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van Lambalgen, M.

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Michiel van Lambalgen

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THE AXIOMATIZATION OF RANDOMNESS

MICHIEL VAN LAMBALGEN

Abstract. We present a faithful axiomatization of von Mises' notion of a random sequence, using an abstract independence relation. A byproduct is a quantifier elimination theorem for Friedman's "almost all" quantifier in terms of this independence relation.

§0. Introduction. Characterizing randomness has always been understood as providing *explicit* definitions of what a random sequence is or should be. The literature contains examples using recursive place selections (Church [1940]), recursive sequential tests (Martin-Löf [1966]) and (variants of) Kolmogorov complexity (e.g. Kolmogorov and Uspensky [1988]). Below, we present an axiomatic treatment of randomness. This approach is motivated by the conviction that randomness is a *mathematical* concept¹ that occurs in many different guises; to mention only the examples that will be treated below: generic sets, indiscernibles, (intuitionistic) lawless sequences, "maximally complex" sequences. The common mathematical core of these examples is formulated by means of the notion of independence, axiomatized in §1.1. Once we have the axioms for independence, we can obtain, e.g., von Mises' concept of random sequence by adding some specific axioms on subsequence selection.

We believe that the notions that are customarily involved in characterizations of randomness, such as irregularity, complexity or independence, are best treated as *primitive*, in the sense that they need not have a unique, privileged, interpretation. These notions can sometimes be paraphrased, i.e. in some cases we can give explicit definitions of the primitive notions in terms of familiar mathematical concepts. But this does not mean that a particular paraphrase exhausts the content of randomness.

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¹For the opposing view, that randomness is a vague notion from daily life with no mathematical content, cf. Freudenthal's remark: "It may be taken for granted that any attempt at defining disorder in a formal way will lead to a contradiction. This does not mean that the notion of disorder is contradictory. It is so, however, as soon as I try to formalize it."

An example is the emphasis on computability in the modern definitions. Without further analysis (cf. Kolmogorov and Uspensky [1988]) it is taken for granted that laws, rules or regularities as they occur in the context of randomness should be identified with rules as studied by recursion theory. By contrast, in our treatment we axiomize one fundamental aspect of randomness (independence) and we then investigate models of these axioms. It turns out that a model for a sizable part of the axioms is definable in recursion theoretic terms (corresponding to the paraphrase of independence given in §5 of van Lambalgen [1987a]). There exist, however, also models of an entirely different kind; and the completeness theorem given in §2 points to a fundamental role for forcing rather than computability.

The most important candidate for axiomatization is the notion of randomness introduced by von Mises [1919].

According to his intuitive characterization, a sequence is random if (a) the limiting relative frequencies within the sequence exist and (b) these limiting relative frequencies are invariant under the operation of “admissible place selection”. An admissible place selection is a procedure for selecting a subsequence of a given sequence x in such a way that the decision to select a term x_n does not depend upon the outcome x_n itself. Evidently, independence is a fundamental concept here, so the main task confronting us will be the axiomatization of independence. We do not wish to imply that the resulting axiom system captures all there is to randomness; there is, e.g., also the only vaguely related idea that random sequences should satisfy “all” strong limit theorems of probability theory. But von Mises’ axioms occupy a special position in that they describe the minimum properties randomness should satisfy to qualify as a foundation for probability theory (for more on this topic, see van Lambalgen [1987b]).

We now proceed to give a brief description of the contents of this paper. It consists of two parts: the first (§§1 and 2) deals with the notion of independence and has a set theoretical emphasis, while the second (§§3, 4 and 5) adds axioms for randomness and has a more probabilistic flavour.

In §1 we state the axioms for independence, and we sketch some interpretations (generic sets, lawless sequences, indiscernibles). In §2 we show that there is a close connection between the independence notion introduced in §1 and Friedman’s “almost all” quantifier (Steinhorn [1985a,b]). We prove, using Boolean valued models and some measure theory, that any first order theory can be conservatively extended with the axioms for independence or, equivalently, the axioms for “almost all”. We then proceed to show that the “almost all” quantifier can be eliminated in a first order theory which has an independence relation satisfying our axioms. This elimination allows us to prove that the theory of independence is complete with respect to a semantics given by forcing with Solovay reals.

In §3 we present the formal analogues of von Mises’ axioms for randomness and we give an informal argument to the effect that the axioms indeed say everything that can be said about admissible place selection. §4 contains the proof that the axioms for randomness and independence can be added consistently to second order arithmetic, thereby showing that a consistent probability theory based on a notion of randomness is possible. In §5 we investigate to what extent some familiar (explicit) definitions of randomness can be used to define models of the axioms; in particular,

we show that the theory of independence developed in van Lambalgen [1987a] (based on relativized prefix complexity) can be used to construct a model for (a considerable part of) the axioms.

§1. Axioms for randomness and independence. Von Mises proposed the following axioms for random infinite binary sequences. For ease of notation, we consider only the case of the uniform distribution on $\{0, 1\}$, i.e. sequences generated by a fair coin.

DEFINITION 1.1. A sequence $x \in 2^\omega$ is *random* if it satisfies the following two conditions.

$$(1) \lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n x_k = \frac{1}{2}.$$

(2) Let $(x_{n_k})_k$ be an admissibly chosen subsequence of x , i.e. a selection of subsequence which proceeds as follows: "Aus der unendliche Folge $[x]$ wird eine unendliche Teilfolge dadurch ausgewählt, daß über die Indizes der auszuwählenden Elemente ohne Benützung der Merkmalunterschiede verfügt wird." Then also $\lim_{m \rightarrow \infty} m^{-1} \sum_{k=1}^m x_{n_k} = \frac{1}{2}$.

The condition of admissibility is slightly enigmatic, so some examples may be helpful:

(a) Choose x_n if n is prime.

(b) Choose x_n if the $(n-9)$ th, ..., $(n-1)$ st terms of x are all equal to 1.

(c) Take a second coin, supposed to be independent of the first in so far as that is possible (no strings connecting the coins, no magnetisation, etc.). Then choose x_n if the outcome of the n th toss with the second coin is heads (we code "heads" as "1").

Intuitively, axiom (2) is satisfied in all three cases, although in (c) a heavy burden is put upon the word "independent". We shall call selections of type (a) and (b) *lawlike* (since they are given by some prescription) and those of type (c) *random*. Evidently combinations of the two types are possible. We shall prove below that, in a sense, all admissible place selections are combinations of random and lawlike selections (see Theorem 3.3.5).

It will be observed that the second axiom consists in fact of two parts. The first part tries to explain admissibility in terms of some concept of independence ("ohne Benützung der Merkmalunterschiede"); the second part says that limiting relative frequencies are invariant under admissible place selection. We shall follow the same procedure: in 1.1, we first axiomatize a notion of independence; in 3.2, we then define admissible place selection in terms of this notion and state axioms on the invariance of the limiting relative frequencies.

1.1. Axioms for independence. The literature contains various discussions of abstract independence relations; e.g. in algebra (van der Waerden [1940]), combinatorial geometry (Welsh [1976]), recursion theory (Metakides and Nerode [1980]) and stability theory (Baldwin [1988]). Another example is furnished by the theory of lawless sequences (Troelstra [1977]), where independence of lawless sequences is a fundamental concept, although it can be shown that in this case independence can be defined explicitly as extensional inequality $\neq$.

We show how to introduce an independence relation $R(x, \vec{y})$ in a countable first-order language L . Here, $\vec{y}$ denotes a vector, of *unspecified length*, of variables; hence R is a relation of indefinite arity. In analogy with linear algebra, the second parameter in $R(x, \vec{y})$ should be thought of as denoting a set; cf. axiom R3 below. (The introduction of R is of course equivalent to the introduction of infinitely many

independence relations R_n , one for each arity n .) One may think of $x, \tilde{y}$ as ranging over (vectors of) sequences in 2^ω , but the setup is in fact completely general. The intended interpretation of $R(x, \tilde{y})$ is: “ x is independent of $\tilde{y}$ ” or “ $\tilde{y}$ has no information about x ”. $\tilde{y}$ may be empty; in that case we write $R(x, \emptyset) =: R(x)$ and we may think of x with $R(x)$ as random sequences (cf. also the fifth remark below).

All independence relations that arise naturally in our context satisfy the properties $R1$ – $R5$ (examples will be given below); furthermore we shall work in classical logic.

$R0$. Axioms and inference rules for classical predicate logic.

$R1$. $\exists x R(x), \forall \tilde{y} \exists x R(x, \tilde{y})$.

$R2$. $R(x, \tilde{y}\tilde{z}) \rightarrow R(x, \tilde{z})$.

$R3$. $R(x, \tilde{y}) \rightarrow R(x, \pi\tilde{y})$ for any permutation π .

$R4$. $\neg R(x, x)$.

$R5$. $R(y, \tilde{z}) \wedge R(x, y\tilde{z}) \rightarrow R(y, x\tilde{z})$.

Before stating the last axiom, we give an example of an independent relation on infinite binary sequences that satisfies the axioms given so far.

Put

$$\text{LLN}(x) := \forall \varepsilon \exists n_0 \forall n \geq n_0 \left| \frac{1}{n} \sum_{k=1}^n x_k - \frac{1}{2} \right| < \varepsilon$$

and define a partial operation $\tilde{\cdot} : 2^\omega \times 2^\omega \rightarrow 2^\omega$ by $(x/y)_n = x_m$ if m is the index of the n th 1 in y , and undefined if there is no such index. (x/y is defined if y has infinitely many ones.) Then define $R(x, \tilde{y})$ as: $\text{LLN}(x) \ \& \ \forall y \in \tilde{y} (x/y \text{ defined} \rightarrow \text{LLN}(x/y))$.

The last axiom, the first to establish a relation between the language L and the independence relation R , is less straightforward. It expresses that random sequences are transcendental objects, not constructible by means available in the language L . To explain the main idea, we first recall a definition from model theory.

Let $\mathbf{M}$ be a model for a language $L \cup \{R\}$, $a \in |\mathbf{M}|$ (the domain of $\mathbf{M}$), $\vec{b}$ a sequence of elements from $|\mathbf{M}|$. We say that a is in the algebraic closure of $\vec{b}$ with respect to L (denoted $a \in \text{cl}_L(\vec{b})$) if there exists a formula $\phi(x, \vec{y})$ in L such that $\mathbf{M} \models \phi(a, \vec{b})$ and $\{c \mid \mathbf{M} \models \phi(c, \vec{b})\}$ is finite. We now want an axiom which says that $\mathbf{M} \models R(a, \vec{b})$ implies $a \notin \text{cl}_L(\vec{b})$. We first give a weak version.

$R6'$. Suppose $\phi(x, \vec{y})$ is in L , and z does not occur free in ϕ . Then

$$\forall x (R(x, z\vec{y}) \rightarrow \phi(x, \vec{y})) \rightarrow \forall x (R(x, \vec{y}) \rightarrow \phi(x, \vec{y})).$$

We now show that $R6'$ indeed implies that random sequences are transcendental with respect to L . Let $\mathbf{M}$ be a model for $R0$ – $R6'$. Let $\phi(x, \vec{y})$ be a formula in L such that $\mathbf{M} \models \phi(a, \vec{b})$ and $\{c \mid \mathbf{M} \models \phi(c, \vec{b})\}$ is finite, say $\{c \mid \mathbf{M} \models \phi(c, \vec{b})\} = \{c_1, \dots, c_k\}$.

By irreflexivity ($R4$) we have $\mathbf{M} \models \forall x (R(x, c_1, \dots, c_k, \vec{b}) \rightarrow \neg \phi(x, \vec{b}))$; hence, by $R6'$, $\mathbf{M} \models \forall x (R(x, \vec{b}) \rightarrow \neg \phi(x, \vec{b}))$, so $a \in \text{cl}_L(\vec{b})$ implies $\mathbf{M} \models \neg R(a, \vec{b})$.

As easy consequences of $R0$ – $R6'$ we have $\forall x (R(x) \rightarrow x \neq a)$, for constants a in L ; and, more generally, $\forall x (R(x, \vec{y}) \rightarrow x \neq t(\vec{y}))$, where t is a term in L .

