of each firm (the first case in the above proposition). In the duopoly on the product market, each firm earns a profit of

$$g_i(c_1, c_2) = \frac{(1 - 2c_i + c_j)^2}{9}. \tag{4.10}$$

Outside the Duopoly region, there is either a monopoly on the product market or there is no production at all. The curve E_1 is the "entry/exit" curve for firm 1. Below it, the market price is lower than firm 1's unit cost, such that it is in the interest of firm 1 not to sell anything. Analogously, E_2 is the "entry/exit" curve for firm 2. Above it, the market price is lower than firm 2's unit cost, such that there firm 2 optimally does not sell anything. In the region "Monopoly of Firm I", firm 2 does not sell anything, whereas firm 1 can sell a positive amount at a price above its unit cost. Consequently, firm 1 is a monopolist on the product market there. Analogously, in the region "Monopoly of Firm II", firm 2 is a monopolist. The two regions – "Monopoly of Firm I" and "Monopoly of Firm II" – correspond to the second case in Proposition 3. Passing from the Duopoly region through E_i in the direction of c_i axis, firm i stops producing ($g_i(c_1, c_2) = 0$), whereas the other, more efficient firm switches to a monopoly output and earns a monopoly profit

$$g_j(c_1, c_2) = \frac{(1 - c_j)^2}{4}. \tag{4.11}$$

In the region "No production", the unit costs of both firms are higher than the choke price ($A = 1$). As firms could sell a positive amount only at negative mark-ups, neither firm produces (the last case in Proposition 3).

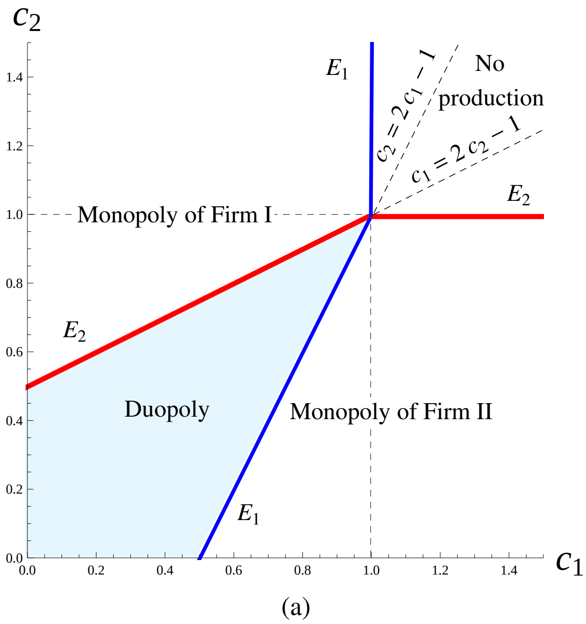
In sum, the sales profit of firm i is given by

$$g_i(c_1, c_2) = \begin{cases} (1 - 2c_i + c_j)^2/9 & \text{if} \quad 2c_1 - c_2 < 1, \quad 2c_2 - c_1 < 1, \\ (1 - c_i)^2/4 & \text{if} \quad 2c_j - c_i \geq 1, \quad c_i < 1, \\ 0 & \quad\quad \text{otherwise,} \end{cases} \tag{4.12}$$

where g_i is a continuous function. The total instantaneous profit is the sales profit g_i diminished by the R&D expenditure k_i^2:

$$\pi_i = g_i(c_1, c_2) - k_i^2. \tag{4.13}$$

Observe that the substitution of equilibrium output levels in firms' profit functions has resulted in the profit function of firm i being dependent only upon unit costs and its R&D effort. Consequently, the problem of the firms is reduced to finding optimal R&D efforts.



(a)

Figure 4.1: *Product-market activity.*

Problem statement

To sum up formally, in our two-firm differential game, each firm maximizes its pay-offs

$$\Pi_i = \int_0^\infty \left[g_i(c_1, c_2) - k_i^2\right] e^{-\rho t} dt \tag{4.14}$$

through its choice of the control $k_i = \gamma_i(t, c_1, c_2) \geq 0$, subject to state equations ($c_i > 0$)

$$\dot{c}_i = c_i \left(1 - (k_i + \beta k_j)\phi\right), \tag{4.15}$$

$i = 1, 2.$

Equilibrium R&D strategies

A feedback strategy of firm i, $k_i = \gamma_i(t, c_1, c_2)$ expresses firm i's R&D efforts as a function of the time variable and both firms' unit costs. Subsequently, we define a feedback Nash equilibrium as a pair of strategies $\gamma^* = (\gamma_1^*, \gamma_2^*)$, such that strategy γ_i^*, if it exists, maximizes the present discounted value of firm i's profits, given that the other firm pursues its strategy

γ_j^*. That is, for firm i, γ_i^* solves

$$\gamma_i^* = \arg\max_{\gamma_i \geq 0} \int_0^\infty \pi_i(\gamma_i, \gamma_j^*) e^{-\rho t} dt, \tag{4.16}$$

subject to the state equations. Introduce the value function $W^i(t, c_1, c_2)$ as the value of the above maximand. That is, let W^i be the maximum present discounted value of profits that can be earned by firm i, given that the other firm pursues its equilibrium strategy. Then, if γ^* is a feedback Nash equilibrium solution to our dynamic game, there exist functions W^i, satisfying the so-called Hamilton-Jacobi-Bellman equations (see Başar and Olsder, 1995, Ch. 6.5) in a suitable sense:

$$\begin{aligned} -W_t^i = \max_{\gamma_i \geq 0} &\left[\left(g_i(c_1, c_2) - \gamma_i^2 \right) e^{-\rho t} + W_{c_i}^i (1 - (\gamma_i + \beta \gamma_j^*(c_1, c_2))\phi) \right. \\ &\left. + W_{c_j}^i (1 - (\gamma_j^*(c_1, c_2) + \beta \gamma_i)\phi) \right], \end{aligned} \tag{4.17}$$

where $i = \{1, 2\}, i \neq j$. We adopt the convention that a subscript to the value function indicates a partial derivative of that function with respect to each subscripted variable.

Introduce the current-time value function $V^i(t, c_1, c_2)$ by setting $W^i = V^i e^{-\rho t}$. That is, V^i equals the profits earned when firm i starts in the state (c_1, c_2) at time t and invests optimally, while firm j pursues its equilibrium strategy. The equations in (4.17) then transform into the following reduced Hamilton-Jacobi-Bellman equations:

$$\begin{aligned} \rho V^i - V_t^i = \max_{\gamma_i \geq 0} &\left[g_i(c_1, c_2) - \gamma_i^2 + V_{c_i}^i (1 - (\gamma_i + \beta \gamma_j^*(c_1, c_2))\phi) \right. \\ &\left. + V_{c_j}^i (1 - (\gamma_j^*(c_1, c_2) + \beta \gamma_i)\phi) \right], \end{aligned} \tag{4.18}$$

where the unknowns are the value functions V^i.

The equations (4.18) above are formulated under the assumption of continuous differentiability of V^i. If this assumption is not valid, V^i does not satisfy the equations (4.18) in a classical sense. In such a case, these equations are understood in the sense of viscosity solution (see Crandall and Lions, 1983). If V^i is differentiable at some (c_1, c_2), then

$$\gamma_i^*(c_1, c_2) = \max\left\{-\frac{1}{2}\phi\left(V_{c_i}^i + \beta V_{c_j}^i\right), 0\right\}. \tag{4.19}$$

It has been shown in Chapter 2 that in the case of a single firm, the value function is not differentiable over the entire space. So-called indifference or Skiba points appear for certain parameter values. These are points where a firm is indifferent between developing and not developing a technology further. At these points, the optimal investment function has a jump, whereas the value function obtains a kink. We expect the same phenomenon to occur in the case of two competing firms. In Chapter 2, we were able to construct the value function using Pontryagin's Maximum Principle. In the present situation, where we are dealing with a dynamic game, we are not able to obtain a solution along these lines. In the next section, we therefore propose a method to obtain a numerical approximation to the value function. In a nutshell, this method consists in considering a related stochastic optimization problem, where the noise intensity σ depends on a parameter ε. As $\varepsilon \to 0$, the Nash equilibria of the stochastic problem tend to those of the deterministic problem.[7]

4.3 Computation

First, we consider the state equation (4.2) with a noise term added. Specifically, let the state of firm i evolve according to the following stochastic differential equation

$$dc_i = c_i\left(1 - (k_i + \beta k_j)\phi\right)dt + c_i\sqrt{2\varepsilon}dB_i, \tag{4.20}$$

where $B_i(t)$ is a standard Brownian motion or Wiener process, and where ε denotes the noise level. Note that this equation is of the form $dc = \mu(c, k)dt + \sigma(c)dB$, where μ and σ are drift and diffusion, respectively, of a controllable Itô process c (see Kloeden and

[7] Kossioris *et al.* (2008) numerically compute a non-linear feedback Nash equilibrium for a differential game with a *single* state variable, limiting themselves to a class of continuous feedback rules. Dockner and Wagener (2006) study necessary conditions for feedback equilibria in games with a single state variable. Through an auxiliary system of differential equations they are also able to find non-continuous feedback strategy equilibria.

Platen, 1995). Hence, we make firms face some randomness in their unit costs.[8] Firms then maximize their expected current and future profits. That is, firm i solves the problem $\max_{\gamma_i \geq 0} \mathbb{E} \int_0^\infty \pi_i(\gamma_i, \gamma_j^*) e^{-\rho t} dt$. The conditions that characterize a feedback Nash equilibrium of the stochastic game are then given by the following coupled second-order parabolic partial differential equations (see Başar and Olsder, 1995, Ch. 6.7)[9]

$$\begin{aligned} \rho V^i - V_t^i = \max_{\gamma_i \geq 0} \big[& g_i(c_1, c_2) - \gamma_i^2 + V_{c_i}^i (1 - (\gamma_i + \beta \gamma_j^*(c_1, c_2))\phi) \\ & + V_{c_j}^i (1 - (\gamma_j^*(c_1, c_2) + \beta \gamma_i)\phi) + c_i^2 \varepsilon V_{c_i, c_i}^i + c_j^2 \varepsilon V_{c_j, c_j}^i \big], \end{aligned} \tag{4.21}$$

$i = \{1, 2\}$. Some motivation for the above formulation is in place. We conjecture on the basis of the results obtained in Chapter 2 that the value functions of the deterministic game are non-smooth. This makes numerical schemes to approximate the value functions hard to implement. For this reason, we exploit the fact that adding stochasticity to the state equations translates into two additional second-order terms being added to the Hamilton-Jacobi-Bellman equations ($c_i^2 \varepsilon V_{c_i, c_i}^i$ and $c_j^2 \varepsilon V_{c_j, c_j}^i$, respectively). We expect these two terms to have the effect of artificially smoothing-up the value functions in (4.18) in the regions where the functions change most rapidly. Then, hopefully, the solution to (4.21) will resemble a solution to (4.18) when the viscosity coefficient $\varepsilon \downarrow 0$. We call in this way obtained solution to (4.18) *a vanishing viscosity solution* (see Başar and Olsder, 1995, Ch. 5.7). A benefit of this formulation is that equation (4.21) is not only an equation for an approximating solution to (4.18), but represents an equilibrium condition for the value function of the related stochastic dynamic game which is interesting in its own right.

In the stochastic process (4.20), we let diffusion σ be a function of c_i. In particular,

[8] Equation (4.20) can be rewritten as

$$\frac{\dot{c}_i}{c_i} = \left(1 + \sqrt{2\varepsilon} \frac{dB_i}{dt}\right) - (k_i + \beta k_j)\phi,$$

where the change in the unit cost depends on two factors: i) the results of the R&D process (the second bracket) and ii) some randomness which is out of a firm's control (the first bracket). In a sense, one can see this as perturbing the depreciation rate δ in the original model (4.2) with white noise.

[9] Notice that there is no stochasticity in the Hamilton-Jacobi-Bellman equations – the expectation operator and randomness have been eliminated by using Itô's lemma.

the variance of noise decreases as c_i decreases. This prevents the system to jump over to negative values of c_i when the values of unit costs get close to zero: the stochastic process (4.20) satisfies the so-called *Feller condition* (Feller, 1951), such that for any $c_i(0) > 0$, the value of $c_i(t)$ remains strictly positive with probability one for all times t.[10] However, when numerically solving (4.21), the dependence of second-order terms on the state variables becomes inconvenient. In the next step, we therefore apply a diffeomorphic transformation of the state variables

$$c_i = e^{-x_i}, \tag{4.22}$$

converting them to more convenient coordinates. Using Itô's formula, $c_i(t)$ in the state equation (4.20) then transforms into a stochastic process $x_i(t)$ with constant diffusion strength[11]

$$dx_i = ((k_i + \beta k_j)\phi - 1 + \varepsilon)\, dt - \sqrt{2\varepsilon} dB_i. \tag{4.23}$$

The Hamilton-Jacobi-Bellman equations take the form:

$$\begin{aligned} \rho V^i - V^i_t = \max_{\gamma_i \geq 0} \big[& g_i(x_1, x_2) - \gamma_i^2 + V^i_{x_i}((\gamma_i + \beta\gamma_j^*(x_1, x_2))\phi - 1 + \varepsilon) \\ & + V^i_{x_j}((\gamma_j^*(x_1, x_2) + \beta\gamma_i)\phi - 1 + \varepsilon) + \varepsilon V^i_{x_i,x_i} + \varepsilon V^i_{x_j,x_j} \big]. \end{aligned} \tag{4.24}$$

To solve the infinite horizon problem, we consider a family of finite horizon problems over $[0, T]$, $T \to \infty$, with a terminal value of zero. That is, $V^i(T) = 0$, as at time T nothing is left for the firms. These finite horizon problems can all be solved simultaneously by reversing the direction of time. Thus, we introduce time to completion, $s = T - t$, as a new time

[10] This formulation is also intuitive. It means that the more efficient a firm gets, the smaller is the probability of large unexpected changes in its unit costs. As an efficient firm has already done a great deal of R&D, it is quite realistic to expect that such an experienced firm is in a better position to avoid undesired, positive shocks to its unit costs. Moreover, as the firm has already reaped many fruits of its R&D endeavors, it is also realistic to expect that unexpected discoveries leading to large further reductions in unit costs are less likely. In fact, in his analysis of the petrochemical industry, Stobaugh (1988) documents that the probability of next process innovation being major innovation decreases over time.

[11] Precisely put, a simple logarithmic transformation $x_i = \ln(c_i)$ would already achieve our aim. We take an inverse of c_i under the logarithm as we find it convenient that the main region of interest (region where $c_i < 1$) is on positive axes and so also the steady states of the state vector field have positive coordinates. On a related note, the applied logarithmic transformation is convenient also for the reason that we can obtain a solution over the same range of the state variable (after the ex-post inverse transformation) by effectively solving over a much smaller range of it (e.g., solving for $x_i \in [a, b]$ enables us to obtain a solution over c_i in the range of $[e^{-b}, e^{-a}]$, $a, b \in \mathbb{R}$. This allows for a significant saving on grid points.

variable. This has the effect of transforming the terminal condition $V^i(T, x_1, x_2) = 0$ into an initial condition $\mathcal{V}^i(0, x_1, x_2) = 0$, where $\mathcal{V}^i(s, x_1, x_2)$ is a value function of the time-reversed problem satisfying

$$\begin{aligned}\rho\mathcal{V}^i + \mathcal{V}^i_s = \max_{\gamma_i \geq 0} \big[& g_i(x_1, x_2) - \gamma_i^2 + \mathcal{V}^i_{x_i}((\gamma_i + \beta\gamma_j^*(x_1, x_2))\phi - 1 + \varepsilon) \\ & + \mathcal{V}^i_{x_j}((\gamma_j^*(x_1, x_2) + \beta\gamma_i)\phi - 1 + \varepsilon) + \varepsilon\mathcal{V}^i_{x_i,x_i} + \varepsilon\mathcal{V}^i_{x_j,x_j}\big].\end{aligned} \tag{4.25}$$

Let $V^i_T(t, x_1, x_2)$ denote the value function of a game with finite horizon $[0, T]$. It then holds that $\mathcal{V}^i(T, x_1, x_2) = V^i_T(0, x_1, x_1)$. Hence, once we have obtained a solution for $\mathcal{V}_i$ over $s \in [0, S]$, we can recover the value function V^i_T for the entire family of finite horizon problems $[0, T]$ with $T \in [0, S]$.

