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New eigenstates of the 1-dimensional Hubbard model

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Carrying out a program proposed by C.N. Yang, we prove that all "regular" (as defined in the paper) Bethe-Ansatz states of the 1-dimensional Hubbard model on a lattice of finite length $L$ are lowest-weight vectors of an SO(4) algebra. Thus new eigenstates can be obtained by acting with the SO(4) raising operators on the Bethe-Ansatz states. In a following publication we will show that the SO(4) structure in combination with the Bethe Ansatz leads to a complete set of eigenfunctions for the 1-dimensional Hubbard model (asymptotically for large but finite lattice lengths $L$).

1. Introduction

1.1. Eigenfunctions and periodic boundary conditions

Interacting 1-dimensional fermionic systems are of great theoretical interest for a variety of reasons. Some of them are of immediate applicability as models for the physics of quasi-one-dimensional systems. Apart from that, these 1-dimensional models are much easier to solve than their higher-dimensional versions and therefore are a first step towards an understanding of interacting systems in higher dimensions. As an example we mention strongly-correlated electrons in 2 dimensions, which are believed to be of great relevance for various properties of high-$T_c$ superconductors. The Hubbard model, being the prototype of all models describing correlated fermions, has the additional feature that its 1-dimensional version is exactly solvable. This model describes electrons that can hop between neighbour-
ing sites in a chain of length $L$ and that repel or attract each other if two of them (with necessarily opposite spins) occupy the same site. In this paper we will focus on an analysis of the 1-dimensional Hubbard model in a finite volume (i.e. $L$ finite).

Electrons on the lattice are described by canonical fermionic operators $c_{n,\sigma}$, where $n$ is the number of the site and $\sigma$ is the value of the $z$-component of the spin ($\sigma = \pm 1$). The anticommutation relations between the creation and annihilation operators are given by $\{c_{n,\sigma}^\dagger, c_{m,\tau}\} = \delta_{n,m} \delta_{\sigma,\tau}$. The Hamiltonian of the model in the form used by Yang and Zhang [1] is

$$H = - \sum_{i=1}^{L} \sum_{\sigma=1,-1} \left( c_{i,\sigma}^\dagger c_{i+1,\sigma} + c_{i+1,\sigma}^\dagger c_{i,\sigma} \right) + U \sum_{i=1}^{L} \left( n_{i,1} - \frac{1}{2} \right) \left( n_{i,-1} - \frac{1}{2} \right),$$

(1.1)

where $n_{i,\sigma} = c_{i,\sigma}^\dagger c_{i,\sigma}$ is the number operator for electrons with spin $\sigma$ on site $i$. $U$ is the coupling constant and can be either positive (repulsive case) or negative (attractive case). For later convenience we define $\mu = U/2i$.

The model was first solved by Lieb and Wu [2] using the nested Bethe Ansatz method [3]. In what follows we will use the form of the wave-functions as given by Woynarovich [4], which is equivalent to the form found by Lieb and Wu. The nested Bethe Ansatz provides us with the following set of eigenstates with $M$ spins down and $N$ spins up:

$$| \Psi_{M,N} \rangle = \sum_{1 \leq x_1, x_2, \ldots \leq L} \psi_{x_1,\ldots,x_M,x_{M+N}}(x_1, \ldots, x_{M+N}) \prod_{j=1}^{M} c_{x_j,-1}^\dagger \prod_{i=M+1}^{M+N} c_{x_j,1}^\dagger |0\rangle,$$

$$= \sum_{1 \leq x_1, \sigma_1, \ldots, \sigma_{M+N} \leq L} \psi_{\sigma_1,\sigma_2,\ldots,\sigma_{M+N}}(x_1, \ldots, x_{M+N}) \prod_{j=1}^{M+N} c_{x_j,\sigma_j}^\dagger |0\rangle,$$

(1.2)

where we have put $\sigma_1 = \ldots = \sigma_M = -1$, $\sigma_{M+1} = \ldots = \sigma_{M+N} = 1$.

The Bethe-Ansatz wave functions explicitly depend on the relative ordering of the $x_j$. We represent this dependence by a permutation $Q$ of $M+N$ elements, which is such that $1 \leq x_Q, \leq x_{Q_2} \leq \ldots \leq x_{Q_{M+N}} \leq L$. In the sector $Q$ the general Bethe wave function for $M$ spins down and $N$ spins up reads

$$\psi_{\sigma_1,\sigma_2,\ldots,\sigma_{M+N}}(x_1, \ldots, x_{M+N})$$

$$= \sum_{P \in S_{M+N}} \text{sgn}(Q) \text{sgn}(P) \exp \left( i \sum_{j=1}^{M+N} k_P x_{Q_j} \right) \varphi(y_1, \ldots, y_M | P).$$

(1.3)
The $P$-summation extends over all permutations of $M + N$ elements and $\text{sgn}(\Pi)$ is the sign of the permutation $\Pi$. The amplitudes $\varphi(y_1, \ldots, y_M | P)$ are of the form

$$\varphi(y_1, \ldots, y_M | P) = \sum_{\pi \in S_M} A_{\pi} \prod_{i=1}^{M} F_P(A_{\pi i}, y_i)$$

with

$$F_P(A_\pi, y) = \left\{ \prod_{i=1}^{y-1} \frac{e^{(i)}(P_i)}{e^{(i)}(P_y)} \right\} \frac{1}{\prod_{i=1}^{y-1} \sin(k_{P_i}) - A_j - (U/4i)} \left( \frac{1}{\sin(k_{P_y}) - A_j + (U/4i)} \right),$$

where we defined

$$e^{(i)}(i) = \sin(k_1) - A_3 \pm (U/2)$$

and

$$A_{\pi \tau} = \frac{\Lambda_{\pi \tau} - \Lambda_{\pi \tau} - u}{\Lambda_{\pi \tau} - \Lambda_{\pi \tau} + u}.$$  

By $\pi = (\pi_1, \pi_2, \ldots, \pi_i, \pi_{i+1}, \ldots, \pi_M)$ we denote a permutation of $M$ elements and $(t, t + 1)\pi = (\pi_1, \pi_2, \ldots, \pi_{i+1}, \pi_i, \ldots, \pi_M)$.

The amplitudes $\varphi(y_1, \ldots, y_M | P)$ depend on $\sigma_1, \ldots, \sigma_{M+N}$ and on $Q$ through the numbers $y_1, \ldots, y_M$, which are defined to be the positions of the down spins among the spins in the series $\sigma_{Q_1}, \sigma_{Q_2}, \ldots, \sigma_{Q_{M+N}}$ in increasing order, i.e.

$$1 \leq y_1 < y_2 < y_3 < \ldots < y_M \leq M + N.$$

For example, for one spin down and one spin up (and $\sigma_1 = -1, \sigma_2 = 1$) we have the two cases $y = 1$ (if the spin down is to the left, which holds in the $Q = (1d)$ sector) and $y = 2$ (if the spin down is to the right, which holds for $Q = (21)$).

As we already indicated in (1.2), we will choose the notation such that the $M$ down spins are at the positions $x_1, \ldots, x_M$, i.e. $\sigma_1 = \ldots = \sigma_M = -1$ and $\sigma_{M+1} = \ldots = \sigma_{M+N} = 1$.

We see that all solutions are characterised by $(M + N)$ momenta $\{k_j | j = 1, \ldots, M + N\}$ of charged spinless excitations (holons), and $M$ rapidities $\{\Lambda_k | k = 1, \ldots, M\}$ of spin waves (spinons).

Imposing the periodic boundary conditions

$$\psi_{\sigma_1, \ldots, \sigma_{M+N}}(x_1, \ldots, x_i, \ldots, x_{M+N}) = \psi_{\sigma_1, \ldots, \sigma_{M+N}}(x_1, \ldots, x_i + L, \ldots, x_{M+N})$$
leads to the following set of Bethe equations:

\[ e^{i k_j L} = \prod_{\alpha=1}^{M} \frac{e^{i (\alpha_j)}}{e^{i (\alpha_j)}} = \prod_{\alpha=1}^{M} \frac{\sin(k_j) - \Lambda_\alpha - (U/4i)}{\sin(k_j) - \Lambda_\alpha + (U/4i)}, \quad j = 1, \ldots, M + N, \quad (1.7) \]

\[ \prod_{i=1}^{M+N} e^{i (\beta j)} = \prod_{i=1}^{M+N} \frac{\sin(k_j) - \Lambda_\beta - (U/4i)}{\sin(k_j) - \Lambda_\beta + (U/4i)} = -\prod_{\alpha=1}^{M} \frac{\Lambda_\alpha - \Lambda_\beta - u}{\alpha - \Lambda_\beta + u}, \quad \beta = 1, \ldots, M. \quad (1.8) \]

Energy and momentum, i.e. the eigenvalues of the hamiltonian (1.1) and the logarithm of the translation operator, of the system in a state corresponding to a solution of eqs. (1.7) and (1.8) are

\[ E_{M+N} = -2 \sum_{i=1}^{M+N} \cos(k_i) + \frac{1}{2} U \left[ \frac{1}{2} L - (M + N) \right], \]

\[ P = \sum_{i=1}^{M+N} k_i. \quad (1.9) \]

As mentioned before, \( N \) and \( M \) denote the numbers of particles with spin up and down, respectively.

Because of the antisymmetry of the product over \( \psi \)'s under interchange of any two of them, the wave functions \( \psi_{\sigma_1 \ldots \sigma_{M+N}}(x_1 \ldots x_{M+N}) \) can be (and have been) chosen to be completely antisymmetric under the simultaneous exchange \( x_k \leftrightarrow x_j \) and \( \sigma_k \leftrightarrow \sigma_j \), i.e.

\[ \psi_{\sigma_1 \ldots \sigma_j \ldots \sigma_k \ldots \sigma_{M+N}}(x_1 \ldots x_j \ldots x_k \ldots x_{M+N}) = -\psi_{\sigma_1 \ldots \sigma_k \ldots \sigma_j \ldots \sigma_{M+N}}(x_1 \ldots x_j \ldots x_k \ldots x_{M+N}). \quad (1.10) \]

**Definition 1.1**

Bethe-Ansatz states on a finite lattice of length \( L \) that have finite momenta \( k_i \) and rapidities \( \Lambda_j \), a nonnegative value of the total spin (i.e. \( N - M \geq 0 \)), and a total number of electrons less than or equal to the length of the lattice (i.e. \( M + N \leq L \)), are called regular.

There exist two symmetries of the model that can be used to obtain additional eigenstates starting from the regular ones [5]. First, there is the obvious invariance of the hamiltonian (1.1) under the transformation that exchanges up and down spins (therefore it is sufficient to know all eigenstates with a nonnegative value of the spin (i.e. \( N - M \geq 0 \)), i.e. \( c_{i,1} \rightarrow c_{i,-1} \) and \( c_{i,1}^\dagger \rightarrow c_{i,-1}^\dagger \). This symmetry does not change the number of particles so that we stay below or at half filling when we transform the regular states. Secondly, there exists the following unitary transfor-
mation around half filling, which also leaves the hamiltonian invariant (for \( \sigma = \pm 1 \) simultaneously):

\[
(-1)^j \tilde{c}_{j,\sigma} \rightarrow b_{j,\sigma}, \tag{1.11}
\]

\[
(-1)^j c_{j,\sigma} \rightarrow b_{j,\sigma}^\dagger. \tag{1.12}
\]

The anticommutation relations remain unchanged under this transformation but the highest weight vector of the Fock space changes from \( |0\rangle \) (which is annihilated by all \( c_{j,\sigma} \)) to the completely filled state \( |\Omega\rangle \) (which is annihilated by all \( c_{j,\sigma}^\dagger \)). Note that due to this symmetry it is possible to obtain the states with \( M + N \geq L \) from the ones with \( M + N \leq L \).

1.2 SO(4) SYMMETRY IN THE HUBBARD MODEL

The hamiltonian of the Hubbard model (in zero magnetic field and without chemical potential terms) exhibits an SO(4) symmetry on lattices with even lengths. Recently Yang and Zhang [1] and independently Pernici [7] gave a representation of this symmetry in terms of pairing operators. Apart from the usual spin SU(2), the SO(4) contains the so called \( \eta \)-pairing SU(2), that was previously discovered by Yang [8]. Application of the \( \eta \)-pairing raising operator creates a pair of (opposite spin) electrons with momentum \( \pi \) on the same site. This could be interpreted as an ultralocalised Cooper pair. The hamiltonian and momentum operator commute with the generators of the SO(4) *.

Thus, this symmetry can be used to characterise the eigenstates of the model. Notice however that application of the SO(4) raising/lowering operators changes the number of up/down spins, so that the spectrum within a sector with fixed number of spin up and spin down electrons is not degenerate.

The SO(4) algebra is given by

\[
\eta = \sum_{i=1}^L (-1)^i c_{i,1}^\dagger c_{i,-1}, \quad \eta^\dagger = (\eta)^\dagger, \quad \eta_z = \frac{1}{2} \sum_{i=1}^L (n_{i,1} + n_{i,-1}) - \frac{1}{2} L, \]

\[
\xi = \sum_{i=1}^L c_{i,1}^\dagger c_{i,-1}, \quad \xi^\dagger = (\xi)^\dagger, \quad \xi_z = \frac{1}{2} \sum_{i=1}^L (n_{i,-1} - n_{i,1}),
\]

\[
[\xi, \xi^\dagger] = -2 \xi_z, \quad [\eta, \eta^\dagger] = +\xi, \quad [\xi^\dagger, \xi_z] = -\xi^\dagger, \quad [\eta^\dagger, H] = 0 = [\xi^\dagger, H]. \tag{1.13}
\]

* The SO(4) does not exhaust all “simple” symmetries of the hamiltonian as shown by Shastry in ref. [9], because the transfer matrix of the model generates an infinite number of conserved charges that commute with the hamiltonian. Bethe-Ansatz states are eigenstates of all of these charges.
All other commutators vanish. Note that in our conventions $\xi_z$ denotes minus the third component of the total spin. Note also that

$$(\xi_z + \eta_z) |\Psi\rangle = \left( \sum_{i=1}^{L} n_{i,-1} - \frac{1}{2} L \right) |\Psi\rangle = (M - \frac{1}{2} L) |\Psi\rangle,$$

where $M - \frac{1}{2} L$ is an integer (because $L$ is even). Therefore the symmetry algebra is $SO(4)$ and not $SU(2) \times SU(2)$.

