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Moments, time-dependent rebates and the put-call parity for barrier options

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Abstract

This paper derives put-call parity relations for barrier options via a probabilistic approach. As in the case of standard options, the difference between call and put prices depends on the stock price and the discounted exercise price. However, these terms now become functions of the moments of the relevant defective distributions and as such correct for the loss of probability that the barrier provisions bring. Moments are also useful building blocks for the valuation of a wide variety of time-dependent rebate specifications that are interesting both from a theoretical as well as a practitioner's point of view. For instance, rebate payments that are larger when the option is knocked out later recognize the higher opportunity costs caused by the option having immobilized funds over a longer period of time.

Keywords: barrier options, first-hitting time, moments, option pricing, put-call parity, rebate, transition density.

JEL classification: G13.

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1 Introduction

The put-call parity of Stoll (1969) establishes a simple arbitrage relation between the value of European call and put options that share the same exercise price and time to maturity. The underlying static hedge portfolio, however, requires that the put and the call exist both at the current point of time as well as at the maturity date. This parity then cannot apply to barrier options as discussed in, for instance, Kunitomo and Ikeda (1992) since knock-out options are cancelled when the underlying asset price hits the barrier and knock-in options only create standard options when the asset touches the barrier.

This paper employs a probabilistic approach in order to derive the relationship between prices of put and call barrier options of the same type (knock-in or knock-out), with identical exercise price, barrier level and time to maturity. The put-call parities for barrier options, similar to the parity for standard options, also express the difference between put and call prices in terms of the discounted exercise price and the stock price. However, the latter terms now are explicitly linked to the zeroth and first moments, respectively, of the relevant defective distribution of the future stock price. This modification of the standard put-call parity has a straightforward intuition. In fact, the connection to the moments essentially corrects the put-call parity of Stoll (1969) for the loss of probability mass (zeroth moment) and conditional expectation (first moment) that is associated with potential option cancellation (knock-out options) and the possible failure to be created (knock-in options). This also implies that the put-call parity for standard options must emerge as a limit case of the parity for barrier options. In fact, letting the lower (upper) barrier decrease (increase) to 0 ($+\infty$) produces the familiar put-call parity as the probability loss is reduced to zero.

Moments are not only useful for the derivation of put-call parities, but also facilitate pricing of the rebate provisions that often accompany barrier options. Rebates compensate the owner of the option for potential cancellation of the option or the lack of creation of a standard option. Typically, rebates are stipulated as fixed cash payments (see, for instance, Rubinstein and Reiner, 1991a), although more elaborate and/or time-depending specifications could also be of interest to investors. For instance, investors may want the rebate of a knock-in option to depend on market conditions at the maturity date in order to let it be larger the further the stock price is situated below the upper barrier. Also, higher rebates when the option is knocked out later would bring a larger compensation in line with the fact that the option has immobilized funds over a longer period. Such alternative rebates are easily valued via moment expressions for the distributions of the future stock price and the first-hitting time.

This paper derives in total 8 put-call parities for single-barrier options that have rebate provisions.

It also specifies the related 8 in-out parities, i.e. the relation between in- and out-options that are otherwise identical. These parities then are used to obtain all 16 single-barrier option prices (call/put, knock-in/knock-out, upper/lower barrier, exercise price above/below the barrier) on the basis of only 4 explicit prices. This allows us to reproduce the barrier option prices for fixed-sum rebates that are derived in Rubinstein and Reiner (1991a) in a simple and intuitive framework and then extend them towards including novel rebate specifications.

The remainder of the paper is organized as follows. Section 2 specifies the relevant densities for the underlying geometric Brownian motion environment and derives all required moments. Section 3 illustrates how the moment expressions naturally emerge as building blocks for the valuation of various (time-dependent) rebates. The put-call parities for barrier options are derived in Section 4. Section 5 extends the familiar in-out parities towards contracts with rebate provisions and Section 6 replicates all 16 barrier option prices via the various moment expressions, put-call and in-out parities. Section 7 concludes. The Appendix derives the closed-form solutions to 4 integrals that are used throughout the paper. The 2 solutions that are based on the largely unknown results of Cho (1971) moreover may well be of interest for various other (option) valuation exercises in which first-hitting time problems arise.

2 The density functions and moments

The stock price is assumed to follow a risk-neutralized geometric Brownian motion with drift and diffusion coefficients rS_t and $\sigma^2 S_t^2$, respectively:

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

where r is the risk-free interest rate and dW denotes the increment of a Wiener process. We restrict attention in this paper to a single constant lower or upper barrier at the level H .

The various parities and rebate prices will be obtained in terms of the moments of the distributions of the future stock price and the time until H is reached, i.e. of the first-hitting time. We first derive the required probabilistic concepts for the future stock price after which we obtain the relevant quantities for the first-hitting time.

2.1 The moments for the probability distribution of the future stock price

The n -th moment around zero, in short the n -th moment, for the continuous stochastic variable x with domain (β_1, β_2) is defined as (see Mood, Graybill and Boes, 1974):

$$M_n = \int_{\beta_1}^{\beta_2} x^n p[x] dx,$$

where $p[x]$ is the transition probability density function. The zeroth moment is the integral of the transition density over (β_1, β_2) . It specifies the total probability mass that is located between β_1 and β_2 and thus excludes the latter two points as we will discuss in more detail below. The first moment defines the conditional mean or conditional expectation over the domain (β_1, β_2) .

The transition probability density function for the future stock price $p[S_T, T; S_t, t]$, in short the transition density, for the unrestricted geometric Brownian motion is the well-known log-normal density function:

$$p[S_T, T; S_t, t] = \frac{1}{S_T \sqrt{2\pi\sigma^2\Delta}} \exp \left[-\frac{(\ln S_T - \ln S_t - (r - \frac{1}{2}\sigma^2)\Delta)^2}{2\sigma^2\Delta} \right], \quad (1)$$

where Δ denotes the time interval $T - t$. The function $p[S_T, T; S_t, t]$ specifies the probability of attaining S_T at time T given that the process currently, i.e. at time t , is at the source point S_t .

Barrier options, however, are to be valued by conditioning the transition density on not reaching the barrier H from below or from above.¹ For instance, down-and-out options are cancelled when the stock price hits H from above such that all paths that touch H must be excluded. Moreover, we also need to ascertain that the underlying process cannot spend time on the barrier H since cancellation of the option at touching H logically implies that the barrier should bring no probability and conditional expectation. We therefore must employ transition densities for which H is a killing barrier such that the stochastic variable is immediately sent to the so-called cemetery state upon first hitting of the barrier H (see Knight, 1981; Borodin and Salminen, 2002). This guarantees both that all paths that touch H are ended and that the underlying cannot spend finite time on the barrier. As a result, we study defective transition densities since the killing boundary prevents the transition density from integrating to unity and thus causes loss of probability mass.²

¹We also use this conditioning on not reaching H within the analysis of knock-in options. Obviously, the relevant transition density for these options should be conditional on hitting H , but the latter density can straightforwardly be obtained by subtracting the density conditional on not hitting H from the unconditional density as noted in for instance Rubinstein and Reiner (1991a) and Rich (1994).

²Absorbing barriers on the contrary keep the stochastic variable at H after it has reached that level. The continued existence of the stochastic variable on H then would contribute to, for instance, the conditional expectation and this clearly is to be avoided when valuing barrier options.

The transition density conditional on not hitting H from above during the interval $T - t$ will be denoted by $p_{nh,d}[S_T, T; S_t, t]$, where the subscripts nh and d refer to the conditioning on not hitting the barrier within a downward movement. Similarly, $p_{nh,u}[S_T, T; S_t, t]$ specifies the transition density conditional on not hitting H from below, i.e. within an upward movement. Applying Itô's lemma to the transform $s_t = \ln S_t$ generates an arithmetic Brownian motion with the barrier being located at $\ln H$. The required density functions for s then can easily be obtained via, for instance, the method of images (see Cox and Miller, 1972). Re-introducing the stock price level then specifies the two required transition densities for the time- T stock price as:

$$p_{nh,u}[S_T, T; S_t, t] = p_{nh,d}[S_T, T; S_t, t] = \frac{1}{S_T \sqrt{2\pi\sigma^2\Delta}} \left\{ \exp \left[-\frac{(\ln S_T - \ln S_t - (r - \frac{1}{2}\sigma^2)\Delta)^2}{2\sigma^2\Delta} \right] - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \exp \left[-\frac{(\ln S_T + \ln S_t - 2\ln H - (r - \frac{1}{2}\sigma^2)\Delta)^2}{2\sigma^2\Delta} \right] \right\} \quad (2)$$

with $\lambda = \frac{2r + \sigma^2}{2\sigma^2}$. The second term in the transition density (2) thus represents the loss of probability that the barrier H brings. As required, $p_{nh,u}[S_T, T; S_t, t]$ and $p_{nh,d}[S_T, T; S_t, t]$ simplify into the transition density for unrestricted geometric Brownian motion in (1) for $H = +\infty$ and $H = 0$, respectively.