Observe now that in our game, profit functions as well as state equations are autonomous. That is, they do not explicitly depend on the time variable. Furthermore, the discount rate and all other parameters are constant throughout the game, the stochastic shocks are independent, the firms are fully informed at the outset of the game and do not learn anything new about the game over the course of time. As a consequence, with infinite horizon, the continuation game at some subsequent instant is identical to the game at the initial instant. In other words, if a firm finds itself in some state (x_1, x_2), the rest of the game is the same whether this situation occurs at some time t_1 or t_2. We therefore expect that the value function of the infinite horizon game is time-invariant. That is, we expect that $V^i_T(t, x_1, x_2) \to \overline{V}^i(x_1, x_2)$ as $T \to \infty$.[12] But then also the time-reversed value function must become time-invariant as the solution proceeds to infinity. That is, as $s \to \infty$, $\mathcal{V}^i$ becomes a function of x_1 and x_2 alone: $\mathcal{V}^i(t, x_1, x_2) \to \overline{\mathcal{V}}^i(x_1, x_2)$ and thus $\overline{\mathcal{V}}^i_s = 0$, where $\overline{\mathcal{V}}^i(x_1, x_2)$ solves the stationary

[12]Notice that $W^i(t, x_1, x_2) = \overline{V}^i(x_1, x_2)e^{-\rho t}$, where W^i is the value of profits discounted to the initial time of 0 (defined as in (4.17)). That is, while $\overline{V}^i$ is time-independent, W^i is not. This is intuitive. While, for a given x_1 and x_2, $\overline{V}^i$ is the same for some t_1 and t_2 $(t_2 > t_1)$, W^i is not as $\overline{V}^i(t_2)$ occurs later and is so worth less than $\overline{V}^i(t_1)$ when evaluated at the initial time of 0.

Hamilton-Jacobi-Bellman equations

$$\begin{aligned}\rho\overline{\mathcal{V}}^i = \max_{\gamma_i\geq 0}\Big[&g_i(x_1,x_2) - \gamma_i^2 + \overline{\mathcal{V}}^i_{x_i}((\gamma_i + \beta\gamma_j^*(x_1,x_2))\phi - 1 + \varepsilon)\\ &+ \overline{\mathcal{V}}^i_{x_j}((\gamma_j^*(x_1,x_2) + \beta\gamma_i)\phi - 1 + \varepsilon) + \varepsilon\overline{\mathcal{V}}^i_{x_i,x_i} + \varepsilon\overline{\mathcal{V}}^i_{x_j,x_j}].\end{aligned} \tag{4.26}$$

To obtain an approximating numerical solution to our infinite horizon game, we therefore let the solution to (4.25) progress in s towards infinity and stop once $\mathcal{V}_s^i$ is sufficiently close to zero.[13] Equilibrium strategies corresponding to the so computed value functions are then also stationary and can be expressed as functions of x_1 and x_2 only: $k_i^* = \gamma_i^*(x_1,x_2)$, where

$$\gamma_i^*(x_1,x_2) = \max\left\{\frac{1}{2}\phi\left(\overline{\mathcal{V}}^i_{x_i} + \beta\overline{\mathcal{V}}^i_{x_j}\right), 0\right\}. \tag{4.27}$$

4.3.1 Numerical Method of Lines

The computation of a feedback Nash equilibrium amounts to solving the Hamilton-Jacobi-Bellman equations given in (4.25). In effect, this leads to determining solutions to a system of two coupled non-linear implicit two-dimensional partial differential equations. We solve this system using the so-called *numerical method of lines* (see Schiesser, 1991), whose main idea is the following. We discretize spatial derivatives of V^i by evaluating their algebraic approximations over the pre-specified grid points, but we leave the time variable continuous. This leads to a system of ordinary differential equations to which usual numerical methods for solving initial value problems can be applied.[14]

We use a finite difference discretization of partial derivatives. For this, we introduce a grid in space, $x_{i,1} < x_{i,2} < ... < x_{i,n}$, where n is the number of grid points in one dimension. We assume constant grid spacing $\Delta x_i = (x_{i,n} - x_{i,1})/(n-1)$, such that $x_{i,\ell} = x_{i,1} + (\ell - 1)\Delta x_i$,

[13]This approach resembles what is in the literature known as a *method of false transients* (see Schiesser, 1991), where a time derivative which is not part of the original problem is added to a partial differential equation in order to transform it into a well-posed initial value (Cauchy) problem. It is then expected that this additional term will have an insignificant effect on the final solution. In our case, we however deal with *true transients* as $\mathcal{V}_s^i$ (or V_t^i) is a true part of the Hamilton-Jacobi-Bellman equation (corresponding to a finite-horizon game) and approaches true zero only in limit (when the horizon of the game approaches infinity and the game itself becomes stationary).

[14]We wrote the code for computations in Fortran 95. The code uses double precision arithmetic. The criterion for the convergence of a solution is that the value of the L^2-norm of $\mathcal{V}_s$ is below 1×10^{-12}. Auxiliary calculations and plots were executed in MATLAB and Mathematica.

for $\ell = 1, ..., n$. The obtained tensor grid is taken to be square ($\Delta x_1 = \Delta x_2 = h$). At each point of the grid, $(x_{1,k}, x_{2,m})$, we then replace first-order and second-order derivatives by second-order central finite differences[15], e.g.,

$$\frac{\partial}{\partial x_1} V^i(t, x_{1,k}, x_{2,m}) \approx \frac{V^i_{(k+1,m)} - V^i_{(k-1,m)}}{2h}, \tag{4.28}$$

$$\frac{\partial^2}{\partial^2 x_1} V^i(t, x_{1,k}, x_{2,m}) \approx \frac{V^i_{(k-1,m)} - 2V^i_{(k,m)} + V^i_{(k+1,m)}}{h^2}. \tag{4.29}$$

This leads to a system of $n \times n$ ordinary differential equations, which we solve using a third-order Runge-Kutta method (Judd, 1998).[16]

Boundary conditions

We already motivated our choice of the initial condition $\mathcal{V}^i(0, x_1, x_2) = 0$. To solve the system of differential equations (4.25), we also need to specify boundary conditions corresponding to the four sides of the grid square.[17] The problem is that the value of a solution at all boundaries is *ex ante* not known to us. We address this delicate matter in Appendix 4.B where we argue that the misspecification of the boundary conditions only results in a significant error in a

[15] We find the second-order scheme a good compromise between accuracy and the minimization of oscillation. That is, while higher-order schemes are in principle expected to increase accuracy, they also increase the possibility of undesired oscillation in the solution (derivatives of increasing order have more roots between which the solution can oscillate).

[16] In presented plots, we set $n = 200$, which leads to a square state space grid of $40,000$ points over $[-2.5, 4.5] \times [-2.5, 4.5]$. The accuracy of the numerical calculation can be increased by increasing the number of grid points, which reduces Δx. To prevent the solution from becoming unstable, we place an upper limit on the time step Δt in the Runge-Kutta method. In particular, we require the time step to satisfy two conditions. The first one is the Courant-Friedrichs-Lewy condition, specifying $\Delta t < \frac{\Delta x}{v}$, where v is a maximum drift velocity. The second one concerns time required for diffusion to be captured $\Delta t < \frac{\Delta x^2}{2\varepsilon}$. We observe that the latter condition is non-binding for small ε.

[17] We already noted that the probability that unit costs reach the value of zero is zero. As the boundary is not attainable, the boundary condition at $c_i = 0$ is not needed from mathematical perspective. However, when solving the system of partial differential equations with a finite differences method, the boundary condition at $c_i \to 0$ is needed despite being mathematically redundant.

small region along the boundaries. In short, we select the following boundary conditions:

$$\frac{\partial}{\partial x_1}\mathcal{V}^1(s, x_{1,1}, x_{2,\ell}) = 0, \quad \frac{\partial}{\partial x_1}\mathcal{V}^2(s, x_{1,1}, x_{2,\ell}) = 0, \tag{4.30}$$

$$\frac{\partial}{\partial x_2}\mathcal{V}^1(s, x_{1,\ell}, x_{2,1}) = 0, \quad \frac{\partial}{\partial x_2}\mathcal{V}^2(s, x_{1,\ell}, x_{2,1}) = 0, \tag{4.31}$$

$$\frac{\partial}{\partial x_1}\mathcal{V}^1(s, x_{1,n}, x_{2,\ell}) = 0, \quad \frac{\partial}{\partial x_1}\mathcal{V}^2(s, x_{1,n}, x_{2,\ell}) = 0, \tag{4.32}$$

$$\frac{\partial}{\partial x_2}\mathcal{V}^1(s, x_{1,\ell}, x_{2,n}) = 0, \quad \frac{\partial}{\partial x_2}\mathcal{V}^2(s, x_{1,\ell}, x_{2,n}) = 0, \tag{4.33}$$

where $\ell = 1, ..., n$.

Time paths

Once we have obtained the numerical approximation of the value functions in (4.25) and from them the equilibrium feedback strategies, we can simulate the investment paths of firms. For low values of ε, equation (4.23) with only the drift term generates a good approximation of the evolution of the state variables over time. Hence, we solve the following system of ordinary differential equations:

$$\dot{x}_i = (\hat{\gamma}_i^* + \beta\hat{\gamma}_j^*)\phi - 1 + \varepsilon, \quad x_i(0) = x_i^0, \tag{4.34}$$

$i = \{1, 2\}$, where $\hat{\gamma}_i^*(x_1, x_2)$ and $\hat{\gamma}_j^*(x_1, x_2)$ are obtained from (4.27) after replacing derivatives of the value functions with their numerical approximations and x_i^0 is firm i's initial value of unit cost. We solve the above system of ordinary differential equations over the time interval $[0, T]$ by a third-order Runge-Kutta method (Judd, 1998), where we limit terminal time T so that the values of x_i remain within the state grid. We calculate any necessary value of variables between the grid points by using cubic splines interpolation (Judd, 1998).

The state vector field assigns to each point in the (by interpolation refined) grid a vector $(\dot{x}_1, \dot{x}_2)$. Steady states are the grid points corresponding to a zero vector $(\dot{x}_1, \dot{x}_2) = (0, 0)$. To analyze the stability of steady states, we approximate the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{pmatrix} \tag{4.35}$$

in each steady-state point of the vector field. We then compute the eigenvalues of the Jacobian matrix and compare their signs.[18] All derivatives are approximated by second-order central differences.

4.4 Equilibrium strategies and industry dynamics

In this section, we present the results of the numerical analysis. We discuss strategic interactions between firms as implied by their value and policy functions. Furthermore, to obtain insight into possible evolutions of the game, we analyze state vector fields and time paths of certain variables of interest.

We begin by examining the case of moderate spillover effects ($\beta = 0.5$). Afterwards, we confront our conclusions with the case of low and high spillovers. The presented plots are all drawn for $\phi = 8$ and $\tilde{\rho} = 1$.[19] The dynamics at this parameterization is representative of all the cases in which firms have an incentive to develop further a technology which requires R&D efforts prior to production. We discuss this at greater length later on, in Section 4.4.8.

4.4.1 Value function

Figure 4.2 shows the value functions and R&D efforts for $\varepsilon_1 = 0.125$ and $\varepsilon_2 = 0.0156$. As the value functions are symmetric in the sense that $\mathcal{V}^2(x_2, x_1) = \mathcal{V}^1(x_1, x_2)$, it is sufficient to consider just $\mathcal{V}^1$. Note that $\varepsilon_1 > \varepsilon_2$ and that ε_1 graphs are smoother. Note also that large values of x_i correspond to small values of c_i.[20] As Figure 4.2 shows, a firm's value function

[18] If the real part of each eigenvalue is negative, the steady state is asymptotically stable. If the real part of at least one eigenvalue is positive, the steady state is unstable. More particular, if one eigenvalue is real and positive and the other one real and negative, the steady state is a saddle.

[19] $\tilde{\rho} = \rho/\delta$ is a rescaled discount factor (see Appendix 4.A).

[20] Recall that $c_i = e^{-x_i}$. Hence, negative (positive) values of x_i correspond to unit costs above (below) the choke price ($A = 1$). At the latter, $x = 0$. As $x_i \to \infty$, $c_i \to 0$.

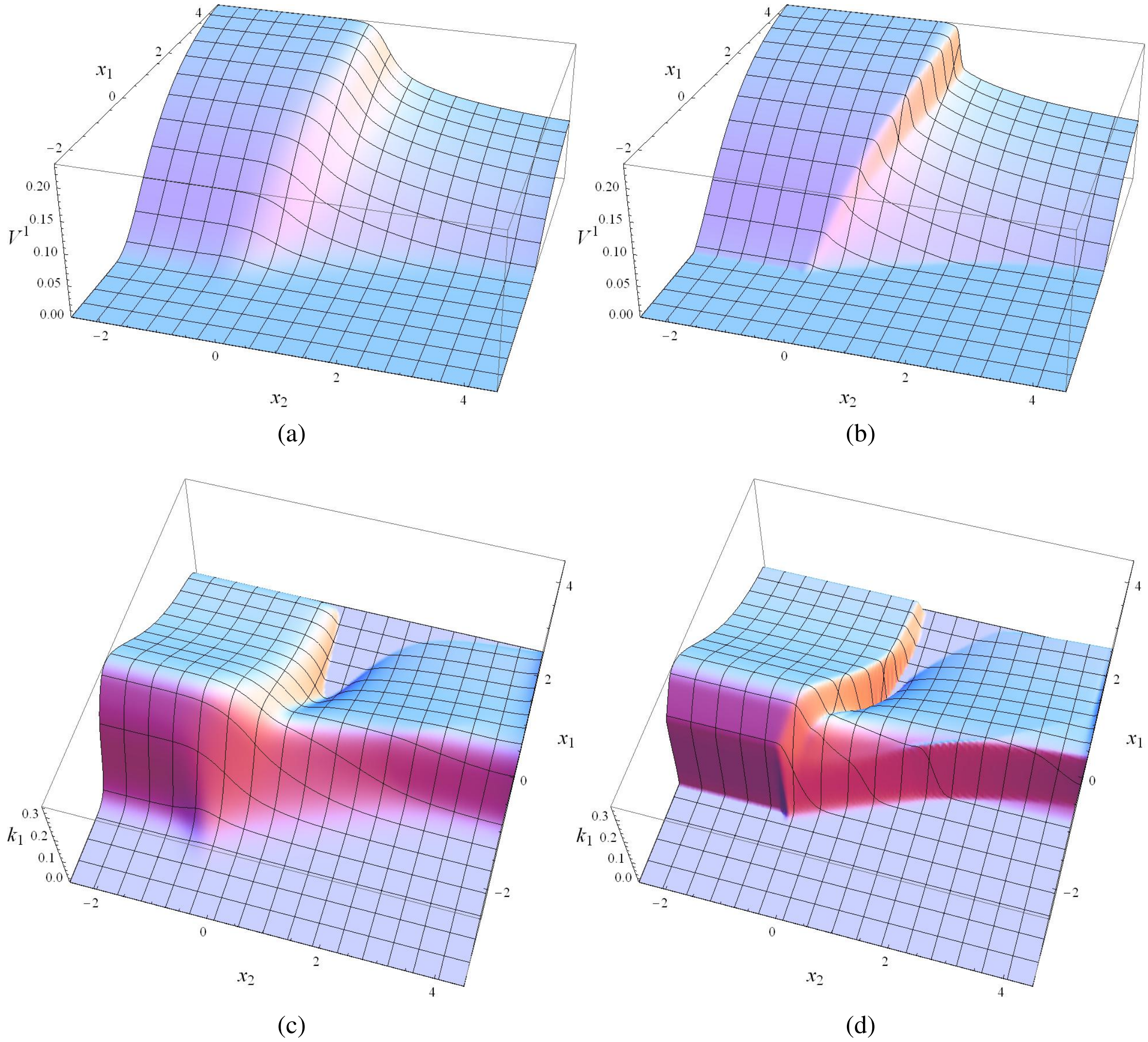
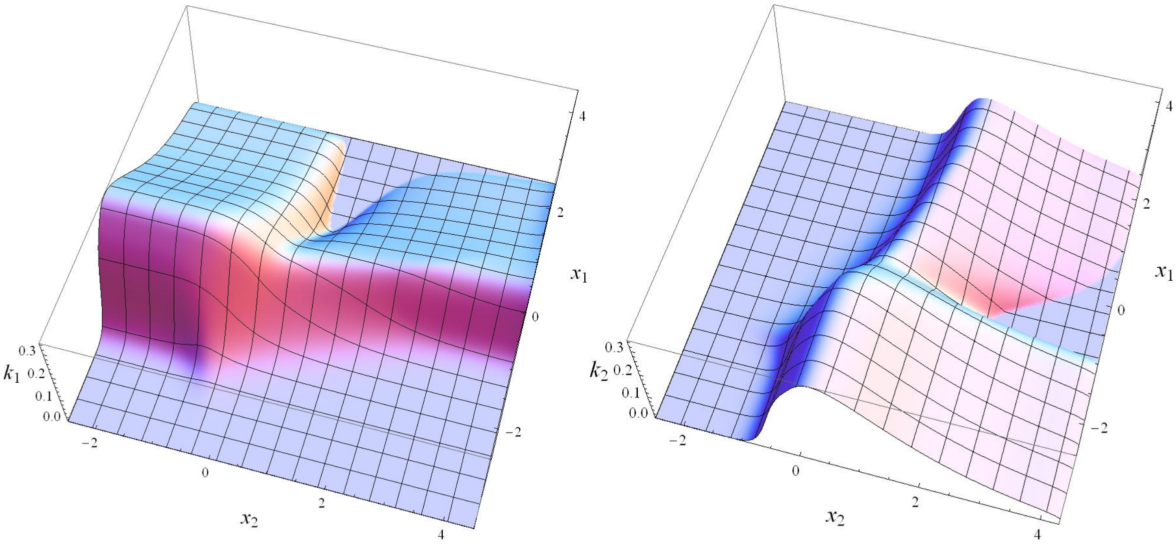