C.N. Yang suggested to us that the Bethe Ansatz for the 1-dimensional Hubbard model does not provide a complete set of eigenstates, but that the spectrum could be completed by using the $SO(4)$ structure. We carried out this program in two steps:

(i) In this paper we show that all regular states are lowest-weight vectors of the $SO(4)$ algebra (1.13), i.e.

$$\eta |\Psi_{M,N}\rangle = 0 \quad \text{and} \quad \xi |\Psi_{M,N}\rangle = 0.$$ 

This shows that the Bethe Ansatz does not provide a complete set of eigenfunctions for the Hubbard model, as additional eigenstates can be obtained by acting with the raising operators on the lowest-weight vectors. The new eigenstates are of the form

$$(\eta^{\uparrow})^\alpha (\xi^{\uparrow})^\beta |\Psi_{M,N}\rangle,$$

where $\alpha$ and $\beta$ are integer numbers.

(ii) In a forthcoming publication [10] we will show that the complete spectrum of the 1-dimensional Hubbard model can be obtained by acting with the $SO(4)$ raising operators on the regular Bethe states. This result is summarised by the following formula:

$$\sum_{0 < \eta < L/2 \atop 0 < \xi < L/2} n_{\text{reg}}(\eta, \xi) (2\eta + 1)(2\xi + 1) = 4^L,$$

where $n_{\text{reg}}(\eta, \xi)$ denotes the number of regular Bethe states with $\eta_z = -\eta$ and $\xi_z = -\xi$.

The proof of the lowest-weight property of the regular Bethe states will be given in sects. 3 (for the $\eta$ symmetry) and 4 (for the $\xi$ symmetry). Before giving the general proofs we shall first, in sect. 2, illustrate the structure of the model on a 2-site lattice, which is particularly simple and can be solved explicitly.

2. Results for the chain with two sites

In this section we solve the model explicitly on a lattice with two sites. We give all $4^2 = 16$ eigenstates and show that the Bethe Ansatz only provides 14 states. We
then prove that one of the two “missing” states lies manifestly outside the Bethe-Ansatz energy spectrum and thus cannot possibly be obtained from the Bethe Ansatz by any well defined limiting procedure. Finally we classify all states according to their transformation properties under the SO(4) algebra. This will make it obvious that for the 2-site model the SO(4) structure can be used to complete the spectrum.

2.1. EXPLICIT FORM OF THE EIGENSTATES

The Hamiltonian of the model for the chain with 2 sites is given by

\[ H = -2\left[ c^\dagger_{1,1}c_{2,1} + c^\dagger_{1,-1}c_{2,-1} + c^\dagger_{2,1}c_{1,1} + c^\dagger_{2,-1}c_{1,-1} \right] \\
+ U\left[ (n_{1,1} - \frac{1}{2})(n_{1,-1} - \frac{1}{2}) + (n_{2,1} - \frac{1}{2})(n_{2,-1} - \frac{1}{2}) \right]. \]

According to eq. (1.9) energy and momentum of Bethe states are given by:

\[ E_{M+N} = -2 \sum_{i=1}^{N+M} \cos(k_i) + \frac{1}{2}U\left[ (N + M) \right], \quad P = \sum_{i=1}^{N+M} k_i, \quad (2.1) \]

where \( M \) and \( N \) are the numbers of spins down and up, respectively.

The regular part of the Bethe Ansatz for the 2-site model is given by the following six normalised states (\( E \) denotes the respective energies):

\[
\begin{align*}
|0\rangle & \quad E = \frac{1}{2}U, \\
|\psi_1(\pi)\rangle & \quad (1/\sqrt{2}) (c^\dagger_{1,1} + c^\dagger_{1,-1}) |0\rangle \\
|\psi_1(0)\rangle & \quad (1/\sqrt{2}) (c^\dagger_{2,1} + c^\dagger_{2,-1}) |0\rangle \\
|\psi_1(\pi, 0)\rangle & \quad c^\dagger_{1,1} c^\dagger_{2,1} |0\rangle \\
|\psi_{-1,1}(k_1^-, k_2^-, | A^-\rangle) & \quad \left( 1/\sqrt{\frac{1}{8}D_+^2 + 2} \right) \left( -\frac{1}{2}D_- |\psi_1\rangle + |\psi_2\rangle \right) \\
|\psi_{-1,1}(k_1^+, k_2^+, | A^+\rangle) & \quad \left( 1/\sqrt{\frac{1}{8}D_+^2 + 2} \right) \left( -\frac{1}{2}D_+ |\psi_1\rangle + |\psi_2\rangle \right)
\end{align*}
\]

where

\[
\begin{align*}
|\psi_1\rangle & = (c^\dagger_{1,1} c^\dagger_{1,-1} + c^\dagger_{2,1} c^\dagger_{2,-1}) |0\rangle, \\
|\psi_2\rangle & = (c^\dagger_{1,1} c^\dagger_{2,-1} - c^\dagger_{1,-1} c^\dagger_{2,1}) |0\rangle, \\
D_\pm & = \frac{U}{2} \pm \sqrt{\frac{U^2}{4} + 16}, \\
e^{ik\hat{r}} & = e^{-ik\hat{z}} = - \frac{D_\pm}{4} + \sqrt{\left( \frac{D_\pm}{4} \right)^2 - 1}, \\
A_\pm & = 0.
\end{align*}
\]
This “regular” part of the Bethe Ansatz can be obtained by explicitly solving (1.7) and (1.8) for \((M + N) \leq 2, N \geq M\), and substituting the solution into (1.2)–(1.5). Here we have used the notation \(|\psi_{\sigma,\tau}(k_1, k_2 | A)\rangle\) and \(|\psi_\sigma(k)\rangle\), where the \(k_j\) and \(\Lambda^\pm\) are solutions of the periodic boundary conditions (1.7) and (1.8), and \(\sigma, \tau = \pm 1\) denote the spin degrees of freedom.

It is easily checked that all of these regular Bethe states are lowest-weight vectors of the \(\text{SO}(4)\) algebra, in agreement with our general claim.

Further eigenstates of the hamiltonian can easily be found by using the symmetries mentioned in sect. 1. By using the symmetry under the transformation that exchanges up and down spins, i.e. \(c_{j,1} \rightarrow c_{j,-1}\) and \(c_{j,1}^\dagger \rightarrow c_{j,-1}^\dagger\), we obtain

\[
|\psi_{-1}(\pi)\rangle = (1/\sqrt{2})(-c_{1,-1}^\dagger + c_{2,-1}^\dagger)|0\rangle = \zeta^\dagger |\psi_1(\pi)\rangle,
\]

\[
|\psi_{-1}(0)\rangle = (1/\sqrt{2})(c_{1,-1}^\dagger + c_{2,-1}^\dagger)|0\rangle = \zeta^\dagger |\psi_1(0)\rangle,
\]

\[
|\psi_{-1,-1}(\pi,0)\rangle = c_{1,-1}^\dagger c_{2,-1}^\dagger |0\rangle = \frac{1}{2}(\zeta^\dagger)^2 |\psi_{1,1}(\pi,0)\rangle.
\]

Using the symmetry (1.11), (1.12) on all of the above states we then obtain five additional eigenstates

\[
|\psi_{-1,1,1}(\pi)\rangle = (1/\sqrt{2})(-c_{1,1}^\dagger c_{2,1}^\dagger c_{2,-1}^\dagger + c_{1,-1}^\dagger c_{1,1}^\dagger c_{2,1}^\dagger)|0\rangle = \eta^\dagger |\psi_1(0)\rangle,
\]

\[
|\psi_{-1,1,-1}(\pi)\rangle = (1/\sqrt{2})(c_{1,1}^\dagger c_{1,-1}^\dagger c_{2,-1}^\dagger - c_{1,-1}^\dagger c_{2,1}^\dagger c_{2,-1}^\dagger)|0\rangle = \zeta^\dagger |\psi_{-1,1,1}(\pi)\rangle
\]

\[
= \eta^\dagger |\psi_{-1}(0)\rangle,
\]

\[
|\psi_{-1,-1,1}(0)\rangle = (1/\sqrt{2})(c_{1,1}^\dagger c_{1,-1}^\dagger c_{2,1}^\dagger + c_{1,-1}^\dagger c_{1,1}^\dagger c_{2,1}^\dagger)|0\rangle = \eta^\dagger |\psi_1(\pi)\rangle,
\]

\[
|\psi_{-1,-1,-1}(0)\rangle = (1/\sqrt{2})(c_{1,-1}^\dagger c_{2,1}^\dagger c_{2,-1}^\dagger + c_{1,1}^\dagger c_{1,-1}^\dagger c_{2,-1}^\dagger)|0\rangle = \zeta^\dagger |\psi_{-1,1,1}(0)\rangle
\]

\[
= \eta^\dagger |\psi_{-1,-1}(\pi)\rangle
\]

\[
|\Omega\rangle = c_{1,1}^\dagger c_{1,-1}^\dagger c_{2,1}^\dagger c_{2,-1}^\dagger |0\rangle = -\frac{1}{2}(\eta^\dagger)^2 |0\rangle.
\]

Altogether we have now found a total of 14 eigenstates, two less than the complete number of \(4^2 = 16\).

The two missing eigenstates are given by the following expressions:

\[
(1/\sqrt{2})\eta^\dagger |0\rangle = (1/\sqrt{2})(-c_{1,-1}^\dagger c_{1,1}^\dagger + c_{2,1}^\dagger c_{2,1}^\dagger)|0\rangle
\]

\[
|\psi_{1,1}(\pi,0)\rangle = (1/\sqrt{2})(c_{1,-1}^\dagger c_{2,1}^\dagger + c_{1,1}^\dagger c_{2,-1}^\dagger)|0\rangle = (1/\sqrt{2})\zeta^\dagger |\psi_{1,1}(\pi,0)\rangle.
\]
The second state is obviously not a lowest-weight state of the $\zeta$-$SU(2)$ algebra as it can be obtained from $|\psi_{i,\pi,0}\rangle$ by acting with a raising operator $\eta^\dagger$.

The fact, that $\eta^\dagger|0\rangle$ is not a Bethe state can be shown as follows:

**Theorem 2.1** $\eta^\dagger|0\rangle$ is not a Bethe state for the lattice with 2 sites and $U \neq 0$.

**Proof:** The energy of $\eta^\dagger|0\rangle$ is clearly equal to $U/2$, as $\eta^\dagger$ commutes with the Hamiltonian. If $\eta^\dagger|0\rangle$ would be a Bethe state, its energy would have to be equal to $-2(\cos(k_1) + \cos(k_2)) - U/2$ according to (2.1). But the transformation properties of $\eta^\dagger|0\rangle$ under translations by one site show that $\exp i(k_1 + k_2) = -1$, i.e. the eigenvalue of the translation operator equals $-1$ as follows from the explicit expression (2.2), and therefore $k_1 + k_2 = \pi$. But this implies that $\cos(k_1) + \cos(k_2) = 0$ and thus we obtain an energy of $-U/2$ which is a contradiction to our previous assertion that $\eta^\dagger|0\rangle$ has energy equal to $U/2$. □

We thus conclude that the Bethe Ansatz, together with the manifest symmetries of the Hamiltonian, does not lead to a complete spectrum of eigenstates for the 2-site model.

The above argument generalises straightforwardly to $L$ sites. Thus the incompleteness of the Bethe Ansatz can already be observed in the two particle sector for arbitrary lattice size, as we do not obtain the total number of states even if we use all symmetries and take the limiting cases for the spectral parameters into account.

2.2. $SO(4)$ Structure on the 2-Site Lattice

We already mentioned that the eigenstates of the 2-site model carry a representation of the $SO(4)$ algebra. In the following diagram we group the states into multiplets of the two $SU(2)$ algebras, such that $\eta^\dagger$ acts horizontally and $\zeta^\dagger$ vertically (among the grouped states). We label all states by a pair of numbers $(i, j)$ which gives the representation of the $\eta$-$SU(2)$ (spin $i$) and $\zeta$-$SU(2)$ (spin $j$).

\[
\begin{align*}
\zeta^\dagger & \downarrow \\
|0\rangle_{1,0} & \quad \eta^\dagger|0\rangle_{1,0} & \quad |\Omega\rangle_{1,0} \\
|\psi_{\pi}(\pi)\rangle_{1/2,1/2} & \quad |\psi_{-1,1}(\pi)\rangle_{1/2,1/2} \\
|\psi_{-\pi}(\pi)\rangle_{1/2,1/2} & \quad |\psi_{-1,-1}(\pi)\rangle_{1/2,1/2} \\
|\psi_{0}(\pi)\rangle_{1/2,1/2} & \quad |\psi_{-1,1}(\pi)\rangle_{1/2,1/2} \\
\end{align*}
\]

* This state can formally be obtained from the Bethe Ansatz by setting $A$ equal to infinity. However there is no well-defined limiting procedure to send $A$ to infinity for finite lattice lengths and we therefore will not pursue this observation any further.
Summarising the results so far, we can see that for the 2-site model
(i) the Bethe Ansatz is not complete,
(ii) all “regular” Bethe states are lowest-weight states of SO(4),
(iii) the complete spectrum can be obtained from the regular Bethe-Ansatz states by using the SO(4) structure.

