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Periodic and quasi-periodic solutions of degenerate modulation equations

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In some circumstances (degenerations) it is essential to add higher-order nonlinear coefficients to a Ginzburg–Landau type modulation equation (which only has one cubic nonlinearity). In this paper we study these degenerate modulation equations. We consider the important situation in which the equation has real coefficients and the case of coefficients with small imaginary parts. First we determine the stability of periodic solutions. The stationary problem is, like in the non-degenerate case, integrable: there exist families of quasi-periodic and homoclinic solutions. This system is perturbed by considering modulation equations with coefficients with small imaginary parts. We establish that there exists an unbounded domain in parameter space in which the modulation equation has quasi-periodic solutions. Moreover, we show that there is a manifold of codimension 1 (in parameter space) on which the homoclinic solutions survive the perturbation.

1. Introduction

Modulation equations govern the evolution of patterns at near-critical conditions in many problems of physical interest (see for example refs. [1, 2, 15–17]). Well known is the so-called Ginzburg–Landau equation, which in the case of supercritical bifurcations is given, in canonical form, by

$$\frac{\partial \Phi}{\partial t} = \left[1 - (1 + iB)|\Phi|^2 \right] \Phi + (1 + iA) \frac{\partial^2 \Phi}{\partial z^2}.$$

Primarily the equation has been used to investigate the stability of bifurcating space-periodic solutions [18]. More recently the Ginzburg–

Landau equation has become an increasingly popular subject of study, in particular since the possibility of chaotic behaviour has been discovered (for example refs. [1, 11, 13, 14]). On the other hand, in the non-chaotic case, Kramer and Zimmerman [12], for the case of real coefficients ($A = B = 0$), have studied stationary quasi-periodic and homoclinic solutions. The mathematical structure of these solutions has an intriguing and intimate relation with the stability boundary of space-periodic solutions. Doelman [4] has shown that the quasi-periodic solutions do not survive the perturbations of the real-coefficients case by small values of A and B and deduced other interesting solutions, which are slow time-periodic.

The procedure to derive (1.1) for a given physical system at near-criticality (i.e. for a bifurcation/control parameter Λ close to a Λ_c at which a stationary laminar solution becomes linearly unstable) has been developed in refs. [2, 15, 17]. However (1.1) is only the $\mathcal{O}(1)$ part of a more complicated equation. Proceeding with the derivation process after deriving the $\mathcal{O}(1)$ part yields higher order corrections (see refs. [5, 7]):

$$\begin{aligned}\Phi_\tau = & a\lambda\Phi + b\Phi|\Phi|^2 + c\Phi_{\xi\xi} \\ & + \hat{\varepsilon}\left(d\lambda\Phi_\xi + e\Phi_{\xi\xi\xi} + f|\Phi|^2\Phi_\xi + g\Phi^2\Phi_\xi^*\right) \\ & + \hat{\varepsilon}^2(\dots + h|\Phi|^4\Phi) + \dots\end{aligned}\quad (1.2)$$

with $0 < \hat{\varepsilon} \ll 1$ defined by $\Lambda = \Lambda_c + \lambda\hat{\varepsilon}^2$, i.e. $\hat{\varepsilon}$ is the measure for the distance between Λ and Λ_c ; λ is the new bifurcation parameter, complex coefficients $a, b, \dots, h, \dots$ can be computed from the underlying physical system. It is not always possible to truncate (1.2) into (1.1): if the real part of the coefficient of the cubic non-linearity, b_r , is small then one has to take into account some of the higher order terms in (1.2): their influence can not be neglected. Computations of the non-linear coefficients of (1.2) for the Blasius boundary layer show that b_r (and h_r) are small [7]. Situations in which b_r is small appear generically in physical systems which have a second control parameter: b_r changes sign by varying this second control parameter. Degenerations of this type can be found, for instance, in binary fluid convection [16], Taylor–Couette flow with counter-rotating cylinders [3] and Jeffrey–Hamel flow [6, 7]. Hence, these are (important) examples in which one has to consider an extension of Ginzburg–Landau equation (1.1) since this truncation of (1.2) degenerates for b_r small (see the appendix to this paper). This extension is called the degenerate modulation equation in this paper.