Figure 4.2: *Value functions (top) and R&D efforts (bottom) for* $\varepsilon = 0.125$ *(left) and* $\varepsilon = 0.0156$ *(right). In both cases* $\beta = 0.5$.

is negatively related to a firm's own unit cost and positively related to the unit cost of its competitor. The smaller a firm's unit costs for a given cost of its competitor, the better a firm's competitive position and so the larger the profits a firm is able to reap. The highest, left part of the value function corresponds to unit costs for which firm 1 is a monopolist. Firm 1's relative cost advantage keeps its competitor out of the market. For lower values of firm 2's unit costs, both firms are (eventually) active in the market (recall Figure 4.1). This change of the regimes is marked by a steep decline in the value function of the incumbent firm. For relatively high values of own unit costs (the region of the southern valley), the value of the game for firm 1 is zero as the firm finds it optimal to stay inactive.



(a) (b)

Figure 4.3: *R&D efforts for firm* 1 *(left) and firm* 2 *(right).* $\beta = 0.5$, $\varepsilon = 0.125$.

4.4.2 Policy function

The profits a firm is able to reap from the product market are determined by a firm's cost efficiency. The latter is costly in the sense that due to a positive rate of technology depreciation, a firm needs to invest in R&D not only to increase its efficiency (relative to its competitor), but also to maintain it. The equilibrium R&D efforts are shown in Figure 4.2. As equation (4.27) indicates, we can decompose two effects underlying R&D efforts. We call the first effect, which corresponds to the relation between the firm's value of the game and its own unit cost, $\mathcal{V}^i_{x_i}$, *a pure cost effect*, and the second one, which corresponds to the relation between the firm's value of the game and its competitor's unit cost, $\beta\mathcal{V}^i_{x_j}$, *a feedback cost effect.*

Observe that, for a given unit cost of its competitor, the R&D effort of a firm increases with decreasing own unit cost over the region of zero production and decreases shortly thereafter. This is driven by the pure cost effect. This effect is always positive and is present whether the competing firm is active or not. It is independent of strategic considerations in the sense that it concerns the relation between a firm's own production costs and its profits. When initial unit costs are high (but still low enough for a firm to pursue further development), there are huge benefits for a firm to exert R&D efforts as this reduces the amount of time needed to reach the production phase. Consequently, R&D efforts are high. The lower the unit costs, the more efforts it takes to reduce them further. This, together with lower tendency of technology to depreciate for lower unit costs (see equation (4.20)), leads the firm to optimally invest the less, the lower its unit costs.

On the contrary, the feedback cost effect corresponds to the fact that due to spillovers, any R&D effort a firm exerts contributes also to the reduction of a competitor's production costs, which retroactively affects the firm's profits through the product market competition. The feedback cost effect is always negative and depends positively on the level of spillovers. In the extreme case of zero spillovers, this effect is null. The feedback cost effect underlies industry dynamics through strategic considerations discussed in what follows.

4.4.3 Vector field and dynamics

The R&D efforts that both firms exert influence the way in which unit costs evolve over time through the drift term (see equation (4.23)). This evolution of costs as governed by the drift term is summarized by the drift vector field in Figure 4.4. Let $x_i^\varepsilon(t)$ be a solution to (4.23). Notice first that if $\varepsilon = 0$, then (4.23) is a deterministic ordinary differential equation with a unique deterministic solution (given k_1 and k_2). If however $\varepsilon > 0$, then (4.23) is a stochastic differential equation whose solution $x_i^\varepsilon(t)$ is a stochastic (random) process. Consequently, in a deterministic game, as approximated by a stochastic game with a small ε, the drift vector field shows how costs evolve for every possible initial position. However, in a stochastic game, the drift vector field only shows the most likely evolution of costs for finite times. This is further illustrated in plot (a) of Figure 4.5. The bold curve indicates a possible path of x_i over time, $x_i = x_i(t)$, as implied by the drift vector field in Figure 4.4. At every *finite* time t, the value $x_i(t)$ converges to the mean of the distribution of trajectories $x_i^\varepsilon(t)$ of the stochastic differential equation (4.23) in the feedback strategy Nash equilibrium as $\varepsilon \to 0$. The variance of the distribution increases over time due to the cumulation of random shocks (see Theorem 2.2.2 in Freidlin and Wentzell, 1998). Plot 4.5b shows how the actual costs fluctuate around the drift path.[21] In what follows, we therefore use a drift path as an approximation of the evolution of the game over time.

[21]The stochastic paths were calculated using the Euler-Maruyama scheme (see Kloeden and Platen, 1995).

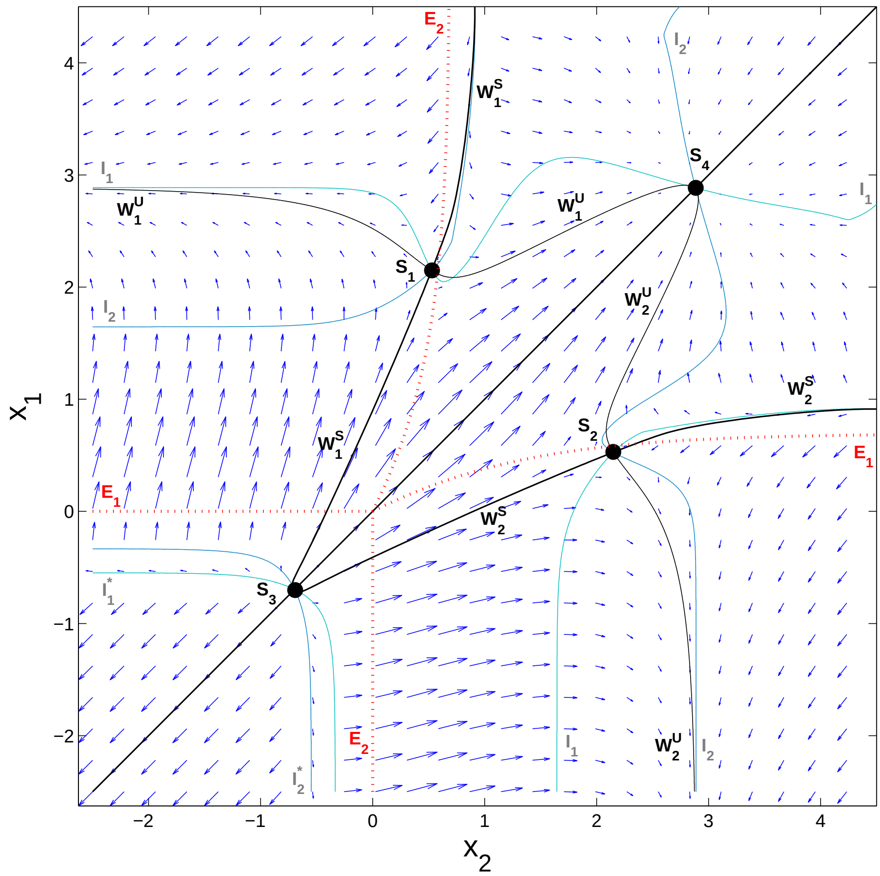


Figure 4.4: *Drift vector field.* $\beta = 0.5$, $\varepsilon = 0.125$.

In Figure 4.4, the $\dot{x}_1 = 0$ loci (labeled I_1) and $\dot{x}_2$ loci (labeled I_2) intersect in four steady states of the drift vector field: S_1 and S_2 are saddles, S_3 is a nodal source, whereas S_4 is a nodal sink. Invariant manifolds of the two saddles are labeled by letter W. Stable and unstable manifolds of S_1 are labeled by W_1^S and W_1^U, respectively. Similarly, W_2^S and W_2^U are, respectively, a stable and an unstable manifold of S_2. E_1 and E_2 are the product market "entry/exit" curves of firm 1 and firm 2, respectively. They were introduced in Figure 4.1 and are here redrawn in the new coordinates. In the region above the indicated 45-degree diagonal, firm 1 has a cost advantage over firm 2, whereas the reverse is true in the region below the diagonal. For combinations of unit costs lying exactly on the diagonal, the firms are equally efficient. Observe that the vector field below the indicated 45-degree diagonal is a mirror image of the field above the diagonal. This follows from the symmetry of the feedback equilibrium which is best visible in Figure 4.3.

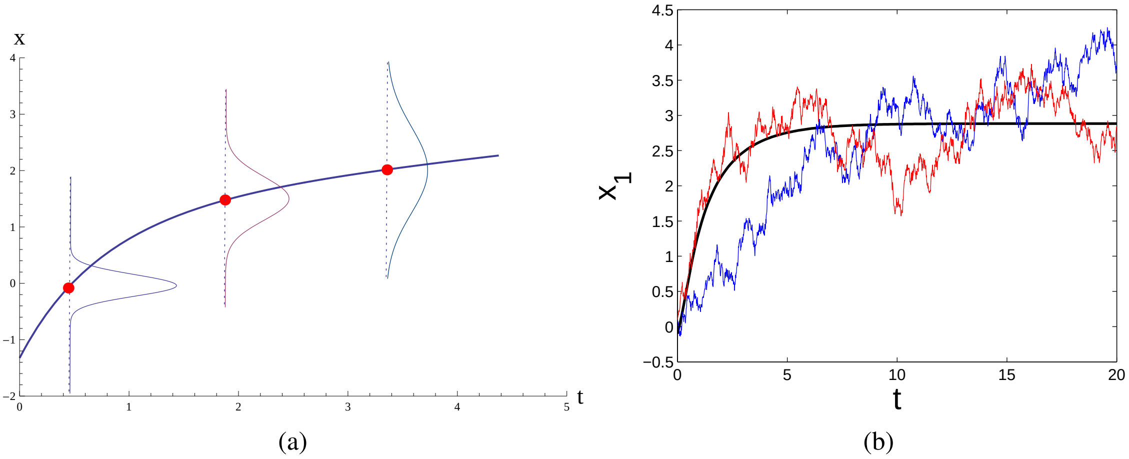


Figure 4.5: *(a) Drift as a change of the mean value of a stochastic process; (b) Drift path (bold line), corresponding to the drift vector field in Figure 4.4, and two different realizations of a stochastic path for unit cost of firm 1* ($x_1(0) = x_2(0) = -0.1, \Delta t = 0.01, T = 20$).

Notice that W_1^S and W_2^S are separatrices which divide the state space into two domains. The first domain is a basin of attraction of the asymptotically stable steady state S_4. Every motion starting in this domain converges to S_4 as $t \to \infty$. In this domain, eventually both firms are active on the product market. In the second domain, the unit cost of at least one firm diverges to infinity; we are left either with a monopoly or no market at all.

We now analyze possible evolutions of the game by jointly looking at Figure 4.3 and Figure 4.4. In the region south-west from S_3, the unit costs of both firms are "very" high and above the choke price, such that both firms decide to refrain from developing further the initial technology. Future expected profits are not high enough to compensate for investments needed to bring technology to the production phase. Technically, unit costs flow towards infinity due to a positive depreciation rate.

Left to the stable manifold of S_1, labeled in Figure 4.4 by W_1^S, the cost advantage of firm 1 over firm 2 is so large that the latter gives up on R&D (see figure 4.3). When cost asymmetries are large, the profits the less efficient firm earns on the product market are low. This reduces the ability of firm 2 to compensate for R&D investments needed to bring its technology to the product market and catch up with firm 1. It turns out that left to W_1^S the cost asymmetries are just so large that firm 2 cannot even afford to battle depreciation of its own technology, thereby succumbing to its more efficient competitor. Firm 2 does produce only when its initial unit costs are already sufficiently low (the region between E_2 and W_1^S curve), such that it can profitably sell a positive quantity in a competitive product market. However, its product market activity is only temporary as the large cost advantage enables the more efficient firm 1 to squeeze firm 2 out of the market. Thus, firm 2 does eventually neither produce nor invest in R&D. Its unit costs in this region always tend to flow towards infinity, which is due to a positive depreciation rate. Intuitively, though, we always interpret any situation in which a firm stays inactive as if this firm has left the market.

The investment of firm 1 depends on its initial unit cost. The firm decides to enter the market for all initial costs that in Figure 4.4 correspond to x_1 above the I_1^* curve which flows through S_3. For initial unit costs above the choke price (x_1 below the horizontal part of E_1 curve), the firm does at first produce nothing but invests increasingly in the reduction of its unit costs. Once its unit costs have been reduced below the choke price, the firm starts producing as it can now sell at positive mark-ups. The level of R&D efforts and unit costs then gradually decrease to their long-run optimal levels (x_1 approaches the unstable manifold W_1^U which asymptotically converges with I_1 isocline). For unit costs above the choke price, instantaneous profits of firm 1 are negative as there is no production yet. Firm 1 initiates R&D as it expects future profits will more than compensate for initial investments. There exists a finite upper bound on unit costs beyond which expected future profits are not enough to compensate for short run losses (unit costs corresponding to x_1 below the I_1^* curve). In this case, the initial technology is not developed further. Observe how the direction of vectors in Figure 4.4 changes its sign when passing through the I_1^* curve. While firm 2 benefits from R&D efforts of firm 1 through spillovers, this effect is not strong enough to bring firm 2 onto the market. It, however, slows down the rise in the discrepancy between the two firms' unit costs.

In the south-eastern part of the state space, below the W_2^S curve, the situation is reversed. It is now firm 2 whose cost advantage leads to its monopoly. For all initial unit costs on the right side of the I_2^* curve passing through S_3, firm 2 brings a technology on the market, while firm 1 is sooner or later forced out of business.