3. Bethe-Ansatz states as lowest-weight vectors of the \( \eta \)-pairing

In this section we give a proof that for any even \( L \) all Bethe states are lowest-weight states of the \( \eta \)-pairing. We first prove this for the case of \( N - 1 \) spins up and one spin down (the case of all spins up is trivial because then the spin down annihilation operator in \( \eta \) can be anticommutated through all creation operators and gives zero on the vacuum). In the course of our proof we derive an important lemma that we will use heavily in the general case. In subsect. 3.2 we then generalise our result to the case of \( N \) spins up and \( M \) spins down.

3.1. \( N - 1 \) SPINS UP AND ONE SPIN DOWN

According to (1.2) the general state with one spin down has the following form:

\[
| \Psi_{1,N-1} \rangle = \sum_{1 \leq x_k \leq L} \Psi_{-1,1,\ldots,1}(x_1, \ldots, x_N) c_{x_{N-1}}^{+} \prod_{i=2}^{N} c_{x_i,1}^{+} | 0 \rangle.
\]

Application of \( \eta \) to this state leads to

\[
\eta | \Psi_{1,N-1} \rangle = \sum_{n=2}^{N} \sum_{\substack{1 \leq x_k \leq L \\kern-10pt \text{for} \ k \neq n \\kern-10pt \text{and} \ n \neq 1 \\kern-10pt \text{for} \ n = 1 \}} \Psi_{-1,1,\ldots,1}(x_1, x_2, \ldots, x_{n-1}, x_1, x_{n+1}, \ldots, x_N)
\]

\[
\times (-1)^{x_1+n} \prod_{\substack{i=2 \\text{for} \ i \neq n \\kern-10pt \text{and} \ n \neq 1}}^{N} c_{x_i,1}^{+} | 0 \rangle.
\]
Using the antisymmetry (1.10) of the wave function under the simultaneous exchange \( x_k \leftrightarrow x_j \) and \( \sigma_k \leftrightarrow \sigma_j \), one can show that all terms in the sum over \( n \) are identical. Thus

\[
\eta | \psi_{1,N-1} \rangle = (N - 1) \sum_{1 \leq x_4 \leq L, k \neq 2} \psi_{-1,1, \ldots, 1}(x_1, x_4, x_3, \ldots, x_N)(-1)^{x_1} \prod_{i=3}^{N} e_{x_i,1}^+ | 0 \rangle.
\]

(3.2)

This vanishes due to the following

**Theorem 3.1.**

\[
\sum_{x=1}^{L} \psi_{-1,1, \ldots, 1}(x, x, x_3, \ldots, x_N)(-1)^{x} = 0.
\]

(3.3)

**Proof.** By using the antisymmetry mentioned above we can always re-arrange the \( x_j \) in such a way that \( x_3 < x_4 < \ldots < x_N \). Then the summation over \( x \) splits up into the \( N-1 \) pieces

\[
\sum_{x=1}^{L} = \sum_{x=1}^{x_4} + \sum_{x=x_3}^{x_4} + \sum_{x=x_4}^{x_5} + \ldots + \sum_{x=x_N}^{L}.
\]

(3.4)

If \( x_j = x_{j+1} \) for some \( j \), then \( \psi_{-1,1, \ldots, 1}(x, x, x_3, \ldots, x_N) = 0 \) and hence (3.3) is trivially valid. Thus we will assume from now on that \( x_3 < x_4 < \ldots < x_N \) holds. In each of the sums on the r.h.s. of (3.4) the wave function lies within the same \( Q \)-sector and is given by (1.3). More explicitly, in \( \Sigma_{x=1}^{x_4} \), we need the expression for the wave function in the sector \( Q_1 := (x, x, x_3, x_4, \ldots, x_N) \), in \( \Sigma_{x=x_3}^{x_4} \) we need it in \( Q_2 := (x_3, x, x, x_4, \ldots, x_N) \), and in \( \Sigma_{x=x_N}^{L} \) in \( Q_{N-1} := (x_3, x_4, \ldots, x_N, x, x) \). With \( x_1, \ldots, x_N \) we mean \( x_1 \leq x_2 \leq \ldots \leq x_N \).

The only explicit \( x \)-dependence of the wave function is in the exponential factor \( \exp(i \sum_{j=1}^{N} k_{p_j} x_{Q_j}) \) as in (1.3). This factor depends on a permutation \( P \) of the momenta \( k_j \). In the expression (1.3) for the wave function we sum over all such permutations \( P \). In (3.3) we therefore can pick a particular \( P^{(j)} \) in each sector \( Q_j \), then perform the summation with respect to \( x \), and finally sum over all possible \( P \)'s. We proceed in this way because it will turn out that, after the \( x \)-summation has been carried out, summation over a small subset of all \( P \)'s will lead to cancellations, which cause the complete sum to vanish as desired. In particular we
will make the following choices:

\[
\begin{align*}
P^{(1)} &= P = (P_1, P_2, P_3, P_4, \ldots, P_N), \\
P^{(2)} &= (P_3, P_1, P_2, P_4, \ldots, P_N), \\
&\vdots \\
P^{(N-1)} &= (P_3, P_4, \ldots, P_N, P_1, P_2).
\end{align*}
\]  

(3.5)

Note that \( \text{sgn}(P^{(1)}) = \text{sgn}(P^{(2)}) = \ldots = \text{sgn}(P^{(N-1)}) \).

Apart from the explicit \( x \)-dependence of the wave function in the exponential factor, there is a hidden dependence on \( x \) in \( F_p(A, y) \), because \( y \) is the position of the down spin (which sits at \( x \)) among all electrons. There is obviously an ambiguity in the determination of \( y \), because we have both a spin up and a spin down at \( x \), and we therefore have the choice of regarding the down spin to be to the left of the up spin or vice versa. This ambiguity is of no consequence, however, because our expression of the wave function is valid not only within a space sector \( x_1 < x_2 < \ldots < x_N \) but also on the sector boundary, i.e. for \( x_j = x_{j+1} \). This means that we get exactly the same form of the wave function no matter which convention we pick for the order of the two spins on the same site. Throughout what follows we will adopt the convention that whenever two spins occupy the same site, the one “to the left” is the spin down. More explicitly this means that for the sector \( Q_1 \), \( y \) equals 1, for \( Q_2 \), \( y \) equals 2 and so on. Having this in mind we can now explicitly work out (3.4), using eqs. (1.3)–(1.5) and (3.5)

\[
\sum_{x=1}^{L} \psi_{-1,1,\ldots,1}(x, x, x_3, \ldots, x_N)(-1)^x
\]

\[
= \frac{1}{2} \sum_{P \in S_N} \text{sgn}(P) \left\{ \sum_{x=1}^{x_N-1} \left( \frac{1}{e_+^{(1)}(P_1)} - \frac{1}{e_+^{(1)}(P_2)} \right) \times \exp \left( i \sum_{j=3}^{N} k_{P_j} x_j \right) \left( -\exp \left( i(k_{P_1} + k_{P_2}) \right) \right)^x \right. \\
+ \sum_{x=x_3}^{x_4-1} \frac{e_+^{(1)}(P_3)}{e_+^{(1)}(P_3)} \left( \frac{1}{e_+^{(1)}(P_1)} - \frac{1}{e_+^{(1)}(P_2)} \right) \exp \left( i \sum_{j=3}^{N} k_{P_j} x_j \right) \left( -\exp \left( i(k_{P_1} + k_{P_2}) \right) \right)^x \\
+ \sum_{x=x_4}^{x_5-1} \frac{e_+^{(1)}(P_4)}{e_+^{(1)}(P_3)} \frac{e_+^{(1)}(P_3)}{e_+^{(1)}(P_4)} \left( \frac{1}{e_+^{(1)}(P_1)} - \frac{1}{e_+^{(1)}(P_2)} \right) \times \exp \left( i \sum_{j=3}^{N} k_{P_j} x_j \right) \left( -\exp \left( i(k_{P_1} + k_{P_2}) \right) \right)^x \\
&\vdots
\]
$F.H.L.\; E\ddot{u}ller\; et\; a!.$

$\langle \psi | H | \psi \rangle = \sum_{x=x_N}^{L} \left( \prod_{j=3}^{N} \frac{e^{(1)}(P_j)}{e^{(1)}_+(P_j)} \right) \left( \frac{1}{e^{(1)}_+(P_1)} - \frac{1}{e^{(1)}_+(P_2)} \right) \times \exp \left( i \sum_{j=3}^{N} k_{P_j} x_j \right) \left( -\exp \left( i (k_{P_1} + k_{P_2}) \right) \right)^{x}$.

Here we have added the two elements of the $P$-summation which are related by the exchange of $P_1$ and $P_2$ and corrected for this by the overall factor of $\frac{1}{2}$. In the next step we perform the $x$-summation using the formula

$$\sum_{x=\alpha}^{\beta} (-e^{ik})^x = \frac{(-1)^{\beta} e^{ik(\beta+1)} + (-1)^{\alpha} e^{ik\alpha}}{1 + e^{ik}}.$$ (3.7)

which holds for $e^{ik} \neq -1$ (this case will be treated separately). The term proportional to $\exp(i(k_{P_1} + k_{P_2})(L + 1))$ cancels against the one proportional to $\exp(i(k_{P_1} + k_{P_2}))$ due to the periodic boundary conditions (1.7) and (1.8); we will therefore omit both of them in (3.8). Assuming for the time being that $\exp(i(k_{P_1} + k_{P_2})) \neq -1$ we find that

$$\sum_{x=1}^{L} \psi_{-1,1,\ldots,1}(x, x, x_3, \ldots, x_N) \left( -1 \right)^{x}$$

$$= -\frac{1}{2} \sum_{P \in S_N} \text{sgn}(P) \left( \frac{\sin(k_{P_1}) - \sin(k_{P_2})}{1 + \exp(i(k_{P_1} + k_{P_2}))} \right) \frac{1}{e^{(1)}_+(P_1) e^{(1)}_+(P_2)} \exp \left( i \sum_{j=3}^{N} k_{P_j} x_j \right)$$

$$\times \left( (-1)^{x_3} \exp(i(k_{P_1} + k_{P_2}) x_3) \left( 1 - \frac{e^{(1)}_+(P_3)}{e^{(1)}_+(P_3)} \right) \right)$$

$$+ (-1)^{x_4} \exp(i(k_{P_1} + k_{P_2}) x_4) \left( 1 - \frac{e^{(1)}_+(P_4)}{e^{(1)}_+(P_4)} \right) \frac{e^{(1)}_+(P_3)}{e^{(1)}_+(P_3)}$$

$$+ (-1)^{x_5} \exp(i(k_{P_1} + k_{P_2}) x_5) \left( 1 - \frac{e^{(1)}_+(P_5)}{e^{(1)}_+(P_5)} \right) \frac{e^{(1)}_+(P_3)}{e^{(1)}_+(P_3)} \frac{e^{(1)}_+(P_4)}{e^{(1)}_+(P_4)}$$

$$:$$

$$+ (-1)^{x_N} \exp(i(k_{P_1} + k_{P_2}) x_N) \left( 1 - \frac{e^{(1)}_+(P_N)}{e^{(1)}_+(P_N)} \right) \frac{\prod_{j=3}^{N-1} e^{(1)}_+(P_j)}{e^{(1)}_+(P_j)}$$
\begin{equation}
\begin{aligned}
&= -\frac{1}{2} \sum_{P \in S_N} \text{sgn}(P) \left( \frac{\sin(k_{P_2}) - \sin(k_{P_1})}{1 + \exp(i(k_{P_1} + k_{P_2}))} \right) \exp \left( i \sum_{j=3}^{N} k_{P_j} x_j \right) \\
&\times \left( -1 \right)^{x_3} \exp \left( i(k_{P_1} + k_{P_2}) x_3 \right) \left( \frac{u}{e^{(1)}(P_3) e^{(1)}(P_1) e^{(1)}(P_2)} \right) \\
&+ (-1)^{x_4} \exp \left( i(k_{P_1} + k_{P_2}) x_4 \right) \left( \frac{u}{e^{(1)}(P_4) e^{(1)}(P_1) e^{(1)}(P_2)} \right) \\
&+ (-1)^{x_5} \exp \left( i(k_{P_1} + k_{P_2}) x_5 \right) \left( \frac{u}{e^{(1)}(P_5) e^{(1)}(P_1) e^{(1)}(P_2)} \right) \\
&+ (-1)^{x_N} \exp \left( i(k_{P_1} + k_{P_2}) x_N \right) \left( \frac{u}{e^{(1)}(P_N) e^{(1)}(P_1) e^{(1)}(P_2)} \right) \left( \prod_{j=3}^{N-1} e^{(1)}(P_j) \right) \\
&= \left( \prod_{j=3}^{N} e^{(1)}(P_j) \right).
\end{aligned}
\end{equation}

(3.8)

To demonstrate the vanishing of this expression we need the following neat lemma:

Lemma 3.1.

\begin{equation}
\sum_{\text{cyclic}} \left( \frac{\sin(k_{P_2}) - \sin(k_{P_1})}{1 + \exp(i(k_{P_1} + k_{P_2}))} \right) = 0,
\end{equation}

(3.9)

where the summation extends over cyclic permutations \( P := (P_1, P_2, P_3) \) of three numbers.