While $R6'$ embodies the main idea, for technical reasons (the desired connection with forcing) we have to modify $R6'$, requiring that random sequences be transcendental in a stronger sense. We first define a subclass of the formulas in the language $L \cup \{R\}$.

DEFINITION 1.1.1. The class **IF** of *independence formulas* is the smallest class of formulas in the language $L \cup \{R\}$ such that

- (i) if ϕ is a formula in L , then ϕ is in **IF**,
- (ii) **IF** is closed under $\wedge$, $\rightarrow$, $\neg$, and $\forall$, and
- (iii) if $\phi(x, \vec{y})$ is in **IF** (all free variables are indicated), then $\forall x(R(x, \vec{z}) \rightarrow \phi(x, \vec{y}))$ is also in **IF**, where $\vec{z}$ contains $\vec{y}$.

We are now in a position to state the last axiom.

R6. Suppose $\phi(x, \vec{y})$ is in **IF**, and z does not occur free in ϕ . Then

$$\forall x(R(x, z\vec{y}) \rightarrow \phi(x, \vec{y})) \rightarrow \forall x(R(x, \vec{y}) \rightarrow \phi(x, \vec{y})).$$

Some remarks on the meaning of these axioms follow.

1. Axiom R3 expresses that the argument $\vec{y}$ behaves as a set.

2. The axioms for independence R1–R5 form a subsystem of the axioms for infinite-dimensional combinatorial geometries. Usually these axioms are formulated in terms of a relation of *dependence*, but if we formulate them instead using the independence relation $R(x, \vec{y})$ we get a system which is almost equal to R1–R5 (e.g. R5 corresponds to the Steinitz exchange principle), the difference being that R2 is replaced by the stronger postulate

$$R2'. R(x, \vec{z}) \rightarrow R(x, \vec{y}) \vee \exists y \in \vec{y} R(y, \vec{z}).$$

The meaning of R2' becomes more transparent if we formulate it in terms of $\neg R$, or dependence; we then get

$$R2'. \neg R(x, \vec{y}) \wedge \forall y \in \vec{y} \neg R(y, \vec{z}) \rightarrow \neg R(x, \vec{z}),$$

in other words, the transitivity of the relation of dependence.

It might seem surprising at first sight that we do not require that the dependence relation corresponding to *our* notion of independence is transitive. The paradigm example of the dependence relation that is axiomatized by combinatorial geometry is linear dependence; so in this context, $\neg R(x, \vec{z})$ means that there exists a functional relationship between x and $\vec{z}$. However, our case is more akin to statistical dependence and allows the following situation. We have a random binary sequence y and subsequences x and z consisting of the odd and even coordinates of y respectively. Then, for any reasonable concept of randomness, we have the following: $R(y)$, $R(x)$, $R(z)$, $\neg R(x, y)$ and $\neg R(z, y)$. By R5 we get $R(z) \& R(y, z) \rightarrow R(z, y)$; hence we also have $\neg R(y, z)$. It follows that, in the presence of R5, R2' fails.

3. Irreflexivity (R4) is slightly implausible from a purely probabilistic point of view; for instance if x represents a two-valued random variable which takes its values with probabilities 0 and 1, then x is (probabilistically) independent of x . We therefore think of our random sequences as sequences in which the limiting relative frequencies are distinct from 0 and 1.

4. The usual probabilistic definitions of independence are symmetric. However, in the presence of monotonicity (R2), the symmetry condition $R(x, y) \rightarrow R(y, x)$ is too strong, since we would then have $R(x, y) \rightarrow R(y)$.

5. The meaning of R6 will be further clarified in the proof of the consistency theorem, Theorem 2.1.3, where it is shown that R6 holds in virtue of the fact that random sequences are “indistinguishable”. We observe that in the presence of R6, $R(x) \equiv R(x, \emptyset)$ really means that x is independent of the set of closed terms of the language.

6. Given $R6'$, the second part of $R1$ becomes redundant: by $R6'$ we have $\exists \bar{y} \forall x (R(x, \bar{y}) \rightarrow x \neq x) \rightarrow \forall x (R(x) \rightarrow x \neq x)$, whence $\exists x R(x) \rightarrow \forall \bar{y} \exists x R(x, \bar{y})$.

Henceforth, the system of axioms $R0$ – $R6$ will be called $\mathfrak{R}$. The proof that these axioms are consistent is nontrivial, especially when we want to establish consistency of $\mathfrak{R}$ with a given theory T (say, second order arithmetic), or even that the addition of $\mathfrak{R}$ to a theory T is conservative. The consistency proof is deferred until §2, where we study the system $\mathfrak{R}$ by means of Friedman's theory of Borel structures and the quantifier "almost all".

EXAMPLE 1.1.2. (a) The paradigm example of an independence relation that satisfies $\mathfrak{R}$ is provided by forcing. Let $\mathbf{M}$ be a countable transitive model for ZFC plus the continuum hypothesis and let $\mathbf{M}[G]$ be obtained by generically adding ω_2 binary sequences. For any sequence $\bar{y}$ of elements of $2^\omega \cap \mathbf{M}[G]$, $\mathbf{M}[\bar{y}]$ is well defined as the smallest transitive submodel of $\mathbf{M}[G]$ containing $\mathbf{M}$ and $\bar{y}$. Observe that $\mathbf{M}[\bar{y}]$ also satisfies the continuum hypothesis.

We say that $x \in 2^\omega \cap \mathbf{M}[G]$ is *Cohen generic over $\mathbf{M}[\bar{y}]$* if x is contained in all comeager Borel sets which have a Borel code in $\mathbf{M}[\bar{y}]$. Put $R(x, \bar{y})$ iff $x \in 2^\omega$ is Cohen generic over $\mathbf{M}[\bar{y}]$. For this interpretation, $R1$ – $R4$ are trivially satisfied and $R5$ expresses the product lemma for forcing. To see this, observe that the partial order for generically adding ω_2 binary sequences is absolute, hence the same in all $\mathbf{M}[\bar{y}]$. The verification of $R6$ is less trivial; we need the machinery of Borel structures developed in the next section.

Similarly, we can interpret R using Solovay random reals (Solovay [1970]). Let $\mathbf{M}[G]$ be obtained from $\mathbf{M}$ by adding ω_2 random reals ($\mathbf{M}[G]$ will be defined in the proof of Theorem 2.2.1). We say that $x \in 2^\omega \cap \mathbf{M}[G]$ is *Solovay random over $\mathbf{M}[\bar{y}]$* if x is contained in all Borel sets of full Lebesgue measure which have a Borel code in $\mathbf{M}[\bar{y}]$. Put $R(x, \bar{y})$ iff $x \in 2^\omega$ is Solovay random over $\mathbf{M}[\bar{y}]$. In this case the validity of $R5$ is not immediate, because the partial order for adding a random real is not absolute. In the proof of Theorem 2.2.1 we shall verify that this interpretation of R indeed satisfies our axioms.

(b) Let L be the language of second order arithmetic (with variables over ω and 2^ω). It is possible to establish the consistency of $\mathfrak{R}$ by means of an interpretation into the intuitionistic theory of lawless sequences LS (for unexplained notions from intuitionism, see Troelstra and van Dalen [1988]).

If $\bar{y}$ denotes the vector $\langle y_1, \dots, y_n \rangle$, $\#(x, \bar{y})$ is defined as $x \neq y_1 \wedge \dots \wedge x \neq y_n$. We interpret the variables $x, y, \dots$ as ranging over lawless sequences. Extend the Gödel-Gentzen negative translation $*$ by means of the following clause for R : $(R(x, \bar{y}))^* := \#(x, \bar{y})$. Then $\mathfrak{R} \vdash \phi$ implies $LS \vdash \phi^*$. All cases are easily checked, except axiom $R6$, for which we need the axioms of density and open data. In this way the consistency of $\mathfrak{R}$ can be established, but since we want consistency of $\mathfrak{R}$ over an arbitrary first order theory we present a different consistency proof below (2.1.4). The details of the extended Gödel-Gentzen interpretation will be given in van Lambalgen and Moerdijk [199?].

(c) (Domenico Zambella). Let T be a theory in L , $\mathbf{M}$ an infinite model for T . Assume that T has built-in Skolem functions. We indicate how to obtain an elementary extension N of T which can be expanded to a model $\langle N, R \rangle$ for $T \cup \mathfrak{R}$, using indiscernibles. Let $\langle I, < \rangle$ be an unbounded dense simple order. Let $\mathbf{K}$ be an

elementary extension of M that contains I as a set of indiscernibles, and let N be the elementary substructure of K generated by $H(I)$, the Skolem hull of I in K .

Put $R(x, \vec{y})$ if $x \in I$ and for some sequence $\vec{z}$ of elements of I , $\vec{y} \in H(\vec{z})$ and $\#(x, \vec{z})$. Then $\langle N, R \rangle$ is a model for $T \cup \mathfrak{R}$. This shows that $T \cup \mathfrak{R}$ is conservative over T . Below, we give a proof of this fact using Boolean valued models, since this technique is more readily adapted to include specific randomness axioms. (We observe in passing that indiscernibles in fact satisfy stronger axioms for independence because we decline to look “inside” them, whereas for random sequences we also consider subsequences.)

§2. Friedman’s quantifier “almost all”. It turns out that the axiom system $\mathfrak{R}$ can be studied profitably in an alternative formulation, by adding the quantifier “almost all” introduced by Harvey Friedman (see Steinhorn [1985a,b]) to the language L . If Q denotes Friedman’s quantifier, and ϕ is a formula of the language $L \cup \{Q\}$, then we can think of the formula $Qx\phi x$ as meaning “ $\{x \mid \phi(x)\}$ has Lebesgue measure 1”. Alternatively, $Qx\phi x$ can be read topologically as “ $\{x \mid \phi(x)\}^c$ is first category”. Friedman exhibits a set $\mathfrak{Q}$ of axioms for Q which is complete for both interpretations.

For us, the interesting point is that there is a close connection between $\mathfrak{Q}$ and the axioms for independence listed in 1.1. Below, we shall define a translation $*$ from $L \cup \{Q\}$ into $L \cup \{R\}$ such that ϕ is derivable in $\mathfrak{Q}$ if and only if ϕ^* is derivable from $\mathfrak{R}$. The advantages of this construction are threefold:

1. We obtain consistency proofs for $\mathfrak{R}$ and $\mathfrak{R}\mathfrak{S}$ (over a suitable base theory) by measure theoretic techniques.
2. Friedman’s completeness proof for $\mathfrak{Q}$ is transformed into a completeness result for $\mathfrak{R}$, relating $\mathfrak{R}$ to Solovay forcing.
3. It brings to the fore an intimate connection between measure on the one hand, and randomness and independence on the other.

2.1. The system $\mathfrak{Q}$. The axioms for Q given below are taken from Steinhorn [1985a], except that Steinhorn uses the existential analogue of Q (“ $\{x \mid \phi(x)\}$ is nonnull”) and does not add the nontriviality condition $Q1$. Again, we start with a countable first order language L .

$Q0$. Axioms and inference rules for classical predicate logic.

$Q1$. $\neg Qx\ x \neq x$.

$Q2$. $Qy\ x \neq y$.

$Q3$. $Qx\phi(\dots, x, \dots) \rightarrow Qy\phi(\dots, y, \dots)$, provided y is free for x in ϕ .

$Q4$. $Qx\phi \wedge \forall x(\phi \rightarrow \psi) \rightarrow Qx\psi$.

$Q5$. $Qx\phi \wedge Qx\psi \rightarrow Qx(\phi \wedge \psi)$.

$Q6$. $QxQy\phi \leftrightarrow QyQx\phi$.