In the north-east region of the state space, between the W_1^S and W_2^S manifolds, the cost asymmetries are moderate. Eventually, a product market duopoly emerges as for all initial costs in this region, each firm sooner or later brings a technology on the product market. It is interesting to observe that the asymptotically stable steady state S_4 lies on the 45-degree diagonal. This implies a kind of a *regression toward the mean* phenomenon, where any initial difference in the unit costs between firms tends to vanish over time.[22] We have noted that above the 45-degree diagonal, firm 1 has a cost advantage, which is to the left of W_1^S large enough to squeeze firm 2 out of the market. However, to the right of W_1^S, this is not the case any more. Notice that W_1^S curve travels along the edge of the precipice in the policy function of firm 2 (see the right plot in Figure 4.3). While left to W_1^S firm 2 gives up on R&D, right to W_1^S, it invests heavily to catch up with firm 1. Firm 1 exerts less R&D efforts than firm 2, however, it prolongs its cost supremacy through positive spillover effects arising from relatively high R&D efforts of firm 2. When its initial costs are very low, firm 1 for some time even sits back on R&D (observe the basin in the northern region of the firm 1's policy function in Figure 4.3) and retards its technology decay optimally by relying mostly on spillovers from the R&D efforts of its zealous counterpart.[23] Namely, when unit costs of firm 1 decrease relative to firm 2, an additional unit of firm 1's R&D effort benefits firm 2 progressively more than firm 1 itself, which diminishes firm 1's incentives for own R&D (this follows directly from the formulation of unit costs in (4.2)). The story is analogous when we are on the other side of the diagonal, where firm 2 has a relative cost advantage. In both cases, a dominant firm gradually loses its lead.

4.4.4 Leader versus follower

We have noted that whenever cost asymmetries are large, the less efficient firm is squeezed out of the market. Only when initial asymmetries are not too large, that is, when we are within the basin of attraction of S_4, both firms steer the evolution of their costs so that they remain active in the product market. In this latter case, any initial asymmetry between the firms tends

[22] We say "tends to" as it *de facto* vanishes only in light of a deterministic game interpretation. In a stochastic interpretation, only the gap between the mean values of the two unit costs narrows and eventually closes.

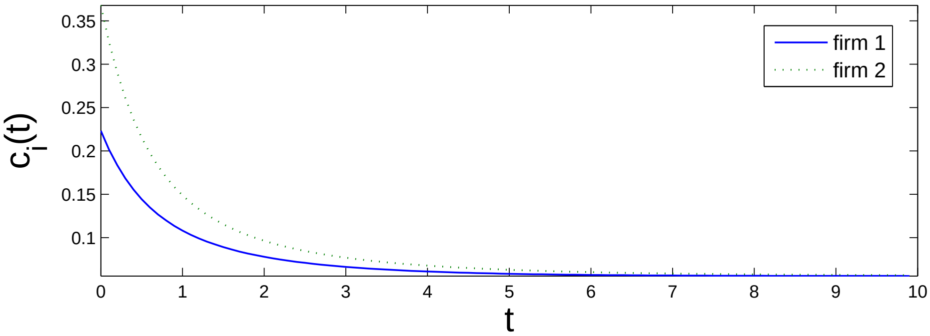
[23] A typical example of a large firm relying on inventions by smaller firms is Microsoft, whose competitors "have long complained that the rest of the industry has served as Microsoft's R&D lab" (*New York Times*, 4th August, 1991, p. 6).

to vanish over time.[24] Figure 4.6 illustrates further how in the latter case, costs and R&D efforts evolve over time along the drift path. In the first two plots, firms already start with relatively low costs, $(x_1, x_2) = (1.5, 1)$ or $(c_1, c_2) = (0.22, 0.37)$. We see that the follower exerts more R&D efforts than the leader, which gradually reduces the gap between the two firms' unit costs. In plots 4.6c and 4.6d, both firms start with unit costs above the choke price, $(x_1, x_2) = (-0.2, -0.3)$ or $(c_1, c_2) = (1.22, 1.35)$. With own unit costs very large, the more efficient firm 1 initially invests a lot in R&D, in particular, more than firm 2. Firm 1 exploits its cost advantage to enter the product market first and earn temporary monopoly profits. The situation changes in the course of time and eventually the follower invests more than the leader. This causes the gap between the unit costs to shrink (the evolution of costs is similar to that in 4.6a and is omitted for brevity). The quantity each firm produces increases over time together with decreasing unit costs. As firm 1 is more efficient than firm 2, firm 1 at all times produces more. However, with the gap between unit costs gradually narrowing, the gap between quantities is narrowing as well.

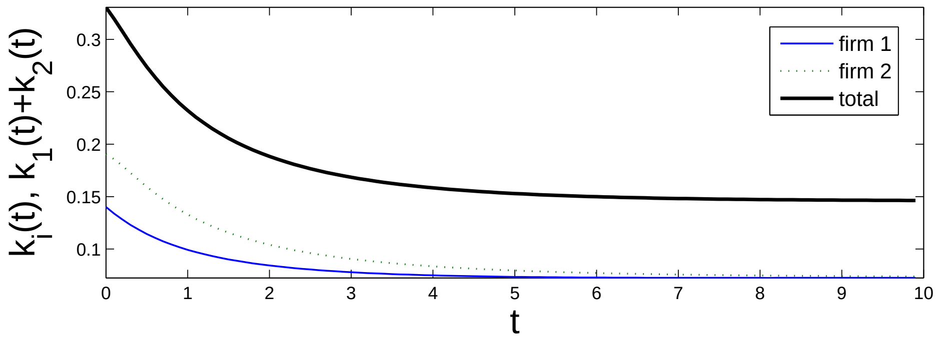
4.4.5 Stochasticity and R&D

It is an interesting question how the R&D efforts of firms relate to uncertainty. A look at Figure 4.2 reveals that both the value function and the policy function are smoother for higher levels of noise in unit costs. To investigate this further, we plot the value function and policy function of firm 1 for different fixed values of firm 2's unit cost. In Figure 4.7, we fix c_2 at such a high value that firm 1 is a monopolist ($c_2 = 11.76$; for reference, $A = 1$). Then, we can directly compare our solution with the deterministic monopoly solution, obtained in Chapter 2. We observe that while the deterministic value function has a kink, the stochastic value function is smooth. The fact that the value function corresponding to a higher noise level lie above the one corresponding to a lower noise level suggests that stochasticity increases expected profits. We see that the stochastic value function converges to the deterministic monopoly value function as $\varepsilon \downarrow 0$. For $\varepsilon = 0.0156$, the stochastic solution is already almost indistinguishable

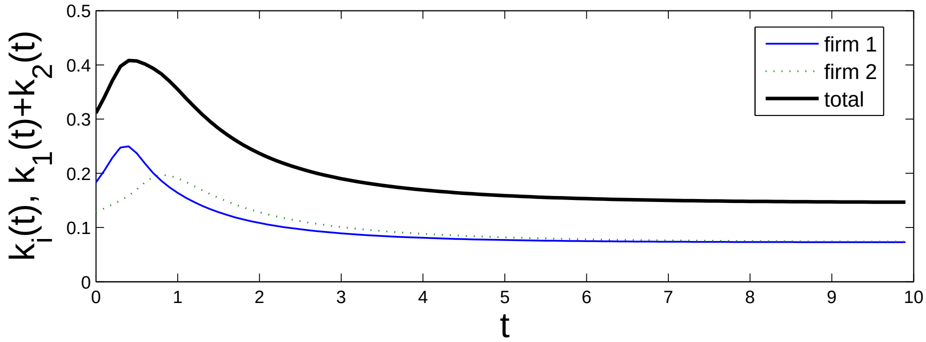
[24] Once the drift path has reached the asymptotically stable steady state S_4, the actual unit costs fluctuates around S_4. Of course, over a very long time, eventually a large shock may occur, driving one of the firms out of the market (its unit cost diverges to infinity).



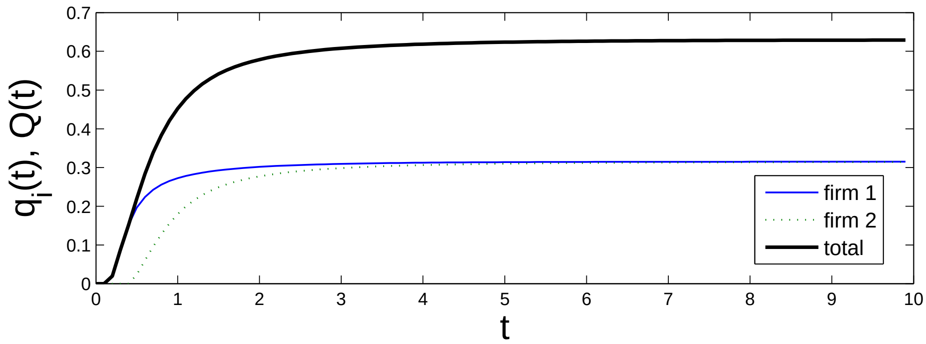
(a)



(b)



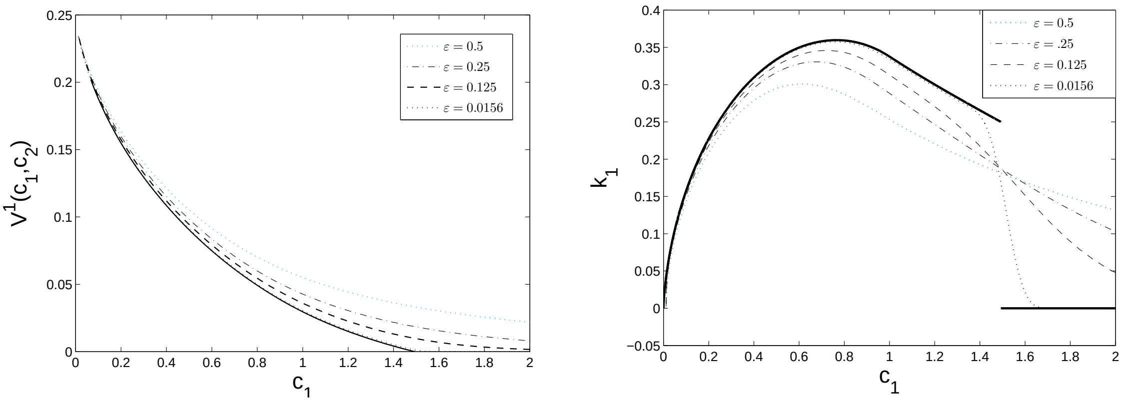
(c)



(d)

Figure 4.6: *Time paths: follower versus leader.*

from the deterministic one, the absolute difference between the two solutions at the kink being 0.0012. The deterministic policy function is discontinuous at the point of indifference,



(a) (b)

Figure 4.7: *Value functions (a) and policy functions (b) of firm 1 for varying levels of noise* ε *when the unit cost of firm 2 is fixed at* $c_2 = 11.76$*. The full line corresponds to the deterministic monopoly solution.*

where the firm is indifferent between developing a technology further or staying out. This discountinuity is smoothed out by stochasticity. The policy function of the stochastic model is smooth and everywhere differentiable. It is interesting to observe that stochasticity makes a firm invest in R&D over the values of unit costs for which a firm in the deterministic setting already gives up. The firm in the stochastic setting still invests a bit at larger costs in hope of a favorable shock, for which it sacrifices some investments at lower unit costs – the R&D efforts are smoothed out. While R&D efforts exerted at large costs might as such not be sufficient to bring a technology to the production phase, they at least retard the decay of a technology for some time during which hopefully a favorable shock arises. Higher uncertainty, therefore, leads to more opportunistic behavior of firms, which increases the chance that the development of expensive technologies will be pursued further. Computations shows that this opportunistic behavior also increases the relative size of the region of the state space for which eventually duopoly tends to appear on the product market (in the drift vector field, the basin of attraction of S_4 spreads out with increasing noise levels).

4.4.6 Deterministic game and indifference curves

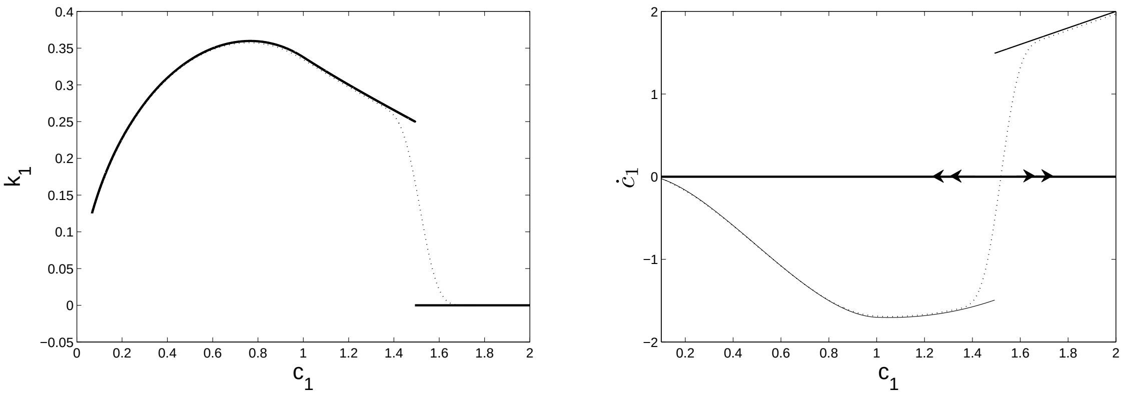
In this section, we take a closer look at the deterministic game. We hope that its solution resembles well the solution to the stochastic game with a small noise level. This hope was in part verified in the previous section, where we saw that the two solutions are close at least at the boundaries where only one of the firms is active.

As shown in Figure 4.7, the policy function in the deterministic monopoly solution is discontinuous at the indifference point. Only for initial costs below the indifference point a monopolist continues to invest in R&D and stays active in the product market. In what follows, we analyze the existence of indifference points in the deterministic competitive game. We define the deterministic indifference point of a firm as a value of its unit cost at which a firm is indifferent between developing a technology further and exiting the market. In general, indifference points do not coincide with points at which the R&D effort of a firm is zero. This is the case only when an indifference point is in the region of zero production.[25] When an indifference point is at the value of a unit cost at which a firm produces, an exiting firm might still invest a bit in order to slow down the speed at which it leaves the product market (cf. the notion of the 'exit trajectory' in previous chapters).[26]

In our stochastic setting, firms steer the evolution of their unit costs through the drift term. The direction of this steering is summarized in the drift vector field (see Figure 4.4). The lower the noise level and so the lower the random shocks to the unit costs, the smaller the deviations of actual costs form their drift path. For zero noise, the drift path is the actual path. We saw in our discussion of the drift vector field above that the isoclines I_1^* and I_2^* around S_3 and the separatrices W_1^S and W_2^S play an important role in the motion of the drift – they are boundaries of different basins of attraction. With noise approaching zero, these curves therefore converge to the boundaries of basins of attraction of actual unit costs. Consequently, our conjecture, on which we further elaborate below, is that these curves converge to the union of indifference points of the deterministic game.

[25]Clearly, when there is no uncertainty involved, a firm has no reason to initiate investment if it never plans to enter the product market.

[26]This is true only for the deterministic game. Whenever $\varepsilon > 0$, the R&D efforts are always positive on both sides of an approximating indifference point due to smoothing-out.

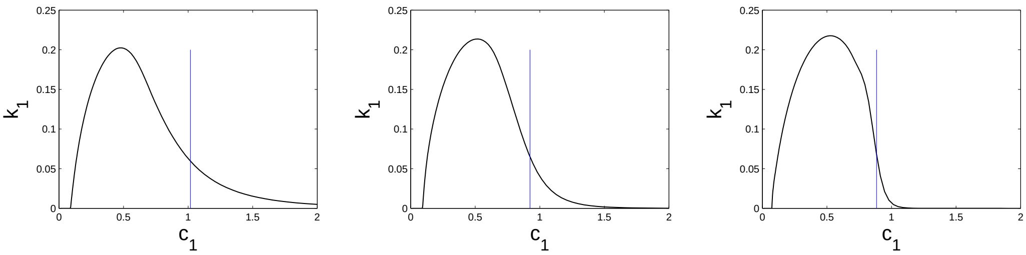


(a) (b)

Figure 4.8: *Policy function of firm 1 (a) and drift of its unit cost (b). The full line corresponds to the deterministic monopoly solution, the dotted line to the stochastic game with* $\varepsilon = 0.0156$. *The unit cost of firm 2 is fixed at* $c_2 = 11.76$.