Eckhaus and Iooss [7] studied the situation in which $b_r = \delta\hat{b}_r$ for some $0 \leq \delta \ll 1$ and derived

the degenerate modulation equation

$$\begin{aligned}\Phi_\tau = & ib_i\Phi|\Phi|^2 \\ & + \delta(a\lambda\Phi + \hat{b}_r\Phi|\Phi| + c\Phi_{zz} + f|\Phi|^2\Phi_z \\ & + g\Phi^2\Phi_z^* + h|\Phi|^4\Phi) + \mathcal{O}(\delta^2).\end{aligned}\quad (1.3)$$

This equation is essentially a rescaled version of (1.2). In the appendix we present a derivation of (1.3) alternative to the approach of ref. [7]. Note that there are two time scales in eq. (1.3): $|\Phi|^2$ evolves on a time scale which is slower than the time scale of the oscillations of Φ . Eckhaus and Iooss [7] studied the stability of periodic solutions of (1.3), assuming that $b_i = \mathcal{O}(1)$, which yielded the phenomenon of strong rejection or selection of periodic patterns.

Complementary to ref. [7] we study in this paper the case that $b_i = 0$ (or small). The situation $b_i = 0$ appears naturally if the underlying physical system has a reflection symmetry: the waves bifurcating at $\Lambda = \Lambda_c$ are stationary. This is for instance the case in (binary) fluid convection problems. By rescaling time and truncating in (1.3) we arrive at

$$\begin{aligned}\Phi_t = & a_r\lambda\Phi + \hat{b}_r\Phi|\Phi| + c_r\Phi_{zz} \\ & + i(f_i|\Phi|^2\Phi_z + g_i\Phi^2\Phi_z^*) + h_r|\Phi|^4\Phi,\end{aligned}\quad (1.4)$$

in which all coefficients are real now, due to the symmetry in the underlying system (see, for instance, ref. [5]). By the conditions on the linear eigenvalue problem which determines the stability of the basal stationary solution of the physical system we have $a_r\lambda > 0$, $c > 0$ (see, for instance, ref. [5]); as in ref. [7] we assume that $h_r < 0$. Furthermore, to simplify the notation we assume $\hat{b}_r = 0$. The restrictions are not essential: the simplified case contains the essence of the general situation.

The aim of this paper is to explore the very rich structure of solutions of (1.4). Since this equation is a generalization of (1.1), with $A = B = 0$, we

search for differences and similarities between this classical case and the “non-classical” degenerate case. We focus our attention on the periodic, the (spatially) quasi-periodic and the (spatially) homoclinic solutions of (1.4) which are known to exist in the classical one. We find large families of quasi-periodic and homoclinic solutions of (1.4), similar to the classical case. A significant part of this paper is devoted to the analysis of the influence of allowing the coefficients of (1.4) to have small imaginary parts. This perturbation analysis for the classical Ginzburg–Landau case has been performed in ref. [4]. We organized this paper so that one can make a direct comparison between the properties of the classical and the non-classical case at any stage of the analysis: every (sub)section is divided in a classical and a non-classical part.

We now give a more detailed description of the results of this paper.

We first study the stability of periodic solutions of (1.4). As in the classical case of the Ginzburg–Landau equation we find in general an interval of stable periodic solutions. However, in the degenerate case there are parameter combinations such that this interval is unbounded or disappears altogether (section 2). In this section we also consider the stationary equation which is integrable in the classical case. The stationary problem for (1.4) is more complicated; nevertheless, and this is remarkable, the problem again is integrable with quasi-periodic and homoclinic solutions similar in structure to the classical case.

