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Fine structure of the Bethe ansatz for the spin- $\frac{1}{2}$ Heisenberg XXX model

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Abstract. We analyse the Bethe ansatz equations for the two-particle sector of the spin- $\frac{1}{2}$ Heisenberg XXX model on a one-dimensional lattice of length N . We show that, beginning at a critical lattice length of $N = 21.86$, new pairs of real solutions develop, whereas complex solutions start to disappear. The integers (that appear in the logarithmic form of the Bethe equations) of the new solutions do not fit into the conventional classification scheme. The total number of solutions in the two-particle sector remains unchanged and is in agreement with the claim that the $SU(2)$ extended Bethe ansatz gives a complete set of 2^N eigenstates.

1. Introduction

The one-dimensional isotropic spin- $\frac{1}{2}$ Heisenberg magnet (XXX spin chain) was the first model solved by means of the Bethe ansatz method [1]. The model describes interacting spins, situated on the sites of a periodic lattice of length N . The Hamiltonian is given by

$$H = J \sum_{i=1}^N (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z - \frac{1}{4}) \quad (1)$$

$$S_{N+1} = S_1$$

where $2S_i$ are the Pauli matrices. The Bethe ansatz provides the following set of eigenfunctions

$$|\Psi\rangle = \sum_{x_1 < x_2 < \dots < x_M} \Psi(x_1, x_2, \dots, x_M) S_{x_1}^- S_{x_2}^- \dots S_{x_M}^- |0\rangle$$

$$\Psi(x_1, x_2, \dots, x_M) = \sum_{P \in S_M} \exp \left(i \sum_{j=1}^M k_{P_j} x_j + i \sum_{\substack{j < l \\ P_j > P_l}} \phi_{P_j, P_l} \right). \quad (2)$$

Here $|0\rangle$ is the ferromagnetic state with all spins up and M is the number of overturned spins. The wavefunction is symmetric under interchange of any two of the

coordinates x_j and $k_1, \dots, k_M \in [0, 2\pi]$ are the spectral parameters, P is a permutation of M elements, and $\phi_{j,l}$ is defined by

$$2 \cot \left(\frac{\phi_{j,l}}{2} \right) = \cot \left(\frac{k_j}{2} \right) - \cot \left(\frac{k_l}{2} \right). \tag{3}$$

Imposing the periodic boundary conditions

$$\Psi(x_1, x_2, \dots, x_{M-1}, N + 1) = \Psi(1, x_1, x_2, \dots, x_{M-1}) \tag{4}$$

leads to the following set of Bethe equations for the spectral parameters k_1, \dots, k_M

$$e^{ik_j N} = \prod_{\substack{l=1 \\ l \neq j}}^M e^{i\phi_{j,l}} \quad j = 1, 2, \dots, M. \tag{5}$$

The usual parametrization for the Bethe equations and eigenfunctions is obtained by the substitution

$$\Lambda_j = \cot(k_j/2). \tag{6}$$

In this parametrization we find that

$$\Psi(x_1, x_2, \dots, x_M) = \sum_{P \in S_M} \prod_{j=1}^M \left(\kappa(\Lambda_{P_j}) \right)^{x_j} \prod_{\substack{i < l \\ P_i > P_l}} \kappa \left(\frac{\Lambda_{P_i} - \Lambda_{P_l}}{2} \right) \tag{7}$$

and the Bethe equations take the form

$$\left(\kappa(\Lambda_j) \right)^N = \prod_{\substack{l=1 \\ l \neq j}}^M \kappa \left(\frac{\Lambda_j - \Lambda_l}{2} \right) \quad j = 1, 2, \dots, M \tag{8}$$

$$\kappa(\Lambda) = \frac{\Lambda + i}{\Lambda - i}.$$

The energy of the state (7) is given by

$$E(\Lambda_1, \dots, \Lambda_M) = \sum_{j=1}^M \frac{-2J}{\Lambda_j^2 + 1}. \tag{9}$$