Consider first the state space left to S_3 in Figure 4.4, where the cost of firm 2 is relatively high. We have already observed that firm 1 steers its unit cost downwards for initial unit costs corresponding to x_1 above the I_1^* curve, and upwards otherwise. In Figure 4.8, we consider the left most part of the state space as, again, there we can compare our solution with the deterministic monopoly solution. We know that the deterministic policy function is discontinuous at the point of indifference. As plot 4.8b shows, the deterministic drift is also discontinuous at the indifference point, it is negative left to the indifference point (the unit cost continues to decrease) and positive to the right of it (the unit cost flows towards infinity as a technology decays due to zero research activity). On the contrary, in the stochastic setting, the policy function is smooth, as is the drift. The point of zero drift in plot 4.8b corresponds to the point on the I_1^* isocline. The difference between the latter and the deterministic indifference point is for $\varepsilon = 0.0156$ already within the second decimal point. This difference decreases further with lowering the noise level as the point on I_1^* converges to the deterministic indifference point. We expect the same relation between the points on the isocline and indifference points to hold also for other parts of the I_1^* isocline. Analogously, for firm 2, indifference points correspond to points on the I_2^* isocline below S_3.

We have already noted that the stable manifold W_2^S act as a separatrix. For unit costs above it, firm 1 invests relatively a lot and steers its unit cost towards S_4. For unit costs below



(a) $\varepsilon = 0.125$ (b) $\varepsilon = 0.0526$ (c) $\varepsilon = 0.0156$

Figure 4.9: *Policy functions of firm 1 for a given value of firm 2's unit cost,* $c_2 = 0.4232$. *The vertical line corresponds to the point on the stable manifold* W_2^S.

it, the firm still invests a bit in hope of a shock (random discovery) that would reduce its unit cost, but progressively less so as its unit cost increases due to technology depreciation, thereby steering its unit cost towards infinity. In the absence of a sufficiently large favorable shock, the firm gradually gives up on R&D. Figure 4.9 shows plots of firm 1's R&D efforts for a given value of firm 2's unit cost. The vertical line corresponds to the point on the W_2^S curve. As plots show, with decreasing ε, the policy function straightens up at the point corresponding to W_2^S as the latter converges to the deterministic indifference point at which the policy function breaks off and the discontinuity arises. Right to the indifference point, the less efficient firm gives up, whereas left to it, the firm invests heavily in R&D in order to catch up with the more efficient competitor.

Analogously, points on W_1^S converge to the indifference points of firm 2 as $\varepsilon \downarrow 0$. We also observe that steady states S_1, S_2, and S_3 can lie in the regions of zero production (see Figure 4.4). Clearly, in a deterministic game, there cannot be steady-state points in such regions as that would imply a situation in which a firm invests at all times but never produces. These steady states are the implication of the continuity of the stochastic drift. They correspond to points at which the drift of both unit costs is zero. Like other points on the two manifolds, when a stochastic game transforms into a deterministic one, these steady states converts into the points of discontinuity. While at a stochastic steady state the drift is zero, at the indifference point the drift is multi-valued and so there can be no steady state corresponding to such a point. This is visible in plot (b) of Figure 4.8 above and fits nicely into our picture of convergence.

We summarize our observations in the following conjecture.

Conjecture 1. *In the deterministic game, the union of the stable manifold W_2^S and the I_1^* isocline to the left of S_3 approximates an indifference curve of firm 1 as $\varepsilon \to 0$. Likewise, the union of the stable manifold W_1^S and the I_2^* isocline below S_3 approximates an indifference curve of firm 2 as $\varepsilon \to 0$.*

This conjecture is illustrated in Figure 4.10. Observe that indifference curves divide the state space into four regions $i)$ the region of eventual duopoly, $ii)$ the region of eventual monopoly of firm 1, $iii)$ the region of eventual monopoly of firm 2, and $iv)$ the no market region, in which initial unit costs are too high for either firm to consider further development of a technology. The indifference curve of a firm roughly outlines the "foothills" of that firm's policy function (see the bottom right plot in Figure 4.2).

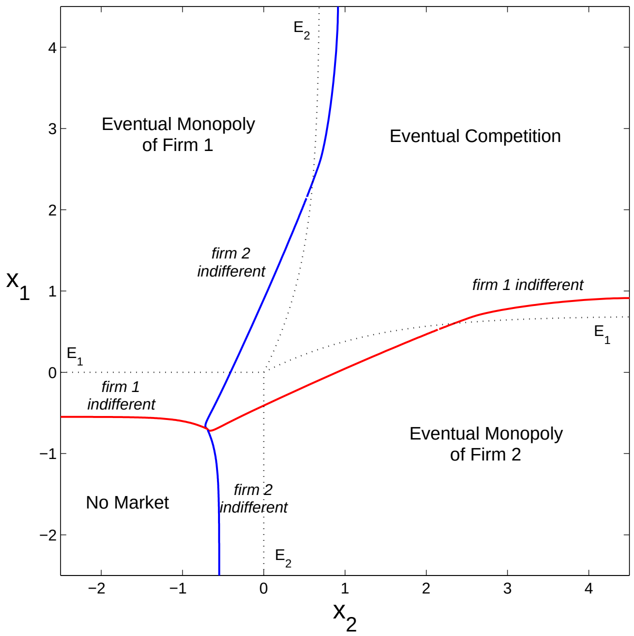
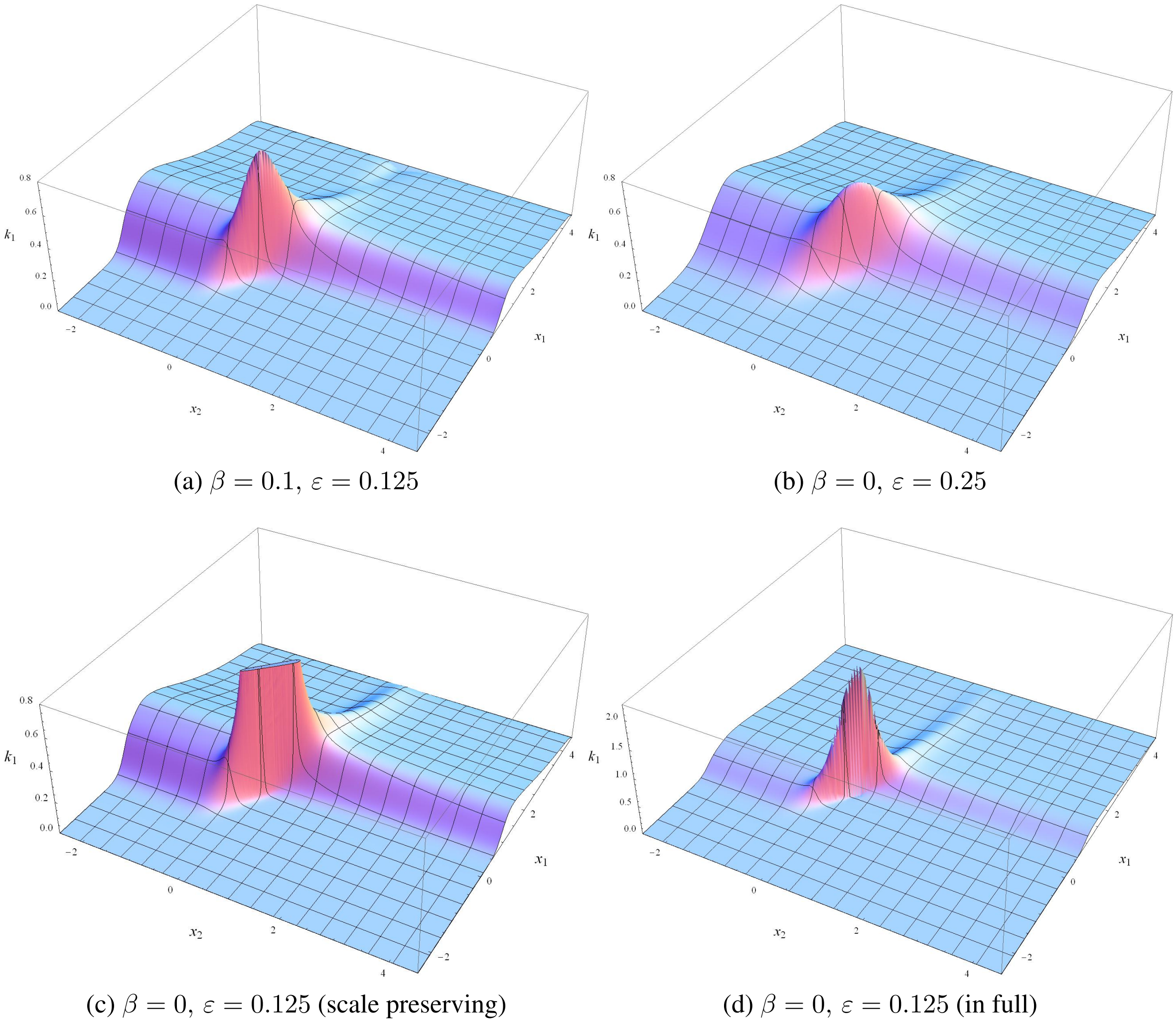


Figure 4.10: *Conjectured indifference curves of the deterministic game.*

4.4.7 Spillover effects and R&D

In this section, we compare the case with moderate spillovers ($\beta = 0.5$) with that of low and high spillovers. The larger the level of spillovers (the larger the β), the more the R&D efforts



(a) $\beta = 0.1$, $\varepsilon = 0.125$ (b) $\beta = 0$, $\varepsilon = 0.25$

(c) $\beta = 0$, $\varepsilon = 0.125$ (scale preserving) (d) $\beta = 0$, $\varepsilon = 0.125$ (in full)

Figure 4.11: *Preemption & predation at low spillovers.* The plots show the policy function of firm 1 for different levels of spillovers and uncertainty.

that a firm exerts benefit its competitor, and thus the larger the role of feedback cost effects in shaping a firm's policy function.

Low spillovers

For low levels of spillovers, the policy function exhibits a sharp and narrow bulge, as visible in Figure 4.11.[27] The lower the level of spillovers and/or noise, the more pronounced and sharp the bulge. Figure 4.12 shows the drift vector field corresponding to the case with $\beta = 0$.

[27]The case with low spillovers is the hardest one to integrate numerically. For small values of ε, the bulge increases and sharpens dramatically, posing problems for the stability of the numerical scheme. Solutions for lower ε require increasing refinements of the grid, which in consequence rather considerably affects the speed of calculations. For this reason, the lowest noise level we currently present for this case is $\varepsilon = 0.125$.

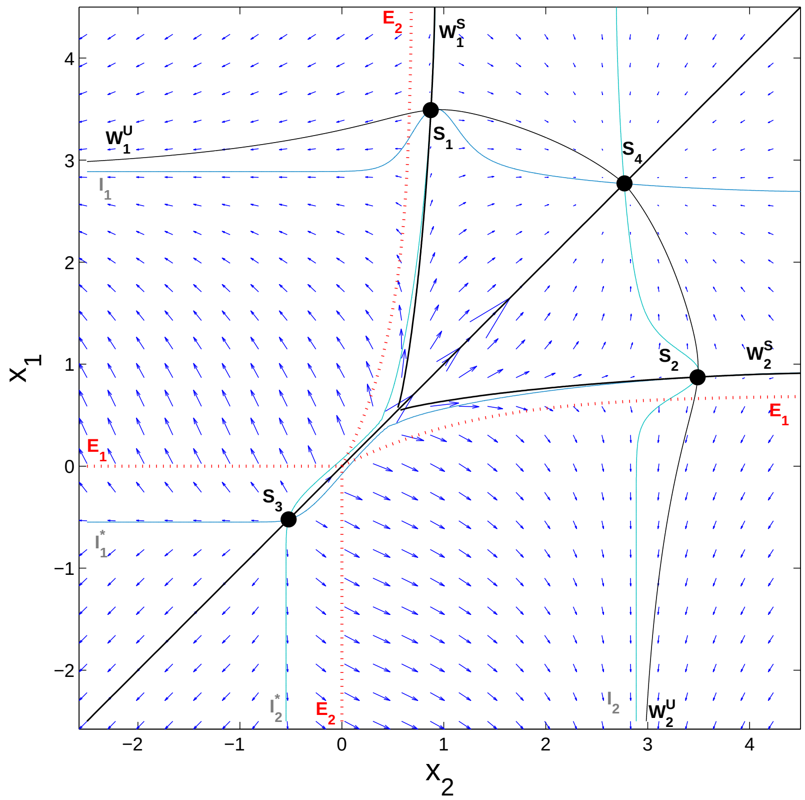


Figure 4.12: *Drift vector field for* $\beta = 0$, $\varepsilon = 0.125$.

Observe that W_1^S and W_2^S separatrices are in the vicinity of S_3 practically indistinguishable from the diagonal – a small difference in unit costs is enough to drive the less efficient firm out of the market. This region of proximity corresponds to the location of the bulge. With increasing levels of spillovers and/or noise, the bulge becomes thicker and lower; the separatrices shift away from the diagonal, implying that a larger cost advantage is needed to drive the opponent out of the market (compare with Figure 4.4).

As Figure 4.13 shows, on the diagonal within the canal, each firm invests a lot trying to reduce its production costs as fast as possible and so increase its chances of survival (the vertical line in the figure corresponds to symmetric costs). For a symmetric initial position, firms are engaged in a preemption race where the one that falls sufficiently behind the other is driven out of the market.

Observe that the bulge attains its top above the diagonal (left to the vertical line in

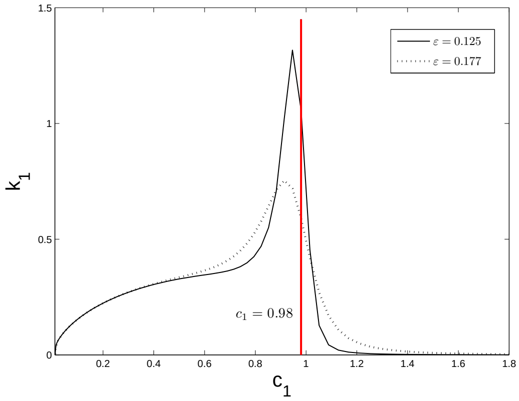


Figure 4.13: *Policy function of firm 1 for a value of firm 2's unit cost corresponding to the region of the bulge.* $\beta = 0$, $c_2 = 0.98$.

Figure 4.13) and sweeps sharply down on the other side of the diagonal (the policy function has a steep slope on the right side of the vertical line in Figure 4.13). The firm with a cost advantage therefore invests heavily (but briefly), whereas the follower is induced to give up. This additional R&D effort of the leader can be considered predatory in a sense that it is profitable only for its effect on the exit decision of the follower, but unprofitable otherwise.[28] The predatory nature of these investments is confirmed by the fact that such large investment asymmetries never occur when the likelihood that a rival remains viable is negligible (e.g., at very high levels of a rival's unit cost) or the ability of a firm to influence this likelihood is negligible (e.g., in the case of large spillovers where large investments would to a great extent benefit the competitor).

The extent of predatory efforts is positively related to the easiness with which the leader can induce the follower to give up. Recall that the bulge grows with the spillovers and noise level approaching zero (see Figure 4.11). At low spillovers, it is easier for the leader to induce the follower to exit as the latter cannot count on catching up with the leader by copying the results of the leader's R&D efforts. Thus, the lower the spillovers, the easier it is for the leader to achieve his dominance by exerting R&D efforts and so the larger are his incentives for extensive predation. Next, when the probability of large unexpected changes in costs is large,

[28] In declaring an action predatory, we follow Cabral and Riordan (1997) who define an action as predatory if "i) a different action would increase the likelihood that rivals remain viable, and ii) the different action would be more profitable under the counterfactual hypothesis that the rival's viability were unaffected" (p. 160). Our interpretation is similar to that of Borkovsky, Doraszelski and Kryukov (2012) who consider predatory investment in a dynamic quality ladder model.