Finally, as in ref. [4], we ask whether quasi-periodic solutions persist when in (1.4) coefficients with small imaginary parts are admitted. The analysis of ref. [4] yields that the stationary quasi-periodic solutions do not survive this perturbation, even if a slow time-periodicity with arbitrary wave number is allowed. The homoclinic solutions of the unperturbed system also disappear due to the perturbation in the case of the Ginzburg–Landau equation. The outcome of the analysis of the degenerate modulation equation with coefficients with small imaginary parts is

quite different from the nondegenerate case: we show that there exists an unbounded, 7-dimensional, region in the 7-dimensional parameter space in which eq. (1.4) has solutions quasi-periodic in space and (slowly) periodic in time. We also establish the existence of a manifold in parameter space (of codimension 1) at which eq. (1.4) has slow time-periodic solutions which exhibit, as a function of the spatial variable z , homoclinic behaviour: the solutions tend to a periodic solution for $z \rightarrow \pm\infty$. Essential in the study of the nondegenerate case are the symmetries of the perturbed problem and the fact that the perturbed ordinary differential equation is divergence free. It is possible to choose the parameters of the degenerate problem such that the degenerate perturbed ordinary differential equation also exhibits these features. In that case one can copy the methods of ref. [4] to prove the disappearance of the quasi-periodic solutions and establish the existence of heteroclinic solutions connecting counter-rotating periodic solutions (and of other special solutions). However, this case is not generic because one has to impose conditions relating coefficients which are essentially independent.

Remark. In section 2.1 we study the periodic solutions of eq. (1.4) of the form Re^{ikz+wt} and also determine the stable periodic solutions of this type. It is, however, a very difficult problem to determine the stability of the quasi-periodic and homoclinic solutions found in sections 2 and 3. Even in the Ginzburg–Landau case with pure real coefficients (i.e. $A = B = 0$ in (1.1)) there are yet no rigorous results on the stability of the (spatially) quasi-periodic and homoclinic solutions, described in section 2.1, *in an unbounded domain* (i.e. $z \in (-\infty, \infty)$). It should also be noted that in most numerical investigations of equations like (1.1) ([11, 13, 14]) the authors study the asymptotic stability (i.e. $t \rightarrow \infty$) of solutions of (1.1) subjected to (some kind of) periodic boundary conditions *on a bounded interval*. The quasi-

Fig. 1. (a) classical; (b) non-classical: $S < 0$; (c) non-classical: $S > 0$.

periodic and the homoclinic solutions can, in general, not satisfy these boundary conditions.

2. Spatially periodic and quasi-periodic solutions in the case of real coefficients

The Ginzburg–Landau equation with real coefficients is given in canonical form by

$$\frac{\partial \Phi}{\partial t} = (1 - |\Phi|^2)\Phi + \frac{\partial^2 \Phi}{\partial z^2}. \quad (2.1)$$

The modulation equation in the degenerate case (1.4), with real coefficients, can be brought in the canonical form

$$\begin{aligned} \frac{\partial \psi}{\partial t} = & (1 - |\psi|^4)\psi - i\left(\check{B}|\psi|^2 \frac{\partial \psi}{\partial z} - \hat{B}\psi^2 \frac{\partial \psi^*}{\partial z}\right) \\ & + \frac{\partial^2 \psi}{\partial z^2}, \end{aligned} \quad (2.2)$$

where the simplification $\hat{b}_r = 0$ has been used and the sign of various coefficients has been chosen as explained in the preceding section. Unlike the classical equation (2.1), the degenerate modulation equation still contains two parameters, which have to be computed in any particular problem under consideration.

2.1. Periodic solutions and their stability

In search of periodic solutions we introduce

$$\phi, \psi = R_0 e^{i(k_0 z + w_0 t)}, \quad R_0, k_0, w_0 \in \mathbb{R}, \quad (2.3)$$

and find for the amplitude R_0 ,

$$\text{in the classical case: } R_0^2 + k_0^2 = 1, \quad (2.4)$$

$$\text{in the non-classical case: } R_0^4 - BR_0^2 k_0 + k_0^2 = 1. \quad (2.5)$$

We consider $B = \hat{B} + \check{B} > 0$; $B < 0$ can be treated analogously. Remark that the curve of periodic solutions does not depend on the value of $\check{B} - \hat{B}$. Furthermore $w_0 = 0$: the periodic solutions are time-independent (see the introduction). Relation (2.4) describes a semi-circle in the (k_0, R_0) plane, while relation (2.5) describes a conic in the (k_0, R_0^2) plane.