Proof. Elementary computations yield

\begin{equation}
\left( \frac{\sin(k_{P_2}) - \sin(k_{P_1})}{1 + \exp(i(k_{P_1} + k_{P_2}))} \right) = \frac{i}{2} \left( \exp(-ik_{P_2}) - \exp(-ik_{P_1}) \right).
\end{equation}

This obviously vanishes upon summation over the three cyclic permutations \( P \). □

Application of Lemma 3.1 now immediately demonstrates the vanishing of (3.8) because each individual term in (3.8) has the form

\begin{equation}
\left( \frac{\sin(k_{P_2}) - \sin(k_{P_1})}{1 + \exp(i(k_{P_1} + k_{P_2}))} \right) \times \text{(something symmetric in \( (P_1, P_2, P_3) \))},
\end{equation}
where \((P_1, P_2, P_3)\) denotes all cyclic permutations of \(P_1, P_2, P_3\). The sum over all permutations \(P\) then can be performed in such a way that we always first sum over all cyclic permutations of \((P_1, P_2, P_3)\), which gives zero due to the Lemma. Thus we have proven the theorem under the assumption that \(\exp(i(k_{P_1} + k_{P_2})) \neq -1\). Inspection of eq. (3.6) however readily takes care of this case, because all terms in (3.6) are proportional to \(((1/\epsilon_{P_1}^f(P_1)) - (1/\epsilon_{P_2}^f(P_2)))\), which is zero if \(\exp(ik_{P_1}) = -\exp(-ik_{P_2})\). This completes the proof of the theorem. □

3.2. \(N\) SPINS UP AND \(M\) SPINS DOWN

In this subsection we give a proof that every regular Bethe state with \(N\) spins up and \(M\) spins down is a lowest weight-state of the \(\eta\)-SU(2) algebra. Acting with \(\eta\) on a general Bethe state (1.2) leads to

\[
\left( \frac{1}{NM} (-1)^{M-1} \right) \eta | \Psi_{M,N} \rangle = \sum_{(x_1, (z_i))} \psi_{-1, ..., -1, ..., 1} (x, z_1, ..., z_{M-1}, x, x_1, ..., x_{N-1}) (-1)^x \times \prod_{i=1}^{M-1} c_{z_i}^{\pi} \prod_{i=1}^{N-1} c_{x_i,1}^{\pi} |0\rangle.
\]

(3.10)

Here we have again used the antisymmetry (1.10) of the wave function to identify identical terms. The vanishing of (3.10) is assured by the following

**Theorem 3.2.**

\[
\sum_{x=1}^{L} \psi_{-1, ..., -1, ..., 1} (x, z_1, ..., z_{M-1}, x, x_1, ..., x_{N-1}) (-1)^x = 0. \quad (3.11)
\]

**Proof.** We first introduce some convenient notations: We denote the union of the set of all \(x_i\) with the set of all \(z_j\) by \(\mathcal{A}\), i.e. \(\{x_i | i = 1, ..., N-1\} \cup \{z_j | j = 1, ..., M-1\} = \mathcal{A} = \{a_k | k = 3, ..., M+N\}\). We impose the ordering \(a_3 \leq a_4 \leq ... \leq a_{M+N}\), which can always be achieved by trivial relabelings. To distinguish the \(x_i\) from the \(z_j\) in \(\mathcal{A}\), we introduce the two subsets \(\mathcal{A}_m = \{a_{m_i} | i = 1, ..., N-1\}\) = \(\{x_i | i = 1, ..., N-1\}\), and \(\mathcal{A}_f = \{a_{f_j} | j = 1, ..., M-1\}\) = \(\{z_j | j = 1, ..., M-1\}\). Using this notation we can write the sum over \(x\) as

\[
\sum_{x=1}^{L} = \sum_{x=1}^{a_3-1} + \sum_{x=a_3}^{a_4-1} + \sum_{x=a_4}^{a_5-1} + \ldots + \sum_{x=a_{M+N}}^{a_{M+1}}
\]

(3.12)

Here we define \(\sum_{x=a_n}^{a_{n+1}-1}\) to be zero if \(a_n = a_{n+1}\). In each sum \(\sum_{x=a_j}^{a_{j+1}-1}\) the wave
function $\psi_{-1,\ldots,-1,\ldots,1}(x, z_1, \ldots, z_{M-1}, x, z_1, \ldots, z_{M-1})$ stays within the same space-sector $Q_j = (a_3, \ldots, a_j, x, x, a_{j+1}, \ldots, a_{M+N})$. As in the proof of theorem 3.1 we interchange the $x$-summation with the sum over all permutations $P$, and consider one specific permutation $P^{(j)}$ in each sector $Q_j$. Upon summing over $x$, using (3.7), we will obtain contributions proportional to $\exp(i(k_{p_1} + k_{p_2} + k_{p_1})a_j)$ and $\exp(i(k_{p_1} + k_{p_2} + k_{p_{j+1}})a_{j+1})$ and contributions proportional to $\exp(i(k_{p_1} + k_{p_2}))$ which are generated by the boundary terms $\Sigma_{x=1}^{a-1}$ and $\Sigma_{x=aM+N}^{L}$. We will then show that the terms proportional to any one of these exponential factors vanish independently when summed over $P$. The situations when two adjacent $a$'s coincide, or when $a_3 = 1$, do not pose any problems since formally the formula (3.7) correctly produces a vanishing result for the summation of the term $\Sigma_{x=a}^{a-1}$. As outlined above, the summation over $x$ yields three different kinds of terms:

(i) terms proportional to $\exp(i(k_{p_1} + k_{p_2} + k_{p_1})a_j)$ ("spin down")

(ii) terms proportional to $\exp(i(k_{p_1} + k_{p_2} + k_{p_{j+1}})a_{j+1})$ ("spin up")

(iii) 'boundary terms' proportional to $\exp(i(k_{p_1} + k_{p_2}))$

We first demonstrate cancellation of the boundary terms (iii), which are generated by $\Sigma_{x=aM+N}$ and $\Sigma_{x=a}^{a-1}$ in (3.12). The two space sectors we have to consider, are $Q_2 = (x, x, a_3, \ldots, a_{M+N})$ and $Q_{M+N} = (a_3, \ldots, a_{M+N}, x, x)$. Note that $\text{sgn}(Q)$ is identical for both sectors. Like in the proof of theorem 3.1 (and as stated above) we pick a particular permutation $P^{(j)}$ in the sector $Q_j$. Our choices for the $P$-permutations are

$$P^{(2)} = P = (P_1, P_2, P_3, \ldots, P_{N+M}),$$

$$P^{(M+N)} = (P_3, \ldots, P_{N+M}, P_1, P_2).$$

We also define for later convenience

$$\tilde{P}^{(j)} = P^{(j)}|_{P_1 \rightarrow P_2}, \quad (3.13)$$

i.e. $\tilde{P}^{(j)}$ is the permutation obtained from $P^{(j)}$ by interchanging $P_1$ with $P_2$. In particular

$$\tilde{P} = (P_2, P_1, P_3, \ldots, P_{N+M}),$$

$$\tilde{P}^{(M+N)} = (P_3, \ldots, P_{N+M}, P_2, P_1).$$

Performing the $x$-summations $\Sigma_{x=1}^{a-1}$ and $\Sigma_{x=aM+N}^{L}$, using the expression (1.3) for the wave function and assuming for the time being that (3.7) holds, we obtain the
following terms proportional to $\exp(i(k_{p_1} + k_{p_2}))$:
\[
\frac{1}{2} \text{sgn}(Q_2) \sum_{p \in S_{M+N}} \text{sgn}(P) \exp \left( i \sum_{j=3}^{M+N} k_p a_j \right) \left( \frac{\exp(i(k_{p_1} + k_{p_2}))}{1 + \exp(i(k_{p_1} + k_{p_2}))} \right) \\
\times \left\{ \exp(i(k_{p_1} + k_{p_2})L) \left( \varphi(l_1 - 2, \ldots, l_{M-1} - 2, N + M - 1 | P^{(M+N)}) \right) \\
- \varphi(l_1 - 2, \ldots, l_{M-1} - 2, N + M - 1 | \tilde{P}^{(M+N)}) \right\} \\
- \left\{ \varphi(1, l_1, \ldots, l_{M-1} | P) - \varphi(1, l_1, \ldots, l_{M-1} | \tilde{P}) \right\} \right\}. \tag{3.14}
\]

Here we have again written the terms with $P_1$ and $P_2$ interchanged explicitly and compensated for this by the factor $\frac{1}{2}$. We have also dropped the factor $(-1)^L$, which is equal to one as $L$ is even. (3.14) vanishes due to the following

**Lemma 3.2.**
\[
\left\{ \exp(i(k_{p_1} + k_{p_2})L) \left( \varphi(l_1 - 2, \ldots, l_{M-1} - 2, N + M - 1 | P^{(M+N)}) \right) \\
- \varphi(l_1 - 2, \ldots, l_{M-1} - 2, N + M - 1 | \tilde{P}^{(M+N)}) \right\} \\
- \left\{ \varphi(1, l_1, \ldots, l_{M-1} | P) - \varphi(1, l_1, \ldots, l_{M-1} | \tilde{P}) \right\} = 0. \tag{3.15}
\]

**Proof.** The periodic boundary conditions (1.7) imply
\[
\exp(i(k_{p_1} + k_{p_2})L) = \prod_{\alpha=1}^{M} \frac{e^{(\alpha)}(P_1)}{e^{(\alpha)}(P_2)}. \tag{3.16}
\]

Using (1.4) and (1.5) we obtain for the difference of the first two $\varphi$’s in (3.15)
\[
\varphi(l_1 - 2, \ldots, l_{M-1} - 2, N + M - 1 | P^{(M+N)}) \\
- \varphi(l_1 - 2, \ldots, l_{M-1} - 2, N + M - 1 | \tilde{P}^{(M+N)}) \\
\sum_{\pi \in S_N} A_{\pi} \left( \prod_{i=1}^{M-1} F_{p^{(M+N)}(A_{\pi_i}, l_i - 2)} \right) \\
\times \left( \prod_{j=3}^{M+N} \frac{\frac{e^{(\pi)}}{e^{(\pi)}}(P_j)}{e^{(\pi)}}(P_1) e^{(\pi)}(P_2) \right) \left( \sin(k_{p_1}) - \sin(k_{p_2}) \right). \tag{3.17}
\]

Here we have used the fact that $F_{p^{(M+N)}(A_{\pi}, l_i - 2)} = F_{p^{(M+N)}(A_{\pi}, l_i - 2)}$ for all $i = 1, \ldots, M - 1$. 


Eqs. (3.16) and (3.17) together yield

\[
\exp(i(k_{P_1} + k_{P_2})L)\{\varphi(l_1 - 2, \ldots, l_{M-1} - 2, N + M - 1 \mid \tilde{P}^{(M+N)}) - \varphi(l_1 - 2, \ldots, l_{M-1} - 2, N + M - 1 \mid \tilde{P}^{(M+N)})\} = \\
\sum_{\pi \in S_M} A_{\pi}\left[ \prod_{i=1}^{M-1} F_{\pi}^{(M+N)}(A_{\pi}, l_i - 2) \left( \prod_{j=1}^{M+N} \frac{e^{(\pi_M)}(P_j)}{e_+^{(\pi_M)}(P_j)} \right) \right] \\
\times \left( \frac{\sin(k_{P_1}) - \sin(k_{P_2})}{e_+^{(\pi_M)}(P_1)e_+^{(\pi_M)}(P_2)} \right) \left( \prod_{\alpha = 1}^{M} \frac{e^{(\alpha)}(P_1) e^{(\alpha)}(P_2)}{e_+^{(\alpha)}(P_1) e_+^{(\alpha)}(P_2)} \right) \\
=: \Delta_1. \tag{3.18}
\]

The difference of the second two \( \varphi \)'s in (3.15) is equal to

\[
\Delta_2 := \varphi(1, l_1, \ldots, l_{M-1} \mid P) - \varphi(1, l_1, \ldots, l_{M-1} \mid \tilde{P}) = \\
\sum_{\sigma \in S_M} A_{\sigma}\left[ \prod_{i=1}^{M-1} F_{\sigma}^{(M+N)}(A_{\sigma}, l_i) \left( \frac{\sin(k_{P_1}) - \sin(k_{P_2})}{e_+^{(\sigma)}(P_1)e_+^{(\sigma)}(P_2)} \right) \right]. \tag{3.19}
\]

where we have used that \( F_{\sigma}(A_{\sigma+1}, l_i) = F_{\sigma}(A_{\sigma+1}, l_i) \) for all \( i = 1, \ldots, M - 1 \). The term with \( \sigma = (\pi_M, \pi_{M-1}, \ldots, \pi_1) \) in (3.19) is seen to be equal to

\[
A_{\sigma}\left( \prod_{i=1}^{M-1} F_{\pi}^{(M+N)}(A_{\pi}, l_i - 2) \left( \prod_{\alpha = 1}^{M} \frac{e^{(\alpha)}(P_1) e^{(\alpha)}(P_2)}{e_+^{(\alpha)}(P_1) e_+^{(\alpha)}(P_2)} \right) \right] \\
\times \left( \frac{\sin(k_{P_1}) - \sin(k_{P_2})}{e_+^{(\pi_M)}(P_1)e_+^{(\pi_M)}(P_2)} \right). \tag{3.20}
\]

Using eqs. (3.18)–(3.20) we now can write down the complete l.h.s. of (3.15) explicitly:

\[
\Delta_1 - \Delta_2 = \sum_{\pi \in S_M} \left( \prod_{i=1}^{M-1} F_{\pi}^{(M+N)}(A_{\pi}, l_i - 2) \left( \prod_{\alpha = 1}^{M} \frac{e^{(\alpha)}(P_1) e^{(\alpha)}(P_2)}{e_+^{(\alpha)}(P_1) e_+^{(\alpha)}(P_2)} \right) \right] \\
\times \left( A_{\pi}\left( \prod_{j=1}^{M+N} \frac{e^{(\pi_M)}(P_j)}{e_+^{(\pi_M)}(P_j)} \right) - A_{\sigma}\left( \frac{\sin(k_{P_1}) - \sin(k_{P_2})}{e_+^{(\pi_M)}(P_1)e_+^{(\pi_M)}(P_2)} \right) \right). \tag{3.21}
\]
The right-hand side of (3.21) vanishes as
\[ A_\sigma \left( \prod_{j=1}^{M+N} \frac{e^{(\pi \sigma)(P_j)}}{e^{(\pi \sigma)(P_j)}} \right) - A_\sigma = 0, \]
which follows from (1.6) and (1.8). This completes the proof of lemma 3.2, because the case of \( \exp(i(k_{p_1} + k_{p_2}) = -1 \) can be treated precisely in the same way as in the proof of theorem 3.1 (i.e. \( \sin(k_{p_1}) - \sin(k_{p_2}) = 0 \) in this case, which causes the difference of the two \( \varphi \)'s in the individual space-sectors \( Q \) to vanish identically, and therefore the sum over \( x \) is trivially zero in both sectors).