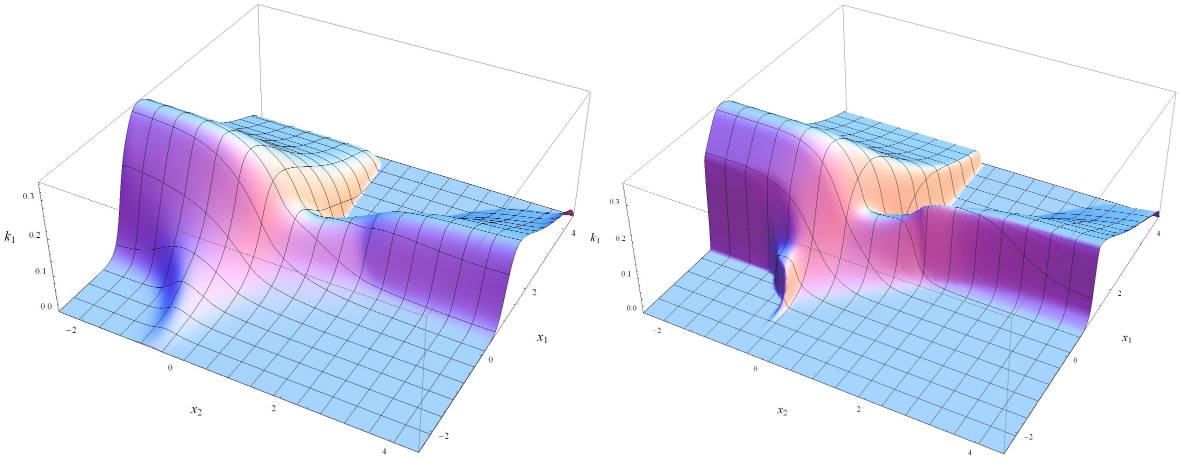
the follower does not give up that fast when falling behind as it is still possible for him to catch up the leader if he has a run of luck. In this case, the leader needs to achieve a relatively large cost advantage to induce the follower to give up. However, due to large randomness in costs, the effect of the leader's R&D efforts on the likelihood of achieving such an advantage is low. Consequently, his incentives for predatory investments are low as well. On the contrary, when there is low uncertainty in cost movements, a small cost advantage is sufficient to drive the other firm out of the market and the effect of the leader's R&D efforts on the likelihood of achieving a needed cost advantage is large. As a consequence, the leader's incentives to engage in extensive predation are large as well.[29]

High spillovers

Figure 4.14 and Figure 4.15 show the policy function and the drift vector field, respectively, for $\beta = 0.9$. Figure 4.16 jointly plots the indifference curves (separatrices) for $\beta = 0$ (L), $\beta = 0.5$ (M), and $\beta = 0.9$ (H).

When the level of spillovers is high, the R&D efforts of one firm benefit the other firm to a large extent. As each firm tries to free-ride on the other firm's R&D efforts, the incentives to exert much R&D efforts can be rather small. This standard conclusion in the literature is in part confirmed by our calculations – R&D efforts decrease over the bulk of the state space as the level of spillovers approach one. However, there is an important exception, depicted in Figure 4.14. The policy function for large spillovers exhibits a pronounced bulge spreading into the "southern" part of the state space (i.e., x_1 low, $x_2 \approx -0.1$). The size of this bulge increases with spillovers (first traces of it appear in the policy function for $\beta = 0.5$ in Figure 4.3). The intuition is the following. Notice that exerting R&D efforts is costly

[29] We have seen that the situation in which one firm invests heavily while the other negligibly small can be a feature of the equilibrium of the stochastic game as large investments are optimal in that they influence the likelihood the rival is induced to give up on R&D and exit the market. Clearly, it is hard to imagine that such a situation could be a feature of the equilibrium of a deterministic game as there is no probability of the rival's market viability involved. We observe that the bulge narrows and sharpens as $\varepsilon \downarrow 0$ (refer also to Figure 4.13). We therefore conjecture that in a deterministic case, the bulge corresponds to the point of discontinuity in the policy function, whereas his left and right foothills determine the R&D efforts of the leader and the follower, respectively. While still asymmetric, the difference in R&D efforts is comparably much smaller. In the region of the bulge, indifference curves are tangent to the diagonal (recall the tangent behavior of W_1^S and W_2^S separatrices in Figure 4.12). For symmetric initial costs, both firms eventually produce, whereas a minuscule asymmetry already leads to a monopoly.



(a) $\varepsilon = 0.125$ (b) $\varepsilon = 0.0156$

Figure 4.14: *Policy function of firm 1 for* $\beta = 0.9$*.*

(recall the quadratic cost function in (4.3)). When spillovers are large and so R&D efforts of the firms complement each other well, the firms facing a convex cost function are able to circumvent diseconomies of scale in R&D to a large extent. This reduction in the costs of R&D enables the two firms to competitively bring on the market a technology which a single firm cannot profitably develop itself. This explains why, provided the cost asymmetries are not too large, a firm in a duopoly market sometimes does invest in further development of a technology whereas a monopolist does not (the bulge). The implication of this is best visible in the drift vector field in Figure 4.15, where we observe that the steady state S_3 (the indifference point of two firms) corresponds to a higher value of the firm 1's unit cost than the leftmost part of the I_1^* isocline (the indifference point of a monopolist). Furthermore, Figure 4.16 shows that S_3^H corresponds to a higher value of unit costs than either S_3^L or S_3^M. The larger the spillovers, the larger the joint savings in R&D costs and so the more infant the initial technology that the firms can afford to develop further. In contrast to the cases with lower spillovers, in the case with high spillovers, W_1^S and W_2^S separatrices also form a wide arc around S_3. Therefore, high spillovers not only increase the range of initial technologies that are developed further, but also the likelihood that the ensuing product market will be competitive.

Comparing the drift vector field of $\beta = 0$ with that of $\beta = 0.5$, we observe that the basin of attraction of the steady state S_4 is wider in the latter case (see Figure 4.16). The W_1^S and

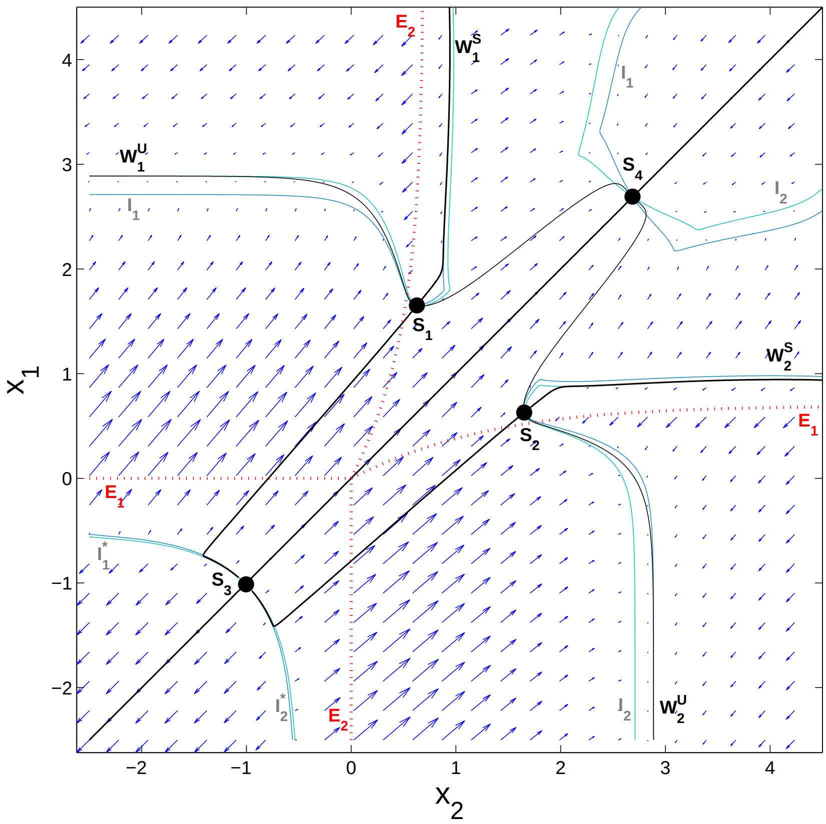


Figure 4.15: *Drift vector field for* $\beta = 0.9$, $\varepsilon = 0.125$.

W_2^S separatrices spread out. This suggests that it takes a larger cost asymmetry for the less efficient firm to leave the market when spillovers are higher. In particular, the exit of any firm is much less likely when both firms already produce (for $\beta = 0.5$, larger parts of separatrices lie outside the production region bounded by E_1 and E_2 curves). The larger the spillovers, the more the follower can benefit from the R&D investments of the leader and so the more disadvantaged it must be to give up. This point was already raised by Petit and Tolwinski (1999) claiming that *"[...] for a duopoly consisting of unequal competitors free diffusion of knowledge may be a way to avoid market concentration"* (p. 204).

In contrast to the aforementioned authors, our analytical framework allows us to draw much more precise conclusions as it makes it possible for us to obtain indifference sets over the entire state space, which in turn enables us to compare entry-exit investment decisions of firms for all possible initial positions. In particular, we show that the pro-competitive

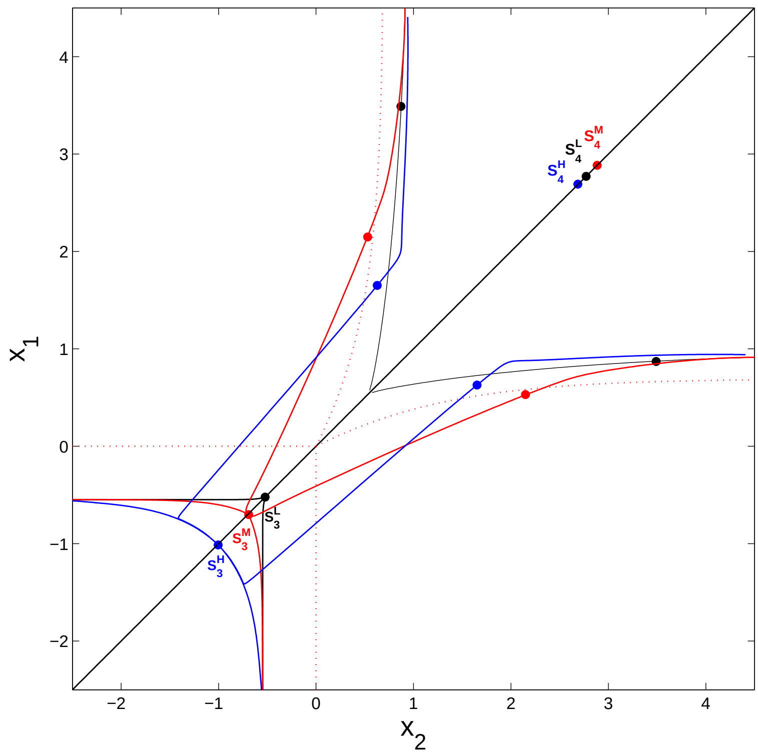


Figure 4.16: *Comparison of indifference curves (separatrices) between* $\beta = 0$ *(black),* $\beta = 0.5$ *(red), and* $\beta = 0.9$ *(blue).* $\varepsilon = 0.125$.

benefit of larger spillovers does not hold for all levels of spillovers and costs. Observe how in Figure 4.16 the separatrices corresponding to $\beta = 0.9$ intersect those corresponding to $\beta = 0.5$. While for high initial unit costs of firms larger spillovers still make duopoly on the ensuing product market more likely, this does not hold for lower values of initial unit costs as there the less efficient firm is sooner squeezed out of the market when spillovers are larger. Behind this result are two countervailing effects of spillovers. The first effect is a pure spillover effect – the larger the spillovers, the more one firm is able to free ride on the other firm's R&D efforts and so the easier it is for the follower to overcome any initial asymmetries. This effect is positively related to the level of spillovers and contributes to widening the region of eventual product market duopoly. The second effect is the feedback cost effect, which is also positively related to the level of spillovers, however, it contributes to narrowing the region of eventual product market duopoly. When the unit cost of a firm is large, an additional unit of R&D effort benefits this firm a lot (pure cost effect dominates). However, when the unit cost of the firm is lower, so is the impact of an additional unit of R&D on its costs (the factor $c_i k_i$ in (4.2) decreases with c_i for a given k_i). If the unit cost of the follower is sufficiently larger, it can well happen that the additional R&D effort of the leader benefits the follower more than the leader himself ($c_i k_i < c_j \beta k_i$). As lower costs of the follower then through the product market competition negatively affects the leader's profits, this reduces the leader's incentives to invest in R&D. This feedback effect, which negatively affects the leader's R&D efforts, is stronger, the larger the spillovers. Consequently, the larger the spillovers, the less asymmetry in costs it takes for the leader to optimally stop his R&D efforts. Observe how the region of zero R&D efforts above the diagonal spreads out in the policy function as the spillovers increase (compare Figure 4.3, Figure 4.11, and Figure 4.14). This explain why larger spillovers might in fact increase the likelihood that the market will be monopolistic. After a certain level, further increases in spillovers decrease the leader's incentives to invest rather significantly, which makes it harder for the follower to catch up with the leader. The follower's possibilities to copy the leader's R&D results do increase further with increasing spillovers, however, the problem is there is now very little or nothing to copy. In the two north-eastern regions between the intersecting separatrices in Figure 4.16, the leader in case of $\beta = 0.9$ invests relatively less than in case of $\beta = 0.5$ and this effect of lower investments by the leader dominates the pure spillover effect. Consequently, while for larger spillovers, the follower is driven out of the market, for smaller spillovers, he continues to catch up with the leader. In sum, *increasing spillovers favors a socially desirable outcome only up to a point*.

Whenever one firm gives up, the unit cost of the remaining firm converges to the same long-run optimal monopolist's level irrespective of the level of spillovers. The separatrices W_1^U for the three different levels of β asymptotically converge as $t \to \infty$ (not shown). The same holds for separatrices W_2^S. However, the long-run steady-state level of unit costs in case of duopoly (S_4) does depend on the level of spillovers. As Figure 4.16 shows, in our rescaled coordinates $S_4^H < S_4^L < S_4^M$. That is, the long-run unit costs are the lowest in case of $\beta = 0.5$, the second lowest in case of $\beta = 0$, and the highest in case of $\beta = 0.9$. This suggests that the spillovers decrease steady-state costs only up to a certain level, beyond which further increases in spillovers start to increase the long-run costs. We illustrate this further by time plots in Figure 4.17. We select an asymmetric initial position which lies in the basin of attraction of S_4 for all the three levels of spillovers: $(x_1, x_2) = (1.68, 0.81)$, corresponding to $(c_1, c_2) = (0.186, 0.440)$.

Plot 4.17a shows total R&D efforts of the two competing firms over time for different levels of spillovers. We see a typical effect of increasing spillovers – the total industry R&D efforts decrease as firms increasingly free-ride on each other. However, due to larger complementarities between R&D efforts at larger spillovers, the effective efforts of the firm i and the firm j ($k_i + \beta k_j$ and $k_j + \beta k_i$, respectively) might be larger at larger spillovers despite the firms' lower *de facto* R&D efforts (k_i and k_j, respectively). Plot 4.17b shows that this is indeed the case when spillovers increase from $\beta = 0$ to $\beta = 0.5$. While for $\beta = 0.5$, the industry R&D efforts are relatively lower at all times, the effective industry R&D efforts are larger for most of the time. This explains why in the latter case, the unit costs converge to a lower long-run level than in the case with $\beta = 0$. We see that among the three regimes, the industry R&D efforts are comparably the lowest for $\beta = 0.9$. At the beginning, the leader in the latter case invests very little as he free-rides on the efforts of the follower. These smaller investments are not offset by larger spillovers, such that the effective efforts are much lower than in the other two cases. This changes over time as the leader himself starts to invest more when the follower gradually reduces his efforts over time. However, as plot 4.17c reveals, lower effective investments at the beginning very much slow down the speed at which unit costs decrease. In the case of $\beta = 0.9$, the unit costs of both the leader (full line) and the follower (dotted line) decrease much slower than in the other two cases. Moreover, the gap between the follower and the leader also closes more slowly. These slower and lower reductions of costs as a consequence of smaller investments are the reason that the total quantity offered in the market is for $\beta = 0.9$ at all times the lowest among the cases considered (see plot 4.17d). The case with $\beta = 0.5$ offers the largest total quantity, whereas the quantity for $\beta = 0$ is close to that for $\beta = 0.5$ but a bit lower.