In the further analysis the following parameter plays an important role:

$$S = \frac{1}{4}B^2 - 1, \quad (2.6)$$

S is the rescaled version of parameter s introduced in ref. [7]. We find:

$S < 0$: (2.5) describes an ellipse in the (k, R^2) plane;

$S > 0$: (2.5) describes a hyperbola in the (k, R^2) plane.

We separate conic (2.5) into two branches Γ_L and Γ_U : Γ_L consists of points (k, R^2) with k such that (2.5) has two solutions, for k fixed; R^2 equals the smallest of these solutions. All other points define branch Γ_U . Γ_U can again be separated into Γ_{U+} , the increasing part, and Γ_{U-} , the decreasing part of Γ_{U+} , see fig. 1. Remark that Γ_L does not exist in the classical case, Γ_{U-} disappears as S becomes positive.

Next, we investigate the stability of periodic solutions. In the classical case the result is well known:

$$R_0 e^{ik_0 z} \text{ is stable if } -\sqrt{\frac{1}{3}} < k_0 < \sqrt{\frac{1}{3}} \\ \text{(or } R_0 > \sqrt{\frac{2}{3}}), \quad (2.7)$$

and unstable otherwise.

To obtain a similar result for the non-classical case we linearize along a periodic solution $R_0 e^{ik_0 z}$ of (2.2), i.e. we set

$$\psi(z, t) = [R_0 + \rho(z, t)] e^{i[k_0 z + \theta(z, t)]}, \quad (2.8)$$

$\psi(z, t)$ a solution of (2.2). The linearized equations for ρ and θ read

$$\rho_t = 2R_0^2(Bk_0 - 2R_0^2)\rho \\ + (BR_0^2 - 2k_0)R_0\theta_z + \rho_{zz}, \\ R_0\theta_t = -2(2HR_0^2 - k_0)\rho_z + R_0\theta_{zz}, \quad (2.9)$$

with

$$H = \frac{1}{4}(\check{B} - \hat{B}). \quad (2.10)$$

(Parameter H has been called “the hidden parameter” in ref. [7], since it cannot be derived from the Landau equation.) Due to the structure of linear system (2.9) one can model a perturbation by

$$(\rho(z, t), R_0\theta(z, t)) = (X(t) e^{ikz}, Y(t) e^{ikz}), \quad (2.11)$$

with $k \in \mathbb{R}$ an arbitrary, spatial, wavenumber. Substituting (2.11) into system (2.9) yields a two-dimensional linear differential equation for $X(t)$ and $Y(t)$, with parameter k . The stability of the zero-solution is determined by the eigenvalues λ

of the linear problem; the eigenvalue equation reads

$$\lambda^2 + 2\lambda[k^2 + (R_0^4 - k_0^2 + 1)] \\ + k^2[k^2 + 4(-1 + R_0^4(2 - BH)) \\ + R_0^2 k_0(2H - B)] = 0. \quad (2.12)$$

Remark that we have been using (2.5) to simplify the expressions. A solution $R_0 e^{ik_0 z}$ of (2.2) is stable if the real parts of the solutions λ of (2.12) are negative for all possible wavenumbers k , i.e.

$$R_0^2 - k_0^2 + 1 > 0, \quad (2.13)$$

$$-1 + R_0^4(2 - BH) + k_0 R_0^2(2H - B) > 0. \quad (2.14)$$

Remark that the arbitrary wavenumber k of the periodic perturbation does not appear in the two stability criteria: this is caused by the fact that we consider here modulation equations with real coefficients; k appears in the stability criteria if the coefficients become complex. Both (2.13) and (2.14) define, for every B and H , a hyperbola in the (k_0, R_0^2) plane; (2.13) is just a reformulation of the trivial stability condition (see ref. [7]): if, for a fixed k_0 , there exist two periodic solutions $R_1 e^{ik_0 z}$ and $R_2 e^{ik_0 z}$, $R_1 < R_2$, then solution $R_1 e^{ik_0 z}$ is unstable, or: solutions on branch Γ_L are always unstable.