The zeroth moment, i.e. the transition probability distribution function or in short the transition distribution, for the unrestricted density in (1) is defined as:

$$M_0 = \int_0^{+\infty} p[S_T, T; S_t, t] dS_T.$$

In order to standardize exposition, the Appendix presents the solution to this definite integral in terms of the integration limits β_1 and β_2 . The solution to this integral is denoted by $\mathcal{J}_{1;y_1,\beta_1,\beta_2}$ for which $y_1 = -\ln S_t - (r - \frac{1}{2}\sigma^2)\Delta$ represents the part of the argument in the exponential function in (1) that is independent of the future stock price S_T . The zeroth moment then can be written as:

$$M_0 = \int_0^{+\infty} p[S_T, T; S_t, t] dS_T = \mathcal{J}_{1;y_1,0,+\infty} = 1, \quad (3)$$

where we used the properties $\Phi[-\infty] = 0$ and $\Phi[+\infty] = 1$ with $\Phi[q] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^q \exp[-\frac{1}{2}x^2] dx$ being the standard normal distribution function (see Zelen and Severo, 1964). The zeroth moment (3) is 1 as indeed no probability mass is located on the inaccessible values 0 and $+\infty$.

The second integral expression in the Appendix allows us to specify the first moment for the

unrestricted geometric Brownian motion as:

$$M_1 = \int_0^{+\infty} S_T p[S_T, T; S_t, t] dS_T = \mathcal{J}_{2;y_1,0,+\infty} = S_t \exp[r\Delta]. \quad (4)$$

The conditional expectation equals $S_t \exp[r\Delta]$ as the stock price is expected to grow in a multiplicative manner at the rate r .

Similarly, the zeroth and first moments conditional on not hitting the upper barrier H from below with $y_1 = -\ln S_t - (r - \frac{1}{2}\sigma^2)\Delta$ and $y_2 = \ln S_t - 2\ln H - (r - \frac{1}{2}\sigma^2)\Delta$ in (2) are:

$$\begin{aligned} M_{0,nh,u} &= \int_0^H p_{nh,u}[S_T, T; S_t, t] dS_T = \mathcal{J}_{1;y_1,0,H} - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \mathcal{J}_{1;y_2,0,H} \\ M_{0,nh,u} &= \Phi\left[\frac{\ln H - \ln S_t - (r - \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}\right] - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \Phi\left[\frac{\ln S_t - \ln H - (r - \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}\right] \end{aligned} \quad (5)$$

and

$$\begin{aligned} M_{1,nh,u} &= \int_0^H S_T p_{nh,u}[S_T, T; S_t, t] dS_T = \mathcal{J}_{2;y_1,0,H} - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \mathcal{J}_{2;y_2,0,H} \\ M_{1,nh,u} &= S_t \exp[r\Delta] \Phi\left[\frac{\ln H - \ln S_t - (r + \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}\right] \\ &\quad - S_t^{1-2\lambda} H^{2\lambda} \exp[r\Delta] \Phi\left[\frac{\ln S_t - \ln H - (r + \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}\right]. \end{aligned} \quad (6)$$

The integration limits in the moments (5) and (6) are 0 and H such that the latter two values gather no probability mass or conditional expectation. As H is a killing barrier, the total probability now is smaller than 1 and the total first moment likewise is below $S_t \exp[r\Delta]$. Indeed, the values 1 and $S_t \exp[r\Delta]$ only arise as limits of (5) and (6) for H to $+\infty$, i.e. when hitting is precluded given that $+\infty$ is not accessible.

Likewise, the two moments when conditioning on not hitting the lower barrier H from above are:

$$\begin{aligned} M_{0,nh,d} &= \int_H^{+\infty} p_{nh,d}[S_T, T; S_t, t] dS_T = \mathcal{J}_{1;y_1,H,+\infty} - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \mathcal{J}_{1;y_2,H,+\infty} \\ M_{0,nh,d} &= \Phi\left[\frac{\ln S_t - \ln H + (r - \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}\right] - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \Phi\left[\frac{\ln H - \ln S_t + (r - \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}\right] \end{aligned} \quad (7)$$

and

$$\begin{aligned}
M_{1,nh,d} &= \int_H^{+\infty} S_T p_{nh,d} [S_T, T; S_t, t] dS_T = \mathcal{J}_{2;y_1,H,+\infty} - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \mathcal{J}_{2;y_2,H,+\infty} \\
M_{1,nh,d} &= S_t \exp[r\Delta] \Phi \left[\frac{\ln S_t - \ln H + (r + \frac{1}{2}\sigma^2) \Delta}{\sigma\sqrt{\Delta}} \right] \\
&\quad - S_t^{1-2\lambda} H^{2\lambda} \exp[r\Delta] \Phi \left[\frac{\ln H - \ln S_t + (r + \frac{1}{2}\sigma^2) \Delta}{\sigma\sqrt{\Delta}} \right]. \tag{8}
\end{aligned}$$

Again, the moments fall below the corresponding moments for the unrestricted process unless the barrier H is effectively removed by lowering it to 0.

2.2 The moments for the probability distribution of the first-hitting time

The first time at which the stock price hits the barrier H , i.e. the first-hitting time τ , is a stochastic variable. The transition probability density function of the first-hitting time, in short the first-hitting time density, will be denoted by $g_u [H, \tau; S_t, t]$ when the stock price is conditioned to hit H from below within an upward movement. This function specifies the probability of each first-hitting time τ given that the process at time t is at S_t . Similarly, the first-hitting time density $g_d [H, \tau; S_t, t]$ conditions on hitting H from above, i.e. in a downward direction. These two first-hitting time densities can be obtained from the transition distributions, i.e. from the above zeroth moments for the distribution of the future stock price, conditional on not hitting the barrier in question (see Cox and Miller (1972) for more detail):

$$\begin{aligned}
g_u [H, \tau; S_t, t] &= -\frac{\partial}{\partial t} [M_{0,nh,u}], \\
g_d [H, \tau; S_t, t] &= -\frac{\partial}{\partial t} [M_{0,nh,d}].
\end{aligned}$$

Straightforward calculations and simplifications then give the well-known first-hitting time densities:

$$g_u [H, \tau; S_t, t] = \frac{\ln H - \ln S_t}{\sigma(\tau - t)^{\frac{3}{2}} \sqrt{2\pi}} \exp \left[-\frac{(\ln H - \ln S_t - (r - \frac{1}{2}\sigma^2)(\tau - t))^2}{2\sigma^2(\tau - t)} \right], \tag{9}$$

$$g_d [H, \tau; S_t, t] = \frac{\ln S_t - \ln H}{\sigma(\tau - t)^{\frac{3}{2}} \sqrt{2\pi}} \exp \left[-\frac{(\ln H - \ln S_t - (r - \frac{1}{2}\sigma^2)(\tau - t))^2}{2\sigma^2(\tau - t)} \right]. \tag{10}$$

The general definition of the moments of the first-hitting time probability in our notation is:

$$M_{n,h}^{+\infty} = \int_t^{+\infty} (\tau - t)^n g[\tau - t] d\tau, \tag{11}$$

where $g[\tau - t]$ specifies the first-hitting time density and the subscript h denotes the conditioning on hitting the barrier H . The superscript $+\infty$ refers to the end point of the time interval that is relevant in calculating the moments. For instance, the first moment or the mean first-hitting time obviously is defined in terms of all possible future τ such that integration is to be performed from t until $+\infty$. As the option contracts in this paper have a finite time to maturity, we need to focus on the time interval $T - t$:

$$M_{n,h}^{T-t} = \int_t^T (\tau - t)^n g[\tau - t] d\tau.$$

The term $M_{n,h}^{T-t}$ then specifies the part of the n -th moment of the first-hitting time distribution that actually accrues between times t and T . All contingent contracts in this paper share the maturity date T such that for notational convenience we drop the reference to the time interval and thus, with some abuse of terminology, discuss $M_{n,h}^{T-t}$ as the n -th moment of the first-hitting time distribution.

However, we need to introduce an additional notion since rebates of knock-out options are typically paid out immediately when H is hit. Such payments must be discounted from the first-hitting time τ to the present date t , which adds an exponential term to the integrand:

$$M_{n,h}^{d,T-t} = \int_t^T (\tau - t)^n \exp[-r(\tau - t)] g[\tau - t] d\tau.$$

We refer to this quantity as the discounted n -th moment of the first-hitting time distribution for which we use the notation $M_{n,h}^{d,T-t}$ where the superscript d refers to the embedded discounting.