Thus we have demonstrated the cancellation of all boundary terms (iii). Now we proceed with the second kind of contribution to \( \eta | \Psi_{M,N} \rangle \), and show that it vanishes, too. Terms of the kind (ii) are generated by \( \sum_{x=1}^{a_{m_k+1}} \) and \( \sum_{x=1}^{a_{m_k+1}} \). The relevant space sectors are
\[ Q_{m_k-1} = (a_3, \ldots, a_{m_k-1}, x, x, a_{m_k}, \ldots, a_{M+N}) \]
and
\[ Q_{m_k} = (a_3, \ldots, a_{m_k}, x, x, a_{m_k+1}, \ldots, a_{M+N}). \]

Our choices for the \( P \)-permutations are
\[ P^{(m_k-1)} = (P_3, \ldots, P_{m_k-1}, P_1, P_2, P_{m_k}, \ldots, P_{M+N}) \]
and
\[ P^{(m_k)} = (P_3, \ldots, P_{m_k}, P_1, P_2, P_{m_k+1}, \ldots, P_{M+N}), \]
both of which give rise to a factor \( \exp(i(k_{p_1} + k_{p_2} + k_{p_{m_k}})a_{m_k}) \). Note that \( \text{sgn}(P^{(m_k)}) = \text{sgn}(P^{(m_k-1)}) = \text{sgn}(P) \), where \( P = (P_1, \ldots, P_{M+N}) \). \( P^{(j)} \) is given by (3.13).

Exchanging once again the \( x \)- and \( P \)-summations, performing the sum over \( x \) in the above two sectors under the assumptions that \( \exp(i(k_{p_1} + k_{p_2})) \neq -1 \), and then isolating the terms proportional to \( \exp(i(k_{p_1} + k_{p_2} + k_{p_{m_k}})a_{m_k}) \), leads to contributions of the form
\[ \frac{1}{2} \text{sgn}(Q_{m_k}) \sum_{P \in S_{M+N}} \text{sgn}(P) \exp \left( i \sum_{j=3}^{M+N} k_{p_j}a_j \right) \]
\[ \times \left( \exp \left( i(k_{p_1} + k_{p_2} + k_{p_{m_k}})a_{m_k} \right) \right) \frac{1}{1 + \exp \left( i(k_{p_1} + k_{p_2}) \right)} (-1)^{a_{m_k-1}} \]
\[ \times \left\{ \varphi(l_1 - 2, \ldots, l_e - 2, m_k - 2, l_{e+1}, \ldots, l_{M-1} \mid P^{(m_k-1)}(P_{e+1}), (3.22) \right\} \]

where \( l_1, \ldots, l_e \) denote the down spins “to the left” of the up spin at \( a_{m_k} \). The desired vanishing of (3.22) is assured by the following

Lemma 3.3.

\[ \sum_{\text{cycl}} \chi(P) = 0, \] (3.23)

where the sum extends over the three cyclic permutations of \( P_1, P_2 \) and \( P_{m_k} \).

Proof.

\[ \varphi(l_1 - 2, \ldots, l_e - 2, m_k - 2, l_{e+1}, \ldots, l_{M-1} \mid P^{(m_k-1)}) \]

\[ - \varphi(l_1 - 2, \ldots, l_e - 2, m_k - 2, l_{e+1}, \ldots, l_{M-1} \mid \tilde{P}^{(m_k-1)}) \]

\[ = \sum_{\pi \in S_M} A_{\pi} \left( \prod_{i=1}^{e} F_{\rho_{m_{k-1}}} \left( \Lambda_{\pi_i}, l_i - 2 \right) \right) \left( \prod_{j=e+1}^{M-1} F_{\rho_{m_{k-1}}} \left( \Lambda_{\pi_{j+1}}, l_j \right) \right) \]

\[ \times \left\{ F_{\rho_{m_{k-1}}} \left( \Lambda_{\pi_{e+1}}, m_k - 2 \right) - F_{\tilde{\rho}_{m_{k-1}}} \left( \Lambda_{\pi_{e+1}}, m_k - 2 \right) \right\}, \] (3.24)

where we have used that \( F_{\rho_{m_{k-1}}} \left( \Lambda_{\xi}, l_1 \right) = F_{\rho_{m_{k-1}}} \left( \Lambda_{\xi}, l_1 \right) \) for all \( i = e + 1, \ldots, M - 1 \), and similarly \( F_{\rho_{m_{k-1}}} \left( \Lambda_{\pi_i}, l_i - 2 \right) = F_{\tilde{\rho}_{m_{k-1}}} \left( \Lambda_{\pi_i}, l_i - 2 \right) \) for all \( i = 1, \ldots, e \). Similarly we find that

\[ \varphi(l_1 - 2, \ldots, l_e - 2, m_k - 1, l_{e+1}, \ldots, l_{M-1} \mid P^{(m_k-1)}) \]

\[ - \varphi(l_1 - 2, \ldots, l_e - 2, m_k - 1, l_{e+1}, \ldots, l_{M-1} \mid \tilde{P}^{(m_k-1)}) \]

\[ = \sum_{\pi \in S_M} A_{\pi} \left( \prod_{i=1}^{e} F_{\rho_{m_{k-1}}} \left( \Lambda_{\pi_i}, l_i - 2 \right) \right) \left( \prod_{j=e+1}^{M-1} F_{\rho_{m_{k-1}}} \left( \Lambda_{\pi_{j+1}}, l_j \right) \right) \]

\[ \times \left\{ F_{\rho_{m_{k-1}}} \left( \Lambda_{\pi_{e+1}}, m_k - 1 \right) - F_{\tilde{\rho}_{m_{k-1}}} \left( \Lambda_{\pi_{e+1}}, m_k - 1 \right) \right\}. \] (3.25)
Inserting the expression (1.5) for $F_p$ in (3.24) and (3.25) and using the fact that except for the terms inside the curly brackets the $F_p(m_k)$ are equal to the $F_p(m_k - 1)$, we then obtain after some elementary algebra

$$\chi(P) = \exp \left\{ \sum_{i=3}^{M+N} \prod_{j=m_k}^{k_1} a_j \right\} \left\{ \frac{\exp\left(i\left(k_{p_1} + k_{p_2} + k_{p_m}\right)a_{m_k}\right)}{1 + \exp\left(i\left(k_{p_1} + k_{p_2}\right)\right)} \right\} \left( -1 \right)^{a_{m_k}-1}$$

$$\times \sum_{\pi \in S_M} \left( A_{\pi} \left( \prod_{i=1}^{e} F_{p^{1m_{k-1}}} (A_{\pi_i}, l_i - 2) \right) \prod_{j=e+1}^{M-1} F_{p^{1m_{k-1}}} (A_{\pi_{j+1}}, l_j) \right)$$

$$\times \left( \prod_{j=3}^{m_k-1} \frac{e^{(\pi_{j+1})}(P_j)}{e^{(\pi_{j+1})}(P_j)} \right) \left( \frac{u\left(\sin(k_{p_1}) - \sin(k_{p_2})\right)}{e^{(\pi_{j+1})}(P_1)e^{(\pi_{j+1})}(P_2)e^{(\pi_{j+1})}(P_{m_k})} \right)$$

$$= \left( \text{something symmetric in } (P_1, P_2, P_{m_k}) \right) \left( \sin(k_{p_1}) - \sin(k_{p_2}) \right) \frac{1}{1 + \exp\left(i\left(k_{p_1} + k_{p_2}\right)\right)}.$$

This means that if we sum $\chi$ over cyclic permutations of $(P_1, P_2, P_{m_k})$, we obtain zero due to lemma 3.1.

Thus we have shown that all terms (ii) cancel, if $\exp(i(k_{p_1} + k_{p_2}) \neq -1$. As before this case can be treated by noting that the difference of the two $\varphi$'s within the same space-sector is proportional to $\sin(k_{p_2}) - \sin(k_{p_1})$, which is identically zero in that case, and therefore the sum over $x$ vanishes trivially.

Last but not at all least we have to take care of the vanishing of all terms of the kind (i). The relevant $Q$-sectors are

$$Q_{l_{k-1}} = (a_3, \ldots, a_{l_{k-1}}, x, \ldots, a_{M+N})$$

and

$$Q_{l_k} = (a_3, \ldots, a_{l_k}, x, \ldots, a_{l_k+1}, \ldots, a_{M+N}).$$

Our choices for the $P$-permutations are

$$P^{(l_{k-1})} = (P_3, \ldots, P_{l_{k-1}}, P_1, P_2, P_{l_k}, \ldots, P_{M+N})$$

and

$$P^{(l_k)} = (P_3, \ldots, P_{l_k}, P_1, P_2, P_{l_k+1}, \ldots, P_{M+N}),$$

both of which give rise to a factor $\exp(i(k_{p_1} + k_{p_2} + k_{p_{l_k}})a_{l_k})$. Note that $\text{sgn}(P^{(l_k)}) = \text{sgn}(P^{(l_{k-1})}) = \text{sgn}(P)$, where $P = (P_1, \ldots, P_{M+N})$. 

Exchanging once again the $x$- and $P$-summations, performing the sum over $x$ in the above two sectors under the assumption that $\exp(i(k_{p_1} + k_{p_2})_i) \neq -1$, and then isolating the terms proportional to $\exp(i(k_{p_1} + k_{p_2} + k_{p_3})a_{i_k})$ leads to contributions of the form

$$\Phi = \frac{1}{2} \text{sgn}(Q_{l_{k-1}}) \sum_{P \in S_{M+N}} \left\{ \text{sgn}(P) \exp \left( i \sum_{j=3}^{M+N} k_{p_j} a_j \right) \right\}$$

$$\times \left( \frac{\exp(i(k_{p_1} + k_{p_2} + k_{p_3})a_{i_k})}{1 + \exp(i(k_{p_1} + k_{p_2}))} \right) (-1)^{a_{i_k} - 1} T(P),$$

where

$$T(P) = \left\{ \{ \varphi(l_1 - 2, \ldots, l_{k-1} - 2, l_k - 2, l_{k+1}, \ldots, l_{M-1} | \tilde{P}^{(l_{k-1})}) \right\}$$

$$- \varphi(l_1 - 2, \ldots, l_{k-1} - 2, l_k - 2, l_{k+1}, \ldots, l_{M-1} | \tilde{P}^{(l_{k-1})}) \right\}$$

$$- \{ \varphi(l_1 - 2, \ldots, l_{k-2}, l_k - 1, l_{k+1}, \ldots, l_{M-1} | \tilde{P}^{(l_k)}) \right\}$$

$$- \varphi(l_1 - 2, \ldots, l_{k-2}, l_k - 1, l_{k+1}, \ldots, l_{M-1} | \tilde{P}^{(l_k)}) \right\} \right\}. \quad (3.26)$$

Using eq. (1.5) $T(P)$ is seen to be equal to

$$T(P) = \sum_{\pi \in S_M} A_{\pi} \left( \prod_{i=1}^{k-1} F_{p^{(l_{i-1})}}(A_{\pi_i}, l_i - 2) \right) \left[ \prod_{j=k+1}^{M-1} F_{p^{(l_{i+1})}}(A_{\pi_{i+1}}, l_j) \right]$$

$$\times \left( F_{p^{(l_{k-1})}}(A_{\pi_{k-1}}, l_k) \left( F_{p^{(l_{k-1})}}(A_{\pi_k}, l_k - 2) - F_{p^{(l_{k-1})}}(A_{\pi_k}, l_k - 2) \right) \right)$$

$$- \sum_{\sigma \in S_M} A_{\sigma} \left( \prod_{i=1}^{k-1} F_{p^{(l_{i-1})}}(A_{\sigma_i}, l_i - 2) \right) \left[ \prod_{j=k+1}^{M-1} F_{p^{(l_{i+1})}}(A_{\sigma_{i+1}}, l_j) \right]$$

$$\times \left( F_{p^{(l_k)}}(A_{\sigma_k}, l_k - 2) \left( F_{p^{(l_k)}}(A_{\sigma_{k+1}}, l_k - 1) - F_{p^{(l_k)}}(A_{\sigma_{k+1}}, l_k - 1) \right) \right).$$