One fundamental assumption in investigating solutions of (8) is the so-called string hypothesis [1–3], which states that in the large N limit (for fixed M), any solution $\Lambda_1, \dots, \Lambda_M$ of (8) consists of a set of strings of the form

$$\Lambda_\alpha^{n,j} = \Lambda_\alpha^n + i(n + 1 - 2j) + O(e^{-\delta N}) \quad j = 1, 2, \dots, n \tag{10}$$

where $n \geq 1$ gives the length of the string, α labels different strings of a given length, j specifies the imaginary part of Λ (Λ_α^n denotes the real part of $\Lambda_\alpha^{n,j}$) and $\delta > 0$. The total number of strings of length n is denoted by M_n . Assuming the validity of this hypothesis, it is possible to reduce the Bethe equations (8) to a set of coupled

equations involving only the real parts Λ_α^n . The logarithmic form of these equations is given by [2, 4]

$$N\theta\left(\frac{\Lambda_\alpha^n}{n}\right) = 2\pi I_\alpha^n + \sum_{(m,\beta)\neq(n,\alpha)} \theta_{n,m}(\Lambda_\alpha^n - \Lambda_\beta^m) \tag{11}$$

where

$$\theta(x) = 2 \arctan(x) \tag{12}$$

and

$$\begin{aligned} \theta_{n,m}(x) &= \theta\left(\frac{x}{|n-m|}\right) + 2\theta\left(\frac{x}{|n-m|+2}\right) + \dots \\ &\quad + 2\theta\left(\frac{x}{n+m-2}\right) + \theta\left(\frac{x}{n+m}\right) \quad \text{if } n \neq m \\ \theta_{n,m}(x) &= 2\theta\left(\frac{x}{2}\right) + 2\theta\left(\frac{x}{4}\right) + \dots \\ &\quad + 2\theta\left(\frac{x}{2n-2}\right) + \theta\left(\frac{x}{2n}\right) \quad \text{if } n = m. \end{aligned} \tag{13}$$

Solutions of (11) are parametrized by integer (for $N - M_n$ odd) or half odd integer (for $N - M_n$ even) numbers I_α^n , which are distributed symmetrically around zero and should satisfy

$$|I_\alpha^n| \leq \frac{1}{2} \left(N - \sum_{m=1}^{\infty} t_{n,m} M_m - 1 \right) \tag{14}$$

where $t_{n,m} = 2 \text{Min}(m, n) - \delta_{m,n}$. It is generally believed that there is a one-to-one correspondence between solutions of the Bethe equations and sets of independent, non-repeating integers I_α^n . (By this we mean that $I_\alpha^n \neq I_\beta^n$ for $\alpha \neq \beta$, such that no two strings of the same length n have the same integer.) This is essential for all proofs of completeness of the (SU(2)-extended) Bethe ansatz solutions for the Heisenberg XXX model [1-4].

In this paper we investigate the validity of these assumptions in the two-particle sector. We find that counting the solutions of (8) using the integer method (based on the assumption that all strings are of the form (10)) [2-4] yields an incorrect result for lattice lengths N greater than $N_{\text{crit}} = 21.86$. For $N > N_{\text{crit}}$ there exist less complex string solutions than those predicted by counting the integers according to (14). These ‘missing’ strings are replaced by additional real solutions, which can be characterized by a pair of repeating integers. The total number of solutions thus remains unchanged, whereas their nature changes drastically. The change in type of solution to the Bethe equations had already been noticed by Bethe himself in [1] but for some reason has been overlooked in the more recent literature.

In section 2 we will give a graphical method for finding all complex two-particle solutions of the Bethe equations (8) and in section 3 we will apply a similar method to determine all real solutions in the two-particle sector. We will find a new type of real solution that is not present in the classification scheme based on (14) (and the assumption of non-repeating integers). In section 4 we classify the new type of real solution in the large N limit and show that the number of new solutions is of order \sqrt{N} .

2. String solutions of the Bethe equations in the two-particle sector

Due to the symmetry of (8) under complex conjugation, complex solutions Λ_1, Λ_2 must be of the form $\Lambda_1 = (\Lambda_2)^*$, where * denotes complex conjugation. This enables us to reduce the Bethe equation (8) to a single equation in the variables x and y with $\Lambda_1 = x + iy$:

$$\left(\frac{x + i(y + 1)}{x + i(y - 1)}\right)^N = \frac{y + 1}{y - 1}. \tag{15}$$

Dividing (15) by its conjugate equation we obtain

$$\left(\kappa\left(\frac{x}{y + 1}\right)\right)^N \left(\kappa\left(\frac{x}{1 - y}\right)\right)^N = 1. \tag{16}$$