The moment $M_{0,h,u}^{T-t}$ is the zeroth moment or the probability of hitting the upper barrier H from below within an upward movement during the time interval $T - t$. Using $y_3 = \ln H - \ln S_t > 0$ and $s = \tau - t$ then gives:

$$\begin{aligned} M_{0,h,u}^{T-t} &= \int_t^T g_u[H, \tau; S_t, t] d\tau = \int_0^{T-t} \frac{y_3}{\sigma\sqrt{2\pi}} \frac{1}{s^{\frac{3}{2}}} \exp\left[-\frac{(y_3 - (r - \frac{1}{2}\sigma^2)s)^2}{2\sigma^2 s}\right] ds \\ M_{0,h,u}^{T-t} &= \frac{y_3}{\sigma\sqrt{2\pi}} \exp\left[\frac{y_3(r - \frac{1}{2}\sigma^2)}{\sigma^2}\right] \int_0^{T-t} \frac{1}{s^{\frac{3}{2}}} \exp\left[-\left\{\frac{r - \frac{1}{2}\sigma^2}{\sigma\sqrt{2}}\right\}^2 s - \left\{\frac{y_3}{\sigma\sqrt{2}}\right\}^2 \frac{1}{s}\right] ds. \end{aligned}$$

This integral can be evaluated using expression $\mathcal{J}_{3,a,b,\beta_1,\beta_2}$ in the Appendix for which $a = \frac{r - \frac{1}{2}\sigma^2}{\sigma\sqrt{2}}$, $b = \frac{y_3}{\sigma\sqrt{2}}$, $\beta_1 = 0$ and $\beta_2 = T - t$. Straightforward simplifications then yield:

$$M_{0,h,u}^{T-t} = \Phi\left[\frac{\ln S_t - \ln H + (r - \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}\right] + \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \Phi\left[\frac{\ln S_t - \ln H - (r - \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}\right]. \quad (12)$$

The zeroth moment (12) can alternatively also be obtained by noting that the integral of the first-hitting time density over t to T , i.e. the probability of first hitting in $T - t$, by definition is the complement of the transition probability of not hitting the barrier H in $T - t$. Or, $M_{0,h,u}^{T-t} = 1 - M_{0,nh,u}$ and this equality can easily be verified from the zeroth moments in (12) and (5) when noting that $1 - \Phi[-x] = \Phi[x]$ (see Zelen and Severo, 1964). As required, lifting H to $+\infty$ reduces the zeroth moment to 0 as $+\infty$ is not accessible.

The first moment gives the mean first-hitting time over $T - t$ conditional on hitting H from below and is defined as:

$$M_{1,h,u}^{T-t} = \int_t^T (\tau - t) g_u [H, \tau; S_t, t] d\tau,$$

which yields:

$$M_{1,h,u}^{T-t} = \frac{y_3}{\sigma\sqrt{2\pi}} \exp\left[\frac{y_3(r - \frac{1}{2}\sigma^2)}{\sigma^2}\right] \int_0^{T-t} \frac{1}{s^{\frac{1}{2}}} \exp\left[-\left\{\frac{r - \frac{1}{2}\sigma^2}{\sigma\sqrt{2}}\right\}^2 s - \left\{\frac{y_3}{\sigma\sqrt{2}}\right\}^2 \frac{1}{s}\right] ds.$$

Using expression $\mathcal{J}_{4,a,b,\beta_1,\beta_2}$ in the Appendix then allows us to write this first moment as:

$$M_{1,h,u}^{T-t} = \frac{\ln H - \ln S_t}{\sigma^2(\lambda - 1)} \left\{ \Phi\left[\frac{\ln S_t - \ln H + (r - \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}\right] - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \Phi\left[\frac{\ln S_t - \ln H - (r - \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}\right] \right\}. \quad (13)$$

The discounted zeroth moment conditional on hitting H from below is specified as:

$$M_{0,h,u}^{d,T-t} = \int_t^T \exp[-r(\tau - t)] g_u [H, \tau; S_t, t] d\tau$$

and calculations similar to those employed for $M_{0,h,u}^{T-t}$ then give:

$$M_{0,h,u}^{d,T-t} = \left(\frac{H}{S_t}\right)^{\lambda-1-q} \Phi\left[\frac{\ln S_t - \ln H + q\sigma^2\Delta}{\sigma\sqrt{\Delta}}\right] + \left(\frac{H}{S_t}\right)^{\lambda-1+q} \Phi\left[\frac{\ln S_t - \ln H - q\sigma^2\Delta}{\sigma\sqrt{\Delta}}\right], \quad (14)$$

with $q = \frac{\sqrt{2\sigma^2r + (r - \frac{1}{2}\sigma^2)^2}}{\sigma^2}$.

Likewise, the discounted first moment is defined as

$$M_{1,h,u}^{d,T-t} = \int_t^T (\tau - t) \exp[-r(\tau - t)] g_u [H, \tau; S_t, t] d\tau,$$

which gives:

$$M_{1,h,u}^{d,T-t} = \frac{\ln H - \ln S_t}{q\sigma^2} \left\{ \left(\frac{H}{S_t}\right)^{\lambda-1-q} \Phi\left[\frac{\ln S_t - \ln H + q\sigma^2\Delta}{\sigma\sqrt{\Delta}}\right] - \left(\frac{H}{S_t}\right)^{\lambda-1+q} \Phi\left[\frac{\ln S_t - \ln H - q\sigma^2\Delta}{\sigma\sqrt{\Delta}}\right] \right\}. \quad (15)$$

The moments conditional on hitting the lower barrier H in a downward direction are obtained along similar lines. For brevity, we omit their definitions and only present the final expressions:

$$M_{0,h,d}^{T-t} = \Phi \left[\frac{\ln H - \ln S_t - (r - \frac{1}{2}\sigma^2) \Delta}{\sigma\sqrt{\Delta}} \right] + \left(\frac{H}{S_t} \right)^{2(\lambda-1)} \Phi \left[\frac{\ln H - \ln S_t + (r - \frac{1}{2}\sigma^2) \Delta}{\sigma\sqrt{\Delta}} \right], \quad (16)$$

$$M_{1,h,d}^{T-t} = \frac{\ln S_t - \ln H}{\sigma^2(\lambda-1)} \left\{ \left(\frac{H}{S_t} \right)^{2(\lambda-1)} \Phi \left[\frac{\ln H - \ln S_t + (r - \frac{1}{2}\sigma^2) \Delta}{\sigma\sqrt{\Delta}} \right] - \Phi \left[\frac{\ln H - \ln S_t - (r - \frac{1}{2}\sigma^2) \Delta}{\sigma\sqrt{\Delta}} \right] \right\}, \quad (17)$$

$$M_{0,h,d}^{d,T-t} = \left(\frac{H}{S_t} \right)^{\lambda-1-q} \Phi \left[\frac{\ln H - \ln S_t - q\sigma^2\Delta}{\sigma\sqrt{\Delta}} \right] + \left(\frac{H}{S_t} \right)^{\lambda-1+q} \Phi \left[\frac{\ln H - \ln S_t + q\sigma^2\Delta}{\sigma\sqrt{\Delta}} \right], \quad (18)$$

$$M_{1,h,d}^{d,T-t} = \frac{\ln S_t - \ln H}{q\sigma^2} \left\{ \left(\frac{H}{S_t} \right)^{\lambda-1+q} \Phi \left[\frac{\ln H - \ln S_t + q\sigma^2\Delta}{\sigma\sqrt{\Delta}} \right] - \left(\frac{H}{S_t} \right)^{\lambda-1-q} \Phi \left[\frac{\ln H - \ln S_t - q\sigma^2\Delta}{\sigma\sqrt{\Delta}} \right] \right\}. \quad (19)$$

3 The valuation of rebates

The above moments allow us to standardize valuation of a wide variety of rebates that also include interesting time-dependent specifications that, to the best of our knowledge, have not been examined in the literature. For instance, Rubinstein and Reiner (1991a) and Rich (1994) restrict attention to constant rebates, i.e. rebate payments that are independent of the actual time at which the barrier is hit. However, time-dependent rebates can be attractive from a theoretical but also from a practitioner's point of view since they, for instance, allow the amount of the rebate of a knock-out option to increase with the time period that has passed since the inception date of the contract. Such specification can offer a more stable value of the rebate-inclusive option position as the decrease in the option's time value can be (partly) offset by the increasing rebate value.

We examine rebates that are paid when the underlying stock fails to hit the barrier H , i.e. when the knock-in option expires without having created a standard option. These payments will be referred to as in-rebates. The notion of out-rebates then is employed for the rebate payments that arise when knock-out options are cancelled after the underlying has hit the barrier H .

3.1 In-rebates

We value two simple types of in-rebates and then briefly discuss alternative specifications such as a path-dependent yet capped rebate payments.

The in-rebate of type 1 is the familiar fixed cash payment of $RI_1^u = RI^u$ where RI^u denotes the constant in-rebate with respect to the upper level H . The value of this rebate at time t , $V[RI_1^u]$, is the discounted value of RI^u multiplied by the probability that the upper barrier H is not hit within the time interval $T - t$.³ The latter probability is the complement of the probability that the first-hitting time lies between t and T or:

$$V[RI_1^u] = \exp[-r\Delta] RI^u \left\{ 1 - \int_t^T g_u[H, \tau; S_t, t] d\tau \right\} = \exp[-r\Delta] RI^u \left\{ 1 - M_{0,h,u}^{T-t} \right\}.$$

Similarly, the current value of the in-rebate written against the lower barrier, $RI_1^d = RI^d$, is:

$$V[RI_1^d] = \exp[-r\Delta] RI^d \left\{ 1 - \int_t^T g_d[H, \tau; S_t, t] d\tau \right\} = \exp[-r\Delta] RI^d \left\{ 1 - M_{0,h,d}^{T-t} \right\}.$$

These values can alternatively also be calculated via the transition density for the future stock price. Indeed, the cumulative likelihood of not hitting H between t and T coincides with the integral over the relevant fluctuation range of the transition density of the future price conditioned on no hitting in the time interval $T - t$:

$$V[RI_1^u] = \exp[-r\Delta] RI^u \int_0^H p_{nh,u}[S_T, T; S_t, t] dS_T = \exp[-r\Delta] RI^u M_{0,nh,u}$$

and

$$V[RI_1^d] = \exp[-r\Delta] RI^d \int_H^{+\infty} p_{nh,d}[S_T, T; S_t, t] dS_T = \exp[-r\Delta] RI^d M_{0,nh,d}.$$

These rebate prices correspond with the values that are derived in for instance Rubinstein and Reiner (1991a). As required, the value of the in-rebate simplifies into $\exp[-r\Delta] RI^u$ ($\exp[-r\Delta] RI^d$) when H is lifted to $+\infty$ (is decreased to 0) since this precludes hitting through which the in-rebate is certain to be paid at time T .