If we now choose $\sigma = (\pi_1, \ldots, \pi_{k-1}, \pi_{k+1}, \pi_{k+1}, \ldots, \pi_M)$ this becomes

$$T(P) = \frac{1}{2} \sum_{\pi \in S_M} \left( \prod_{i=1}^{k-1} F_{p^{(l_{i-1})}}(A_{\pi_i}, l_i - 2) \right) \left[ \prod_{j=k+1}^{M-1} F_{p^{(l_{i+1})}}(A_{\pi_{i+1}}, l_j) \right] \Xi_{\pi}(P),$$

(3.28)
where

\[ \Xi_\pi(P) = A_\pi \left\{ F_{\sigma(k+1)}(A_{\sigma_{k+1}}, l_k) \left[ F_{\sigma(k)}(A_{\sigma_{k}}, l_k - 2) - F_{\bar{\sigma}(k)}(A_{\bar{\sigma}_{k}}, l_k - 2) \right] \right\} 
+ A_\sigma \left\{ F_{\sigma(k)}(A_{\sigma_{k}}, l_k) \left[ F_{\sigma(k+1)}(A_{\sigma_{k+1}}, l_k - 1) - F_{\bar{\sigma}(k+1)}(A_{\bar{\sigma}_{k+1}}, l_k - 1) \right] \right\} 
- A_\pi \left\{ F_{\sigma(k)}(A_{\sigma_{k}}, l_k - 2) \left[ F_{\sigma(k+1)}(A_{\sigma_{k+1}}, l_k - 1) - F_{\bar{\sigma}(k+1)}(A_{\bar{\sigma}_{k+1}}, l_k - 1) \right] \right\} \]

(3.29)

Here we have written the terms with \( \pi \) and \( \sigma \) in the sum over all permutations explicitly and compensated for that by the factor of \( \frac{1}{2} \). Using again eq. (1.5) in (3.29) leads to

\[ \Xi(P) = \left( \prod_{j=1}^{l_k} \frac{e^{-i\pi(P_j)}}{e^{i\pi(P_j)}} \right) (\sin(k_{P_1}) - \sin(k_{P_2})) \xi(P), \quad (3.30) \]

where

\[ \xi(P) = \frac{A_\pi}{e^{i\pi(P_1)}e^{i\pi(P_2)}e^{i\pi(P_{l_k})}} \left\{ \frac{e^{i\pi(P_1)}}{e^{i\pi(P_1)}} \frac{e^{i\pi(P_2)}}{e^{i\pi(P_2)}} - \frac{e^{i\pi(P_{l_k})}}{e^{i\pi(P_{l_k})}} \right\} \]

(3.31)

If we now combine eqs. (3.26), (3.28), (3.30) with (3.31) we find that all terms of kind (i) combine into

\[ \Phi = \left( \text{something symmetric in } (P_1, P_2, P_{l_k}) \right) \times \left( \frac{\sin(k_{P_2}) - \sin(k_{P_1})}{1 + \exp(i(k_{P_1} + k_{P_2}))} \right) \xi(P). \]

This vanishes due to the following
Lemma 3.4.

\[
\sum_{\text{cyl}} \left( \frac{\sin(k_{P_1}) - \sin(k_{P_2})}{1 + \exp(i(k_{P_1} + k_{P_2}))} \right) \xi(P) = 0. \tag{3.32}
\]

Proof. We first notice that due to lemma 3.1 we can drop all terms in \( \xi(P) \) that are symmetric under cyclic permutations of \((P_1, P_2, P_{l_k})\). We will now evaluate the second term in (3.31), noting that the first term can be obtained from it by the replacement \( \pi_k \leftrightarrow \pi_{k+1} \). The main identity for evaluating the second curly bracket in (3.31) is

\[
e^{(\pi_k)}(P_{h_k}) - e^{(\pi_{k+1})}(P_{h_k}) = \Lambda_{\pi_{k+1}} - \Lambda_{\pi_k} - u. \tag{3.33}
\]

This leads to

\[
\left\{ \frac{e^{(\pi_k)}(P_1) e^{(\pi_k)}(P_2) - e^{(\pi_{k+1})}(P_1) e^{(\pi_{k+1})}(P_2)}{e^{(\pi_{k+1})}(P_1) e^{(\pi_{k+1})}(P_2) - e^{(\pi_k+1)}(P_1) e^{(\pi_k+1)}(P_2)} \right\} - 1 + \frac{(\Lambda_{\pi_{k+1}} - \Lambda_{\pi_k} + u)(e^{(\pi_k)}(P_1) e^{(\pi_k)}(P_2) + e^{(\pi_{k+1})}(P_1) e^{(\pi_{k+1})}(P_2))}{e^{(\pi_{k+1})}(P_1) e^{(\pi_{k+1})}(P_2) e^{(\pi_k+1)}(P_{l_k})}.
\]

Here the first two terms are symmetric under permutations of \((P_1, P_2, P_{l_k})\) and can be dropped due to lemma 3.1. Using eq. (3.34) in (3.30), the remaining part of \( \xi(P) \) is found to be equal to

\[
\frac{(e^{(\pi_k)}(P_1) e^{(\pi_k)}(P_2) + e^{(\pi_{k+1})}(P_1) e^{(\pi_{k+1})}(P_2))}{e^{(\pi_{k+1})}(P_1) e^{(\pi_{k+1})}(P_2) e^{(\pi_k+1)}(P_{l_k})} \times \left\{ \Lambda_{\pi_k} - \Lambda_{\pi_{k+1}} + u \right\} + \frac{(\Lambda_{\pi_{k+1}} - \Lambda_{\pi_k} + u)(e^{(\pi_k)}(P_1) e^{(\pi_k)}(P_2) + e^{(\pi_{k+1})}(P_1) e^{(\pi_{k+1})}(P_2))}{e^{(\pi_{k+1})}(P_1) e^{(\pi_{k+1})}(P_2) e^{(\pi_k+1)}(P_{l_k})} \times \left\{ \Lambda_{\pi_k} - \Lambda_{\pi_{k+1}} + u \right\}. \tag{3.35}
\]

Here we have dropped additional terms stemming from the first term in (3.31), which are symmetric under permutations of \((P_1, P_2, P_{l_k})\).
The curly bracket in (3.35) vanishes because (1.6) implies that
\[ \frac{A_\pi}{A_\sigma} = \frac{A_{\pi_k} - A_{\pi_{k+1}} + u}{A_{\pi_k} - A_{\pi_{k+1}} - u}. \]
This completes the proof of lemma 3.4. \( \square \)

Thus we have proven the vanishing of all terms of kind (i), provided that \( \exp(i(k_{P_1} + k_{P_2})) \neq -1 \). This case however can be dealt with in the usual manner. This completes the proof of the theorem 3.2. \( \square \)

We have thus proven rigorously that all Bethe states (1.2) are lowest-weight states of the \( \eta \)-SU(2) algebra.

4. Bethe-Ansatz states as lowest-weight states of the \( \zeta \)-pairing

In this section we will prove that
\[ \zeta | \Psi_{M,N} \rangle = 0 \] (4.1)
for any regular eigenstate (as defined in definition 1.1) \( | \Psi_{M,N} \rangle \) of the Hamiltonian (1.1). In our proof, which is divided into a sequence of lemmata, we shall first gradually reduce (4.1) to other, simpler identities. In subsect. 4.1 we will reduce (4.1) to an identity for the functions \( \varphi \) (as defined in (1.4)), which describe the spin part of the Bethe Ansatz. This will enable us to deal with only the spin degrees of freedom; the charge degrees of freedom will turn out to be nonessential. After a short review of the central notions of the Algebraic Bethe Ansatz in subsect. 4.2 we then will reformulate the spin part of the Bethe Ansatz in terms of the Algebraic Bethe Ansatz in subsect. 4.3, and finally prove our central identity for the functions \( \varphi \) using this formulation.

4.1. REDUCTION TO SPIN DEGREES OF FREEDOM

We start with the following

Lemma 4.1. In order to prove (4.1) it is sufficient to prove that
\[ \sum_{y=1}^{z_1-1} \varphi(y, z_1, z_2, \ldots, z_{M-1} | P) + \sum_{y=z_1+1}^{z_2-1} \varphi(z_1, y, z_2, \ldots, z_{M-1} | P) \]
\[ + \ldots + \sum_{y=z_{M-1}+1}^{L} \varphi(z_1, z_2, \ldots, z_{M-1}, y | P) = 0. \] (4.2)
Here $z_j, j = 1, \ldots, M - 1$ is a set of ordered integers $z_1 < z_2 < \ldots < z_{M-1}$ and $\varphi$ is defined by (1.4).

**Proof.** In order to prove this lemma let us first re-arrange the summations in expression (1.2) for the eigenfunctions. We want to divide up the sums in (1.2) such that the outer summation is with respect to the positions of the electrons (without specifying the values of their spins) and the inner summation is with respect to different configurations of spins for a given configuration of the positions of the electrons. We will denote the space positions of the electrons by $X_j, j = 1, \ldots, M + N$. For a fixed set $\{X\}$ we then sum over all possibilities of distributing $M$ spins down among the $M + N$ electrons. Thus we sum over $y_i, i = 1, \ldots, M$, where each of the $y$’s takes values between 1 and $M + N$, and where all $y$’s are distinct as the wave function in (1.2) vanishes if two spins down occupy the same site.

To elucidate the change of summation more explicitly let us consider two different configurations $x^{(1)}_1, x^{(1)}_{M+N}$ and $x^{(2)}_1, x^{(2)}_{M+N}$ in the summation over $x_1, \ldots, x_{M+N}$ in (1.2), with corresponding permutations $Q^1$ and $Q^2$, such that

$$x^{(1)}_Q = x^{(2)}_Q, \quad j = 1, 2, \ldots, M + N$$

(without demanding $\sigma_Q^1 = \sigma_Q^2$). This means that the space positions of all electrons are the same and the two configurations differ only by the positions of the down spins, i.e. the values of the $y$’s. We will group together all such configurations and sum explicitly over the distributions of the $y$’s. Furthermore, we re-order the product of the $c^\dagger$’s in (1.2) according to the space ordering, i.e. for the sector $Q$: $x^{(1)}_{Q_1} \leq x^{(1)}_{Q_2} \leq \ldots \leq x^{(1)}_{Q_{M+N}}$

$$\prod_{j=1}^{M} c^\dagger_{x_{Q_1j+1}} \prod_{i=M+1}^{M+N} c^\dagger_{x_{Q_i1}} = \text{sgn}(Q) \prod_{i=1}^{M+N} c^\dagger_{y_{Q_i}, \sigma_{Q_i}}$$

(4.4)

Recall that $Q$ is a permutation of $M + N$ elements and that $\sigma_1 = \ldots = \sigma_M = -1$, $\sigma_{M+1} = \ldots = \sigma_{M+N} = 1$. Using the notation

$$X_l = x_{Q_l}, \quad l = 1, \ldots, M + N$$

(4.5)

we now can rewrite (1.2) as

$$|\Psi_{M,N}\rangle = \sum_{(X)} \sum_{(y)} \tilde{\psi}_{M,N}(X_1, \ldots, X_{M+N} | y_1, \ldots, y_M) \prod_{i=1}^{M+N} c^\dagger_{X_{l}, \sigma_{l}} |0\rangle.$$  

(4.6)

Here the ranges of summation are

$$1 \leq X_1 \leq X_2 \leq \ldots \leq X_{M+N} \leq L,$$

$$1 \leq y_1 < y_2 < \ldots < y_M \leq M + N,$$
and
\[ \sigma_l = \begin{cases} -1 & \text{if } l \in \{y_1, \ldots, y_M\} \\ +1 & \text{else}. \end{cases} \]

The wave function \( \tilde{\psi}_{M,N} \) can be obtained from (1.3)
\[
\tilde{\psi}_{M,N}(X_1, \ldots, X_{M+N} | y_1, \ldots, y_M) = \sum_{P \in S_{M+N}} \text{sgn}(P) \exp \left( i \sum_{l=1}^{M+N} k_p X_l \right) \varphi(y_1, \ldots, y_M | P),
\]
where
\[
\varphi(y_1, \ldots, y_M | P) := \sum_{\pi \in S_M} A_\pi \prod_{j=1}^{M} F_p(A_{\pi_j}, y_j)
\]
with
\[
F_p(A_j, y) := \prod_{i=1}^{y-1} \frac{\sin(k_{p_i}) - A_j - (U/4i)}{\sin(k_{p_i}) - A_j + (U/4i)} \frac{1}{\sin(k_{p_i}) - A_j + (U/4i)}
\]
and
\[
\frac{A_\pi}{A_{(i,l+1)\pi}} = \frac{A_{\pi_{i+1}} - A_{\pi_i} - u}{A_{\pi_{i+1}} - A_{\pi_i} + u}.
\]

In what follows we will make extensive use of the periodic boundary conditions (1.8), which can be written as
\[
\prod_{i=1}^{M+N} \left( \frac{A_n - \sin(k_i) + (iU/4)}{A_n - \sin(k_i) - (iU/4)} \right) = -\prod_{j=1}^{M} \left( \frac{A_n - A_j + (iU/2)}{A_n - A_j - (iU/2)} \right), \quad n = 1, \ldots, M.
\]

Let us now return to the expression (4.1), which consists of a large number of terms. It is sufficient to demand that smaller subsums are vanishing. We will make use of this fact considering only the terms with fixed set \( X_1, X_2, \ldots, X_{M+N} \) of \( X \)'s, and fixed permutation \( P \) in (4.7). If we now apply \( \zeta \) to (4.6) and isolate the coefficient of the term where the \( M - 1 \) remaining down spins are located at \( X_{z_1}, \ldots, X_{z_{M-1}} \), i.e. the coefficient of
\[
\prod_{j=1}^{M+N} c_{X_j, \sigma_j}^\dagger |0\rangle, \quad \text{where } \sigma_j = \begin{cases} -1 & \text{if } j \in \{z_1, \ldots, z_{M-1}\} \\ +1 & \text{else} \end{cases}
\]
we find it to be proportional to

\[
\sum_{y=1}^{z_1-1} \varphi(y, z_1, z_2, \ldots, z_{M-1} | P) + \sum_{y=z_1+1}^{z_2-1} \varphi(z_1, y, z_2, \ldots, z_{M-1} | P) + \cdots + \sum_{y=z_{M-1}+1}^{L} \varphi(z_1, z_2, \ldots, z_{M-1}, y | P).
\]