In the further stability analysis we distinguish between three cases:

I. $S < 0$

The hyperbola defined by (2.14) intersects the (2.5) ellipse in two points, $P_- = (k_-, R_-^2)$ and $P_+ = (k_+, R_+^2)$. P_- , P_+ depend on the values of B and H :

$$BH < 1: \quad k_- < 0, \quad P_+ \in \Gamma_{U-},$$

$$BH > 1: \quad k_- > 0, \quad P_- \in \Gamma_{U+}, \quad P_+ \in \Gamma_L.$$

Fig. 2. (a) $BH < 1$; (b) $BH > 1$. Fig. 3. (a) $2H - B > 0$: no stable solutions; (b) $2H - B < 0$.

Periodic solutions on Γ_U , between P_- and P_+ , are stable, see fig. 2.

II. $S > 0$, $H^2 - BH + 1 > 0$

Now there is only one intersection point $P_- = (k_-, R_-^2)$ or $P_+ = (k_+, R_+^2)$:

for $2H - B > 0$ it is on $\Gamma_L (= P_+)$,

for $2H - B < 0$ it is on $\Gamma_U (= P_-)$.

And thus: $2H - B > 0$: no stable periodic solutions;

$2H - B < 0$: periodic solutions on Γ_U with $R^2 > R_1^2$ are stable.

II defines a region in (B, H) space: $\{\frac{1}{4}(2H - B)^2 > S > 0\}$. This region consists of two disjunct subregions, distinguished by the sign of $2H - B$. The sign of $BH - 1$ is not important in this case: in case I it decided whether intersection point P_+ is part of Γ_U - or part of Γ_L ; for $S > 0$ branch Γ_U - does not exist (see fig. 3).

III. $S > 0$, $H^2 - BH + 1 < 0$

No intersection points, all periodic solutions are unstable.

Remark. The case $S < 0$ resembles the classical situation (i.e. condition (2.7)) in the sense that within the k -interval of possible periodic solutions there is a smaller subinterval of stable periodic solutions. It should further be noted that the parameter H plays an important role.

2.2. Quasi-periodic solutions

2.2.1. The classical case

The stationary problem associated to (2.1) is integrable [12, 14]. Introducing $\phi(z, t) = \rho(z)e^{i\theta(z)}$ we find

$$\rho_{zz} - \rho\theta_z^2 + \rho - \rho^3 = 0,$$

$$2\rho_z\theta_z + \rho\theta_{zz} = 0, \quad (2.15)$$

Fig. 4. Phase portrait of (2.15) in the $\Omega = \Omega_0$ plane, $0 < |\Omega_0| < \sqrt{\frac{4}{27}}$.

with integrals

$$\Omega = \rho^2 \theta_z, \quad (2.16)$$

$$K = \rho_z^2 + \rho^2 - \frac{1}{2}\rho^4 + \frac{\Omega^2}{\rho^2}, \quad (2.17)$$

Ω and K constants in $\mathbb{R}$.

For Ω_0 fixed, $0 < |\Omega_0| < \sqrt{\frac{4}{27}}$ we find, in a certain range of K (depending on Ω_0), solutions (of (2.5)) periodic in ρ : in the phase-plane $\Omega = \Omega_0$, there is a saddle point $\rho_2(\Omega)$ with a homoclinic loop, the periodic orbits and the centerpoint $\rho_1(\Omega)$ are situated inside this loop, see fig. 4.

The solution ϕ , $\phi = \rho e^{i\theta}$ of (2.1) is quasi-periodic for values of K and Ω such that $\rho(z)$ is periodic (with $\theta_z = \Omega/\rho^2$). The critical points $\rho_1(\Omega)$ and $\rho_2(\Omega)$ correspond to periodic solutions of (2.1). It is easy to check that:

- The saddles $\rho_2(\Omega)$ correspond to stable periodic solutions (with respect to space-periodic perturbations). The centers $\rho_1(\Omega)$ correspond to unstable periodic solutions.

- Consulting (2.7) we observe that $R = \sqrt{\frac{2}{3}}$ is the boundary of the region of stable periodic solutions: as $|\Omega| \uparrow \sqrt{\frac{4}{27}}$ then $\rho_1(\Omega) \downarrow \sqrt{\frac{2}{3}}$ and $\rho_2(\Omega) \downarrow \sqrt{\frac{2}{3}}$.

- $|\Omega| > \sqrt{\frac{4}{27}}$: no critical points, no bounded solutions.