Calculations show that total profits monotonically increase with spillovers. This is the effect of higher complementarities in R&D outputs that allow for significant savings on R&D

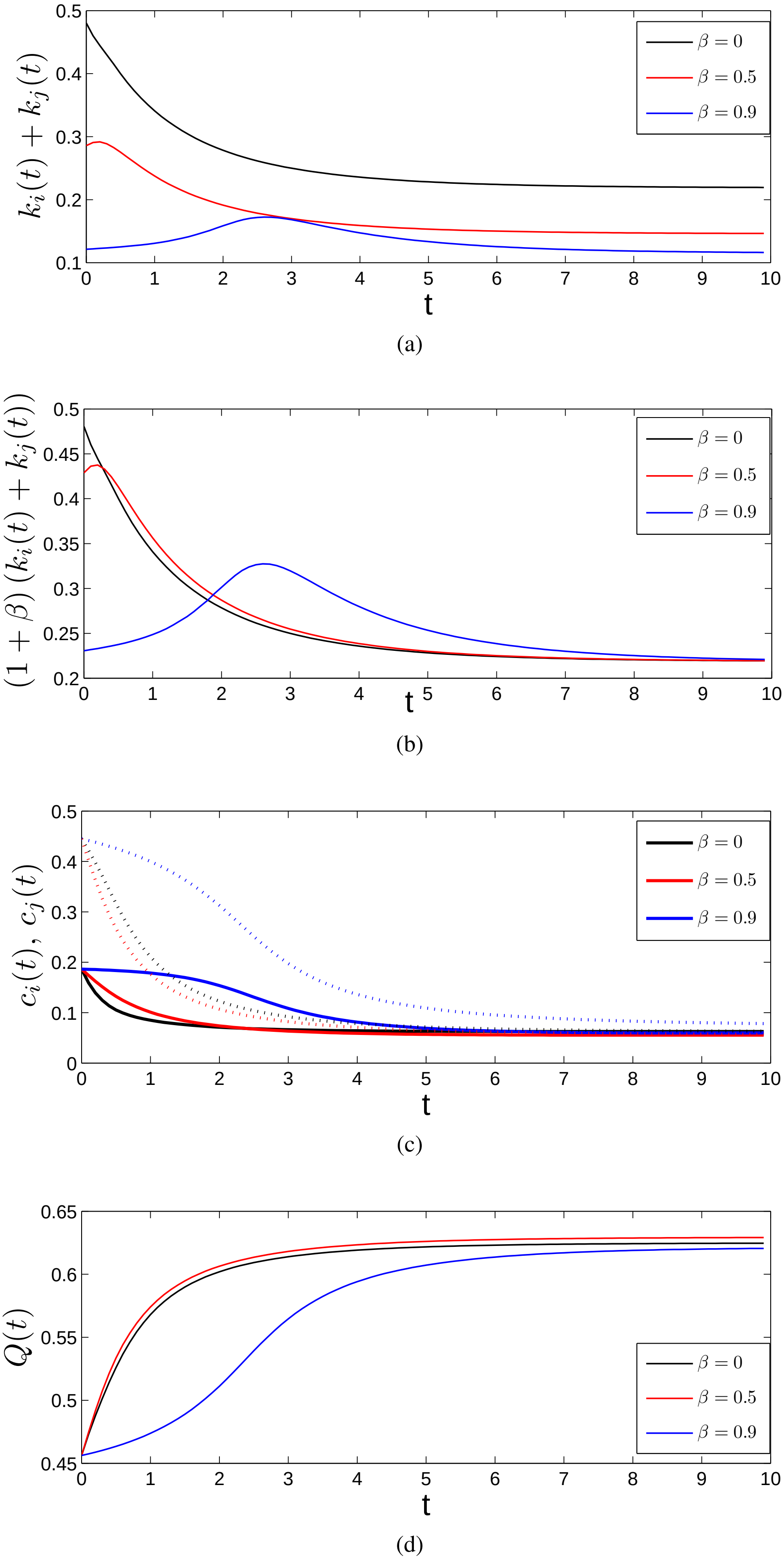


Figure 4.17: *Time paths for varying levels of spillovers.* $\varepsilon = 0.125$. *In plot* (c)*, the full lines correspond to the leader, whereas the dotted lines correspond to the follower.*

costs. However, the consumers are not necessarily any better for it. As our comparisons indicate, there exists a threshold level of spillovers after which further increases in spillovers do not benefit consumers. At large spillovers, the free riding effect induces firms to invest less and the consequent lower production efficiency, to the detriment of consumers, also induces them to produce less.

4.4.8 Market size and industry dynamics

The presentation so far has been focused on the dynamics that occurs at parameters $(\phi, \tilde{\rho}) = (8, 1)$. Our calculations over a wide range of parameterizations show that dynamics at this particular parameterization is representative of the subset of the parameter space for which both coordinates of the unstable steady state S_3 are negative, i.e., S_3 corresponds to unit costs above the choke price. This means that it can be profitable for firms to develop further a technology which requires R&D efforts before the production can profitably start. We focused on this subset of the parameter space, which we label "promising technology", as we find it most relevant – for great many new technologies, research starts long before a prototype sees the light. We now briefly consider two other possibilities suggested by our solution to the monopoly case (cf. the bifurcation diagram in Chapter 2). In the first one, which we call "strained market", a technology corresponding to initial unit costs above the choke price is never developed further, whereas that already in the production phase is developed further only if it is already sufficiently developed, such that it does not require "too much" additional R&D efforts. If a technology is still relatively undeveloped such that firms need to exert a lot of R&D efforts to maintain and develop it further, it is gradually discontinued by all firms in the market. In the second case, which we call "obsolete technology", it is always in the interest of any firm to exit the market at some optimal speed as a low demand makes it unprofitable to maintain a decaying technology.

Consider first the case of a strained market, represented by $(\phi, \tilde{\rho}) = (5, 1)$. Notice that a lower ϕ for a given discount rate corresponds to a lower demand and/or higher costs of R&D (recall the rescaling in Lemma 10). In general, lowering ϕ moves S_3 and S_4 closer together,

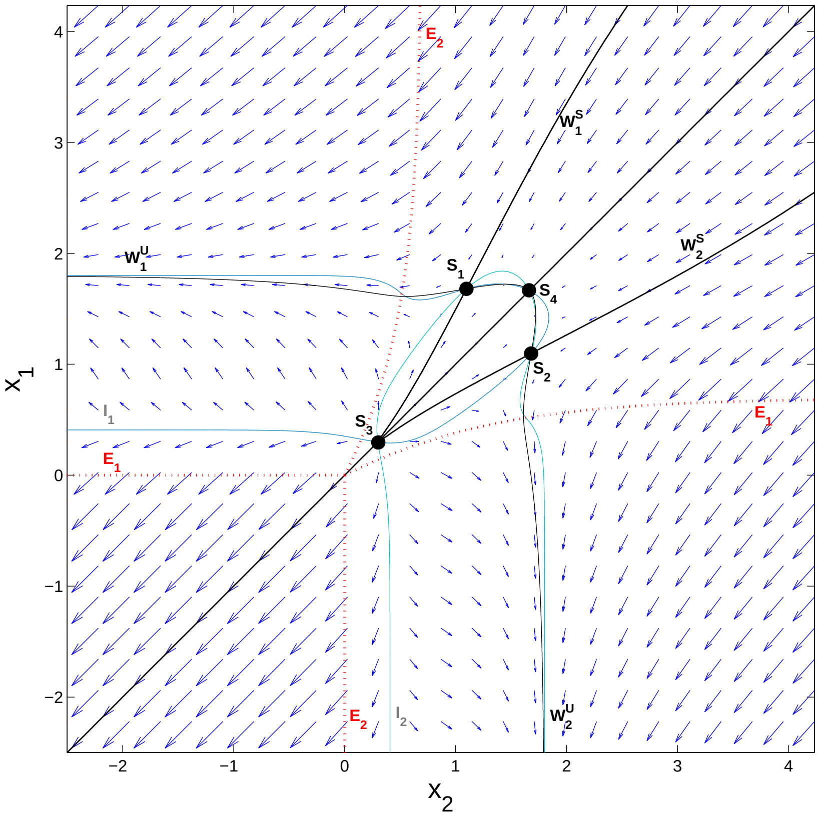


Figure 4.18: *Drift vector field for* $\beta = 0.5$, $(\phi, \tilde{\rho}, \varepsilon) = (5, 1, 0.125)$.

contracting the region of the state space for which there is a duopoly on the product market (the region between the W_1^S and W_2^S separatrices). When demand decreases or R&D costs rise, a much smaller lead is needed to induce the follower to give up. Figure 4.18 indicates this for the case of $\beta = 0.5$. Observe that neither a monopolist nor any of the two competing firms develop further a technology which would require investments prior to production – the basin of attraction of S_4 is compressed and fully contained within the production area bounded by the E_1 and E_2 curves (compare with Figure 4.4).

Figure 4.19 shows the drift vector field for $\beta = 0.1$ and $(\phi, \tilde{\rho}) = (5, 1)$. There are now only two steady states – a nodal source S_3 and a saddle point steady state S_1^*. The two saddles (S_1 and S_2) and the nodal sink (S_4) have colluded and formed a new steady state S_1^*. This new saddle has two manifolds – the unstable manifold W_1^U and the stable manifold W_1^S which lies on the diagonal of the state space. The implication of this is that the region of duopoly is now compressed into a line segment which originates in S_3, passes through S_1^* and continues to infinity. Only for symmetric initial position lying on this line both firms keep producing

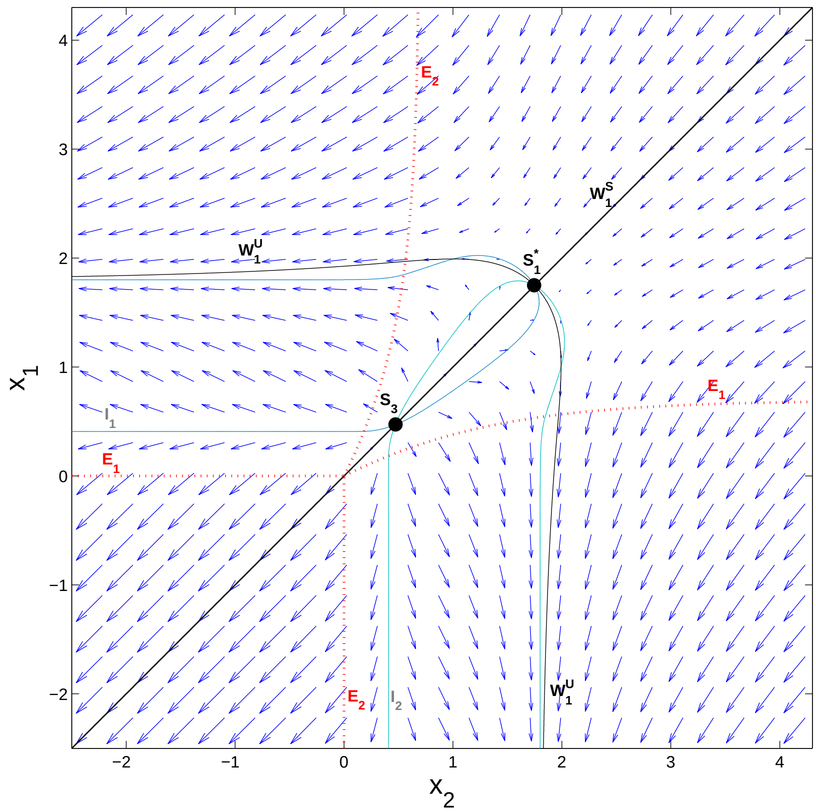


Figure 4.19: *Drift vector field for* $\beta = 0.1$, $(\phi, \tilde{\rho}, \varepsilon) = (5, 1, 0.125)$.

and steer their unit costs towards the long-run equilibrium level of S_1^*.[30] However, any initial asymmetry makes the less efficient firm to gradually exit the market. At low spillovers, a small lead is enough to induce the follower to give up. Observe how the diagonal acts like a repeller – on each side of it, the motion is away from it. Clearly, in a stochastic game interpretation, any symmetry is only temporary, such that for most of the time, one firm always diverges out of the market.

In sum, it is noteworthy that for less favorable market and R&D conditions, the asymmetry emerge for low spillovers – initial asymmetries lead to asymmetric outcomes (a firm with an initial cost advantage becomes a monopolist, whereas the other firm exits the market).

As spillovers increase, S_1^* transforms into two saddles and a nodal sink. The region of duopoly becomes a proper region, as in the case of $\beta = 0.5$. With increasing spillovers, the two saddles (S_1 and S_2) move aside and the region of duopoly enlarges. After some point,

[30] If an initial position happens to lie on the diagonal above S_1^*, firms find it optimal to decrease their efficiency towards a higher long-run level which is less costly to maintain.

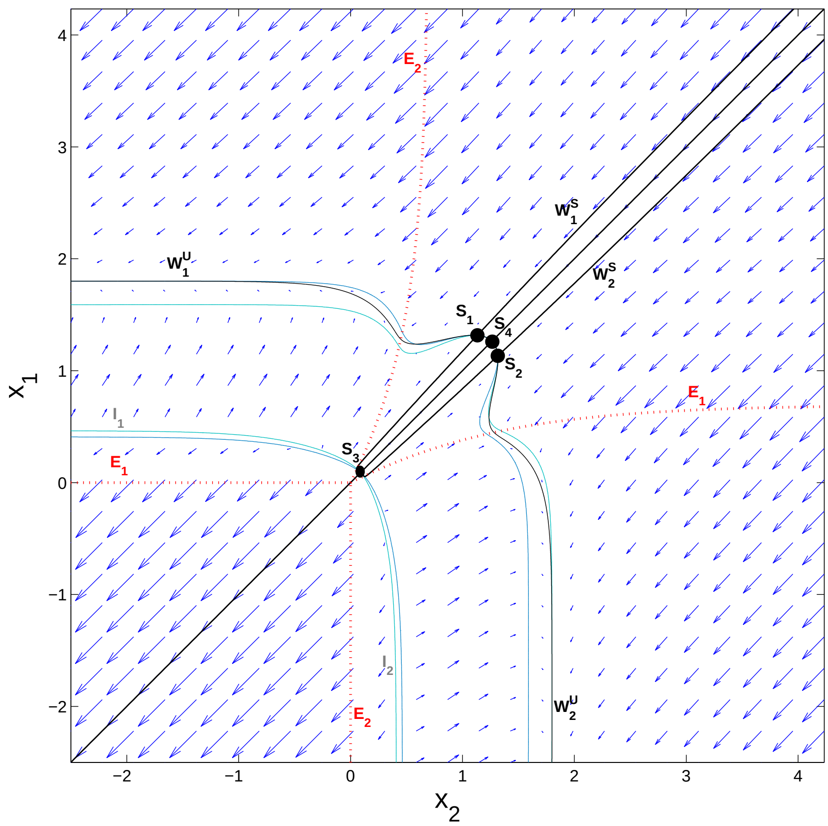


Figure 4.20: *Drift vector field for* $\beta = 0.9$, $(\phi, \tilde{\rho}, \varepsilon) = (5, 1, 0.125)$.

however, the saddles start approaching each other and so the region of duopoly starts to contract. This effect is visible when comparing the drift vector fields for $\beta = 0.1$ and $\beta = 0.5$ with the drift vector field for $\beta = 0.9$ (see Figure 4.20).

Figure 4.21, which shows a zoomed comparison between $\beta = 0.5$ and $\beta = 0.9$, indicates that this contraction of the duopoly region at higher spillovers is again not universal as the region of duopoly for $\beta = 0.9$ remains wider at larger levels of unit costs. All in all, our conclusion is similar as before – after a certain level, larger spillovers start to reduce the duopoly region (the region of regression toward the mean) as smaller investments of the unmotivated leader makes it harder for the follower to catch up.