After taking the logarithm this yields the analogue to (11) for arbitrary string width y :

$$N \left[\theta\left(\frac{x}{y + 1}\right) + \theta\left(\frac{y - 1}{x}\right) \right] = 2\pi I_\alpha^2. \tag{17}$$

Clearly this reduces to (11) for $y = 1$. The cut of $\tan^{-1}(z)$ has been chosen from i to $i\infty$ and from $-i$ to $-i\infty$ along the imaginary axis. Note that I_α^2 is integer (half odd integer) for even (odd) lattice length N . Taking the magnitude of (15) we arrive at

$$\left(\frac{x^2 + (y + 1)^2}{x^2 + (y - 1)^2}\right)^N = \left(\frac{y + 1}{y - 1}\right)^2. \tag{18}$$

The curve B_N that solves (18) has two branches as shown in figure 1 and includes the whole x -axis ($y = 0$) as well. The ‘inner’ branch intersects the x -axis for $N \gg 1$ at

$$x^{(1)} = \pm\sqrt{(N - 1)}. \tag{19}$$

The ‘outer’ branch has the asymptotes [5]

$$y = \pm \frac{1}{\sqrt{(N - 1)}} x. \tag{20}$$

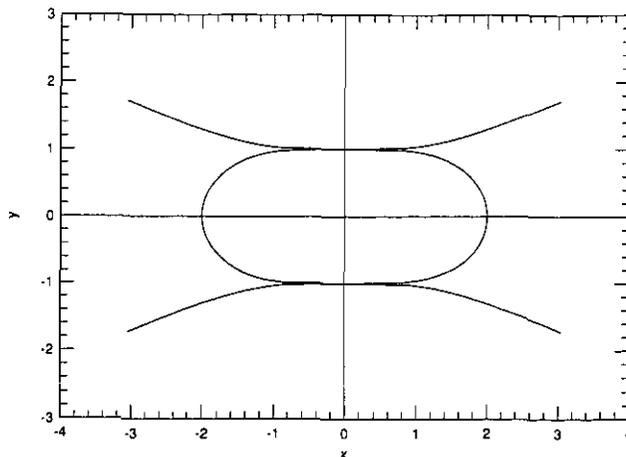


Figure 1. The curve B_N for $N = 5$.

To obtain solutions of (15) we also have to discuss the phases. Noting that the right-hand side of (15) is real we can write

$$\left(\frac{x + i(y + 1)}{x + i(y - 1)}\right)^N = \frac{y + 1}{y - 1} = \pm r^N \tag{21}$$

where $r \in [0, \infty)$. Taking the N th root of this equation we arrive at

$$\left(\frac{x + i(y + 1)}{x + i(y - 1)}\right) = e^{im\pi/N} r \quad m = 0, 1, \dots, 2N - 1. \tag{22}$$

We denote the solutions of (22) without restriction on r for given m by C_m . Looking at (21) we see that there is a constraint on the allowed range of r in (22). Even (odd) m correspond to the + (-) sign on the right-hand side of (21) and thus to $|y| > 1$ ($|y| < 1$). This constraint selects a certain part K_m of the curve C_m and only points (x, y) that lie on K_m are solutions of (21).

A parametrization of C_m is given by

$$x(r) = \frac{-2r \sin(\pi m/N)}{1 + r^2 - 2r \cos(\pi m/N)} \quad y(r) = \frac{1 - r^2}{1 + r^2 - 2r \cos(\pi m/N)} \tag{23}$$

where $r \in [0, \infty)$. The constraint mentioned earlier has to be imposed on these solutions. The curves C_m intersect the x -axis at ($r = 1$)

$$x_m^{(2)} = \pm \cot(\pi m/2N). \tag{24}$$

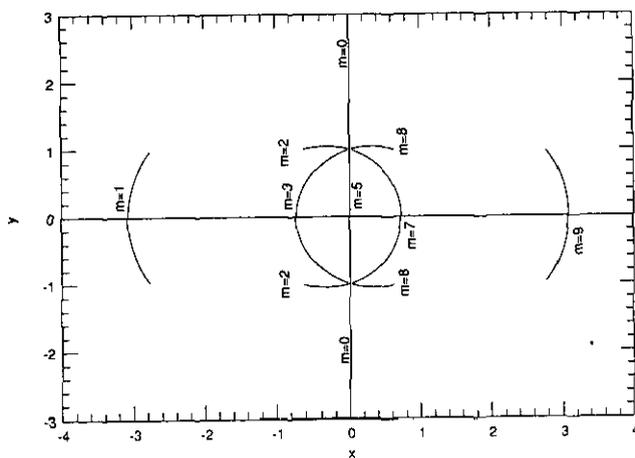


Figure 2. The curves K_m for lattice length $N = 5$.