The existential analogues of axioms $Q1$ – $Q5$ describe the quantifier “there are many” studied by Krivine and McAloon [1973]. $Q1$ – $Q5$ are clearly satisfied for the intended interpretation(s); but the consistency of the full system is a delicate matter. Without the nontriviality condition $Q1$, consistency is easily established: interpret $Qx\phi$ as $\forall x(\perp \rightarrow \phi)$, where $\perp$ denotes falsum. $Q1$, however, excludes this possibility. The intended interpretation of Q , be it measure theoretic or topological, is not very helpful in establishing consistency either. For if L is the language of second order arithmetic, L is sufficiently rich to define projective sets. But it is consistent to

assume that some projective sets are nonmeasurable and in that case we run into trouble with $Q6$, which is more or less a definable analogue of Fubini's theorem (but not quite, since in Fubini's theorem the measurability of $\{\langle x, y \rangle \mid \phi(x, y)\}$ is an *assumption*). For instance, assuming $V = L$ there is a projective $\phi(x, y)$ defining the graph of a wellordering $<$ of 2^ω of length ω_1 ; but then we have $QxQy\phi(x, y)$ and $QyQx\neg\phi(x, y)$. Continuing this line of thought, one would seem to need a model in which all definable (with parameters) sets of reals are measurable; but this implies that ω_1 is inaccessible in the constructible universe, a most unwelcome assumption.

One way out of this predicament is to switch to the topological interpretation. Shelah has shown that if ZF is consistent, so is ZF + "every set of reals has the Baire property". We then get the consistency of $\mathfrak{Q}$ by applying the Kuratowski-Ulam theorem, the topological analogue of Fubini's theorem (see Theorem 15.1 in Oxtoby [1980]). While this result might set the reader's mind temporarily at rest, it is not very helpful in settling the consistency of additional assumptions with $\mathfrak{Q}$. For instance, we would like to have that $QxLLN(x)$ is consistent with $\mathfrak{Q}$; but of course this is false if Q is interpreted topologically (and $LLN(x)$ has the standard interpretation).

There is, however, yet a third way to interpret Q , and that interpretation is the main reason for our interest in Q : read $Qx\phi(x, \bar{y})$ as "for all x independent of $\bar{y}$, ϕ ". A little reflection shows that $Q6$ is now obvious (if x and y are supposed to be independent, then it does not matter which one is chosen first), whereas $Q4$ is more problematic. (This interpretation of Q is formalized by the Elimination Theorem 2.2.1, which shows that there is an embedding of $\mathfrak{Q}$ into $\mathfrak{R}$.) Furthermore, using a statistical concept of independence, it becomes fairly easy to prove the consistency of both $\mathfrak{Q}$ and $\mathfrak{R}$ over arbitrary first order theories (2.1.3 and 2.1.4). The proofs use Friedman's theory of *Borel structures*.

Let M be a model for L , with domain a subset of 2^ω of positive Lebesgue measure. On the measure theoretic interpretation the satisfaction clause for Q is as follows:
 $M \models Qx\phi(x, a_1, \dots, a_n)$ iff $\lambda\{x \mid M \models \phi(x, a_1, \dots, a_n)\} = 0$, where $\lambda = (\frac{1}{2}, \frac{1}{2})^\omega$ is Lebesgue measure on 2^ω .

The other logical constants have standard interpretation. A structure whose domain is a subset of 2^ω and all of whose relations and functions are Borel is called a *Borel structure* (or model). A Borel structure for $L \cup \{Q\}$ is called *totally Borel* if any relation defined using a formula from $L \cup \{Q\}$ (possibly with parameters) is Borel. Evidently the axioms for Q are valid on a totally Borel model. Henceforth we shall consider only totally Borel structures whose domain is a *full* subset of 2^ω ; this is justified by the completeness Theorem 2.1.2 below. Interestingly, Friedman has shown that if a theory in the language $L \cup \{Q\}$ is consistent with $\mathfrak{Q}$, then it has a totally Borel model. It is somewhat hard to construct such models, since it is usually the case (e.g. for second order arithmetic) that the primitive relations must have a nonstandard interpretation. We therefore prefer a Boolean valued approach, where the primitive relations can be interpreted straightforwardly, while by contrast Q has a nonstandard interpretation, in terms of an independence relation.

THEOREM 2.1.1 (See Steinhorn [1985a,b]). *Let T be a consistent theory in L with an infinite model. Then T has a totally Borel model with domain 2^ω .*

THEOREM 2.1.2 (See Steinhorn [1985a]). *Let T be a theory in $L \cup \{Q\}$. Then T has a totally Borel model whose domain is a full subset of 2^ω iff T is consistent with $\mathfrak{Q}$.*

Using 2.1.1 and a Boolean valued interpretation of $\forall$, $\exists$ and Q , we show that any consistent theory T in L , with at least one infinite model, can be consistently extended with $\mathfrak{Q}$, or, equivalently, with $\mathfrak{R}$. (This technique will allow us to show the consistency of $\mathfrak{RS}$ in §4.) The proof of the consistency theorem proceeds by defining an independence relation that satisfies the Boolean valued analogue of $\mathfrak{R}$ and showing that Q can be defined explicitly using this relation. This procedure suggests a quantifier elimination theorem for Q that will be stated and proved in 2.2.

THEOREM 2.1.3. *Let T be a consistent theory in L with an infinite model. Then $T \cup \mathfrak{Q}$ is conservative over T .*

PROOF. We construct a Boolean valued model on a universe defined as follows. (For general information on Boolean valued models, see Bell [1977].) Consider $(2^\omega)^\kappa$, where $\kappa \geq \omega_1$. We equip $(2^\omega)^\kappa$ with the product topology and the product measure λ^κ defined on the Borel σ -algebra $B((2^\omega)^\kappa)$. Let I denote the σ -ideal of λ^κ nullsets; then the quotient algebra $B := B((2^\omega)^\kappa)/I$ is a complete Boolean algebra. B will be our space of truth values. The domain of the Boolean valued model consists of the (Borel) measurable functions $f: (2^\omega)^\kappa \rightarrow 2^\omega$.

Let A be a totally Borel model for T with universe 2^ω . If ψ is in L and atomic, we put

$$[[\psi(f_1, \dots, f_n)]]_B = \{\xi \in (2^\omega)^\kappa \mid A \models \psi(f_1(\xi), \dots, f_n(\xi))\}/I;$$

the interpretation of the Boolean connectives is obvious, and $\forall$ and $\exists$ are interpreted using the infimum $\bigwedge$ and the supremum $\bigvee$, respectively.

On the space of (Borel) measurable functions $(2^\omega)^\kappa \rightarrow 2^\omega$ there are several natural independence relations that can be used to interpret Q . We choose the following option (but see Remark 1 after Corollary 2.1.4). Say that $\alpha < \kappa$ is in the support of f (denoted $\alpha \in \text{supp}(f)$) if there are $\xi, \xi' \in (2^\omega)^\kappa$ which differ only at coordinate α and are such that $f(\xi) \neq f(\xi')$. If f is Borel measurable, $\text{supp}(f)$ is countable. (Observe that the notion of support is not preserved under equivalence modulo nullsets; but we may always consider representatives with least support.) For $\alpha < \kappa$, let $\pi_\alpha: (2^\omega)^\kappa \rightarrow 2^\omega$ denote the projection on the α th coordinate. It now makes sense to define a relation (of indefinite arity) $R(g, f_1, \dots, f_n)$ as follows: $R(g, f_1, \dots, f_n)$ if there is $\alpha < \kappa$ such that $g = \pi_\alpha$ and $\alpha \notin \text{supp}(f_1) \cup \dots \cup \text{supp}(f_n)$.

We now define $[[Qx\psi(x, f_1, \dots, f_n)]]_B$ as

$$[[Qx\psi(x, f_1, \dots, f_n)]]_B = \bigwedge \{ [[\psi(g, f_1, \dots, f_n)]]_B \mid R(g, f_1, \dots, f_n) \},$$

and we have to show that the Boolean truth value of the sentences T and the axioms of $\mathfrak{Q}$ is 1_B .

1. T is valid in B . Since A is a model for T , it suffices to show that for any formula ψ in L ,

$$[[\psi(f_1, \dots, f_n)]]_B = \{\xi \in (2^\omega)^\kappa \mid A \models \psi(f_1(\xi), \dots, f_n(\xi))\}/I.$$

The proof is by induction, and the only case of interest is where ψ is $\exists x\phi$. By the maximum principle,

$$[[\exists x\phi(x, f)]]_B = \bigvee_g [[\phi(g, f)]]_B = [[\phi(g, f)]]_B,$$

for some particular measurable $g: (2^\omega)^\kappa \rightarrow 2^\omega$. Hence, by the induction hypothesis,

$$\begin{aligned} [[\phi(g, f)]]_B &= \{\xi \in (2^\omega)^\kappa \mid A \models \phi(g(\xi), f(\xi))\} / I \\ &\leq \{\xi \in (2^\omega)^\kappa \mid A \models \exists x \phi(x, f(\xi))\} / I. \end{aligned}$$

For the converse we need a uniformity principle, the von Neumann measurable selection theorem (see, e.g., Parthasarathy [1972]).

Since $\{\langle x, \xi \rangle \in 2^\omega \times (2^\omega)^\kappa \mid A \models \phi(x, f(\xi))\}$ is Borel, we can apply the selection theorem to obtain a uniformizing measurable $g: (2^\omega)^\kappa \rightarrow 2^\omega$ which satisfies

$$\text{for } \lambda^\kappa \text{ a.a. } \xi \in (2^\omega)^\kappa, \quad A \models \phi(g(\xi), f(\xi)) \text{ iff } A \models \exists x \phi(x, f(\xi)).$$

We then have

$$\begin{aligned} \{\xi \in (2^\omega)^\kappa \mid A \models \exists x \phi(x, f(\xi))\} / I &= \{\xi \in (2^\omega)^\kappa \mid A \models \phi(g(\xi), f(\xi))\} / I \\ &= [[\phi(g, f)]]_B \leq [[\exists x \phi(x, f)]]_B. \end{aligned}$$

2. $\mathfrak{Q}$ is valid in B . All axioms are trivially satisfied except (somewhat surprisingly) $\mathfrak{Q}4$. We have to show that

$$[[Qx\phi(x, g)]]_B \wedge \bigwedge_f ([[\phi(f, g)]])_B \rightarrow [[\psi(f, h)]]_B \leq [[Qx\psi(x, h)]]_B.$$

Evidently, it suffices to show that, if g does not occur in ψ ,

$$\bigwedge \{ [[\psi(\pi_\alpha, h)]]_B \mid \alpha \notin \text{supp}(g) \cup \text{supp}(h) \} = \bigwedge \{ [[\psi(\pi_\alpha, h)]]_B \mid \alpha \notin \text{supp}(h) \}.$$

Before we commence the proof, a word on motivation. Suppose we had interpreted Q , not in the measure algebra B , but in the category algebra, that is, the Borel σ -algebra divided by the σ -ideal of first category sets. As is well known, this algebra is completely isomorphic to the complete Boolean algebra of regular open sets on $(2^\omega)^\kappa$. On the latter algebra, one may think of $[[Qx\phi(x, g)]]_B$ as being given by a kind of interior operator (applied to $[[\phi(\pi_\alpha, g)]]_B$, for $\alpha \notin \text{supp}(g)$), with the essential difference that the interior operator is considered only with respect to the topology on $(2^\omega)^\kappa$ induced by g^{-1} . It will be seen that in the case of measure, one can construct an analogue of this notion of interior using conditionalization on the complete subalgebra generated by g . The analogy between Q and interior operators will be spelled out in 2.1.5 and 2.1.6.

For notational convenience we do not distinguish between an element of the Boolean algebra and a representative Borel set. Choose $\alpha < \kappa$ with $\alpha \notin \text{supp}(h)$. Let E be a Borel set such that the support of the characteristic function of E is contained in $\text{supp}(h)$ and such that

$$(*) \quad \lambda^\kappa([[\psi(\pi_\alpha, h)]]_B \mid E) = 1, \quad \lambda^\kappa([[\psi(\pi_\alpha, h)]]_B \mid E^c) < 1.$$

That such an E exists can be seen as follows. We may consider $[[\psi(\pi_\alpha, h)]]_B$ to be a Borel subset of $(2^\omega)^\omega \times (2^\omega)^\omega$, where h is supported by the second coordinate and π_α lives on the first coordinate. (One has to be careful here: usually (a representative of) $[[\psi(g, h)]]_B$ is not measurable with respect to the σ -algebra generated by g and h . This is the analogue of the nonmeasurability of $\{\langle x, y \rangle \mid \psi(x, y)\}$ in the Boolean valued case.) By differentiating $(\lambda^\omega)^2$, we obtain the conditional probability $(\lambda^\omega)^2(F \mid h)$, where F ranges over Borel subsets of $(2^\omega)^\omega$. Then define E as

$$E := \{y \in (2^\omega)^\omega \mid (\lambda^\omega)^2([[\psi(\pi_\alpha, h)]]_B \mid h(y)) = 1\}.$$

E is determined up to a nullset, and its support is contained in $\text{supp}(h)$. Furthermore we may assume that $1 > \lambda^* E > 0$, for otherwise we are done. Hence the conditional probabilities in $(*)$ are well defined.