The transition density of the future stock price also straightforwardly allows for the valuation of rebate payments that are not constant but depend on the stock price at time T . In particular, the

³In order to save on subscripts, the time subscript t in the value of rebates and later on in the option prices will be omitted.

rebate payments of type 2 are $RI_2^u = S_T RI^u$ and $RI_2^d = S_T RI^d$ for which the current value is given by:⁴

$$V [RI_2^u] = \exp [-r\Delta] RI^u \int_0^H S_T p_{nh,u} [S_T, T; S_t, t] dS_T = \exp [-r\Delta] RI^u M_{1,nh,u}$$

and

$$V [RI_2^d] = \exp [-r\Delta] RI^d \int_H^{+\infty} S_T p_{nh,d} [S_T, T; S_t, t] dS_T = \exp [-r\Delta] RI^d M_{1,nh,d}.$$

The in-rebates of types 1 and 2 can be linearly combined into specifications that can offer tailor-made solutions to investors. For example, the rebate payment with respect to the upper barrier H can take the general form $[aH + bS_T]$. With $a = 1$ and $b = -1$, for instance, the payment at time T increases in the positive distance between H and S_T such that volatility in the option holder's total position can be lowered. The value of such rebate is easily expressed in terms of the above moments as $\exp [-r\Delta] (aHM_{0,nh,u} + bM_{1,nh,u})$. A capped version of the in-rebate of type 2 still offers path-dependency but sets a maximum on this type of insurance in order to reduce its cost. For example, the rebate payment could be $S_T RI^u$ when $S_T < 0.8H$ and 0 otherwise. Such rebate then can easily be valued by setting the upper integration limit in the pricing formula for $V [RI_2^u]$ at $0.8H$ and then evaluate $\mathcal{J}_{2,y1,0,0.8H}$ and $\mathcal{J}_{2,y2,0,0.8H}$.

3.2 Out-rebates

Typically, out-rebates are specified as fixed cash payments that are to be paid at the first-hitting time τ , i.e. when the knock-out option is cancelled. This type of rebate payment, for which we employ the subscript 1, yields $RO_1^u = RO^u$ and $RO_1^d = RO^d$. The time- t value of such rebate requires discounting the payment at time τ to the present such that the above discounted moments are to be used:

$$V [RO_1^u] = \int_t^T RO^u \exp [-r(\tau - t)] g_u [H, \tau; S_t, t] d\tau = RO^u M_{0,h,u}^{d,T-t}$$

and

$$V [RO_1^d] = \int_t^T RO^d \exp [-r(\tau - t)] g_d [H, \tau; S_t, t] d\tau = RO^d M_{0,h,d}^{d,T-t}.$$

Some re-arranging confirms that these prices correspond with the well-known expressions for fixed-value out-rebates in Rubinstein and Reiner (1991a). Note that the above moments can also value rebates under deferred payment. In fact, payment of the rebate at time T requires replacing $\exp [-r(\tau - t)]$

⁴Note that these values actually are the complements of the corresponding up- or down-and-in asset-(at-expiration)-or-nothing binary barrier options in Rubinstein and Reiner (1991b).

in the above pricing equations by $\exp[-r\Delta]$ such that we obtain $V[RO_1^u] = \exp[-r\Delta] RO^u M_{0,h,u}^{T-t}$ and $V[RO_1^d] = \exp[-r\Delta] RO^d M_{0,h,d}^{T-t}$. For brevity, we restrict attention in what follows to rebates that are paid immediately at time τ .

The above fixed-sum rebate has the advantage of clarity concerning the rebate's level. However, the rebate is independent of the time period that has elapsed until τ or that separates τ from the maturity date T . The following 4 specifications introduce different types of time-dependency that can straightforwardly be handled by the above moment expressions. The rebate payment of type 2 is $RO_2^u = RO^u \exp[r(\tau - t_0)]$ and $RO_2^d = RO^d \exp[r(\tau - t_0)]$ where t_0 denotes the time of inception of the rebate contract that is situated before or at the time of valuation, i.e. $t \geq t_0$. This rebate grows in function of the period between the time of first hitting and the creation of the rebate contract or investors are compensated more heavily the later the option is cancelled. Such specification may be attractive for investors as the opportunity cost in holding an option that ultimately is cancelled increases with the time until first hitting. The link to time here is exponential but a Taylor approximation shows that such function for small r and time intervals, which typically apply to options, is virtually identical to a linear specification.⁵ The current value of this rebate is:

$$V[RO_2^u] = \int_t^T RO^u \exp[r(\tau - t_0)] \exp[-r(\tau - t)] g_u[H, \tau; S_t, t] d\tau = \exp[r(t - t_0)] RO^u M_{0,h,u}^{T-t}$$

and

$$V[RO_2^d] = \int_t^T RO^d \exp[r(\tau - t_0)] \exp[-r(\tau - t)] g_d[H, \tau; S_t, t] d\tau = \exp[r(t - t_0)] RO^d M_{0,h,d}^{T-t}$$

The rebate of type 3 shares the exponential relation to time but instead connects the payment to the period that separates the first-hitting time from the maturity date with $RO_3^u = RO^u \exp[-r(T - \tau)]$ and $RO_3^d = RO^d \exp[-r(T - \tau)]$. Also here the rebate is larger when first hitting takes place later. Its current value is:

$$V[RO_3^u] = \int_t^T RO^u \exp[-r(T - \tau)] \exp[-r(\tau - t)] g_u[H, \tau; S_t, t] d\tau = \exp[-r\Delta] RO^u M_{0,h,u}^{T-t}$$

and

$$V[RO_3^d] = \int_t^T RO^d \exp[-r(T - \tau)] \exp[-r(\tau - t)] g_d[H, \tau; S_t, t] d\tau = \exp[-r\Delta] RO^d M_{0,h,d}^{T-t}$$

⁵The below rebates of types 4 and 5 instead posit a linear relation to time.

The type-4 out-rebate postulates a linear relation between the time of first hitting and the rebate payment of the following form: $RO_4^u = RO^u (\tau - t_0)$ and $RO_4^d = RO^d (\tau - t_0)$. This specification offers a positive link between the rebate and the length of time until first hitting, yet ensures that the payment is small when first hitting comes about quickly. The current value of this rebate is:

$$\begin{aligned} V [RO_4^u] &= \int_t^T RO^u (\tau - t_0) \exp [-r (\tau - t)] g_u [H, \tau; S_t, t] d\tau \\ V [RO_4^u] &= RO^u \int_t^T (\tau - t) \exp [-r (\tau - t)] g_u [H, \tau; S_t, t] d\tau \\ &\quad + RO^u (t - t_0) \int_t^T \exp [-r (\tau - t)] g_u [H, \tau; S_t, t] d\tau \\ V [RO_4^u] &= RO^u \left\{ M_{1,h,u}^{d,T-t} + (t - t_0) M_{0,h,u}^{d,T-t} \right\} \end{aligned}$$

and

$$\begin{aligned} V [RO_4^d] &= \int_t^T RO^d (\tau - t_0) \exp [-r (\tau - t)] g_d [H, \tau; S_t, t] d\tau \\ V [RO_4^d] &= RO^d \int_t^T (\tau - t) \exp [-r (\tau - t)] g_d [H, \tau; S_t, t] d\tau \\ &\quad + RO^d (t - t_0) \int_t^T \exp [-r (\tau - t)] g_d [H, \tau; S_t, t] d\tau \\ V [RO_4^d] &= RO^d \left\{ M_{1,h,d}^{d,T-t} + (t - t_0) M_{0,h,d}^{d,T-t} \right\}. \end{aligned}$$