Solutions of the Bethe equations (15) are given by intersection points of the curves K_m with the curves B_N . In figures 3 and 4 we show the resulting curves for lattice lengths $N = 5$ and $N = 21$. Inspection of figure 3 shows that there exist two string solutions on the lattice with five sites. This is in agreement with the counting according to (14). Numerically one finds two string solutions with spectral parameters

$$\begin{aligned} \Lambda_1^{2,1} &= -0.63 + 1.04i = -\Lambda_2^{2,2} \\ \Lambda_1^{2,2} &= -0.63 - 1.04i = -\Lambda_2^{2,1} \end{aligned} \tag{25}$$

which is in perfect agreement with figure 3 and the standard string picture. We see that for even and sufficiently small N every curve K_m with $1 < m < N/2$ and $3N/2 < m < 2N - 1$ has exactly two intersection points with B_N , which leads to one string solution for any such m and, in addition, we have one string solution at zero $(x, y) = (0, \pm i)$, which can be shown to correspond to the wavefunction

$$|\Psi\rangle = \sum_{y=1}^N (-1)^y S_y^- S_{y+1}^- |0\rangle. \quad (26)$$

(Clearly this state only makes sense on a lattice with an even number of sites.) This gives a total of $N - 3$ solutions.

For odd and small enough N every curve K_m with $1 < m < (N + 1)/2$ and with $(3N - 1)/2 < m < 2N - 1$ gives one solution which also adds up to a total of $N - 3$.

Using the standard string counting (14) we find that $|I_\alpha^2| \leq (N - 4)/2$ which leads to a total of $N - 3$ solutions for N either even or odd. Thus one would think that (14) indeed predicts the correct number of solutions to the Bethe equations. This is not true, however, as can be seen in figure 5 which shows all string solutions for $N = 25$. Clearly the curves K_3 and K_{47} do not lead to any intersection points and we are thus left with a total of $25 - 5 = 20$ solutions, two less than predicted by (14). The situation for 21 sites (see figure 4) is different, because the curves K_3 and K_{39} still intersect with the curve B_{21} (although the width of the corresponding strings is visibly much smaller than the 'ideal' value of 2).

Thus the following picture emerges: The strings corresponding to intersection points of K_3 and K_{2N-3} with B_N get narrower and narrower until they disappear for $N \geq 22$. We find that for $22 \leq N \leq 61$ there are only $N - 5$ string solutions to the Bethe equations. Starting at $N = 62$ we 'lose' an additional two solutions. Using (19) and (24) we see that whenever $|x_m^{(2)}| \geq |x^{(1)}|$ for any odd m with $3 \leq m \leq [N/2]$ (where $[z]$ denotes the integer part of z), we lose two string solutions of the Bethe equations, i.e. we find (an additional) two fewer solutions than predicted by (14).

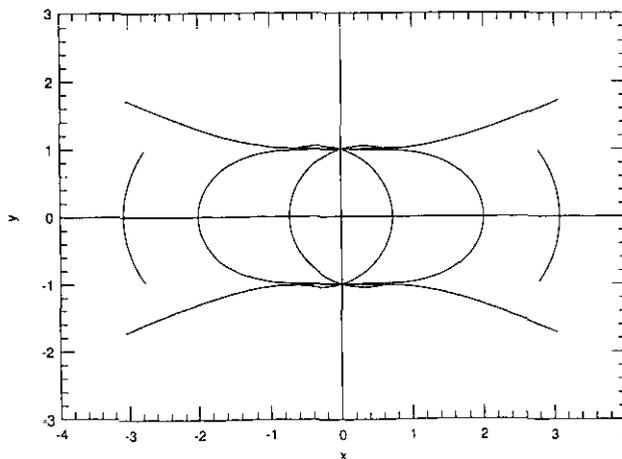


Figure 3. String solutions for lattice length $N = 5$.