Obviously $(*)$ is true for any $\alpha \notin \text{supp}(h)$, since any two such projections can be transformed into each other by an automorphism of $\langle (2^\omega)^\kappa, B((2^\omega)^\kappa), \lambda^* \rangle$ that is the identity on $\text{supp}(h)$.

It follows that for any $\alpha \notin \text{supp}(h)$ we have $E \leq [[\psi(\pi_\alpha, h)]]_B$.

By distributivity, we have

$$\begin{aligned} & \bigwedge \{ [[\psi(\pi_\alpha, h)]]_B \mid \alpha \notin \text{supp}(g) \cup \text{supp}(h) \} \\ &= (\bigwedge \{ [[\psi(\pi_\alpha, h)]]_B \mid \alpha \notin \text{supp}(g) \cup \text{supp}(h) \} \wedge E) \\ & \quad \vee (\bigwedge \{ [[\psi(\pi_\alpha, h)]]_B \mid \alpha \notin \text{supp}(g) \cup \text{supp}(h) \} \wedge E^c). \end{aligned}$$

However, we can show that the second disjunct must equal zero. We may suppose that $\text{supp}(h)$ and $\text{supp}(g)$ are disjoint. Write $E^c \wedge [[\psi(\pi_\alpha, h)]]_B = F \cap F_\alpha$, where $\text{supp}(F) = \text{supp}(h)$ is disjoint from $\text{supp}(F_\alpha)$. We may choose α such that $\text{supp}(F_\alpha)$ is also disjoint from $\text{supp}(g)$.

By the maximality of E , either $\lambda^*(F) = 0$ or $\lambda^*([[\psi(\pi_\alpha, h)]]_B \mid F) < 1$. In the first case we are done, so suppose $\lambda^*([[\psi(\pi_\alpha, h)]]_B \mid F) < 1$. Then we must have $\lambda^*(F_\alpha) < 1$. Choose a countable set of automorphisms $\{\tau_n\}$ which are fixed on $\text{supp}(h)$, such that the $\text{supp}(\tau_n(F_\alpha))$ are pairwise disjoint and also disjoint from $\text{supp}(g)$.

By independence, for a countable set of projections $\pi_{\alpha_1}, \dots, \pi_{\alpha_n}, \dots$ on coordinates not contained in $\text{supp}(g) \cup \text{supp}(h)$ we obtain

$$\begin{aligned} 0_B &= \bigwedge_n \tau_n [[\psi(\pi_{\alpha_n}, h)]]_B \wedge E^c = \bigwedge_n [[\psi(\pi_{\alpha_n}, h)]]_B \wedge E^c \\ &\geq \bigwedge \{ [[\psi(\pi_\alpha, h)]]_B \mid \alpha \notin \text{supp}(g) \cup \text{supp}(h) \} \wedge E^c. \end{aligned}$$

Therefore

$$\begin{aligned} & \bigwedge \{ [[\psi(\pi_\alpha, h)]]_B \mid \alpha \notin \text{supp}(g) \cup \text{supp}(h) \} \\ &= (\bigwedge \{ [[\psi(\pi_\alpha, h)]]_B \mid \alpha \notin \text{supp}(g) \cup \text{supp}(h) \} \wedge E) \leq E \leq [[\psi(\pi_\alpha, h)]]_B, \end{aligned}$$

for any $\alpha \notin \text{supp}(h)$. □

The proof of 2.1.3 in effect constructs an explicit independence relation satisfying R0–R6. Hence if we put $[[R(g, f_1, \dots, f_n)]]_B = 1_B$ if $R(g, f_1, \dots, f_n)$ and equal to 0_B otherwise, we get

COROLLARY 2.1.4. *Let T be a consistent theory in L with an infinite model. Then $T \cup \mathfrak{R}$ is conservative over T .*

The proof of Theorem 2.1.3 can be seen as establishing that Q is a kind of interior operator:

COROLLARY 2.1.5. *For any $\alpha \notin \text{supp}(f_1) \cup \dots \cup \text{supp}(f_n)$,*

$$\begin{aligned} & [[Qx\psi(x, f_1, \dots, f_n)]]_B \\ &= \bigvee \{ E \mid E \leq [[\psi(\pi_\alpha, f_1, \dots, f_n)]]_B \text{ and } \text{supp}(f_1) \cup \dots \cup \text{supp}(f_n) \supseteq \text{supp}(E) \}. \end{aligned}$$

PROOF. The proof of 2.1.3 gives, for the specific E with $\text{supp}(E) \subseteq \text{supp}(h)$ constructed there,

$$\begin{aligned} & [[Qx\psi(x, h)]]_B = \bigwedge \{ [[\psi(\pi_\alpha, h)]]_B \mid \alpha \notin \text{supp}(g) \cup \text{supp}(h) \} \\ &= (\bigwedge \{ [[\psi(\pi_\alpha, h)]]_B \mid \alpha \notin \text{supp}(g) \cup \text{supp}(h) \} \wedge E) \leq E \leq [[\psi(\pi_\alpha, h)]]_B, \end{aligned}$$

for any $\alpha \notin \text{supp}(h)$; hence

$$[[Qx\psi(x, h)]]_B \leq E \leq \bigwedge \{ [[\psi(\pi_\alpha, h)]]_B \mid \alpha \notin \text{supp}(h) \} = [[Qx\psi(x, h)]]_B.$$

Observe that, whereas (representatives of) $[[\forall x\psi(x, h)]]_B$ need not be measurable with respect to the σ -algebra generated by h , $[[Qx\psi(x, h)]]_B$ does have a representative (namely, E) which is measurable with respect to that σ -algebra. $\square$

We have chosen the above representation of Q to show that there is an intimate connection between the “almost all” quantifier and independence. An alternative approach would be to define the algebraic interpretation of Q directly in terms of an interior operator and “generic” elements (in this case projections). While on this interpretation of Q there is no straightforward connection with independence, the verification of the axioms is easier. An additional advantage is that the relationship between the validity of the axioms and the homogeneity of the Boolean algebra becomes clearer.

THEOREM 2.1.6. *Choose $\alpha \notin \text{supp}(f_1) \cup \dots \cup \text{supp}(f_n)$. If we interpret Q in B as*

$$[[Qx\psi(x, f_1, \dots, f_n)]]_B = \bigvee \{ E \mid E \leq [[\psi(\pi_\alpha, f_1, \dots, f_n)]]_B \text{ and } \text{supp}(f_1) \cup \dots \cup \text{supp}(f_n) \supseteq \text{supp}(E) \},$$

then the $\mathfrak{Q}$ axioms hold.

PROOF. Only (4) and (5) are nontrivial. First observe that the supremum in the definition of $[[Q]]$ is in fact attained for some E . For the monotonicity axiom (4) we have to verify that

$$\bigvee \{ F \mid F \leq [[\phi(\pi_\beta, g)]]_B \text{ and } \text{supp}(g) \supseteq \text{supp}(E) \} \wedge \bigwedge_f ([[\phi(f, g)]]_B \rightarrow [[\psi(f, h)]]_B) \leq \bigvee \{ E \mid E \leq [[\psi(\pi_\alpha, h)]]_B \text{ and } \text{supp}(h) \supseteq \text{supp}(E) \}$$

for suitable α and β . Choose $F \leq [[\phi(\pi_\beta, g)]]_B$ with $\text{supp}(F) \subseteq \text{supp}(g)$. We may suppose that $\alpha \notin \text{supp}(F)$. By hypothesis $F \leq [[\psi(\pi_\alpha, h)]]_B$. Let τ range over the automorphisms of $(2^\omega)^\kappa$ that are fixed on α and $\text{supp}(h)$. Then $\tau(F) \leq [[\psi(\pi_\alpha, h)]]_B$; hence also $F \leq \bigvee_\tau \tau(F) \leq [[\psi(\pi_\alpha, h)]]_B$. By applying the independence of the coordinates of $(2^\omega)^\kappa$ to $\bigwedge_\tau \tau(F^c)$, using the fact that $\alpha \notin \text{supp}(F)$, we obtain as in the proof of 2.1.3 that $\bigvee_\tau \tau(F)$ has a representative E such that $\text{supp}(E) \subseteq \text{supp}(h)$.

We now consider (5). Let $F \leq [[\phi(\pi_\alpha, \pi_\beta, g)]]_B$ be maximal such that $\text{supp}(F) \subseteq \text{supp}(g) \cup \{\beta\}$ and let $E \leq F$ be maximal such that $\text{supp}(E) \subseteq \text{supp}(g)$. Using an automorphism τ of $(2^\omega)^\kappa$ that is fixed on $\text{supp}(h)$ and exchanges α and β , we obtain the required maximal $\tau(F) \leq [[\phi(\pi_\alpha, \pi_\beta, g)]]_B$ with $\text{supp}(\tau(F)) \subseteq \text{supp}(g) \cup \{\alpha\}$ and $E = \tau(E) \leq \tau(F)$. $\square$

REMARKS. 1. While the above simple interpretation of R works for the purpose of showing consistency, it has an important drawback. Since mixtures (see Bell [1977]) of projections are not themselves projections, the Boolean valued model for $T \cup \mathfrak{R}$ constructed above does not satisfy a closure principle: if ϕ is a formula in $L \cup \{R\}$, then we do not necessarily have $[[\exists x\phi(x, f)]]_B = [[\phi(g, f)]]_B$, for some measurable $g: (2^\omega)^\kappa \rightarrow 2^\omega$. This means that the (Boolean valued) universe of random “sequences” is not closed under nontrivial operations. For instance, a statement like

$$\forall x \forall y (R(x) \wedge R(x, y) \rightarrow \exists z (z = x/y \wedge R(z)))$$

is false (observe that this statement is not an **IF** formula). Models which do satisfy the closure principle will be studied in van Lambalgen and Moerdijk [199?].

2. It is perhaps worth emphasizing that while the monotonicity axiom $Q4$ is innocuous on the “almost all” interpretation of Q , it is far from obvious on the “independence” interpretation. On the latter interpretation, $Q4$ turns out to be true because an independence structure has many symmetries.

3. It can be shown that $\mathfrak{R}$ implies the nonexistence of a definable well-ordering.

2.2. Eliminating Q . We define a translation $*$ of $L \cup \{Q\}$ into $L \cup \{R\}$ as follows: $*$ is the identity on L formulas; $*$ commutes with $\wedge, \neg, \forall$; and

$$(Qx\phi(x, \tilde{y}))^* := \forall x(R(x, \tilde{y}) \rightarrow \phi(x, \tilde{y})^*).$$

THEOREM 2.2.1. $*$ is a faithful relative interpretation of $L \cup \{Q\}$ into $L \cup \{R\}$, i.e., for all ϕ in $L \cup \{Q\}$, $\mathfrak{Q} \vdash \phi$ iff $\mathfrak{R} \vdash \phi^*$.

PROOF. $\Rightarrow$. $R1$ ensures that $*$ is correctly defined as a relative interpretation. The remainder of the argument proceeds by a routine induction on the length of proofs in $\mathfrak{Q}$. That the axioms for Q are derivable in $\mathfrak{R}$ can be seen as follows:

— $(\neg Qx\ x \neq x)^* = \neg \forall x(R(x) \rightarrow x \neq x) \equiv \exists x R(x)$; hence $Q1$ corresponds to $R1$ under $*$.