Type 5 specifies a rebate that linearly decreases over time, namely $RO_5^u = RO^u (T - \tau)$ and $RO_5^d = RO^d (T - \tau)$. Such specification may be attractive for investors that expect a sharp decline in volatility as it allows them to economize on the cost of the rebate. The value at time t is:

$$\begin{aligned} V [RO_5^u] &= \int_t^T RO^u (T - \tau) \exp [-r (\tau - t)] g_u [H, \tau; S_t, t] d\tau \\ V [RO_5^u] &= RO^u (T - t) \int_t^T \exp [-r (\tau - t)] g_u [H, \tau; S_t, t] d\tau \\ &\quad - RO^u \int_t^T (\tau - t) \exp [-r (\tau - t)] g_u [H, \tau; S_t, t] d\tau \\ V [RO_5^u] &= RO^u \left\{ (T - t) M_{0,h,u}^{d,T-t} - M_{1,h,u}^{d,T-t} \right\} \end{aligned}$$

and

$$\begin{aligned}
V \left[RO_5^d \right] &= \int_t^T RO^d (T - \tau) \exp [-r (\tau - t)] g_d [H, \tau; S_t, t] d\tau \\
V \left[RO_5^d \right] &= RO^d (T - t) \int_t^T \exp [-r (\tau - t)] g_d [H, \tau; S_t, t] d\tau \\
&\quad - RO^d \int_t^T (\tau - t) \exp [-r (\tau - t)] g_d [H, \tau; S_t, t] d\tau \\
V \left[RO_5^d \right] &= RO^d \left\{ (T - t) M_{0,h,d}^{d,T-t} - M_{1,h,d}^{d,T-t} \right\}.
\end{aligned}$$

These 5 types of out-rebates present a subset of specifications that can be handled via the above moment expressions and the integral expressions in the Appendix. In fact, various other specifications may also appeal to investors. For instance, linear combinations of the above 5 types yield rebates in which time-dependent payments are superimposed on constant rebates. The rebates of types 2 and 5 can also be merged into $RO^u (T - \tau) \exp [r (\tau - t_0)]$ such that the negative link between the rebate payment and time is mitigated. Furthermore, the integral expressions in the Appendix can handle growth rates in the exponential term in the rebates of types 2 and 3 that differ from r . Investors that expect high short-term but lower medium- and long-term volatility can reduce costs by opting for rebates that are only active in the initial period until t_1 with $t < t_1 < T$. In the case of a fixed payment as in type 1, the current value of such rebate would then simply be $RO^u M_{0,h,u}^{d,t_1-t}$.

4 The put-call parities

The static hedge portfolio of Stoll (1969) for standard options starts from the sale of a call and the purchase of a put at time t with both options having the same exercise price and time until maturity. This portfolio then is completed by buying a stock at the price S_t and borrowing $K \exp [-r (T - t)]$. It guarantees a zero payoff at time T and to ensure that it does not dominate nor is dominated, its value at time t must be 0. The put-call parity for standard options then is:

$$P - C = \exp [-r \Delta] K - S_t. \quad (20)$$

The put-call parity (20) cannot apply to barrier options. Indeed, hitting the barrier cancels the option's payoff at time T (knock-out option) or introduces it (knock-in option) such that the portfolio value at time T becomes uncertain. However, a probabilistic approach will allow us to specify also put-call parities for barrier options that share the same barrier, exercise price and expiration date.

This section first derives 8 put-call parities: for up-and-out, down-and-out, up-and-in and down-and-in options with in each case the additional distinction in function of the exercise price K being smaller or larger than the barrier H . We then show that our probabilistic approach is actually equivalent to a strategy in which the static hedge of Stoll (1969) is extended by a binary barrier option. All options in this section contain a rebate provision. The 2 in-rebates will be indexed by i with $i = \{1, 2\}$ and the 5 specifications for the out-rebates are indexed by j with $j = \{1, \dots, 5\}$. Their current value for the upper (lower) barrier H will be denoted by $V [RI_i^u]$ and $V [RO_j^u]$ ($V [RI_i^d]$ and $V [RO_j^d]$), respectively. For brevity, the rebates are assumed to be of the same type within each put-call parity such that, for instance, the out-rebate RO_j^d applies to both the knock-out call as well as the knock-out put. Or, the value of i and j is identical across the options pairs in each of the 8 parities.⁶

We start with the parity between puts and calls that are cancelled when the barrier H is reached from below with the exercise price K being located below H . These knock-out call and put prices are denoted by $C_{u,o,K < H}$ and $P_{u,o,K < H}$ where the subscripts u and o refer to their up-and-out nature. The derivation of the put-call parity starts with subtracting the risk-neutral call price inclusive of the out-rebate from the corresponding rebate-inclusive put price. The integrals in the option prices then are combined into the above moment expressions:

$$\begin{aligned}
P_{u,o,K < H} - C_{u,o,K < H} &= \exp[-r\Delta] \int_0^K [K - S_T] p_{nh,u} [S_T, T; S_t, t] dS_T + V [RO_j^u] \\
&\quad - \exp[-r\Delta] \int_K^H [S_T - K] p_{nh,u} [S_T, T; S_t, t] dS_T - V [RO_j^u] \\
P_{u,o,K < H} - C_{u,o,K < H} &= \exp[-r\Delta] \int_0^H [K - S_T] p_{nh,u} [S_T, T; S_t, t] dS_T \\
P_{u,o,K < H} - C_{u,o,K < H} &= \exp[-r\Delta] \left\{ K \int_0^H p_{nh,u} [S_T, T; S_t, t] dS_T - \int_0^H S_T p_{nh,u} [S_T, T; S_t, t] dS_T \right\} \\
P_{u,o,K < H} - C_{u,o,K < H} &= \exp[-r\Delta] \{KM_{0,nh,u} - M_{1,nh,u}\}. \tag{21}
\end{aligned}$$

The discounted exercise price in the parity (20) for standard options now is multiplied by the zeroth moment conditioned on H not being reached from below during the option's life. The remaining term in the put-call parity (21) is the discounted value of the first moment, i.e. of the conditional expectation, again conditional on H not being reached from below. Intuitively, the presence of moments expressions

⁶This restriction is not necessary and can easily be dropped. However, it considerably shortens the below expressions with the advantage of not obscuring the basic intuition.

in the parity (21) corrects the right-hand side terms of the put-call parity (20) for the loss of probability and expected value that option cancellation brings. Such correction obviously no longer will be required when lifting the barrier H to $+\infty$ since this precludes option cancellation. The terms $M_{0,nh,u}$ and $M_{1,nh,u}$ then simplify into 1 and $S_t \exp[r\Delta]$, respectively, and the familiar put-call parity of Stoll (1969) results.

The moments that emerge in the put-call parity (21) relate to the defective density and thus do not depend on the relative position of K versus H . Or, exactly the same expression for the put-call parity must arise for $K > H$. This intuition is confirmed when calculating the difference between risk-neutral option prices whilst noting that the call now can never obtain intrinsic value:

$$\begin{aligned}
P_{u,o,K>H} - C_{u,o,K>H} &= \exp[-r\Delta] \int_0^H [K - S_T] p_{nh,u}[S_T, T; S_t, t] dS_T + V[RO_j^u] - 0 - V[RO_j^u] \\
P_{u,o,K>H} - C_{u,o,K>H} &= \exp[-r\Delta] \left\{ K \int_0^H p_{nh,u}[S_T, T; S_t, t] dS_T - \int_0^H S_T p_{nh,u}[S_T, T; S_t, t] dS_T \right\} \\
P_{u,o,K>H} - C_{u,o,K>H} &= \exp[-r\Delta] \{KM_{0,nh,u} - M_{1,nh,u}\}. \tag{22}
\end{aligned}$$

Similarly, the down-and-out put for $K < H$ has zero intrinsic value or its value equals the current price of the rebate:

$$\begin{aligned}
P_{d,o,K<H} - C_{d,o,K<H} &= 0 + V[RO_j^d] - \exp[-r\Delta] \int_H^{+\infty} [S_T - K] p_{nh,d}[S_T, T; S_t, t] dS_T - V[RO_j^d] \\
P_{d,o,K<H} - C_{d,o,K<H} &= \exp[-r\Delta] \{KM_{0,nh,d} - M_{1,nh,d}\}. \tag{23}
\end{aligned}$$

Again, the same put-call parity emerges for the case in which the exercise price exceeds H or:

$$P_{d,o,K>H} - C_{d,o,K>H} = \exp[-r\Delta] \{KM_{0,nh,d} - M_{1,nh,d}\}. \tag{24}$$

Note that the put-call parity (24) collapses into the expression for standard options when decreasing the barrier to 0. Indeed, the moments $M_{0,nh,d}$ and $M_{1,nh,d}$ in (7) and (8), respectively, then simplify into the moments for the unrestricted process since $\lim_{H \rightarrow 0} [M_{0,nh,d}] = M_0 = 1$ and $\lim_{H \rightarrow 0} [M_{1,nh,d}] = M_1 = S_t \exp[r\Delta]$.