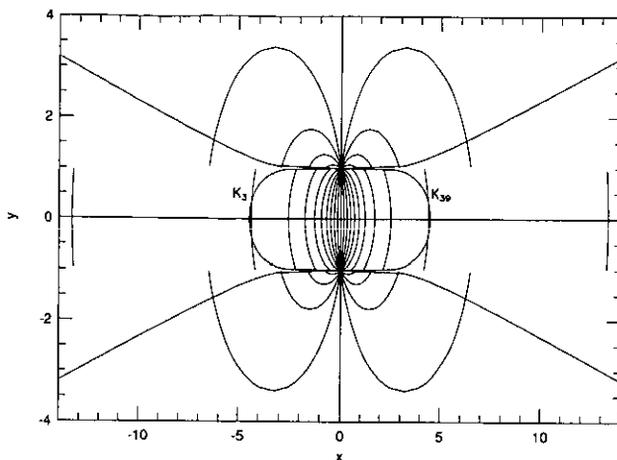


Figure 4. String solutions for $N = 21$.

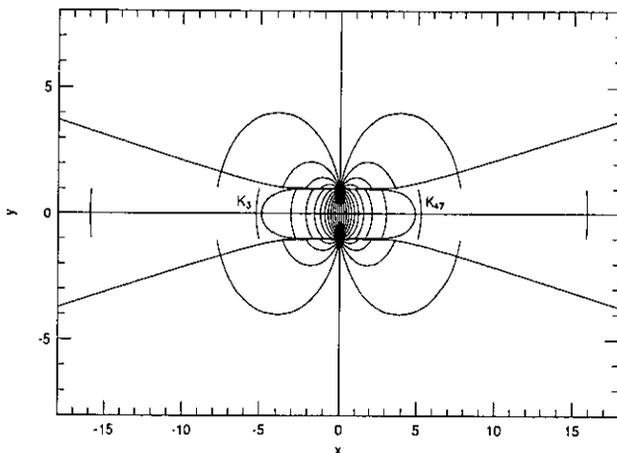


Figure 5. String solutions for $N = 25$.

3. Real solutions of the Bethe equations in the two-particle sector

The Bethe equations (5) in the two-particle sector are of the form

$$e^{ik_2 N} = e^{i\phi_{2,1}}$$

and

$$e^{i(k_1+k_2)N} = 1 \tag{27}$$

where $\phi_{1,2}$ is given by (3) and the second equality is obtained by multiplying the equations for k_1 and k_2 . Taking the logarithm and solving for k_1 one arrives at

$$\begin{aligned} k_1 &= 2 \cot^{-1}(\cot(k_2/2) - 2 \cot(k_2 N/2)) \\ k_1 &= -k_2 + (2\pi l/N) \bmod 2\pi \quad l = 1, \dots, N. \end{aligned} \tag{28}$$

Plotting k_1 as a function of k_2 and taking into account that the k s range between 0 and 2π , one arrives at the graphs shown in figures 6 and 7 for $N = 4$.

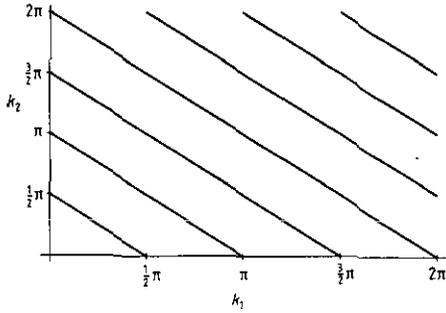


Figure 6. Plot of $k_1 = -k_2 + 2\pi l/N \text{ mod } 2\pi$ against k_2 for $N = 4$.

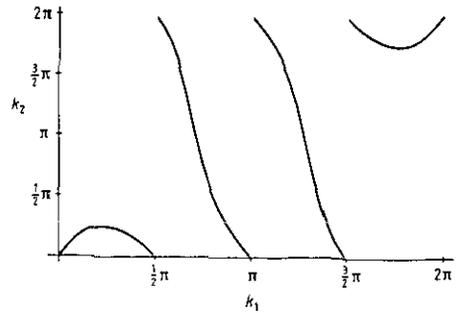


Figure 7. Plot of $k_1 = 2\cot^{-1}(\cot(k_2/2) - 2\cot(k_2N/2))$ against k_2 for $N = 4$.