(4.9)

The factor of proportionality is \( \text{sgn}(P) \exp(i \sum_{j=1}^{M+N} k_j X_j) \) which is constant because \( P \) and all \( X \)'s are fixed. This proves the lemma. \( \Box \)

In the next step of reducing the original problem to a simpler one, we define an auxiliary spin model (later we will identify it with the inhomogeneous Heisenberg XXX antiferromagnet). Let us consider a lattice with \( M + N \) sites. With each site we associate an independent two-dimensional linear space. This linear space can be identified with a spin-\( \frac{1}{2} \) representation space of SU(2), with a natural basis consisting of spin up (\( +1 \)) and spin down (\( -1 \)). The tensor product of all the independent two-dimensional spaces sitting on the respective lattice sites forms a linear space \( \mathcal{K} \). In \( \mathcal{K} \) we define the three operators \( \sigma_i^+, \sigma_i^-, \sigma_i^z \) to act trivially in all spin spaces at sites different from \( l \), and to act like the Pauli matrices on the basis states at site \( l \). The pseudovacuum (reference state) \( |0\rangle \) in \( \mathcal{K} \) is purely ferromagnetic, i.e., at every site there is an up spin. All other states have some overturned spins. The local operator that flips a spin at site \( l \) from up to down is \( \sigma_l^- \) and the spin raising operator in \( \mathcal{K} \) is defined to be

\[
S^+ := \sum_{l=1}^{M+N} \sigma_l^+.
\]

(4.10)

The state with \( M \) spins down located at the sites \( y_1, y_2, \ldots, y_M \) is

\[
\prod_{j=1}^{M} \sigma_{y_j}^- |0\rangle.
\]

(4.11)

Let us now use the functions \( \varphi \) given by (1.4) to define the state \( | \varphi_M \rangle \)

\[
| \varphi_M \rangle := \sum_{\{y_j\}} \varphi(y_1, \ldots, y_M | P) \prod_{j=1}^{M} \sigma_{y_j}^- |0\rangle
= \sum_{\{y_j\}} \sum_{\pi \in S_M} A_\pi \prod_{l=1}^{M} F_p(\Lambda_{\pi_l}, y_l) \prod_{j=1}^{M} \sigma_{y_j}^- |0\rangle
\]
where the summation extends over

\[ 1 \leq y_1 < y_2 < y_3 < \ldots < y_M \leq M + N, \]

\( P \) is a permutation of \( M + N \) elements, and the \( A_{\pi} \) satisfy (4.7).

Now we are ready to formulate

**Lemma 4.2.** Equality (4.2) is equivalent to

\[ S^z | \varphi_M \rangle = 0. \tag{4.13} \]

**Proof.** The proof is straightforward. \( \square \)

We have thus shown that in order to prove (4.1) it is sufficient to prove (4.13). We have separated charge and spin degrees of freedom and reduced the problem to proving an equation that only involves spin degrees of freedom. The wave functions (1.4) are similar to the eigenfunctions of the isotropic Heisenberg XXX antiferromagnet for spin \( \frac{1}{2} \) (they formally coincide only if all \( \sin(k_i) \) are the same). For the spin- \( \frac{1}{2} \) Heisenberg XXX model equality (4.13) is well known [11,12].

We will prove (4.13) by means of the Algebraic Bethe Ansatz.

### 4.2. ALGEBRAIC BETHE ANSATZ

The main concepts of the Algebraic Bethe Ansatz in relation with the Heisenberg XXX model are summarised in the articles by L.D. Faddeev and L. Takhtajan [11,13,14]. We will restrict ourselves to a short review here and mainly just set up the notations. The \( L \)-operator of the XXX-model is given by

\[ L(l | \Lambda) = \Lambda + \frac{iU}{4} \sigma \cdot \sigma_l = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} + \frac{iU}{4} \begin{pmatrix} \sigma_l^- & 2\sigma_l^- \\ 2\sigma_l^+ & -\sigma_l^- \end{pmatrix}. \tag{4.14} \]

This matrix is a \( 2 \times 2 \) matrix in an auxiliary space with operator valued entries \( \sigma_l \), where \( \sigma_l \) is the set of Pauli matrices acting nontrivially only on site number \( l \). The \( L \)-operators fulfills the Yang–Baxter equation, which can be written in the following form

\[ R(\Lambda_1, \Lambda_2)(L(l | \Lambda_1) \otimes L(l | \Lambda_2)) = (L(l | \Lambda_2) \otimes L(l | \Lambda_1))R(\Lambda_1, \Lambda_2), \tag{4.15} \]
The \( R \)-matrix is given by

\[
R(A_1, A_2) = \begin{pmatrix}
 f(A_2, A_1) & 0 & 0 & 0 \\
 0 & g(A_2, A_1) & 1 & 0 \\
 0 & 1 & g(A_2, A_1) & 0 \\
 0 & 0 & 0 & f(A_2, A_1)
\end{pmatrix},
\]

(4.16)

where

\[
f(A_2, A_1) = 1 - \frac{iU}{2(A_2 - A_1)}, \quad g(A_2, A_1) = -\frac{iU}{2(A_2 - A_1)}.
\]

(4.17)

The monodromy matrix is defined as a product of the \( L \)-operators on all lattice-sites

\[
T(A) = L(M + N | A - p_{M+N}) L(M + N - 1 | A - p_{M+N-1}) \ldots L(1 | A - p_1).
\]

(4.18)

Here we have shifted the spectral parameters of the \( L \)-operators by different amounts \( p_i \) on different lattice sites. We make the following choice for the \( p \)'s:

\[
p_i = \sin(k_p),
\]

(4.19)

where \( P \) is a permutation of \( M + N \) elements. The monodromy matrix \( T(A) \) is a \( 2 \times 2 \) matrix in the auxiliary space

\[
T(A) = \begin{pmatrix}
 A(A) & B(A) \\
 C(A) & D(A)
\end{pmatrix},
\]

(4.20)

The matrix elements \( A, B, C, D \) depend on all \( \sigma_i \) and act nontrivially in all sites of the lattice. The commutation relations between monodromy matrices for different spectral parameters are given by

\[
R(A_1, A_2)(T(A_1) \otimes T(A_2)) = (T(A_2) \otimes T(A_1))R(A_1, A_2),
\]

(4.21)

where \( R \) is the same \( R \)-matrix as in (4.15). The matrix elements of the monodromy matrix act in the following simple way on the ferromagnetic vacuum \( |0\rangle \):

\[
C(A) |0\rangle = 0, \quad A(A) |0\rangle = a(A) |0\rangle, \quad D(A) |0\rangle = d(A) |0\rangle,
\]

where

\[
a(A) = \prod_{l=1}^{M+N} \left( A - p_l + \frac{iU}{4} \right), \quad d(A) = \prod_{l=1}^{M+N} \left( A - p_l - \frac{iU}{4} \right).
\]

(4.22)
The transfer matrix $\tau(A)$ is defined as the trace of the monodromy matrix
\begin{equation}
\tau(A) = A(A) + D(A).
\end{equation}

(4.23)

The framework of the Algebraic Bethe Ansatz provides us with the following expression for the eigenfunctions of the transfer matrix (note that eq. (4.21) implies that $[B(A_1), B(A_2)] = 0$):
\begin{equation}
\prod_{j=1}^{M} B(A_j) |0\rangle, \quad (4.24)
\end{equation}

provided that the spectral parameters fulfill the Bethe equations
\begin{equation}
\frac{a(A_n)}{d(A_n)} \prod_{j \neq n}^{M} \left( \frac{f(A_n, A_j)}{f(A_j, A_n)} \right) = 1 \quad n = 1, \ldots, M. \quad (4.25)
\end{equation}

Substituting for $a(A)$ and $d(A)$ this becomes
\begin{equation}
\prod_{n=1}^{M+N} \left( \frac{A_n - \sin(k_n) + (iU/4)}{A_n - \sin(k_n) - (iU/4)} \right) = -\prod_{j=1}^{M} \left( \frac{A_n - A_j + (iU/2)}{A_n - A_j - (iU/2)} \right), \quad n = 1, \ldots, M, \quad (4.26)
\end{equation}

which coincides with eq. (4.8).

Now we shall prove that
\begin{equation}
S^+ \prod_{j=1}^{M} B(A_j) |0\rangle = 0. \quad (4.27)
\end{equation}

Here $S^+ = \sum_{l=1}^{M+N} \sigma_l^+$ and $\prod_{l=1}^{M} B(A_j) |0\rangle$ is an eigenfunction of the transfer-matrix (4.23), i.e. the $A_j$ fulfill the system of equations (4.25). Let us introduce
\begin{equation}
S = \sum_{l=1}^{M+N} \sigma_l. \quad (4.28)
\end{equation}

The proof of (4.27) is based on the following facts:
\begin{equation}
S^+ |0\rangle = 0, \quad (4.29)
\end{equation}

and
\begin{equation}
[(S + \sigma), L(l | A - p_l)] = 0, \quad (4.30)
\end{equation}

which follow from
\begin{equation}
[(\sigma_l + \sigma), (\sigma_l \cdot \sigma)] = 0.
\end{equation}
The monodromy matrix $T$ also commutes with $(S + \sigma)$

$$[(S + \sigma), T(A)] = \sum_{l=1}^{M+N} L(M+N | \Lambda - p_{M+N}) \times \ldots [(S + \sigma), L(l | \Lambda - p_{l})] \ldots L(1 | \Lambda - p_{1}) = 0. \quad (4.31)$$

The $+$ component of this vector identity reads

$$[(S^{+} + \sigma^{+}), T(A)] = 0, \quad (4.32)$$

and the $(12)$-component of this matrix equation is

$$[S^{+}, B(A)] = A(A) - D(A). \quad (4.33)$$

Using eqs. (4.33) and (4.29) in (4.27) we obtain

$$S^{+} \prod_{j=1}^{M} B(A_{j}) |0\rangle = \sum_{i=1}^{M} \left( \prod_{k=1}^{i-1} B(A_{k}) \right) \left[ A(A_{i}) - D(A_{i}) \right] \left( \prod_{j=i+1}^{M} B(A_{j}) \right) |0\rangle. \quad (4.34)$$

Standard Algebraic Bethe Ansatz techniques give the following result:

$$S^{+} \prod_{j=1}^{M} B(A_{j}) |0\rangle = \sum_{i=1}^{M} M_{i} \prod_{j=1}^{i-1} B(A_{j}) |0\rangle,$$

where

$$M_{i} = a(A_{i}) \left( \prod_{j=1}^{i-1} f(A_{j}, A_{j}) \right) - d(A_{i}) \left( \prod_{j=1}^{i-1} F(A_{j}, A_{j}) \right). \quad (4.35)$$

All coefficients $M_{i}$ vanish due to the Bethe equations (4.26). This finally proves (4.27), i.e. all eigenfunctions of the transfer matrix $\tau(A)$ with finite spectral parameter $\Lambda$ are highest-weight vectors of the spin rotational $SU(2)$.

In the last part of our proof we will now show that the state $|\varphi_{M}\rangle$ is proportional to the eigenfunction $\Pi_{j=1}^{M} B(A_{j}) |0\rangle$ of the transfer matrix, which will establish (4.13) and thus (4.1) as desired.

4.3. PROPORTIONALITY OF COORDINATE- AND ALGEBRAIC- BETHE-ANSATZ WAVE FUNCTIONS

In this subsection we will prove that

$$|\varphi_{M}\rangle = \text{const.} \prod_{j=1}^{M} B(A_{j}) |0\rangle \quad (4.36)$$
and explicitly evaluate the constant factor. In order to prove (4.36) we will make use of the results obtained in refs. [15] and [16].