The derivation of the put-call parity for up-and-in options is somewhat lengthier. In fact, we desire to work with transition densities conditional on no hitting, rather than hitting, in order to be able to employ the above moment expressions. Hereto, we start from the property that the cumulative transition probability for the stock price at time T per definition can be split up as

follows: $P[S_T, T; S_t, t] = P_{nh,u}[S_T, T; S_t, t] + P_{h,u}[S_T, T; S_t, t]$, where the term $P[S_T, T; S_t, t]$ denotes the unrestricted transition probability and where $P_{nh,u}[S_T, T; S_t, t]$ and $P_{h,u}[S_T, T; S_t, t]$ condition the transition probability on hitting not taking place and hitting taking place, respectively. Evaluating the derivative to S_T preserves the same identity but allows us to move towards the transition densities with $p[S_T, T; S_t, t] = p_{nh,u}[S_T, T; S_t, t] + p_{h,u}[S_T, T; S_t, t]$. The transition density of the future stock price conditional on hitting taking place at the upper barrier H thus is $p_{h,u}[S_T, T; S_t, t] = p[S_T, T; S_t, t] - p_{nh,u}[S_T, T; S_t, t]$.⁷ Using the latter relation and carefully accounting for the domain then gives the following put-call parity for the up-and-in case with $K < H$:

$$\begin{aligned}
P_{u,i,K < H} - C_{u,i,K < H} &= \exp[-r\Delta] \int_0^K [K - S_T] p_{h,u}[S_T, T; S_t, t] dS_T + V[RI_i^u] \\
&\quad - \exp[-r\Delta] \int_K^{+\infty} [S_T - K] p_{h,u}[S_T, T; S_t, t] dS_T - V[RI_i^u] \\
P_{u,i,K < H} - C_{u,i,K < H} &= \exp[-r\Delta] \left\{ \int_0^K [K - S_T] p[S_T, T; S_t, t] dS_T \right. \\
&\quad \left. - \int_0^K [K - S_T] p_{nh,u}[S_T, T; S_t, t] dS_T \right\} - \exp[-r\Delta] \left\{ \int_K^{+\infty} [S_T - K] p[S_T, T; S_t, t] dS_T \right. \\
&\quad \left. - \int_K^H [S_T - K] p_{nh,u}[S_T, T; S_t, t] dS_T \right\} \\
P_{u,i,K < H} - C_{u,i,K < H} &= \exp[-r\Delta] \left\{ \int_0^{+\infty} [K - S_T] p[S_T, T; S_t, t] dS_T \right. \\
&\quad \left. - \int_0^H [K - S_T] p_{nh,u}[S_T, T; S_t, t] dS_T \right\} \\
P_{u,i,K < H} - C_{u,i,K < H} &= \exp[-r\Delta] \{K(M_0 - M_{0,nh,u}) - (M_1 - M_{1,nh,u})\}. \tag{25}
\end{aligned}$$

The put-call parity for up-and-in options in (25) thus is expressed in terms of the difference between the zeroth and first moments for the unrestricted and the no-hitting distributions. Lifting the barrier H to $+\infty$ ensures that hitting cannot occur. The difference between put and call prices then must be zero since the non-rebate part of both options will have no value since creation of standard options is precluded. This is confirmed by the limit of the right-hand side of the parity (25) since $M_{0,nh,u}$ and $M_{1,nh,u}$ simplify into M_0 and M_1 .

⁷This argument can also be seen as the probabilistic underpinning of the in-out parity that states that the up-and-in call price equals the price of the standard call minus the up-and-out call price.

As the moments do not depend on the relative position of K to H , the same right-hand side emerges for up-and-in options with $K > H$:

$$P_{u,i,K>H} - C_{u,i,K>H} = \exp[-r\Delta] \{K (M_0 - M_{0,nh,u}) - (M_1 - M_{1,nh,u})\}. \quad (26)$$

The derivation of the put-call parity for down-and-in options with $K < H$ is similar to that for the up-and-in case:

$$\begin{aligned} P_{d,i,K<H} - C_{d,i,K<H} &= \exp[-r\Delta] \int_0^K [K - S_T] p_{h,d}[S_T, T; S_t, t] dS_T + V [RI_i^d] \\ &\quad - \exp[-r\Delta] \int_K^{+\infty} [S_T - K] p_{h,d}[S_T, T; S_t, t] dS_T - V [RI_i^d] \\ P_{d,i,K<H} - C_{d,i,K<H} &= \exp[-r\Delta] \left\{ \int_0^K [K - S_T] p[S_T, T; S_t, t] dS_T \right. \\ &\quad \left. - \int_0^K [K - S_T] p_{nh,d}[S_T, T; S_t, t] dS_T \right\} - \exp[-r\Delta] \left\{ \int_K^{+\infty} [S_T - K] p[S_T, T; S_t, t] dS_T \right. \\ &\quad \left. - \int_K^{+\infty} [S_T - K] p_{nh,d}[S_T, T; S_t, t] dS_T \right\} \\ P_{d,i,K<H} - C_{d,i,K<H} &= \exp[-r\Delta] \left\{ \int_0^K [K - S_T] p[S_T, T; S_t, t] dS_T - 0 \right\} \\ &\quad - \exp[-r\Delta] \left\{ \int_K^{+\infty} [S_T - K] p[S_T, T; S_t, t] dS_T - \int_H^{+\infty} [S_T - K] p_{nh,d}[S_T, T; S_t, t] dS_T \right\} \\ P_{d,i,K<H} - C_{d,i,K<H} &= \exp[-r\Delta] \left\{ \int_0^{+\infty} [K - S_T] p[S_T, T; S_t, t] dS_T \right. \\ &\quad \left. - \int_H^{+\infty} [K - S_T] p_{nh,d}[S_T, T; S_t, t] dS_T \right\} \\ P_{d,i,K<H} - C_{d,i,K<H} &= \exp[-r\Delta] \{K (M_0 - M_{0,nh,d}) - (M_1 - M_{1,nh,d})\} \end{aligned} \quad (27)$$

with the same right-hand side also emerging for the case of $K > H$:

$$P_{d,i,K>H} - C_{d,i,K>H} = \exp[-r\Delta] \{K (M_0 - M_{0,nh,d}) - (M_1 - M_{1,nh,d})\}. \quad (28)$$

The limit for H decreasing to 0 removes the possibility of ever creating standard options and the right-hand side in the parity (28) indeed reduces to zero.

The moment expressions for the relevant (defective) distributions thus allow us to derive 8 put-call parities by combining risk-neutral call and put prices. Note that these parity relations also can be obtained via the static hedge argument of Stoll (1969) when extending the portfolio with a binary barrier option. We illustrate this for the case of up-and-out options with $K < H$. The potential of option cancellation and the presence of rebate provisions generate two differences when compared with the portfolio of Stoll (1969). First, intermediate payments, namely the out-rebates, will arise when the stock price hits H before date T .⁸ Given our assumption of calls and puts having identical rebates, these intermediate payments cancel out.⁹ Second, the overview of the value of the portfolio positions at time T in Table 1 now must take account of an additional dimension, namely whether or not options have been knocked out prior to time T . The option positions do not create value at T if hitting takes place before T , but the loan is to be paid back and the long position in the stock still exists. To ensure a combined value of 0 under all eventualities at time T , the investment strategies at time t must include the sale of a binary barrier option, BI , that pays $S_T - K$ conditional on the stock price having hit H prior to T . The value at time t of this claim will be denoted by $V[BI]$.

Table 1: Value of the extended static hedge portfolio for the put-call parity in the case of up-and-out options with the exercise price located below the barrier.

Value at time t	Payments between t and T		Value at time T		
	Hit prior to T	No hit prior to T	Hit prior to T	No hit prior to T	
				$S_T > K$	$S_T < K$
$C_{u,o,K < H}$	$-RO_j^u$	0	0	$-(S_T - K)$	0
$-P_{u,o,K < H}$	$+RO_j^u$	0	0	0	$+(K - S_T)$
$K \exp[-r\Delta]$	0	0	$-K$	$-K$	$-K$
$-S_t$	0	0	$+S_T$	$+S_T$	$+S_T$
$V[BI]$	0	0	$-(S_T - K)$	0	0
0	0	0	0	0	0

Table 1 shows that the put-call parity then can be written as:

$$P_{u,o,K < H} - C_{u,o,K < H} = \exp[-r\Delta] K - S_t + V[BI]. \quad (29)$$

The put-call parity (29) is identical to the parity in (21) as will be illustrated by expressing the value of the binary barrier option in terms of the above moments. The value of the binary barrier option

⁸First hitting, and thus payment of out-rebates, can also take place precisely at time t or at time T . For brevity, we assume that first hitting does not occur at these two points of time. This restriction greatly simplifies the below exposition, but is obviously not essential to the argument.

⁹Differences in rebate provisions for puts and calls can easily be dealt with via the inclusion of an additional binary barrier option in the hedge portfolio.

measures its payoff with respect to the probability of hitting H between t and T :

$$\begin{aligned}
V [BI] &= \exp [-r\Delta] \int_0^H [S_T - K] p_{h,u} [S_T, T; S_t, t] dS_T \\
V [BI] &= \exp [-r\Delta] \left\{ \int_0^{+\infty} [S_T - K] p [S_T, T; S_t, t] dS_T - \int_0^H [S_T - K] p_{nh,u} [S_T, T; S_t, t] dS_T \right\} \\
V [BI] &= \exp [-r\Delta] \{K (M_{0,nh,u} - M_0) - (M_{1,nh,u} - M_1)\}.
\end{aligned}$$

Plugging the latter value into the put-call parity (29) yields:

$$\begin{aligned}
P_{u,o,K < H} - C_{u,o,K < H} &= \exp [-r\Delta] K - S_t + \{\exp [-r\Delta] (K \{M_{0,nh,u} - M_0\} - \{M_{1,nh,u} - M_1\})\} \\
P_{u,o,K < H} - C_{u,o,K < H} &= \exp [-r\Delta] K - S_t + \{\exp [-r\Delta] (K \{M_{0,nh,u} - 1\} - \{M_{1,nh,u} - S_t \exp [r\Delta]\})\} \\
P_{u,o,K < H} - C_{u,o,K < H} &= \exp [-r\Delta] \{K M_{0,nh,u} - M_{1,nh,u}\},
\end{aligned}$$

which, as required, corresponds with the parity (21).