The real solutions of the periodic boundary conditions for two particles are given by the intersection points of the graphs in figures 6 and 7. It is a well known fact that the wavefunction (2) vanishes if any two spectral parameters coincide ('Pauli principle'). Therefore we can ignore intersection points on the line $k_1 = k_2$, and, due to the symmetry of the problem under interchange of k_1 and k_2 , restrict our attention to the region $k_2 > k_1$. The requirement of all Λ s to be finite [2, 3] leads to the exclusion of the axis $k_1 = 0$. Figure 8 shows the resulting graph for $N = 4$, and we can clearly identify one solution, as predicted by the integer method ((14) gives $|I_\alpha^1| \leq \frac{1}{2}(4 - 2 - 1) = \frac{1}{2}$ and thus there are the two permitted values ± 1 for the two integers $I_{1,2}^1$, thus one solution).

The qualitative behaviour of the graphs remains unchanged for $N \leq 21$ and the integer method predicts the right number, which is $\binom{N-2}{2}$, of solutions. A close investigation of the situation for a larger lattice length like $N = 25$ reveals an interesting development, however. The dotted line in figure 9 is the line $k_1 = k_2$. As mentioned earlier there is a clear symmetry in the graph with respect to the interchange of k_1 and k_2 . Due to this symmetry and the 'Pauli principle' (the vanishing of the wavefunction for $k_1 = k_2$) we only have to consider the region below the dotted line in figure 9. As can be seen in figure 9, the line with $l = 3$ of (28) approaches the \cot^{-1} curve rather closely in the regions $k_1 \approx k_2 \approx 0.4$ and $k_1 \approx k_2 \approx 5.9$.

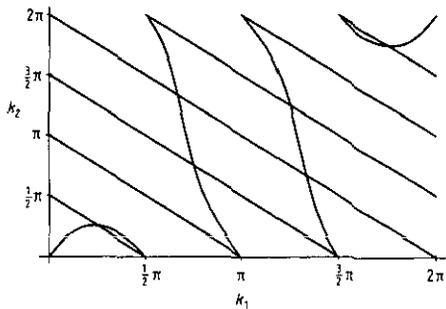


Figure 8. Real solutions for $N = 4$.

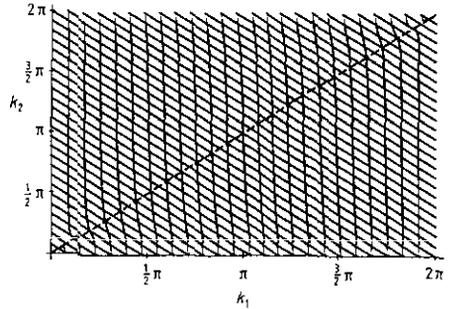


Figure 9. Real solutions for $N = 25$.

Magnification of these regions reveals that instead of one intersection point with $k_1 = k_2$ (as was the case for $N \leq 21$), there now exist two additional intersection points (related by the interchange $k_1 \leftrightarrow k_2$, so that we must count only one of them) in each region (see figure 10), so that we obtain a total of 255 real solutions. The integer method predicts two real solutions less, as (14) gives $|I_\alpha^1| \leq \frac{1}{2}(25-2-1) = 11$ and thus there are 23 permitted values for $I_{1,2}^1$, which leads to $\binom{23}{2} = 253$ solutions.

A numerical solution of the Bethe equations (8) for $N = 25$ (see appendix I) is in complete agreement with the situation described here and the new solutions are indeed found in the regions predicted by the graphical method.

The critical lattice length N_{crit} , for which new solutions start to appear is found by equating the first derivatives of the two intersecting curves. The numerical value is found to be $N_{crit} = 21.86$. From figure 9 it becomes clear that for large lattice lengths N further additional solutions will develop, as the situation that occurs for the second 'branch' of the \cot^{-1} curve and the line with $l = 3$ at $N_{crit} = 21.86$, will eventually have an analogue in the third, fourth, etc, 'branch' of the \cot^{-1} curve. Numerical studies reveal that the next two additional real solutions appear for $N = 61.34$ (the line with $l = 5$ intersecting the third branch of the \cot^{-1} curve).