Fig. 5. The degenerate phase portrait of (2.15) at $\Omega_0 = 0$.

It is very remarkable that as one increases Ω towards the boundary $\sqrt{\frac{4}{27}}$, the two critical points ρ_1 and ρ_2 in fig. 4 move towards each other and coalesce at the stability boundary of periodic solutions. We shall find similar behaviour in the non-classical case.

We finally consider the degenerate case $\Omega = 0$: centerpoint $\rho_1(\Omega) \downarrow 0$ as $\Omega \rightarrow 0$ ($\rho_2(\Omega) \uparrow 1$), there is a centerpoint at the origin for $\Omega = 0$, hence $\rho(z)$ oscillates with positive and negative values, see fig. 5. This phase portrait is the fusion of the non-degenerated phase portrait and its mirror image. The periodic solutions $\rho(z)$ correspond to periodic solutions $\psi = \rho e^{i\theta_0}$, θ_0 phase-constant (since $\theta_z \equiv 0$), which oscillate through the origin of the complex plane.

2.2.2. The non-classical case

The basic equation (2.2) is much more complicated than the classical (2.1). Nevertheless, and this is remarkable, for stationary solutions the equation is again integrable. Setting $\psi(z) = \rho(z) e^{i\theta(z)}$ one gets the system

$$\begin{aligned} \rho_{zz} - \rho \theta_z^2 + \rho - \rho^5 + B \rho^3 \theta_z &= 0, \\ 2\rho_z \theta_z + \rho \theta_{zz} - 4H \rho^2 \rho_z &= 0. \end{aligned} \quad (2.18)$$

The two integrals are given by

$$\Omega = \rho^2 \theta_z - H \rho^4, \quad (2.19)$$

$$\begin{aligned} K &= \rho_z^2 + \rho^2 [1 - \Omega(2H - B)] \\ &\quad - \frac{1}{3} \rho^6 (H^2 - BH + 1) + \frac{\Omega^2}{\rho^2}. \end{aligned} \quad (2.20)$$

Fig. 6. $S < 0$: (a) periodic solutions as critical points of (2.18); (b) periodic solutions Re^{ikz} of (2.2) (see fig. 1b). The dashed line represents (in both figures) Ω_0 fixed; critical stability curve (2.14) is plotted in both figures.

As in the analysis of the stability of periodic solutions of (2.2) we distinguish between three cases:

I. $S < 0$

We find that there are values $\Omega_-(H, B) < 0 < \Omega_+(H, B)$ such that eq. (2.18) exhibits, in a $\Omega = \Omega_0$ plane, with $\Omega_0 \in (\Omega_-, \Omega_+)$, $\Omega_0 \neq 0$, a phase portrait similar to the one sketched in fig. 4: a center point $\rho_1(\Omega)$, a family of periodic solutions (corresponding to quasi-periodic solutions of (2.2)), a homoclinic loop and a saddle point. No critical points exist for $\Omega_0 \notin [\Omega_-, \Omega_+]$. As in the classical case we observe:

- As $\Omega \uparrow \Omega_+$ then $\rho_j(\Omega) \rightarrow R_+$, $j = 1, 2$, with $P_+ = (k_+, R_+^2)$ a boundary point of the region of stable periodic solutions (see fig. 2). Analogous behaviour occurs for $\Omega \downarrow \Omega_-$.
- Centerpoints $\rho_1(\Omega)$ corresponds to unstable periodic solutions.
- Saddlepoints $\rho_2(\Omega)$ are stable, unless they correspond to periodic solutions on Γ_L (see fig. 2b).
- The phase portrait in a $\Omega = \Omega_0$ plane degenerates, as in the classical case, at $\Omega_0 = 0$.

In fig. 1b we plotted all periodic solutions of (2.2) in the (k, R^2) plane; these periodic solutions also emerge as the critical points $\rho_1(\Omega_0), \rho_2(\Omega_0)$, $\Omega_0 \in (\Omega_-, \Omega_+)$, of (2.18). In fig. 6 we draw the ellipse of all critical points $\rho_1(\Omega_0), \rho_2(\Omega_0)$ in the

(ρ^4, Ω) plane and compare this ellipse (via the transformation $k = \theta_z = \Omega_0/\rho^2 + H\rho^2$) with the ellipse of (2.5) in the (k, R^2) plane.