5 The in-out parities

The in-out parities express the relation between in- and out-options of the same type (call or put) that furthermore share the same exercise price, time to maturity and barrier. It is well-known for the case without rebates that the sum of the value of an in- and out-option gives the corresponding Black and Scholes (1973) price (see for instance Rubinstein and Reiner, 1991a). As all option contracts in this paper have rebate provisions, the familiar in-out parity must be slightly modified. Table 2 illustrates the hedge argument for the in-out parity for up-and-out and up-and-in calls with $K < H$, where C^{BS} is the Black and Scholes (1973) call price.

In order to simplify exposition in Table 2, we again assume that first hitting cannot come about precisely at time t or at time T . The first three rows give the values that emerge from selling an up-and-out call as well as an up-and-in call, termed up-options, and buying a Black and Scholes (1973) call at time t . The rebate provisions can require the payment of an out-rebate by the seller of the barrier option between times t and T or the payment of an in-rebate at time T . Hence, the familiar strategy must be extended by buying an out- as well as an in-rebate contract at time t . Then, intermediate payments cancel out and values at time T are always 0 such that the in-out parity can be written as:

$$C_{u,o,K < H} + C_{u,i,K < H} = C^{BS} + V [RI_i^u] + V [RO_j^u]. \quad (30)$$

Table 2: Value of the extended static hedge portfolio for the in-out parity in the case of up-options with the exercise price located below the barrier.

Value at time t	Payments between t and T		Value at time T			
	Hit prior to T	No hit prior to T	Hit prior to T		No hit prior to T	
			$S_T > K$	$S_T < K$	$S_T > K$	$S_T < K$
$C_{u,o,K < H}$	$-RO_j^u$	0	0	0	$-(S_T - K)$	0
$C_{u,i,K < H}$	0	0	$-(S_T - K)$	0	$-RI_i^u$	$-RI_i^u$
$-C^{BS}$	0	0	$+(S_T - K)$	0	$+(S_T - K)$	0
$-V [RI_i^u]$	0	0	0	0	$+RI_i^u$	$+RI_i^u$
$-V [RO_j^u]$	$+RO_j^u$	0	0	0	0	0
0	0	0	0	0	0	0

As the rebates carry no reference to the relative position of K to H , the right-hand side of the in-out parity (30) also holds for the case of $K > H$ or:

$$C_{u,o,K > H} + C_{u,i,K > H} = C^{BS} + V [RI_i^u] + V [RO_j^u]. \quad (31)$$

Similar arguments apply to the remaining cases and for brevity we only give the final form of these 6 in-out parities:

$$C_{d,o,K < H} + C_{d,i,K < H} = C^{BS} + V [RI_i^d] + V [RO_j^d], \quad (32)$$

$$C_{d,o,K > H} + C_{d,i,K > H} = C^{BS} + V [RI_i^d] + V [RO_j^d], \quad (33)$$

$$P_{u,o,K < H} + P_{u,i,K < H} = P^{BS} + V [RI_i^u] + V [RO_j^u], \quad (34)$$

$$P_{u,o,K > H} + P_{u,i,K > H} = P^{BS} + V [RI_i^u] + V [RO_j^u], \quad (35)$$

$$P_{d,o,K < H} + P_{d,i,K < H} = P^{BS} + V [RI_i^d] + V [RO_j^d], \quad (36)$$

$$P_{d,o,K > H} + P_{d,i,K > H} = P^{BS} + V [RI_i^d] + V [RO_j^d], \quad (37)$$

where P^{BS} denotes the put price of Black and Scholes (1973).

6 Retrieving barrier option prices via the parities and moments

The above moments together with the put-call and in-out parities can also express barrier option prices in a simple and intuitive manner. In fact, all 16 barrier option prices can be obtained via the parities obtained in this paper when starting from only 4 explicit price equations. In what follows, we distinguish 4 cases: barrier options written on an upper barrier (up-options) and a lower barrier

(down-options) with in each case two set-ups depending on the relative position of the exercise price with respect to the barrier.

6.1 Up-options with $K > H$

We have 4 possible up-options, namely the up-and-out call and put as well as the up-and-in call and put. In order to value these 4 barrier options, we need only one explicit price equation and we, arbitrarily, choose for the value of $P_{u,o,K>H}$ that, inclusive of an out-rebate, emerges as:

$$\begin{aligned} P_{u,o,K>H} &= \exp[-r\Delta] \int_0^H [K - S_T] p_{nh,u}[S_T, T; S_t, t] dS_T + V[RO_j^u] \\ P_{u,o,K>H} &= \exp[-r\Delta] (KM_{0,nh,u} - M_{1,nh,u}) + V[RO_j^u], \end{aligned} \quad (38)$$

where it is to be remembered that also the value of the rebate can be expressed in terms of moments. Rubinstein and Reiner (1991a) specify this put price for $j = 1$, i.e. for the case of a constant out-rebate, which can easily be confirmed by inserting the moment expressions (5) and (6) and the value of the rebate $V[RO_1^u]$ into the put price (38).

The prices of the other 3 up-options then can be obtained from $P_{u,o,K>H}$ together with the parities (22), (35) and (31), respectively:

$$C_{u,o,K>H} = P_{u,o,K>H} - \exp[-r\Delta] \{KM_{0,nh,u} - M_{1,nh,u}\}, \quad (39)$$

$$P_{u,i,K>H} = P^{BS} - P_{u,o,K>H} + V[RI_i^u] + V[RO_j^u],$$

$$C_{u,i,K>H} = C^{BS} - C_{u,o,K>H} + V[RI_i^u] + V[RO_j^u]. \quad (40)$$

These 3 option values correspond with the expressions, for $i = j = 1$, that are derived in Rubinstein and Reiner (1991a). Note that the above 3 expressions still allow for some simplifications. For instance, the up-and-out call for $K > H$ can never attain intrinsic value since the stock price cannot move above K without first touching the barrier H . The price of this barrier option then must be $V[RO_j^u]$. This is immediately confirmed when plugging the put price (38) into the expression of the call price in (39). Similarly, the value of the up-and-in call must equal the call price of Black and Scholes (1973) augmented by the current value of the in-rebate. Indeed, both give the same payoff at time T , namely RI_i^u when no hitting takes place and $\max[0, S_T - K]$ when hitting takes place since moving above K can only take place after hitting H . As argued earlier, $C_{u,o,K>H} = V[RO_j^u]$ such that indeed the call price (40) simplifies into $C_{u,i,K>H} = C^{BS} + V[RI_i^u]$.

6.2 Up-options with $K < H$

The put price $P_{u,o,K < H}$ can be taken from the extant literature, e.g. from Rubinstein and Reiner (1991a), when $j = 1$. This price as well as the prices for other values of j can be calculated via the integral solutions in the Appendix as:

$$\begin{aligned}
P_{u,o,K < H} &= \exp[-r\Delta] \int_0^K [K - S_T] p_{nh,u}[S_T, T; S_t, t] dS_T + V [RO_j^u] \\
P_{u,o,K < H} &= \exp[-r\Delta] \left\{ K \int_0^K p_{nh,u}[S_T, T; S_t, t] dS_T - \int_0^K S_T p_{nh,u}[S_T, T; S_t, t] dS_T \right\} + V [RO_j^u] \\
P_{u,o,K < H} &= \exp[-r\Delta] \left\{ K \left(\mathcal{J}_{1;y_1,0,K} - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \mathcal{J}_{1;y_2,0,K} \right) - \left(\mathcal{J}_{2;y_1,0,K} - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \mathcal{J}_{2;y_2,0,K} \right) \right\} \\
&\quad + V [RO_j^u].
\end{aligned}$$

The other 3 up-option prices for $K < H$ then can be retrieved by employing the parities (21), (34) and (30):

$$\begin{aligned}
C_{u,o,K < H} &= P_{u,o,K < H} - \exp[-r\Delta] \{ KM_{0,nh,u} - M_{1,nh,u} \}, \\
P_{u,i,K < H} &= P^{BS} - P_{u,o,K < H} + V [RI_i^u] + V [RO_j^u], \\
C_{u,i,K < H} &= C^{BS} - C_{u,o,K < H} + V [RI_i^u] + V [RO_j^u].
\end{aligned}$$

6.3 Down-options with $K > H$

The put price $P_{d,o,K > H}$ is documented in the literature for $j = 1$. However, for all values of j also the following derivation can be employed:

$$\begin{aligned}
P_{d,o,K > H} &= \exp[-r\Delta] \int_H^K [K - S_T] p_{nh,d}[S_T, T; S_t, t] dS_T + V [RO_j^d] \\
P_{d,o,K > H} &= \exp[-r\Delta] \left\{ K \int_H^K p_{nh,d}[S_T, T; S_t, t] dS_T - \int_H^K S_T p_{nh,d}[S_T, T; S_t, t] dS_T \right\} + V [RO_j^d] \\
P_{d,o,K > H} &= \exp[-r\Delta] \left\{ K \left(\mathcal{J}_{1;y_1,H,K} - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \mathcal{J}_{1;y_2,H,K} \right) - \left(\mathcal{J}_{2;y_1,H,K} - \left(\frac{H}{S_t}\right)^{2(\lambda-1)} \mathcal{J}_{2;y_2,H,K} \right) \right\} \\
&\quad + V [RO_j^d].
\end{aligned}$$