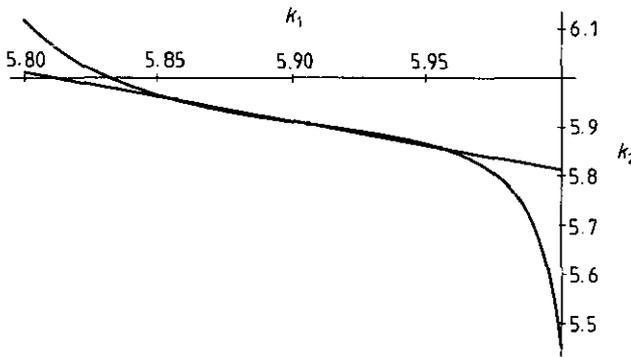


Figure 10. Magnification of the upper right-hand corner of figure 9.

4. Large- N analysis

The new real solutions always develop in the vicinity of the curve $k_1 = k_2$. Therefore one can solve the equations (28) perturbatively in powers of $1/N$ around this line. One finds that for a given N

$$\begin{aligned}
 k_1 &= \frac{2\pi(n-1)}{N} \left[1 + \frac{2n}{N} + \frac{4n(n^2 + (n-1)^2)}{N^2} + O(N^{-3}) \right] \\
 k_2 &= \frac{2\pi n}{N} \left[1 - \frac{2(n-1)}{N} - \frac{4(n-1)(n^2 + (n-1)^2)}{N^2} + O(N^{-3}) \right]
 \end{aligned}
 \tag{29}$$

$$n = 2, 3, \dots, n_{max} = \left[\frac{\sqrt{N}}{\pi} + \frac{1}{2} \right]$$

where $[x]$ denotes the integer part of x . This means that there exist $n_{max} - 1$ pairs of new real solutions for a given N . The 'critical' lattice length, at which new real

solutions of the form (29), with parameter n , first develop is given by the solution of the equation

$$N_{\text{crit}}^{(n)} \sin^2 \left(\frac{\pi(n - \frac{1}{2})}{N_{\text{crit}}^{(n)}} \right) = 1. \quad (30)$$

Perturbatively one finds that

$$N_{\text{crit}}^{(n)} = \pi^2(n - \frac{1}{2})^2. \quad (31)$$

The results for $n = 2, 3$ are in very good agreement with the numerically found values given earlier. Using (6), (11) and (29) we can compute the integers for the new real solutions. Elementary calculations yield for the solution with parameter n

$$I_1^1 = I_2^1 = \frac{N+1}{2} - n. \quad (32)$$

Thus the new real solutions are characterized by a pair of *repeating* integer numbers. This is a new phenomenon, which contradicts the usual assumption that all integers are independent and non-repeating.

5. Conclusions

In this paper we have shown that the total number of solutions of the Bethe equations in the two-particle sector is in agreement with the result obtained by counting integers according to (14) as carried out in [2–4]. We saw, however, that the integer method fails to predict the correct distribution of solutions among real and complex solutions. Furthermore we demonstrated that the ‘Pauli principle for the integers’, i.e. the assumption that all solutions of the Bethe equations can be characterized by a set of independent *non-repeating* (half-odd) integer numbers, is invalid. The new states we found are described by two identical integers and the physical effects of such excitations above the ferromagnetic vacuum can be studied along the lines described in [6].

Acknowledgments

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Appendix. Some numerical results for $N = 25$

In this appendix we give parts of the numerical solution of the Bethe equations in the two-particle sector. Table 1 lists the two additional real solutions as well as their integers, which are found to be repeating. Table 3 lists all (complex) string solutions together with their integers. We find that the solutions with ‘integer’ ± 9.5 are missing. Tables 2 and 4 give samples of the numerical results for the ‘regular’ real solutions and the corresponding integers.

Table 1. Extra real solutions.

Λ_1^1	Λ_2^1	(I_1^1, I_2^1)
6.076 130351 764 672	4.604 119967 471 151	(11, 11)
-6.076 130351 764 672	-4.604 119967 471 151	(-11, -11)

Table 2. Real solutions with integers of the form $(-I_a^1, I_a^1)$.