— $(\forall x Qy\ x \neq y)^* = \forall x \forall y(R(y, x) \rightarrow x \neq y) \equiv \forall x \neg R(x, x)$; hence $Q2$ corresponds to $R4$ under $*$.

— $Q3$ holds trivially.

— $(Qx\phi(x, \tilde{y}) \wedge \forall x(\phi(x, \tilde{y}) \rightarrow \psi(x, \tilde{z})) \rightarrow Qx\psi(x, \tilde{z}))^* = \forall x(R(x, \tilde{y}) \rightarrow \phi(x, \tilde{y})^*) \wedge \forall x(\phi(x, \tilde{y})^* \rightarrow \psi(x, \tilde{z})^*) \rightarrow \forall x(R(x, \tilde{z}) \rightarrow \psi(x, \tilde{z})^*)$; the antecedent implies that $\forall x(R(x, \tilde{y}\tilde{z}) \rightarrow \psi(x, \tilde{z})^*)$; hence by $R6$ also $\forall x(R(x, \tilde{z}) \rightarrow \psi(x, \tilde{z})^*)$.

— $(Qx\phi(x, \tilde{y}) \wedge Qx\psi(x, \tilde{z}) \rightarrow Qx(\phi(x, \tilde{y}) \wedge \psi(x, \tilde{z})))^* = \forall x(R(x, \tilde{y}) \rightarrow \phi(x, \tilde{y})^*) \wedge \forall x(R(x, \tilde{z}) \rightarrow \psi(x, \tilde{z})^*) \rightarrow \forall x(R(x, \tilde{y}\tilde{z}) \rightarrow \phi(x, \tilde{y})^* \wedge \psi(x, \tilde{z})^*)$; hence $(Q5)^*$ can be derived in $\mathfrak{R}$ using $R2$.

— $(Qx Qy\phi(x, y, \tilde{z}) \leftrightarrow Qy Qx\phi(x, y, \tilde{z}))^* = \forall x \forall y(R(x, \tilde{z}) \wedge R(y, x, \tilde{z}) \rightarrow \phi(x, y, \tilde{z})^*) \leftrightarrow \forall y \forall x(R(y, \tilde{z}) \wedge R(x, y, \tilde{z}) \rightarrow \phi(x, y, \tilde{z})^*)$; hence $(Q6)^*$ can be derived in $\mathfrak{R}$ using $R5$ and $R2$.

The induction step is trivial, since both in the case of $\mathfrak{Q}$ and $\mathfrak{R}$ the inference rules are those of classical predicate logic.

$\Leftarrow$. We present two proofs. The first proof leads to a two valued interpretation for R and a completeness theorem for $\mathfrak{R}$ with respect to a forcing semantics. The second proof exploits the construction in the proof of 2.1.3.

First proof. Let $\mathbf{M}$ be a model of $ZFC + CH$, B the Boolean algebra constructed in the proof of 2.1.3 with $\kappa = \omega_2$, G an $\mathbf{M}$ -generic ultrafilter on B and $\mathbf{M}[G]$ the resulting generic extension. For general information on this model see Friedman [1980]. Suppose that $\mathfrak{Q} \not\vdash \phi$; then by the Completeness Theorem 2.1.2 applied in $\mathbf{M}$ there exists a totally Borel model $\mathcal{A}$ in $\mathbf{M}$ for $\mathfrak{Q} \cup \{\neg \phi\}$. We may assume that the domain of $\mathcal{A}$ is a full subset of 2^ω . We shall freely use the absoluteness properties of Borel structures (see, e.g., Steinhorn [1985a, pp. 586–588]). In $\mathbf{M}[G]$, we expand $\mathcal{A}$ to a structure $\langle \mathcal{A}, R \rangle$ by defining: $R(x, \tilde{y})$ iff $x \in 2^\omega$ is Solovay random over $\mathbf{M}[\tilde{y}]$.

We show that $\langle \mathcal{A}, R \rangle$ satisfies $\mathfrak{R}$ and $\neg \phi^*$. $R2$ – $R4$ are trivially valid. To establish the validity of $R1$, we use an observation of Solovay: if c is the cardinality of the continuum in $\mathbf{M}$, then in $\mathbf{M}[G]$ the intersection of c Borel sets of measure 1 has

outer measure 1. Since the cardinality of the continuum is not changed in passing from $\mathbf{M}$ to $\mathbf{M}[\tilde{y}]$, R1 follows.

We next consider R5. Suppose y is random over $\mathbf{M}$, and x is random over $\mathbf{M}[y]$ (the proof of the general case is exactly similar). Suppose y is nonrandom over $\mathbf{M}[x]$; then there exists a Borel set B with code in $\mathbf{M}[x]$ such that $\lambda B = 0$ and $y \in B$. Observe that the code for B is arithmetical in x and a real in $\mathbf{M}$, and that the formula “ α codes the Borel set B_α and $x \in B_\alpha$ ” is Π_1^1 . Hence there is a Borel set C with code in $\mathbf{M}$ such that $\langle x, y \rangle \in C$, $B = \{u \in 2^\omega \mid \langle x, u \rangle \in C\}$ and $\mathbf{M}[x] \models \lambda\{u \in 2^\omega \mid \langle x, u \rangle \in C\} = 0$. Then there must exist a Borel set D , with code in $\mathbf{M}$, such that, for all random $v \in D$,

$$\mathbf{M}[v] \models \lambda\{u \in 2^\omega \mid \langle v, u \rangle \in C\} = 0.$$

Define $E := \{\langle u, v \rangle \mid v \in D \ \& \ \langle v, u \rangle \in C\}$, then $\langle x, y \rangle \in E$. By Fubini, $\{u \mid \lambda E^u = 0\}$ (where E^u denotes the section of E along the second coordinate) is a Borel set of full Lebesgue measure with code in $\mathbf{M}$. Since y is random over $\mathbf{M}$, $\lambda E^y = 0$. However, E^y has code in $\mathbf{M}[y]$ and $x \in E^y$, so x is not random over $\mathbf{M}[y]$. This proves R5.

To show the validity of R6 and $\neg \phi^*$, we need

LEMMA 2.2.2. *For all ψ and $\tilde{y} \in 2^\omega \cap \mathbf{M}[G]$, $A \models \psi(\tilde{y})$ iff $\langle A, R \rangle \models \psi^*(\tilde{y})$.*

PROOF OF THE LEMMA. The cases of the ordinary logical operations are trivial, since $*$ commutes with these. So let $\psi(\tilde{y})$ be of the form $Qx\theta(x, \tilde{y})$; the induction hypothesis gives

$$A \models \theta(\tilde{y}) \quad \text{iff} \quad \langle A, R \rangle \models \theta^*(\tilde{y}).$$

Assume $A \models Qx\theta(x, \tilde{y})$; we then have $\lambda\{x \mid A \models \theta(x, \tilde{y})\} = 1$. But since $\{x \mid A \models \theta(x, \tilde{y})\}$ is a Borel set with code in $\mathbf{M}[\tilde{y}]$, we have, by the definition of Solovay randomness, $\forall x(R(x, \tilde{y}) \Rightarrow A \models \theta(x, \tilde{y}))$.

Hence, by the induction hypothesis, $\langle A, R \rangle \models \forall x(R(x, \tilde{y}) \rightarrow \theta^*(x, \tilde{y}))$.

On the other hand, if $A \not\models Qx\theta(x, \tilde{y})$, then $\lambda\{x \mid A \models \theta(x, \tilde{y})\} < 1$. Since the set of Solovay random reals over $\mathbf{M}[\tilde{y}]$ has outer measure 1, we have $\exists x(R(x, \tilde{y}) \ \& \ A \models \neg \theta(x, \tilde{y}))$, whence by the induction hypothesis $\langle A, R \rangle \models \exists x(R(x, \tilde{y}) \wedge \neg \theta^*(x, \tilde{y}))$. The lemma is proved.

We are now in a position to verify the validity of R6 in the structure $\langle A, R \rangle$. Suppose $\langle A, R \rangle \models \forall x(R(x, \tilde{y}z) \rightarrow \psi(x, \tilde{y}))$, where ψ is an **IF** formula and z is not free in ψ . Let θ be the formula such that $\theta^* = \psi$. Then, by the lemma, for all x such that $R(x, \tilde{y}z)$ we have $A \models \theta(x, \tilde{y})$. Since $\lambda\{x \mid R(x, \tilde{y}z)\} = 1$, it follows that $A \models Qx\theta(x, \tilde{y})$, whence again by the lemma $\langle A, R \rangle \models (Qx\theta(x, \tilde{y}))^*$; and this is equivalent to $\langle A, R \rangle \models \forall x(R(x, \tilde{y}) \rightarrow \psi(x, \tilde{y}))$. Hence $\langle A, R \rangle$ satisfies $\mathfrak{R}$.

Applying 2.2.2 to $A \models \neg \phi$, we obtain $\langle A, R \rangle \models \neg \phi^*$, and we are done.

Second proof. Let A be a totally Borel model for $\mathfrak{Q} \cup \{\neg \phi\}$. As in the proof of the Consistency Theorem 2.1.3, we lift A to a Boolean valued model on $(2^\omega)^\kappa$, where $\kappa \geq \omega_1$.

Let B be the complete Boolean algebra on $(2^\omega)^\kappa$, where the latter set is equipped with the product topology and product Lebesgue measure λ^κ . Let I be the σ -ideal of λ^κ nullsets. If ψ is a quantifier free formula and $f_1, \dots, f_n$ a sequence of measurable functions: $(2^\omega)^\kappa \rightarrow 2^\omega$, then

$$[[\psi(f_1, \dots, f_n)]]_B = \{\xi \in (2^\omega)^\kappa \mid A \models \psi(f_1(\xi), \dots, f_n(\xi))\}/I.$$

The quantifiers $\forall$ and $\exists$ are interpreted using $\bigwedge$ and $\bigvee$ respectively.

In the proof of 2.1.3, we saw that $[[Qx\psi(x, f_1, \dots, f_n)]]_B$ could be interpreted as

$$[[Qx\psi(x, f_1, \dots, f_n)]]_B = \bigwedge_g \{ [[\psi(g, f_1, \dots, f_n)]]_B \mid R(g, f_1, \dots, f_n) \},$$

where $R(g, f_1, \dots, f_n)$ is defined as follows: $R(g, f_1, \dots, f_n)$ if there is $\alpha < \kappa$ such that $g = \pi_\alpha$ and $\alpha \notin \text{supp}(f_1) \cup \dots \cup \text{supp}(f_n)$, where $\pi_\alpha: (2^\omega)^\kappa \rightarrow 2^\omega$ denotes the projection on the α th coordinate. Hence, if we put $[[R(g, f_1, \dots, f_n)]]_B = 1_B$ if $R(g, f_1, \dots, f_n)$ and equal to 0_B otherwise, we get

$$[[Qx\psi(x, f_1, \dots, f_n)]]_B = \bigwedge_g ([R(g, f_1, \dots, f_n)]_B \rightarrow [[\psi(g, f_1, \dots, f_n)]]_B).$$

Applying the von Neumann measurable selection theorem as in the proof of 2.1.3, we obtain, for any formula ψ in $L \cup \{Q\}$,

$$[[\psi(f_1, \dots, f_n)]]_B = \{ \xi \in (2^\omega)^\kappa \mid \mathcal{A} \models \psi(f_1(\xi), \dots, f_n(\xi)) \} / I.$$

An easy induction then shows that, for ψ in $L \cup \{Q\}$,

$$[[\psi(f_1, \dots, f_n)]]_B = [[\psi^*(f_1, \dots, f_n)]]_B.$$

Hence we have obtained a B -valued model of $\mathfrak{R}$ and $\neg\phi^*$. □

In fact, the second proof of Theorem 2.2.1 proves a slightly stronger result.

COROLLARY 2.2.3. *The addition of*

$$R7: R(x, \tilde{z}) \wedge R(x, \tilde{y}) \rightarrow R(x, \tilde{y}\tilde{z})$$

is conservative over $\mathfrak{R}$ for IF formulas, i.e. it leads to no new identities for Q .