The other 3 down-option prices for $K > H$ then follow from the parities (24), (37) and (33) as:

$$\begin{aligned} C_{d,o,K>H} &= P_{d,o,K>H} - \exp[-r\Delta] \{KM_{0,nh,d} - M_{1,nh,d}\}, \\ P_{d,i,K>H} &= P^{BS} - P_{d,o,K>H} + V \left[RI_i^d \right] + V \left[RO_j^d \right], \\ C_{d,i,K>H} &= C^{BS} - C_{d,o,K>H} + V \left[RI_i^d \right] + V \left[RO_j^d \right]. \end{aligned}$$

6.4 Down-options with $K < H$

The expression for the put price $P_{d,o,K<H}$ is particularly simple since the option part of the contract can never attain intrinsic value such that:

$$P_{d,o,K<H} = V \left[RO_j^d \right].$$

The parities in (23), (36) and (32) then specify the other 3 down-option prices as:

$$\begin{aligned} C_{d,o,K<H} &= P_{d,o,K<H} - \exp[-r\Delta] \{KM_{0,nh,d} - M_{1,nh,d}\}, \\ P_{d,i,K<H} &= P^{BS} - P_{d,o,K<H} + V \left[RI_i^d \right] + V \left[RO_j^d \right], \\ C_{d,i,K<H} &= C^{BS} - C_{d,o,K<H} + V \left[RI_i^d \right] + V \left[RO_j^d \right]. \end{aligned}$$

7 Conclusions

The put-call parity for standard options establishes a simple relationship between the prices of European put and call options that share the same exercise price and time to maturity. However, the static hedge argument of Stoll (1969) relies on the assumption that options exist at the present point of time and remain alive until the maturity date. It then cannot apply to barrier options since knock-out options are cancelled when the underlying asset price touches a prespecified barrier and knock-in options only create standard options when the underlying touches the barrier.

However, a probabilistic argument allows us to derive put-call parities also for barrier put and call options that have the same exercise price, the same barrier and identical time until maturity. The resulting parity is very similar to the familiar put-call parity and has an intuitive, probabilistic interpretation. In fact, the difference between put and call prices still is expressed in terms of the stock price and the discounted exercise price, but these will now be linked to the moments of the relevant defective distributions. Intuitively, the link to the zeroth and first moments acts as a probability adjustment when compared with the traditional put-call parity. Such adjustment is required since option cancellation and the potential lack of option creation in barrier options bring a reduction in probabilities and expectations. This argument also offers a probabilistic interpretation of the put-call

parity of Stoll (1969). Indeed, lifting (decreasing) the upper (lower) barrier to $+\infty$ (0) in the put-call parities for barrier options removes the possibility of hitting the barrier. The need for the probability adjustment then vanishes which causes the put-call parities for barrier options to collapse into the parity for standard options.

The moment expressions derived in this paper are also useful building blocks for the valuation of rebates. In fact, we employ them to easily value various novel types of for instance time-dependent rebate specifications. For instance, we allow the rebates in knock-out options to linearly and exponentially depend upon the time that remains between the time of first hitting and the expiration date of the option. Indeed, investors may well be interested in (combinations of) such time-dependent rebates in order to bring the rebate payment in line with the higher opportunity costs involved in holding an option that is cancelled at a later date and/or to finetune the rebate in function of cash management and risk profile. The moments together with the put-call parities and the in-out parities also express all published 16 single-barrier option values in a simple and intuitive manner. In fact, only 4 explicit pricing equations are required as the other 12 prices then can be retrieved via the barrier put-call and in-out parities.

The methods and results in this paper allow for several interesting extensions. The above relations can easily be modified for alternative underlying instruments such as foreign exchange and dividend-paying stocks. The results also can be extended towards double-barrier set-ups both for constant barriers as well as the curved boundaries in Kunitomo and Ikeda (1992) for which the required allocation of first-hitting time density across the barriers can be handled via Lemma 5.2.9 in Knight (1981). The integral expressions of Cho (1971) that are at the center of part of the solutions in the Appendix actually can be used to value a wide variety of additional time-dependent rebates. For instance, rebates that grow over time but that are only active within one or more subperiods of the option's life can easily be valued via these integral expressions. Given the close relation of our results to the valuation of binary barrier options, the results in Rubinstein and Reiner (1991b) can likewise be extended in various useful directions.

Appendix: Useful integral expressions

The solutions to the definite integrals $\mathcal{J}_{1;y_i,\beta_1,\beta_2}$ and $\mathcal{J}_{2;y_i,\beta_1,\beta_2}$ are obtained via repeated substitutions that transform the integrand into the standard normal density function. This yields:

$$\mathcal{J}_{1;y_i,\beta_1,\beta_2} = \frac{1}{\sqrt{2\pi\sigma^2\Delta}} \int_{\beta_1}^{\beta_2} \frac{1}{S_T} \exp \left[-\frac{(\ln S_T + y_i)^2}{2\sigma^2\Delta} \right] dS_T = \Phi \left[\frac{\ln \beta_2 + y_i}{\sigma\sqrt{\Delta}} \right] - \Phi \left[\frac{\ln \beta_1 + y_i}{\sigma\sqrt{\Delta}} \right]$$

and

$$\begin{aligned} \mathcal{J}_{2;y_i,\beta_1,\beta_2} &= \frac{1}{\sqrt{2\pi\sigma^2\Delta}} \int_{\beta_1}^{\beta_2} \exp \left[-\frac{(\ln S_T + y_i)^2}{2\sigma^2\Delta} \right] dS_T = \exp \left[-y_i + \frac{1}{2}\sigma^2\Delta \right] \\ &\times \left\{ \Phi \left[\frac{\ln \beta_2 + y_i - \sigma^2\Delta}{\sigma\sqrt{\Delta}} \right] - \Phi \left[\frac{\ln \beta_1 + y_i - \sigma^2\Delta}{\sigma\sqrt{\Delta}} \right] \right\}, \end{aligned}$$

where $\Phi[q] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^q \exp[-\frac{1}{2}x^2] dx$ is the standard normal distribution function.

The solutions to the definite integrals $\mathcal{J}_{3;a,b,\beta_1,\beta_2}$ and $\mathcal{J}_{4;a,b,\beta_1,\beta_2}$ are obtained on the basis of two expressions that were derived in Cho (1971) and later were reproduced as integrals 12 and 13 on p. 428 in Beck et al. (1992):¹⁰

$$\begin{aligned} \mathcal{J}_{3;a,b,\beta_1,\beta_2} &= \int_{\beta_1}^{\beta_2} s^{-\frac{3}{2}} \exp \left[-a^2s - \frac{b^2}{s} \right] ds = \frac{\sqrt{\pi}}{b} \exp[-2ab] \left\{ \Phi \left[\sqrt{2} \left\{ a\sqrt{\beta_2} - \frac{b}{\sqrt{\beta_2}} \right\} \right] - \right. \\ &\Phi \left[\sqrt{2} \left\{ a\sqrt{\beta_1} - \frac{b}{\sqrt{\beta_1}} \right\} \right] \left. \right\} - \frac{\sqrt{\pi}}{b} \exp[2ab] \left\{ \Phi \left[\sqrt{2} \left\{ a\sqrt{\beta_2} + \frac{b}{\sqrt{\beta_2}} \right\} \right] - \right. \\ &\Phi \left[\sqrt{2} \left\{ a\sqrt{\beta_1} + \frac{b}{\sqrt{\beta_1}} \right\} \right] \left. \right\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_{4;a,b,\beta_1,\beta_2} &= \int_{\beta_1}^{\beta_2} s^{-\frac{1}{2}} \exp \left[-a^2s - \frac{b^2}{s} \right] ds = \frac{\sqrt{\pi}}{a} \exp[2ab] \left\{ \Phi \left[\sqrt{2} \left\{ a\sqrt{\beta_2} + \frac{b}{\sqrt{\beta_2}} \right\} \right] - \right. \\ &\Phi \left[\sqrt{2} \left\{ a\sqrt{\beta_1} + \frac{b}{\sqrt{\beta_1}} \right\} \right] \left. \right\} + \frac{\sqrt{\pi}}{a} \exp[-2ab] \left\{ \Phi \left[\sqrt{2} \left\{ a\sqrt{\beta_2} - \frac{b}{\sqrt{\beta_2}} \right\} \right] - \right. \\ &\Phi \left[\sqrt{2} \left\{ a\sqrt{\beta_1} - \frac{b}{\sqrt{\beta_1}} \right\} \right] \left. \right\}. \end{aligned}$$

¹⁰The expressions in Cho (1971) and Beck et al. (1992) are presented in terms of the error function $\text{Erf}[x]$ that is related to the standard normal distribution function via the relationship $\text{Erf}[x] = 2\Phi[\sqrt{2}x] - 1$ (see Zelen and Severo, 1964).

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