Λ_1^1	Λ_2^1	(I_1^1, I_2^1)
-0.131 652 483 159 000	0.131 652 483 159 000	(-0.999 999 9, 0.999 999 9)
-0.267 949 207 552 647	0.267 949 207 552 647	(-2.000 000 1, 2.000 000 1)
-0.414 213 554 923 667	0.414 213 554 923 667	(-3, 3)
-0.577 350 271 101 900	0.577 350 271 101 900	(-4, 4)
-0.767 326 987 742 179	0.767 326 987 742 179	(-5, 5)
-1.000 000 000 010 941	1.000 000 000 010 941	(-6, 6)
-1.303 225 372 841 047	1.303 225 372 841 047	(-7, 7)
-1.732 050 807 568 888	1.732 050 807 568 888	(-8, 8)
-2.414 213 562 373 096	2.414 213 562 373 096	(-9, 9)
-3.732 050 807 568 634	3.732 050 807 568 634	(-9, 9)
-7.595 754 112 725 638	7.595 754 112 725 638	(-11, 11)

Table 3. Spectral parameters and integers of the string solutions.

$\Lambda^{2,j}$	I_a^2
7.550 697 954 902 596 ± 1.674 220 479 211 304 i	10.5
3.603 532 496 872 190 ± 1.060 262 279 024 803 i	8.5
2.760 451 929 520 186 ± 0.989 331 795 228 972 i	7.5
2.129 094 407 420 760 ± 1.000 733 494 880 967 i	6.5
1.654 575 100 754 238 ± 0.999 974 181 007 305 i	5.500 000 000 000 001
1.269 238 065 274 572 ± 1.000 000 336 229 817 i	4.500 000 000 000 002
0.941 128 564 704 973 ± 0.999 999 998 926 892 i	3.499 999 999 999 998
0.649 839 392 464 719 ± 1.000 000 000 000 355 i	2.499 999 999 999 996
0.381 520 404 437 133 ± 1.000 000 000 000 000 i	1.499 999 999 999 998
0.125 829 380 000 000 ± 1.000 000 000 000 000 i	0.500 000 180 296 046
-0.125 829 380 000 000 ± 1.000 000 000 000 000 i	-0.500 000 180 296 046
-0.381 520 404 437 138 ± 1.000 000 000 000 000 i	-1.500 000 000 000 017
-0.649 839 392 464 719 ± 1.000 000 000 000 355 i	-2.499 999 999 999 996
-0.941 128 564 704 973 ± 0.999 999 998 926 892 i	-3.499 999 999 999 998
-1.269 238 065 274 572 ± 1.000 000 336 229 817 i	-4.500 000 000 000 002
-1.654 575 100 754 238 ± 0.999 974 181 007 305 i	-5.500 000 000 000 001
-2.129 094 407 420 760 ± 1.000 733 494 880 967 i	-6.5
-2.760 451 929 520 186 ± 0.989 331 795 228 972 i	-7.5
-3.603 532 496 872 190 ± 1.060 262 279 024 803 i	-8.5
-7.550 697 954 902 596 ± 1.674 220 479 211 304 i	-10.5

Table 4. Real solutions with one integer (but not both) equal to 11.

Λ_1^1	Λ_2^1	(I_1^1, I_2^1)
2.660818949240510	6.937316937832509	(10, 11)
1.884097488224177	7.077265200079445	(9, 11)
1.416020427893359	7.142422144443372	(8, 11)
1.092353837732522	7.181455030337195	(7, 11)
0.848008820447725	7.208239378457300	(6, 11)
0.651308925902454	7.228334310008103	(5, 11)
0.484695845133505	7.244430628292553	(4, 11)
0.337407004916716	7.258009100128192	(3, 11)
0.202216795753967	7.269970770157027	(2, 11)
0.073793040214152	7.280916005878645	(1, 11)
-0.052243320518647	7.291284166897977	(0, 11)
-7.595754112725151	7.595754112725151	(-11, 11)
-3.784235077366884	7.492364916365747	(-10, 11)
-2.468071197228016	7.438644066117615	(-9, 11)
-1.782616957891460	7.404677027965097	(-8, 11)
-1.350629918256678	7.380566510373989	(-7, 11)
-1.045151822554743	7.362036505894714	(-6, 11)
-0.811201425781150	7.346914476556023	(-5, 11)
-0.620884146211136	7.333960813623058	(-4, 11)
-0.458329400981613	7.322397412814130	(-3, 11)
-0.313613511426339	7.311689634578599	(-2, 11)
-0.179952252749314	7.301432693193145	(-1, 11)

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