We remark that the critical stability curve (2.14), derived in the stability analysis of section 2.1, is a hyperbola in the k, R^2 plane and transforms (via $\Omega = R^2k - HR^4$) to a straight line, connecting (R_+, Ω_+) and (R_-, Ω_-) , in the (ρ^4, Ω) plane: in fig. 6a, line (2.14) decides whether a critical point of (2.18) is of saddle or of center type.

This is remarkable since one would not expect such a direct correspondence between the stability analysis and the analysis of stationary solutions.

The second stability criterion (2.13) does not appear in the analysis of stationary solutions: using only (2.19) and (2.20) it is not possible to decide whether a saddle point is stable or unstable (i.e. whether it corresponds to a periodic orbit on Γ_U or on Γ_L).

II. $S > 0$, $H^2 - BH + 1 > 0$

There is one critical value $\Omega_{\pm} = \Omega_{\pm}(B, H)$ corresponding to the critical point $P_{\pm}$ found in section 2.1, II. For $\Omega_0 > \Omega_-$ or $\Omega_0 < \Omega_+$ depending on the sign of $2H - B$, and $\Omega_0 \neq 0$ a phase portrait as in fig. 4 can be found in the $\Omega = \Omega_0$ plane. Again we can compare between the hyper-

Fig. 7. $S > 0$, $H^2 - BH + 1 > 0$: (a) periodic solutions as critical points of (2.18); (b) periodic solutions of (2.2) (see fig. 1c). Critical stability curve (2.14) is plotted in both figures, $\rho_1(\Omega_0)$ and $\rho_2(\Omega_0)$ are indicated for a fixed Ω_0 .

bola of periodic solutions in the (k, R^2) plane (fig. 1c)) and the hyperbola of critical points $\rho_1(\Omega_0), \rho_2(\Omega_0)$, for $\Omega_0 > \Omega_-$ (or $\Omega_0 < \Omega_+$) in the (ρ^4, Ω) plane. Critical stability curve (2.14) plays the same role as above, case I (see fig. 7).

III. $S > 0$, $H^2 - BH + 1 < 0$

For all values of Ω we find one critical point $\rho_1(\Omega)$, of center type. All points $\rho_1(\Omega)$ correspond to an unstable periodic solution of (2.2). All solutions of (2.18) in a $\Omega = \Omega_0$ plane are periodic.

Concluding remark. We have found that the existence of spatially quasi-periodic solutions is not a peculiar property of the classical case.

3. Slow time-periodic solutions in the case of complex coefficients

In this section we study solutions of the two types of modulation equations, admitting in the coefficients small imaginary parts. For the classical case the analysis has been given in ref. [4]. We recapitulate here the main points in as much as needed for the sequel.

3.1. The classical case

The classical modulation equation with small complex coefficients is given, in a canonical form, by

$$\frac{\partial \Phi}{\partial t} = [1 - (1 + i\epsilon b)|\phi|^2]\phi + (1 + i\epsilon a)\frac{\partial^2 \phi}{\partial z^2} \quad (3.1)$$

with $0 < \epsilon \ll 1$ and $a, b \in \mathbb{R}$, $a \neq b$ (for $a = b$ one can eliminate the perturbation by $\phi = e^{-i\epsilon at}\tilde{\phi}$). We intend to

- investigate the effect of this perturbation on the special solutions (quasi-periodic, homoclinic) found in section 2;
- search for other types of special solutions.