(i) Generalised two-site model.
Let us first group the product of $L$-operators in (4.18) into two strings

$$T_1(A) = L(n \mid A - p_n - p_n) \ldots L(2 \mid A - p_2) L(1 \mid A - p_1),$$

$$T_{II}(A) = L(M + N \mid A - p_{M+N}) \ldots L(n + 2 \mid A - p_{n+2}) L(n + 1 \mid A - p_{n+1}),$$

(4.37)
in such a way that

$$T(A) = T_{II}(A) T_1(A).$$

(4.38)

Both $T_{II}(A)$ and $T_1(A)$ are $2 \times 2$ matrices

$$T_1(A) = \begin{pmatrix} A_1(A) & B_1(A) \\ C_1(A) & D_1(A) \end{pmatrix}, \quad T_{II}(A) = \begin{pmatrix} A_{II}(A) & B_{II}(A) \\ C_{II}(A) & D_{II}(A) \end{pmatrix}.$$  

(4.39)
The matrix elements of $T_1(A)$ commute with the matrix elements of $T_{II}(A)$. The commutation relations of the matrix elements of the same $T$-operator are as in (4.21), i.e.

$$R(A_1, A_2)(T_1(A_1) \otimes T_1(A_2)) = (T_1(A_2) \otimes T_1(A_1))R(A_1, A_2).$$

(4.40)

A similar equation holds for $T_{II}$. The matrix elements of $T$ can be expressed in terms of the matrix elements of $T_1$ and $T_{II}$, e.g.

$$B(A) = A_{II}(A) B_1(A) + B_{II}(A) D_1(A).$$

(4.41)

It is also possible to express the eigenfunctions $\prod_{j=1}^{M} B(A_j) \mid 0 \rangle$ in terms of $B_1$ and $B_{II}$. In order to do this we will use that

$$C_{II}(A) \mid 0 \rangle = 0, \quad A_{II}(A) \mid 0 \rangle = a_{II}(A) \mid 0 \rangle, \quad D_{II}(A) \mid 0 \rangle = d_{II}(A) \mid 0 \rangle$$

and

$$C_1(A) \mid 0 \rangle = 0, \quad A_1(A) \mid 0 \rangle = a_1(A) \mid 0 \rangle, \quad D_1(A) \mid 0 \rangle = d_1(A) \mid 0 \rangle.$$  

(4.42)

Here

$$a_1(A) = \prod_{l=1}^{n} (A - p_l + (iU/4)), \quad d_1(A) = \prod_{l=1}^{n} (A - p_l - (iU/4))$$
and

\[ a_{11}(\lambda) = \prod_{l=n+1}^{M+N} (\lambda - p_l + (iU/4)), \quad d_{11}(\lambda) = \prod_{l=n+1}^{M+N} (\lambda - p_l - (iU/4)). \]  

(4.43)

In refs. [15,16] it was proved that

\[ \prod_{j=1}^{M} B(\lambda_j) |0\rangle = \sum_{\mathcal{S}_1, \ldots, \mathcal{S}_m} \prod_{k=1}^{m} \prod_{l=1}^{n} a_{11}(A_{k})d_{11}(A_{m})f(A_{k}, A_{m})B_{11}(A_{m})B_{1}(A_{k}) |0\rangle. \]  

(4.44)

On the r.h.s. we have summations with respect to partitions of the set of all \( \lambda_i \) into two subsets \( \mathcal{S}_1 = \{A_{k} \} \) and \( \mathcal{S}_2 = \{A_{m} \} \). \( k \) labels different elements in the subset \( \mathcal{S}_1 \) and \( m \) labels different elements in the subset \( \mathcal{S}_2 \). The above model is called generalised two-site model because \( T \) is represented as a product of two factors. This is not sufficient for our purposes however. Let us therefore now consider the so-called (ii) \( k \)-site generalised model.

Now \( T \) is represented by a product of \( k \) factors:

\[ T(\lambda) = T_k(\lambda) \ldots T_2(\lambda)T_1(\lambda). \]  

(4.45)

Here each \( T_\alpha(\lambda) \) is a string of \( L \)-operators like in (4.37). The commutation relations of the matrix elements of each of the \( T_\alpha(\lambda) \) is given by the same \( R \)-matrix as in eq. (4.40). The matrix elements of \( T_\beta(\lambda) \) commute with the matrix elements of \( T_\alpha(\lambda) \) if \( \alpha \neq \beta \) and like for the two-site model we have

\[ T_\alpha(\lambda) = \begin{pmatrix} A_\alpha(\lambda) & B_\alpha(\lambda) \\ C_\alpha(\lambda) & D_\alpha(\lambda) \end{pmatrix}, \]  

(4.46)

\[ C_\alpha(\lambda) |0\rangle = 0, \quad A_\alpha(\lambda) |0\rangle = a_\alpha(\lambda) |0\rangle, \quad D_\alpha(\lambda) |0\rangle = d_\alpha(\lambda) |0\rangle. \]  

(4.47)

By explicitly multiplying the matrices in eq. (4.45) we can express \( B(\lambda) \) in terms of the matrix elements of the \( T_\alpha(\lambda) \). Iteration of eq. (4.44) leads to the following expression for the eigenfunctions of the transfer matrix \( T(\lambda) \):

\[ \prod_{j=1}^{M} B(\lambda_j) |0\rangle = \sum_{\mathcal{S}_1, \ldots, \mathcal{S}_m} \left( \prod_{\alpha=1}^{k} \prod_{m_\alpha \in \alpha} B_\alpha(A_{m_\alpha}) |0\rangle \right) \times \left( \prod_{1 \leq \alpha < \beta \leq k} \prod_{m_\alpha \in \alpha} \prod_{k_\beta \in k} a_\beta(A_{m_\alpha})d_\alpha(A_{k_\beta})f(A_{m_\alpha}, A_{k_\beta}) \right). \]  

(4.48)
Here the summation is with respect to the partitions of the set of all \( \Lambda_i \) into \( k \) disjoint subsets \( \mathcal{S}_{\beta} \), \( \beta = 1, \ldots, k \). The index \( m_\alpha \) enumerates different \( \Lambda \) in the subset \( \mathcal{S}_\alpha \) and the index \( k_\beta \) enumerates different \( \Lambda \) in the subset \( \mathcal{S}_\beta \). Eq. (4.48) was first proved in ref. [16].

(iii) Coordinate Bethe Ansatz.

Let us now identify each factor \( T_\alpha(A) \) in our multi-site generalised model with an individual \( L \)-operator in (4.18). This evidently implies that \( k = M + N \) and each string of \( L \)-operators consists of only one factor. Replacement of the index \( \alpha \) by \( l \) leads to

\[
a_l(A) = \Lambda - p_l + (iU/4), \quad d_l(A) = \Lambda - p_l - (iU/4),
\]

and

\[
B_l(A) = \frac{1}{2}iU \sigma_l^-.
\]

The \( B \)-operators have two important features: (1) They are independent of \( \Lambda \), (2) \((B_l(A))^2 = 0\). Property (2) implies that each set \( \mathcal{S}_\alpha \) in (4.48) consists of maximally one element.

(a) One spin down.

Let us first consider the eigenfunctions in the sector with one spin down. Application of eq. (4.48) leads to

\[
B(A) |0\rangle = \frac{1}{2}iU \sum_{y=1}^{N+1} \sigma_y^- |0\rangle \prod_{k=y+1}^{N+1} (\Lambda - p_k + (iU/4)) \prod_{j=1}^{y-1} (\Lambda - p_j - (iU/4)).
\]

(4.50)

In this particular case all sets \( \mathcal{S}_\alpha \) except one are empty. This one (one-element) set is \( \mathcal{S}_y = \{A\} \). Comparison of (4.50) with (4.12) yields

\[
|\varphi_1\rangle = - \sum_{y=1}^{N+1} \sigma_y^- |0\rangle \prod_{i=1}^{y-1} \left( \frac{\Lambda - p_i - (iU/4)}{\Lambda - p_i + (iU/4)} \right) \frac{1}{\Lambda - p_y + (iU/4)}
\]

\[
= (2i/U) \prod_{l=1}^{N+1} (\Lambda - p_l + (iU/4))^{-1} B(A) |0\rangle.
\]

(4.51)

Here we have used that \( p_l = \sin(k_{\rho_l}) \), and we have put \( A = 1 \).
(b) Two spins down

Let us consider the eigenfunctions in the sector with two spins down. Application of eq. (4.48) leads to

\[ B(A_1) B(A_2) |0\rangle \]

\[ = \sum_{1 \leq y_1 < y_2 \leq N + 2} \sum_{\pi \in S_2} (iU/2)^2 \sigma^-_{\alpha_1} \sigma^-_{\alpha_2} |0\rangle \]

\[ \times \prod_{j_1 = y_1 + 1}^{N+2} (A_{\pi_1} - p_{j_1} + (iU/4)) \prod_{i_1 = 1}^{y_1-1} (A_{\pi_1} - p_{i_1} - (iU/4)) \]

\[ \times \prod_{j_2 = y_2 + 1}^{N+2} (A_{\pi_2} - p_{j_2} + (iU/4)) \prod_{i_2 = 1}^{y_2-1} (A_{\pi_2} - p_{i_2} - (iU/4)) f(A_{\pi_1}, A_{\pi_2}). \] (4.52)

Here \( \pi \) is a permutation of two elements 1, 2 and

\[ f(A_{\pi_1}, A_{\pi_2}) = \frac{(A_{\pi_1} - A_{\pi_2}) - (iU/2)}{(A_{\pi_1} - A_{\pi_2})}. \]

In this case only two subsets \( S_{y_1} \) are nonempty. Each of them consists of one element \( \{A_{\pi_1}\} \) and \( \{A_{\pi_2}\} \). Comparison of eq. (4.52) with the expression for \( |\varphi_2\rangle \) in (4.12) yields (recall that \( p_y = \sin(k_y) \))

\[ \left\{ (2i/U)^2 \prod_{l=1}^{N+2} (A_1 - p_l + (iU/4))^{-1} (A_2 - p_l + (iU/4))^{-1} \right\} B(A_1) B(A_2) |0\rangle \]

\[ = \sum_{1 \leq y_1 < y_2 \leq N + 2} \sum_{\pi \in S_2} \sigma^-_{\alpha_1} \sigma^-_{\alpha_2} |0\rangle \left\{ \prod_{l_1 = 1}^{y_1-1} \left( \frac{A_{\pi_1} - p_{l_1} - (iU/4)}{A_{\pi_1} - p_{l_1} + (iU/4)} \right) \frac{1}{A_{\pi_1} - p_{y_1} - (iU/4)} \right\} \]

\[ \times \left\{ \prod_{l_2 = 1}^{y_2-1} \left( \frac{A_{\pi_2} - p_{l_2} - (iU/4)}{A_{\pi_2} - p_{l_2} + (iU/4)} \right) \frac{1}{A_{\pi_2} - p_{y_2} + (iU/4)} \right\} \left( \frac{A_{\pi_1} - A_{\pi_2} - (iU/2)}{A_{\pi_1} - A_{\pi_2}} \right). \] (4.53)

Note that the expressions in the curly brackets coincide with \( -F_\rho (A | y) \) as defined in (1.5) and

\[ A_\pi = \frac{A_{\pi_1} - A_{\pi_2} - (iU/2)}{A_{\pi_1} - A_{\pi_2}}. \] (4.54)
Clearly eq. (4.54) implies that

\[ \frac{A_{(\text{id})}}{A_{(21)}} = \frac{\Lambda_1 - \Lambda_2 - (iU/2)}{\Lambda_1 - \Lambda_2 + (iU/2)}, \tag{4.55} \]

which is in complete agreement with eq. (4.7). Thus we have proved that

\[ |\varphi_2\rangle = (2i/U)^2 \prod_{l=1}^{N+2} (\Lambda_1 - p_l + (iU/4))^{-1} (\Lambda_2 - p_l + (iU/4))^{-1} B(\Lambda_1) B(\Lambda_2) |0\rangle. \tag{4.56} \]

(c) \textit{M} spins down

Let us finally consider the eigenfunctions of the transfer matrix for \textit{M} overturned spins

\[ (2i/U)^M \left\{ \prod_{l=1}^{M+N} \prod_{j=1}^{M} \left( A_j - p_l + (iU/4) \right)^{-1} \right\} \prod_{j=1}^{M} B(A_j) |0\rangle \]

\[ = \sum_{1 \leq y_1 < y_2 < \ldots < y_M \leq M+N} \sum_{\pi \in S_M} \prod_{i=1}^{M} \sigma_{y_i}^{-1} |0\rangle \left( \prod_{j=1}^{M} F_{\pi}(A_{\pi_{y_j}}, y_j) \right) A_{\pi}. \tag{4.57} \]

Again we have used eq. (4.48) to obtain this result. The nonempty subsets \( \mathcal{S}_{\pi} \) (each of which consists of exactly one element) are \( \mathcal{S}_{y_1} = A_{\pi_{y_1}}, \mathcal{S}_{y_2} = A_{\pi_{y_2}}, \ldots, \mathcal{S}_{y_M} = A_{\pi_{y_M}} \), where \( \pi \) is some permutation of \( M \) elements. \( F_{\pi}(A, y) \) is given by (1.5) and

\[ A_{\pi} = \prod_{1 \leq l < k \leq M} f(A_{\pi_l}, A_{\pi_k}) = \prod_{1 \leq l < k \leq M} \left( \frac{\Lambda_{\pi_l} - \Lambda_{\pi_k} - (iU/2)}{\Lambda_{\pi_l} - \Lambda_{\pi_k}} \right), \tag{4.58} \]

which again satisfies eq. (4.7).

Using eq. (4.58) in (4.57) and comparing the result with (4.12) shows that

\[ |\varphi_M\rangle = (2i/U)^M \left\{ \prod_{l=1}^{M+N} \prod_{j=1}^{M} \left( A_j - p_l + (iU/4) \right)^{-1} \right\} \prod_{j=1}^{M} B(A_j) |0\rangle. \tag{4.59} \]

The periodic boundary conditions (4.8) for \( |\varphi_M\rangle \) coincide with the periodic boundary conditions (4.26) for \( \Pi_j B(A_j) \). Eq. (4.27) together with (4.59) yield

\[ S^+ |\varphi_M\rangle = 0, \tag{4.60} \]
which, due to (4.13) and Lemma (4.1), is sufficient to prove (4.1), i.e.

\[ \zeta |\Psi_{M,N}\rangle = 0. \]

This completes the proof of our assertion that all regular Bethe-Ansatz states for the one-dimensional Hubbard model are lowest weight-vectors of the \( \zeta \)-SU(2) algebra.

5. Conclusions

In this paper we have shown that the regular Bethe-Ansatz states for the one-dimensional Hubbard model are lowest-weight vectors of an SO(4) symmetry algebra. Application of the SO(4) raising operators to the Bethe-Ansatz states leads to new eigenstates of the hamiltonian (1.1). It will be shown in a forthcoming paper [10] that the set of all regular Bethe-Ansatz states together with all states that are obtained from them by using the SO(4) structure form a complete set of eigenstates (asymptotically for large but finite lattice lengths \( L \)).

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