PROOF. The R defined in the second proof satisfies R7. □

COROLLARY 2.2.4. *The addition of*

$$R8: R(x, \tilde{y}) \rightarrow R(x, \tilde{y}\tilde{t}) \text{ (where } \tilde{t} \text{ is a sequence of terms of } L \text{ in variables } \tilde{y})$$

is conservative over $\mathfrak{R}$ for IF formulas, i.e., it leads to no new identities for Q .

PROOF. The R defined in the first proof satisfies R8. □

REMARK. Although we shall not pursue the matter here, we remark in passing that Theorem 2.2.1 can be used to set up a natural deduction system for Q . In natural deduction systems for the predicate calculus, the rules for introduction of $\forall$ and elimination of $\exists$ have side conditions (on the nonoccurrence of variables or constants) which are primitive recursive. By contrast, both the introduction and elimination rules for Q have also side conditions in terms of an independence relation R satisfying $\mathfrak{R}$, but otherwise unspecified. In a sense, then, the rules for Q are more complicated than the rules for $\forall$ and $\exists$. This fact gives some substance to the intuition that probabilistic reasoning (for Q read: “almost surely”) is more complex than ordinary logical reasoning. The subject will be explored in greater detail in van Lambalgen and Moerdijk [199?]. There we shall also comment on the parallel efforts of Fine [1985], who formulates various systems for classical predicate logic in terms of a dependence relation on variables.

2.3. Completeness. The elimination Theorem 2.2.1, together with the Borel completeness Theorem 2.1.2 imply the following completeness theorem for $\mathfrak{R}$.

THEOREM 2.3.1. *Let T be a consistent theory in $L \cup \{R\}$ and suppose the formulas of T are either $\mathfrak{R}$ axioms or IF formulas. Let $\mathbf{M}$ be a countable model of $\mathbf{ZF} + (V = L)$ which contains T . If T has an infinite model, then it has a model in which R is interpreted as Solovay forcing over $\mathbf{M}$.*

PROOF. By hypothesis, T can be identified with a theory in $L \cup \{Q\}$. This theory has a totally Borel model $\mathcal{A}$ by 2.1.2, and we may apply the construction of 2.2.1 to obtain the desired interpretation of R . $\square$

§3. Randomness. Having obtained a theory of independence, we may now formalize von Mises' axioms for random sequences as stated in Definition 1.1. We do this in two stages: first we give a simplified version which highlights the role of independence (3.1), and then we present the full version (3.2). In 3.3 we consider the resulting system $\mathfrak{R}\mathfrak{S}$ from the viewpoint of stochastic processes.

3.1. Axioms for randomness: simplified version. In this subsection, L is a language for second order arithmetic with function variables. In this language, we can express that a sequence satisfies the law of large numbers:

DEFINITION 3.1.1. $\text{LLN}(x) := \forall \varepsilon \exists n_0 \forall n \geq n_0 |n^{-1} \sum_{k=1}^n x_k - \frac{1}{2}| < \varepsilon$.

We could of course define a more general predicate $\text{LLN}(x, p)$, but we take $p = \frac{1}{2}$ throughout for notational convenience. To formulate place selection, we have to introduce a defined operation in the language.

DEFINITION 3.1.2. The partial operation $/: 2^\omega \times 2^\omega \rightarrow 2^\omega$ is defined as follows: $(x/y)_n = x_m$ if m is the index of the n th 1 in y , and undefined if there is no such index. x/y is defined if y has infinitely many ones.

Von Mises' first axiom is expressed by

RS1. $\forall x(R(x) \rightarrow \text{LLN}(x))$.

We first give a weak form of the second axiom, to emphasize the role of independence.

RS2'. $\forall x \forall y \forall z (R(x) \wedge R(x, y) \wedge x/y = z \rightarrow \text{LLN}(z))$.

Observe that, by axiom R2, $R(x)$ can be dropped. This axiom generalizes the case that $\text{LLN}(x/y)$ for random x and *recursive* y ; here, we require only that y has no information about x . Note that RS2' already implies (using R5)

$$\forall x \forall y (R(x) \wedge R(y, x) \rightarrow \text{LLN}(x/y)),$$

which is the formal expression of the third example of admissible place selection (selection by means of a second coin independent of the first).

It may be of interest to observe that the transcendentality axiom R6 allows the formal expression of the first type of admissible place selection, namely, by means of definable y .

For we have, by the same reasoning that was used to motivate R6':

—if t is a closed term of L , $\forall x \forall z (R(x) \wedge x/t = z \rightarrow \text{LLN}(z))$; and more generally,

—if T is a theory in the language of second order arithmetic, and $\phi(x)$ is algebraic in T (that is, in any model of T $\phi(x)$ has only finitely many solutions), then $\forall x \forall y \forall z (R(x) \wedge \phi(y) \wedge x/y = z \rightarrow \text{LLN}(z))$. For example, if T is RCA_0 , we can prove in $T + \mathfrak{R} + \text{RS1}, 2'$ that if x is random and y recursive, then x/y satisfies the law of large numbers.

3.2. Axioms for randomness. Von Mises' description of admissible place selection, "Auswahl ohne Benützung der Merkmalunterschiede", is purely negative, hence difficult to formalize. The main idea of the axiomatization given here is that the negative condition "the decision to select x_n does not depend upon the value of x_n " is replaced by the positive condition "the decision to select x_n depends

on (at most) the initial segment $x(n-1)$ and (perhaps) on some data *which do not contain information about x* . The italicized phrase can be made precise by means of the independence relation introduced in 1.1.

In other words, admissible place selection can be viewed as a “machine” which takes as input the random sequence x and additional (*independent*) information represented by an infinite binary sequence z ; this machine produces an output y in such a way that the “computation” of $y(n)$ uses at most the first $n-1$ bits of x , and an arbitrary, but finite, number of bits of z . The output y determines the subsequence x/y of x . We may think of the oracle z as produced by a stochastic mechanism such as coin tossing, but there are other possibilities, e.g. choice sequences.

The essential property of these “machines” is isolated in part (1) of the following definition:

DEFINITION 3.2.1. A (possibly partial) continuous function $F: 2^\omega \times 2^\omega \rightarrow 2^\omega$ is *admissible* if (1) for all n and all x, x' and y : $x(n-1) = x'(n-1)$ and $F(x, y), F(x', y)$ defined implies $F(x, y)(n) = F(x', y)(n)$; and (2) $\lambda\{y \mid F(x, y) \text{ defined}\} = 1$ for all x , where $\lambda = (\frac{1}{2}, \frac{1}{2})^\omega$ is Lebesgue measure on 2^ω .

Part (2) of the definition might seem strange at first sight, but it is not difficult to see that an F not satisfying (2) can be extended to an F' which does satisfy (2) and such that $\text{range}(F) = \text{range}(F')$.

The three examples of admissible selection mentioned after Definition 1.1 can be represented as admissible functions. There exist, however, more general ways to construct admissible place selections.

Let $\phi: 2^{<\omega} \rightarrow \{0, 1\}$ be any function; then define an admissible function $F: 2^\omega \times 2^\omega \rightarrow 2^\omega$ as follows:

$$F(x, z)_n = \begin{cases} 0 & \text{if } \phi(x(n-1)) = 0, \\ z_n & \text{otherwise.} \end{cases}$$

F may be used to represent the following process: to decide whether to choose x_n , we first “compute” $\phi(x(n-1))$; if the result is 1, we then toss a fair coin and choose x_n if the outcome is heads. The resulting subsequence of x is $x/F(x, z)$ (when $F(x, z)$ has infinitely many ones), and the selection procedure may be viewed as a combination of lawlike and random selection. Intuitively it seems clear that the combination of an admissible function with some mechanism generating y is a general description of admissible place selection, since all information additional to x can be coded into one infinite binary sequence y . For instance, it would not make a difference if the decision to select x_n depends on tosses with several coins. There is, however, one possibility of generalizing the description, by varying the order in which the x_n are chosen, so that, for some $k > n$, x_k may be chosen before x_n . We shall not consider this possibility here. Barring this possibility, we have in fact a kind of completeness theorem: in a sense to be made precise in §3.3, *any* admissible selection procedure can be seen as a combination of an admissible function together with a random oracle.

To formulate the final axiom, we need a slight extension of the language of second order arithmetic used in the previous subsection. In addition to the sorts for ω and 2^ω we use a subsort of 2^ω consisting of the “lawlike” or “given” elements of 2^ω . One

may think of the latter sort as being of the form $2^\omega \cap \mathbf{M}$, where $\mathbf{M}$ is a countable model of second order arithmetic. Corresponding to these three sorts, we have quantifiers $\forall n, m, \dots$ over the natural numbers, $\forall x, y, \dots$ over 2^ω , and $\forall a, \forall F$ over the lawlike objects. The choice of this language is intended to bring out that we regard admissible selection procedures as “given”, whereas random sequences are considered to be “incomplete” objects (cf. also the Boolean valued model constructed in §4).

Observe that admissible functions in the sense of 3.2.1, being continuous, can be coded into an infinite binary sequence; furthermore the condition of admissibility itself can be formulated in second order arithmetic. We may therefore state the final axiom as follows:

RS2. $\forall F (F \text{ admissible} \rightarrow \forall x \forall y \forall z \forall z' (F(x, z) = y \wedge R(x, z) \wedge x/y = z' \rightarrow \text{LLN}(z')))$.

The system consisting of the axioms R0–R6, RS1 and RS2 will be denoted $\mathfrak{R}\mathfrak{S}$.

3.3. Admissible place selections as stochastic processes. In 3.2 we gave an informal argument to the effect that admissible place selection can be represented by an admissible function in the sense of Definition 3.2.1, together with an oracle satisfying an independence condition. Here, we prove a kind of informal completeness theorem: we show that, *in so far as admissible place selection can be represented as a stochastic process*, RS2 is indeed an exhaustive description of the situation.

An admissible selection procedure, viewed as a stochastic process, can be represented as a measure on 2^ω . Because the condition of admissibility entails the dependence of the measure on the random sequence x , the admissible place selection must in fact be represented as a parametrized measure.

DEFINITION 3.3.1. For $w \in 2^{<\omega}$, $[\omega] := \{x \mid x(|w|) = w\}$, where $|w|$ is the length of w .

DEFINITION 3.3.2. A *random measure* on 2^ω is a family of finite measures $\{\mu_x \mid x \in 2^\omega\}$, where μ_x is defined on 2^ω , such that for each w the function $x \rightarrow \mu_x[w]$ is Borel measurable.

There seems to be only one reasonable definition for a measure to represent an admissible selection process, if the decision to select x_n is determined only by past values of x .

DEFINITION 3.3.3. A family of probability measures $\{\nu_x \mid x \in 2^\omega\}$ is called *admissible* if for all x and x' and all n the following holds: if $x(n-1) = x'(n-1)$, then for all w , $|w| = n$, we have $\nu_x[w] = \nu_{x'}[w]$. (If a family of measures $\{\nu_x \mid x \in 2^\omega\}$ is admissible, we have automatically that for all w , the function $x \rightarrow \nu_x[w]$ is Borel measurable. Hence admissibility is a strengthening of the random measure condition.)

The intuitive picture behind the definition is that the properties of the process at time n are completely determined by the values of x up to time $n-1$. A few examples will make this clearer.

EXAMPLE 3.3.4. The admissible place selections mentioned after Definition 1.1 can be represented by admissible random measures:

(a) ν_x is concentrated on the sequence $z \in 2^\omega$ defined by $z_n = 1$ iff n is prime (that is, $\nu_x[z(n)] = 1$ for all n and x).

(b) ν_x is concentrated on the sequence $z^x \in 2^\omega$ defined by: $z_n^x = 1$ iff $x_{n-9}, \dots, x_{n-1}$ are all equal to 1 (this means that $\nu_x[z^x(n)] = 1$ for all n and x).

(c) $\nu_x = \lambda$ for all x .

(d) More interesting examples are constructed as follows. If F is an admissible function in the sense of 3.2.1, then $\{v_x \mid x \in 2^\omega\}$ defined by $v_x := \lambda F_x^{-1}$ is an admissible random measure. (There is nothing special about λ here: any probability measure would do.)