The drift vector field in Figure 4.22 shows the drift motion for the case of obsolete technology represented by $(\phi, \tilde{\rho}) = (2, 1)$. We see that both unit costs diverge towards infinity. At this parameters, the demand is so low and the R&D process so costly that both firms find it optimal to eventually leave the market. They might still invests in R&D at some smaller rate that retards the decay of the technology optimally, but eventually both the R&D and

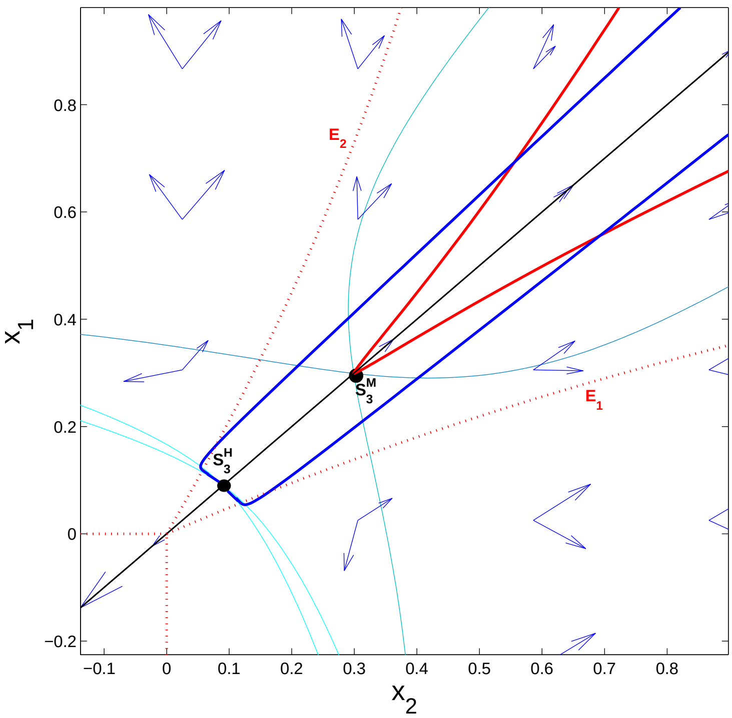


Figure 4.21: *Comparison between* $\beta = 0.5$ *(red) and* $\beta = 0.9$ *(blue).*

production will terminate and the firms will exit the market.

4.5 Concluding remarks

In this chapter, we study feedback Nash equilibria of a dynamic game in which firms enhance their production efficiency through R&D endeavors. The model allows for the possibility that initial unit costs of firms are above the choke price. Firms' product market participation constraints are also explicitly taken into account. As a result, R&D efforts and production do not necessarily coexist at all times. Furthermore, by allowing for asymmetric initial positions, the model provides insight into the investment relations between the market leader and the follower and their entry-exit decisions in relation to the level of spillovers, market size, the efficiency of R&D, and uncertainty in unit costs.

Our results qualify the indication in the previous literature that higher spillovers might be socially beneficial as they might obstruct the monopolization of the industry by preventing

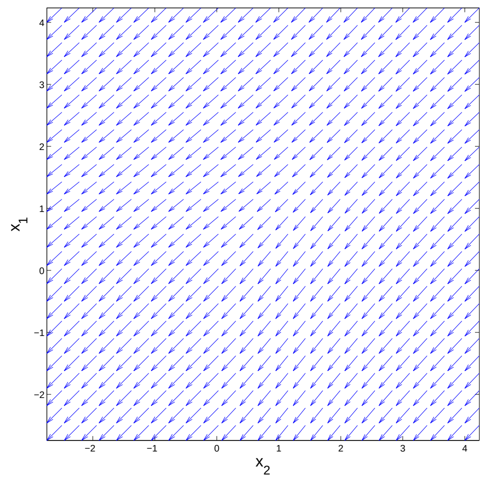


Figure 4.22: *Drift vector field when exit is always optimal.* $(\beta, \phi, \tilde{\rho}, \varepsilon) = (0.5, 2, 1, 0.125)$.

the lagging firm from falling too much behind the leader. We show that this pro-competitive effect of larger spillovers holds only up to a certain point, after which further increases in spillovers start reducing the region of the state space for which there is duopoly on the product market. While higher levels of spillovers indeed make it easier for the follower to copy the R&D results of the leader, the latter might have very little incentives to invest knowing that its R&D efforts will benefit its competitor to a large extent (possibly even more than the leader himself). As the follower has then little to copy, smaller initial asymmetries can induce the monopolization of the industry at larger spillovers. The pro-competitive effect of larger spillovers however remains at high initial unit costs as there the leader himself has high incentives to invest.

We show that larger spillovers always increase the likelihood that some initial technology which requires investments in advance of production will be developed further. With a convex R&D cost function, larger spillovers enable the firms to save more on R&D costs, which in consequence enables the firms to bring on the market expensive technologies which the firms with lower complementarities in R&D cannot afford to. In this sense, larger spillovers are always conducive to R&D. The other thing, however, is to which level these technologies are developed. The long-run steady-state unit costs decrease by increasing spillovers only up to a certain point, after which they start increasing. At first, larger spillovers more than compensate for lower actual R&D efforts of firms, such that effective R&D efforts increase with increasing spillovers. In consequence, larger spillovers lead to duopoly with lower steady-state unit costs and larger quantity produced. After a point, however, larger spillovers are not anymore enough to compensate for lower actual efforts of firms that try to free-ride on each other, leading to lower effective R&D efforts along the equilibrium path. Consequently, further increases in spillovers lead to higher steady-state unit costs and lower quantity produced.

We show that the region for which there is duopoly on the product market is characterized by regression toward the mean phenomenon, where asymmetries between the firms tend to vanish over time.

Next, we find that at low spillovers, a preemption race occurs at relatively high initial unit costs where initially symmetric firms invest a lot trying to win the race in which a small lead suffices for monopolizing the industry. We also find that at low spillover levels, the leader may engage in large investments that can be considered predatory in the sense that they are profitable only in inducing the less efficient opponent to surrender.

We find that larger noise in unit costs induces the firms to develop further the technologies which would be left intact at smaller noise. This investment of firms is stimulated by a higher chance of a large favorable shock to unit costs at larger noise. Larger noise also widens the duopoly region as the follower is less inclined to exit the market when there is a higher chance of a large favorable shock in the future. As both favorable and unfavorable shocks are always equally likely, the firms appear cautious in the sense that they rather invests more than be later sorry for giving up on a technology too early.

Appendix 4.A Proof of Lemma 10

A rescaled variable or parameter is distinguished by a tilde: for instance, if π denotes profit, then $\tilde{\pi}$ denotes profit in rescaled variables. Set $A = 1$, $b = 1$, $\delta = 1$, and define the following conversion: $q_i = A\tilde{q}_i$, $q_j = A\tilde{q}_j$, $k_i = \frac{A}{\sqrt{b}}\tilde{k}_i$, $k_j = \frac{A}{\sqrt{b}}\tilde{k}_j$, $c_i = A\tilde{c}_i$, $c_j = A\tilde{c}_j$, $\pi_i = A^2\tilde{\pi}_i$, $\pi_j = A^2\tilde{\pi}_j$, $\phi = \frac{A}{\delta\sqrt{b}}$, $t = \frac{\tilde{t}}{\delta}$, $\tilde{\rho} = \frac{\rho}{\delta}$. The state equation in the rescaled variables is then

$$\dot{\tilde{c}}_i(\tilde{t}) = \tilde{c}_i(\tilde{t}) \left(1 - \left(\tilde{k}_i(\tilde{t}) + \beta\tilde{k}_j(\tilde{t})\right)\phi\right). \tag{4.36}$$

The new instantaneous profit function is:

$$\tilde{\pi}_i(\tilde{t}) = (1 - \tilde{q}_i(\tilde{t}) - \tilde{q}_j(\tilde{t}) - \tilde{c}_i(\tilde{t}))\tilde{q}_i(\tilde{t}) - \tilde{k}_i(\tilde{t})^2, \tag{4.37}$$

whereas the total discounted profit is

$$\tilde{\Pi}_i = \int_0^\infty \tilde{\pi}_i(\tilde{t}) e^{-\tilde{\rho}\tilde{t}} d\tilde{t}. \tag{4.38}$$

To prove this, consider first the profit function of firm i, which is in the original (non-rescaled) model given by:

$$\pi_i = (A - q_i - q_j - c_i)\, q_i - bk_i^2.$$

Using the conversion rules given above, we obtain:

$$\begin{aligned}
\pi_i &= (A - q_i - q_j - c_i)\, q_i - bk_i^2 \\
&= (A - A\tilde{q}_i - A\tilde{q}_j - A\tilde{c}_i)\, A\tilde{q}_i - b\left(\frac{A}{\sqrt{b}}\tilde{k}_i\right)^2 \\
&= A^2\left((1 - \tilde{q}_i - \tilde{q}_j - \tilde{c}_i)\, \tilde{q}_i - \tilde{k}_i^2\right) \\
&= A^2\tilde{\pi}_i
\end{aligned}$$

The equation for the evolution of the unit cost over time is in original variables given by:

$$\dot{c}_i(t) = c_i(t)\left(-k_i(t) - \beta k_j(t) + \delta\right).$$

Write $c_i(t) = c_i\left(\frac{1}{\delta}\tilde{t}\right)$. Then,

$$\begin{aligned}
\frac{dc_i}{d\tilde{t}} &= \frac{dc_i}{dt}\frac{dt}{d\tilde{t}} \\
&= \frac{1}{\delta}\dot{c}_i \\
&= c_i\left(1 - \frac{1}{\delta}k_i - \beta\frac{1}{\delta}k_j\right).
\end{aligned}$$

Setting $k_i = \frac{A}{\sqrt{b}}\tilde{k}_i$ and $k_j = \frac{A}{\sqrt{b}}\tilde{k}_j$, and substituting them in the previous equation, we obtain:

$$\frac{dc_i}{d\tilde{t}} = c_i \left(1 - \left(\tilde{k}_i + \beta \tilde{k}_j\right) \frac{A}{\sqrt{b}\delta}\right).$$

It is now natural to introduce $\phi = \frac{A}{\delta\sqrt{b}}$. Notice that if $\tilde{c}_i = c_i/A$, then $\dot{\tilde{c}}_i = \dot{c}_i/A$ and

$$\dot{\tilde{c}}_i = \tilde{c}_i \left(1 - \left(\tilde{k}_i + \beta \tilde{k}_j\right) \phi\right). \tag{4.39}$$

Observe finally that if $t = \frac{\tilde{t}}{\delta}$, then $e^{-\tilde{\rho}\tilde{t}} = e^{-\rho t}$ if and only if $\tilde{\rho} = \frac{\rho}{\delta}$. Despite dealing with the rescaled model, we omit tildes in the main text so not to blur the exposition. □

Appendix 4.B Boundary conditions

To solve our system of differential equations, we need to specify eight boundary conditions (four for each player's value function), corresponding to the four sides of the grid square.[31] The problem is that the true value of a solution at all boundaries is *ex ante* not known to us. In what follows, we show that the solution in the interior of the state space is unaffected by the precise specification of the boundary conditions, excepting a small strip along the boundaries. The reason for this is that the characteristics of the associated first-order Hamilton-Jacobi-Bellman partial differential equations leave the state space at the boundaries.

As explained in the main text, the grid square follows from discretizing each of the two state variables. That is, we introduce $x_{i,1} < x_{i,2} < ... < x_{i,n}$, where n is the number of grid points in one dimension, $i = 1, 2$. At each boundary value of x_i, we impose *Neumann conditions*, which we motivate as follows.

From a deterministic solution to a monopolist's problem (see Chapter 2), we know that a firm pursues R&D only if its initial unit cost is small enough. We therefore conjecture that

[31] We observe *ex post* that the Nash equilibrium is symmetric; that is, $\mathcal{V}^1(x_1, x_2) = \mathcal{V}^2(x_2, x_1)$ and $\gamma_1^*(x_1, x_2) = \gamma_2^*(x_2, x_1)$. This information can be used to halve the dimension of the problem and thereby speed up (reparameterized) calculations. Define $\mathcal{V}^1(x_1, x_2) = \mathcal{V}(x_1, x_2)$ and $\gamma_1^*(x_1, x_2) = \gamma^*(x_1, x_2)$. Then $\mathcal{V}^2(x_1, x_2) = \mathcal{V}(x_2, x_1)$ and $\gamma_2^*(x_1, x_2) = \gamma^*(x_2, x_1)$. In this reformulation, the unknown are then only $\mathcal{V}$ and γ^*. We do not pursue this here as our code is written so that it allows for asymmetric equilibria. While this way we do sacrifice speed a bit, the solution itself is not affected.

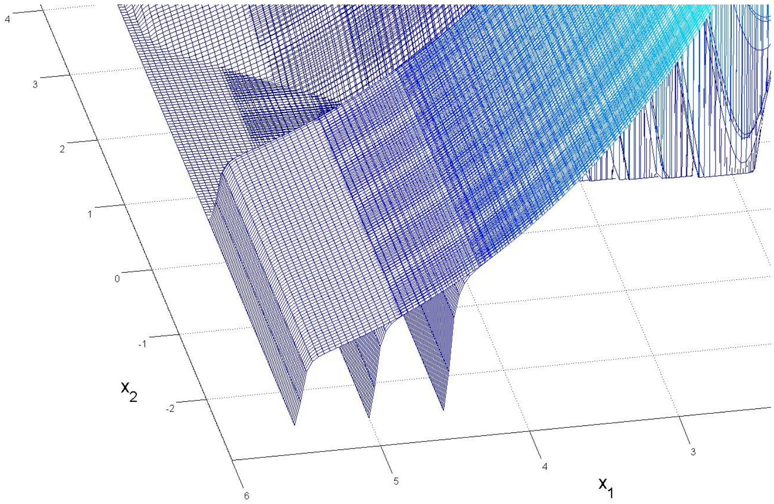


Figure 4.23: *Solution at the boundary.*

in the competitive case, for sufficiently large own unit costs (small x_i), firm i stays inactive, whereas firm j is a potential monopolist. The value of the game is then zero for firm i and small changes in its unit costs have no effect on either its own or its competitor's value function. The situation at sufficiently large unit costs of firm j is analogous. From this, the following boundary conditions follow:

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\mathcal{V}^1(s, x_{1,1}, x_{2,\ell}) = 0, \quad && \frac{\partial}{\partial x_1}\mathcal{V}^2(s, x_{1,1}, x_{2,\ell}) = 0,\\
&\frac{\partial}{\partial x_2}\mathcal{V}^1(s, x_{1,\ell}, x_{2,1}) = 0, \quad && \frac{\partial}{\partial x_2}\mathcal{V}^2(s, x_{1,\ell}, x_{2,1}) = 0,
\end{aligned}$$

where $\ell = 1, ..., n$.

The situation at low unit costs is more subtle. We expect, however, that whenever one firm has very low unit cost, the effect of this unit cost on each firm's value of the game is small at the margin. We approximate this effect by zero, from which the following conditions follow:

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\mathcal{V}^1(s, x_{1,n}, x_{2,\ell}) = 0, \quad && \frac{\partial}{\partial x_1}\mathcal{V}^2(s, x_{1,n}, x_{2,\ell}) = 0,\\
&\frac{\partial}{\partial x_2}\mathcal{V}^1(s, x_{1,\ell}, x_{2,n}) = 0, \quad && \frac{\partial}{\partial x_2}\mathcal{V}^2(s, x_{1,\ell}, x_{2,n}) = 0,
\end{aligned}$$

where $\ell = 1, ..., n$.

In calculations presented in the main text, we usually set $x_{i,1} = -2.5$ and $x_{i,n} = 4.5$, corresponding to $c_i = 12.1825$ and $c_i = 0.0111$, respectively. To see how boundary conditions affect the solution, we vary the range of the grid and compare the so obtained solutions. It turns out that the interior solution is very much unaffected by varying the grid. Figure 4.23 compares the solution to the policy function of player 1 at the upper boundary of x_1 for different values of $x_{i,n}$: $x_{i,n} = 4.5$, $x_{i,n} = 5$, and $x_{i,n} = 5.5$, respectively. We see that consecutive solutions diverge only in the very close proximity of the boundary, where the solution corresponding to a smaller grid range sweeps sharply down to zero. Hence, by specifying a large enough range of the grid, we can always obtain a good approximation over the interior region of interest. The correctness of our solution is further confirmed by the first plot in Figure 4.7, which shows that for a given large value of firm j's unit cost, the solution for firm i converges to the deterministic monopoly solution as $\varepsilon \downarrow 0$.