The next theorem is a converse to Example 3.3.4(d).

THEOREM 3.3.5. *Let $\{v_x\}$ be an admissible random measure. Then there exists an admissible function $F: 2^\omega \times 2^\omega \rightarrow 2^\omega$ such that $v_x = \lambda F_x^{-1}$.*

PROOF. By admissibility, $v_x[0]$ and $v_x[1]$ do not depend on x . Choose disjoint open sets $A_0^{\langle \cdot \rangle}, A_1^{\langle \cdot \rangle}$ such that $\lambda A_0^{\langle \cdot \rangle} = v_x[0]$ and $\lambda A_1^{\langle \cdot \rangle} = v_x[1]$. (One of the $A_0^{\langle \cdot \rangle}$ or $A_1^{\langle \cdot \rangle}$ may be empty, namely when either $v_x[0]$ or $v_x[1]$ equals zero.) Then F will be defined such that $\forall i \forall x \forall y \in A_i^{\langle \cdot \rangle} (F(x, y) \in [i])$. Again by admissibility, for $|w| = 2$, $v_x[w]$ depends only on x_1 . Choose disjoint open subsets A_{j0}^i, A_{j1}^i of $A_j^{\langle \cdot \rangle}$ such that $\lambda A_{jk}^{x_1} = v_x[jk]$, for i, j, k in $\{0, 1\}$. F will be defined such that

$$\forall i \forall x \in [i] \forall w (|w| = 2 \rightarrow \forall y \in A_w^i (F(x, y) \in [w])).$$

Continuing in this way, we construct a sequence of open sets $\{A_w^v \mid v, w \in 2^{<\omega}, |v| = |w| - 1\}$ with the following properties:

1. For $i \in \{0, 1\}$ and $|v| = |w| - 1$, A_{w0}^{vi} and A_{w1}^{vi} are disjoint open subsets of A_w^v .
2. If $m = |w| - 1$, $\lambda A_w^{x(m)} = v_x[w]$.

Put $W := \{\langle x, y, w \rangle \mid \exists n (y \in A_w^{x(n)})\}$; then we can define F by

$$F(x, y) = z \equiv \forall n (\langle x, y, z(n) \rangle \in W).$$

Obviously F is continuous on a domain of full measure and satisfies

3. $\forall v \forall x \in [v] \forall w (|w| = |v| + 1 \rightarrow \forall y \in A_w^v (F(x, y) \in [w]))$.

It follows from 3 that F is admissible. □

The preceding theorem shows that the examples of admissible place selection given after Definition 1.1 in a sense constitute a complete description: every admissible selection can be viewed as a combination of lawlike and random selection. It also follows from the theorem that the oracle can be taken to be independent of the random sequence in a strong sense: we can think of the oracle as being generated by a fair coin. It must be emphasized, however, that this is true only on the assumption that admissible selections can be represented stochastically.

§4. Consistency of $\mathcal{R}\mathfrak{S}$ in second order arithmetic. A probability theory in the style of von Mises consists of a theory of random sequences together with a base theory which allows one to manipulate infinite sequences and real numbers. For the base theory one may take second order arithmetic.

Consider again the three-sorted language L of §3, with sorts for ω , 2^ω and the lawlike elements of 2^ω . In this language, second order arithmetic Z_2 has the following form:

- basic arithmetic, i.e. defining equations for 0, S , $+$ and $\cdot$;
- an induction scheme: $\phi(0) \wedge \forall n (\phi(n) \rightarrow \phi(n+1)) \rightarrow \forall n \phi(n)$, where ϕ is a formula of L ; and
- 2 comprehension schemes:
 - $\exists a \forall n (a_n = 1 \leftrightarrow \phi(n))$ if ϕ has only numerical and lawlike parameters,
 - $\exists x \forall n (x_n = 1 \leftrightarrow \phi(n))$ for arbitrary ϕ in L .

We now construct a Boolean valued model for $Z_2 + \mathcal{R}\mathfrak{S}$. We slightly alter the construction used in the proof of 2.1.3. 2^ω is again represented by the set of Borel

measurable functions $(2^\omega)^\kappa \rightarrow 2^\omega$. The sort of lawlike elements is represented by the *constant* functions $(2^\omega)^\kappa \rightarrow 2^\omega$. ω is represented by itself (alternatively we could have taken functions $(2^\omega)^\kappa \rightarrow \omega$).

The Boolean interpretations of the basic arithmetical relations are as follows.

- $[[n + m = k]] = 1_B$ if $n + m = k$ and 0_B otherwise; likewise for 0 , S and $\cdot$;
- if $a: (2^\omega)^\kappa \rightarrow 2^\omega$ is constant (i.e. lawlike), we put $[[a_n = 1]] = 1_B$ if n is in the range of a ; and 0_B otherwise; and
- if $f: (2^\omega)^\kappa \rightarrow 2^\omega$ is measurable, then $[[f_n = 1]] = \{\xi \in (2^\omega)^\kappa \mid f(\xi)_n = 1\}/I$.

The independence relation R is interpreted as in the proof of 2.1.3; and the quantifiers and connectives have the obvious interpretation.

The Boolean validity of Z_2 now follows. The validity of basic arithmetic and the induction scheme is trivial. To establish lawlike comprehension, it suffices to note that the Boolean algebra B is homogeneous, so that for any ϕ with only numerical and lawlike parameters, $[[\phi]]$ is either 0 or 1 ; for then we may define a lawlike $a: (2^\omega)^\kappa \rightarrow 2^\omega$ by putting, for any ξ , $a(\xi)_n = 1$ iff $[[\phi(n)]]_B = 1_B$. Finally, the validity of the second comprehension scheme is shown by defining a measurable $f: (2^\omega)^\kappa \rightarrow 2^\omega$ such that $[[f_n = 1]]_B = [[\phi(n)]]_B$.

We now turn to the validity of $\mathfrak{R}\mathfrak{S}$ in this model. That $\mathfrak{R}$ is valid was shown in 2.1. Consider first $\mathfrak{R}\mathfrak{S}1$:

$$[[\forall x(Rx \rightarrow \text{LLN}(x))]]_B = \bigwedge_{\pi} [[\text{LLN}(\pi)]]_B,$$

where π ranges over the projections $(2^\omega)^\kappa \rightarrow 2^\omega$. But since the quantifiers in the defining expression for LLN have standard interpretation and the projections are measure preserving, we have

$$[[\text{LLN}(\pi)]]_B = \{\xi \in (2^\omega)^\kappa \mid \text{LLN}(\pi\xi)/I = 1_B\} \text{ for any } \pi.$$

The verification of $RS2$ is more involved. We have to show that

$$[[\forall F(F \text{ admissible} \rightarrow \forall x \forall y \forall z \forall z'(F(x, z) = y \wedge R(x, z) \wedge x/y = z' \rightarrow \text{LLN}(z')))]_B = 1_B.$$

This follows if we can show that, for any admissible function F and any projection π and $f: (2^\omega)^\kappa \rightarrow 2^\omega$ such that $\text{supp}(\pi) \cap \text{supp}(f) = \emptyset$,

$$[[\text{LLN}(\pi/F(\pi, f))]]_B = 1_B.$$

LEMMA 4.1. *Let $f: (2^\omega)^\kappa \rightarrow 2^\omega$ be measurable, so that f can be identified with a function $(2^\omega)^\omega \rightarrow 2^\omega$. Then $\lambda \times \lambda^\omega \{\langle x, \zeta \rangle \in 2^\omega \times (2^\omega)^\omega \mid F(x, f(\zeta)) \text{ defined, } x/F(x, f(\zeta)) \text{ defined and } \in A\} \leq \lambda A \text{ for all Borel sets } A$.*

PROOF. For the sake of brevity we write

$$\lambda \times \lambda^\omega \{\langle x, \zeta \rangle \in 2^\omega \times (2^\omega)^\omega \mid x/F(x, f(\zeta)) \in A\}$$

instead of $\lambda \times \lambda^\omega \{\langle x, \zeta \rangle \in 2^\omega \times (2^\omega)^\omega \mid F(x, f(\zeta)) \text{ defined, } x/F(x, f(\zeta)) \text{ defined and } \in A\}$. We then have

$$\lambda \times \lambda^\omega \{\langle x, \zeta \rangle \in 2^\omega \times (2^\omega)^\omega \mid x/F(x, f(\zeta)) \in A\} \leq \int_{(2^\omega)^\omega} \lambda \{x \mid x/F(x, f(\zeta)) \in A\} d\lambda^\omega(\zeta).$$

By the properties of F , for fixed ζ the function $\bullet/F(\bullet, f(\zeta)): 2^\omega \rightarrow 2^\omega$ is a place selection in the sense that the choice of x_n depends only on $x(n-1)$. Hence, by a

result of Doob (see van Lambalgen [1987a, p. 758]), we have $\lambda\{x \mid x/F(x, f(\zeta)) \in A\} \leq \lambda A$ for each ζ ; and the result follows by integrating over $(2^\omega)^\omega$. $\square$

Since $\lambda\{x \mid \text{LLN}(x)\} = 1$, we have

$$\begin{aligned} \lambda^\kappa\{\xi \in (2^\omega)^\kappa \mid F(\pi\xi, f(\xi)) \text{ defined, } \pi\xi/F(\pi\xi, f(\xi)) \text{ defined and } \text{LLN}(\pi\xi/F(\pi\xi, f(\xi)))\} \\ = \lambda \times \lambda^\omega\{\langle x, \zeta \rangle \in 2^\omega \times (2^\omega)^\omega \mid \text{LLN}(x/F(x, f(\zeta)))\} = 1. \end{aligned}$$

Hence $Z_2 + \mathfrak{R}\mathfrak{S}$ is consistent.

§5. Recursion theoretic interpretations of $\mathfrak{R}$ and $\mathfrak{R}\mathfrak{S}$. With the axiom system $\mathfrak{R}\mathfrak{S}$, we have tried to isolate some essential features of randomness and independence. We will now investigate how these axioms are related to the most widely known explicit definitions: in terms of recursive sequential tests (Martin-Löf [1966]) and the definition using the variant of Kolmogorov complexity called prefixcomplexity (cf. Kolmogorov and Uspensky [1988] and Chaitin [1987]). Since these approaches are essentially equivalent (compare Theorem 5.4 below), we shall concentrate on prefixcomplexity.

The definitions of Martin-Löf and Kolmogorov differ essentially from that of von Mises in that they do not presuppose a notion of independence. Our first task will therefore be to introduce an explicit notion of independence using prefixcomplexity. It turns out that we can take the notion defined in van Lambalgen [1987a]. We first give a quick introduction to prefixcomplexity and Martin-Löf's definition of randomness. For more information see van Lambalgen [1987a].

DEFINITION 5.1 (KOLMOGOROV AND USPENSKY [1988]; CHAITIN [1987]). A set $E \subseteq 2^{<\omega}$ is called *prefixfree* if no $w \in E$ is an initial segment of some $v \in E$. A *prefixalgorithm* is a partial recursive function $A: 2^{<\omega} \rightarrow 2^{<\omega}$ which has prefixfree domain. Let $A: 2^{<\omega} \rightarrow 2^{<\omega}$ be a prefixalgorithm with Gödelnumber $\ulcorner A \urcorner$. We define a universal prefixalgorithm U by the following condition: on inputs of the form $q = 0^{\ulcorner A \urcorner}1p$, U simulates the action of A on p . The *prefixcomplexity* $I(w)$ is defined as $I(w) := \{|p| \mid U(p) = w\}$.

DEFINITION 5.2. $x \in 2^\omega$ is *irregular* if $\exists m \forall n I(x(n)) > n - m$.

DEFINITION 5.3 (MARTIN-LÖF [1966]). A *recursive sequential test* N is a Π_2^0 subset of 2^ω , $N = \bigcap_n O_n$, O_n in Σ_1^0 , such that $\lambda O_n \leq 2^{-n}$ for all n . A sequence $x \in 2^\omega$ is *random* (denoted $x \in R$) if it is not contained in any recursive sequential test. (Obviously $\lambda R = 1$.)

THEOREM 5.4. $\exists m \forall n I(x(n)) > n - m$ iff $x \in R$ (in other words, randomness (5.3) and irregularity (5.2) coincide).

REMARK. It should be noted that the analogue of 5.4 for ordinary Kolmogorov complexity K is false; as Martin-Löf [1971] has shown, no sequence x satisfies $\exists m \forall n K(x(n)) > n - m$.

The following alternative characterization of Martin-Löf randomness is due to Solovay.

THEOREM 5.5. $x \in R$ iff for any r.e. sequence of Σ_1^0 sets O_n such that $\sum \lambda O_n < \infty$ one has $x \notin \bigcap_n O_n$.

We are now in a position to define a complexity-theoretic independence relation. Let I^y be prefixcomplexity as defined above, except that we now use the oracle y , thus replacing the universal Turing machine U by U^y . Similarly, we may relativize Martin-Löf's randomness concept R (4.3) to R^y . Relativizing the proof of Theorem 5.4, we obtain

THEOREM 5.4'. (a) $\exists m \forall n \ I^y(x(n)) > n - m$ iff $x \in R^y$; and hence (b) $\lambda\{x \mid \exists m \forall n \ I^y(x(n)) > n - m\} = 1$.

The promised complexity-theoretic characterisation of independence is given by

DEFINITION 5.6. If x is random (or irregular, cf. 5.4), then y has no information about x if $\exists m \forall n \ I^y(x(n)) > n - m$. Similarly for $\bar{y}$.

The following two theorems are proved in van Lambalgen [1987a]. The first shows that almost no sequence contains information about a given random sequence x , whereas the second shows that axiom RS2' (cf. §1.2) is valid for the notion of independence introduced above.

THEOREM 5.7. If x is random, then $\lambda\{y \mid y \text{ has no information about } x\} = 1$.

THEOREM 5.8. If x is random and y has no information about x and contains infinitely many ones, then x/y is also random. A fortiori, $\text{LLN}(x/y)$.

The main question now confronting us is: does the relation of independence defined in 5.6 satisfy $\mathfrak{R}\mathfrak{S}$? R1–R4 are trivially satisfied, but R5 is not so easy. In §2 we saw that R5 could be used to interpret the “Fubini axiom” Q6 for Friedman’s quantifier “almost all”. The validity of R5 for our present notion of independence is also connected to Fubini’s theorem, but in this case we have to use the effective analogue of Fubini proved in van Lambalgen [1987b].

First some notation. If A is a subset of $2^\omega \times 2^\omega$, we define $(A)^y := \{x \mid \langle x, y \rangle \in A\}$ and $(A)_x := \{y \mid \langle x, y \rangle \in A\}$ (the more customary notation A^y being used for relativization).

THEOREM 5.9. Let $N \subseteq 2^\omega \times 2^\omega$ be a recursive sequential test with respect to $\lambda \times \lambda$, where $N = \bigcap_n O_n$, O_n in Σ_1^0 , such that $\lambda O_n \leq 2^{-n}$ for all n . Then we have:

(a) $y \in R$ implies $\sum_{n=0}^\infty \lambda(O_n)^y < \infty$.

(b) (Effective Fubini) $\{y \mid \lambda(N)^y > 0\}$ is contained in a recursive sequential test (with respect to λ).

PROOF. We first prove (b) from (a). It suffices to show that if $y \in R$, then $\lambda(\bigcap_n O_n)^y = 0$. We have

$$\left(\bigcap_n O_n\right)^y = \left(\bigcap_n \bigcup_{k \geq n} O_k\right)^y = \bigcap_n \bigcup_{k \geq n} (O_k)^y$$

and

$$\lambda \bigcup_{k \geq n} (O_k)^y \leq \sum_{k=n}^\infty \lambda(O_k)^y \rightarrow 0 \quad \text{for } k \rightarrow \infty.$$

It remains to prove (a). Let $O_n = \bigcup_i [w_{ni}] \times [v_{ni}]$. Define for each n a sequence of functions f_{nk} , where $k \geq 0$, as follows: $f_{n0}(y) = 0$ for all y , and

$$f_{nk}(y) = \sum \{\lambda[w_{ni}] \mid y \in [v_{ni}], i \leq k\}, \quad k \geq 1.$$

The f_{nk} are computable step functions, and $f_{nk} \leq f_{n(k+1)}$; moreover $f_{n(k+1)}(y) - f_{nk}(y) = \lambda[w_{n(k+1)}]$ for $y \in [v_{n(k+1)}]$, and $f_{n(k+1)}(y) - f_{nk}(y) = 0$ otherwise.

Clearly

$$\int f_{nk} d\lambda = \sum_{i=1}^k \int (f_{ni} - f_{n(i-1)}) d\lambda = \sum_{i=1}^k \lambda[w_{ni}] \cdot \lambda[v_{ni}] \leq 2^{-n}.$$

Define $B_{nm} = \{y \mid \exists k \leq m \ f_{nk}(y) > (n+1)^{-2}\}$ and $B_n = \{y \mid \exists k \ f_{nk}(y) > (n+1)^{-2}\}$. Obviously $B := \bigcap_n B_n$ is Π_2^0 . Moreover $\lambda B_n \leq (n+1)^2 \cdot 2^{-n}$: since $f_{nk} \leq f_{n(k+1)}$, we

must have, for all m ,

$$\lambda B_{nm} = \int 1_{B_{nm}} d\lambda < (n+1)^2 \int f_{nm} d\lambda \leq (n+1)^2 \cdot 2^{-n}.$$

Hence B is (contained in) a recursive sequential test.

Since the functions f_{nk} are nondecreasing in k , we can define

$$f_n(y) = \lim_{k \rightarrow \infty} f_{nk}(y).$$

We then have

$$\sum_{n=0}^{\infty} \lambda(O_n)^y \leq \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \{\lambda[w_{ni}] \mid y \in [v_{ni}]\} = \sum_{n=0}^{\infty} f_n(y).$$

Now if $y \in R$, then $y \notin B$; hence

$$\sum_{n=0}^{\infty} \lambda(O_n)^y \leq \sum_{n=0}^{\infty} f_n(y) \leq \sum_{n=1}^{\infty} n^{-2} < \infty. \quad \square$$

THEOREM 5.10. *If $\exists m \forall n I^y(x(n)) > n - m$ and y is random, then*

$$\exists m \forall n I^x(y(n)) > n - m.$$

PROOF. Let R^y denote Martin-Löf's randomness notion relativized to y ; hence (5.4') $x \in R^y$ iff $\exists m \forall n I^y(x(n)) > n - m$. Let R^2 denote Martin-Löf's randomness notion in $2^\omega \times 2^\omega$ (with respect to $\lambda \times \lambda$), and let $(R^2)^y$ and $(R^2)_x$ be the sections of R^2 as defined above.

It suffices to show that $x \in R^y$ & $y \in R \rightarrow \langle x, y \rangle \in R^2$. For if $\langle x, y \rangle \in R^2$, then $y \in (R^2)_x$, and the following argument shows that we must have $(R^2)_x \subseteq R^x$.

The proof of Theorem 5.4 establishes that $\{y \mid \forall m \exists n I(y(n)) \leq n - m\}$ is a recursive sequential test (and in fact the *universal* recursive sequential test). It follows by relativizing that proof to y that $\{\langle y, x \rangle \mid \forall m \exists n I^x(y(n)) \leq n - m\}$ is a recursive sequential test with respect to $\lambda \times \lambda$: clearly, $\{\langle y, x \rangle \mid \forall m \exists n I^x(y(n)) \leq n - m\}$ is Π_2^0 , and, for each m ,

$$\begin{aligned} \lambda \times \lambda \{\langle y, x \rangle \mid \exists n I^x(y(n)) \leq n - m\} &= \int \lambda \{y \mid \exists n I^x(y(n)) \leq n - m\} d\lambda \\ &\leq \int 2^{-n} d\lambda = 2^{-n}. \end{aligned}$$

Hence $\{\langle y, x \rangle \mid \forall m \exists n I^x(y(n)) \leq n - m\}$ is contained in $(R^2)^c$, and therefore $(R^2)_x \subseteq R^x$.

So suppose $y \in R$ and $\langle x, y \rangle \notin R^2$. Since $(R^2)^c$ is a universal recursive sequential test (see Martin-Löf [1966]), it is of the form $\bigcap_n O_n$, O_n in Σ_1^0 , such that $\lambda O_n \leq 2^{-n}$ for all n . Now if $\langle x, y \rangle \notin R^2$, then $x \in ((R^2)^c)^y = \bigcap_n (O_n)^y$. We have assumed $y \in R$; hence by part (a) of the preceding theorem $\sum_{n=0}^{\infty} \lambda(O_n)^y$ converges. By using the relativized version of 5.5, we obtain $x \notin R^y$. $\square$

Theorem 5.10 is perhaps interesting in its own right. As an illustration, we shall give an application to Turing degrees. Let $\leq_T$ denote Turing reducibility. We prove a weaker version of Sacks' result that $\lambda\{x \mid y \leq_T x\} = 0$ for nonrecursive y . Suppose y

is a random sequence and $y \leq_T x$. We then have $\forall m \exists n I^y(x(n)) \leq n - m$, for if $\exists m \forall n I^y(x(n)) > n - m$, in virtue of 5.10 we would also have $\exists m \forall n I^x(y(n)) > n - m$, which is impossible because $y \leq_T x$. Hence it follows from 5.4' that $\lambda\{x \mid y \leq_T x\} = 0$ for random y .

In particular, we have $\lambda\{x \mid \emptyset' \leq_T x\} = 0$, for $\emptyset'$ contains the random sequence Ω , the halting probability of the universal Turing machine as defined in Chaitin [1987].

We now return to the relation between independence as defined in 5.6 and $\mathfrak{RS}$. The validity of RS1 and RS2' for our notion of independence follows from Theorems 5.4 and 5.8 respectively. Moreover, if the admissible function F is Σ_1^0 on its domain, we also have $[x \in R \ \& \ y \text{ has no information about } x \ \& \ F(x, y) \text{ is defined and has infinitely many ones} \rightarrow x/F(x, y) \in R]$. This is so because for such F , $F(\cdot, y)$ is a place selection recursive in y . Summarizing the discussion so far, we have

THEOREM 5.11. *The independence relation defined in 5.6 satisfies R1–R5, RS1 and RS2 when the quantifier $\forall F$ is restricted to F which are Σ_1^0 on their domain.*

It can also be shown that R6 is true when restricted to the Σ_2^e formulas as defined by Gaifman and Snir [1983], but we shall not go into the details.

5.12. Concluding remarks. In accordance with von Mises' slogan "Erst das Kollektiv [i.e. random sequence], dann die Wahrscheinlichkeit", $\mathfrak{RS}$ does not refer to a probability measure, only to limiting relative frequency. This contrasts with the (more common) approach that takes random sequences to be sequences that satisfy all ("effective") strong limit laws of probability theory. It will be observed that $\mathfrak{R}$, the independence part of $\mathfrak{RS}$, is complete with respect to a semantics in which randomness is indeed defined as satisfaction of a large class of properties of Lebesgue measure 1 (2.3.1). Does this mean that, after all, the proper concept of randomness is "satisfaction of all effective strong limit laws of probability theory"? It does not, because $\mathfrak{RS}$ by no means singles out a unique measure. $\mathfrak{R}$ can be interpreted using any continuous Borel measure, and to interpret RS1 and RS2 we only need a product measure $\prod_n (1 - p, p_n)$ such that $p_n \rightarrow \frac{1}{2}$ for $n \rightarrow \infty$. Therefore Ville's theorem, which says that Church random sequences need not satisfy the law of the iterated logarithm, has an analogue in our situation: it does not follow from $\mathfrak{RS}$ that $\forall x (R(x) \rightarrow x \text{ satisfies the law of the iterated logarithm})$, as can be shown using the arguments in §4 of van Lambalgen [1987a].

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FACULTEIT WISKUNDE EN INFORMATICA

UNIVERSITEIT VAN AMSTERDAM

1018 TV AMSTERDAM, THE NETHERLANDS