The periodic solutions of the form $\phi(z, t) = R e^{i(kz + wt)}$ must satisfy

$$R^2 + k^2 = 1, \quad w = -\epsilon(bR^2 + ak^2).$$

Hence, spatially periodic solutions are now also slow time-periodic. One is thus led to consider solutions of the form

$$\phi(z, t) = \rho(z) e^{i[\theta(z) - \epsilon wt]} \quad (3.2)$$

with $w \in \mathbb{R}$ a free parameter. Substituting (3.2) into eq. (3.1) and using the integral Ω of the

unperturbed system (2.15), given by $\Omega(z) = \rho^2 \theta_z$, we obtain a three-dimensional system

$$\begin{aligned}\rho_z &= V, \\ V_z &= -\rho + \rho^3 + \frac{\Omega^2}{\rho^3} \\ &\quad + \frac{a\varepsilon^2}{1 + \varepsilon^2 a^2} \rho [(a - w) - \rho^2(a - b)], \\ \Omega_z &= \frac{\varepsilon}{1 + \varepsilon^2 a^2} \rho^2 [(a - w) - \rho^2(a - b)].\end{aligned}\quad (3.3)$$

System (3.3) is a perturbed version of integrable system (2.15). Due to the special character of the perturbation, some properties of the unperturbed system do not vanish:

(i) The flow induced by (3.3) is volume preserving.

(ii) There is a symmetry: $z \rightarrow -z$, $V \rightarrow -V$, $\Omega \rightarrow -\Omega$.

(iii) For values of the free parameter w between a and b ($a < w < b$ or $b < w < a$) system (3.3) has two critical points: $(\rho_*, 0, \pm \Omega_*)$. The position of these critical points do not depend on ε ; these critical points are also critical points of the unperturbed system: $(\rho_*, 0, \Omega_*)$ is a perturbed saddle or a perturbed center.

For finite ($= \mathcal{O}(1)$) time, solutions of system (3.3) remain $\mathcal{O}(\varepsilon)$ close to solutions of the unperturbed system. Since we are only interested in bounded solutions we study solutions of (3.3) with initial data $\mathcal{O}(\varepsilon)$ close to the region of periodic solutions of the unperturbed system in (ρ, V, Ω) space. Hence we define a Poincaré map P for the perturbed system. This map is the main tool to handle the flow of (3.3). We now give a short description of the Poincaré map P .

As is remarked in section 2.2.1: for all Ω_0 satisfying $0 < |\Omega_0| < \sqrt{\frac{4}{27}}$ there is an interval $(K_1(\Omega_0), K_2(\Omega_0))$ such that a solution of the unperturbed system with integral values K_0, Ω_0 , $K_0 \in (K_1(\Omega_0), K_2(\Omega_0))$, is periodic: this describes a bounded symmetric area E in (K, Ω) space. Integrals K and Ω of the unperturbed system are

slow variables in the perturbed system: $K_z, \Omega_z = \mathcal{O}(\varepsilon)$. Return map P , with “time” $= z$, is defined on an ε -neighbourhood of E , E_ε : let $(K_0, \Omega_0) \in E_\varepsilon$, consider $\Gamma_\varepsilon(z)$ the solution of (3.3) with initial data $V(0) = 0$, $\Omega(0) = \Omega_0$ and $\rho(0)$ such that $K(0) = K_0$. Let $(\bar{\rho}, 0, \bar{\Omega})$ be the next intersection point of Γ_ε with the $V = 0$ plane, $dV/dz < 0$ (if such a point exists). Then, realizing that $\bar{K} = \bar{\rho}^2 - \frac{1}{2}\bar{\rho}^4 + \bar{\Omega}^2/\bar{\rho}^2$ (see 2.17)

$$\begin{aligned}P(K_0, \Omega_0) &= (\bar{K}, \bar{\Omega}) \\ &= (K_0 + \Delta K(K_0, \Omega_0), \Omega_0 + \Delta \Omega(K_0, \Omega_0)).\end{aligned}$$

The essential observation is that one can compute the expressions ΔK and $\Delta \Omega$, which are $\mathcal{O}(\varepsilon)$, up to $\mathcal{O}(\varepsilon^2)$ using the solutions of the unperturbed flow (see section 3.2 for the computation of ΔK and $\Delta \Omega$ in the non-classical case).

Using return map P , one can prove (see ref. [4]):

Proposition 3.1. The quasi-periodic solutions of unperturbed equation (2.2) break open as a consequence of the complex perturbation. However, for every value of free parameter w between a and b , there exists one degenerated periodic solution (see fig. 5) which “survives” the perturbation.

Proposition 3.2. There is a subset $W = \{w_1, \dots, w_N\} \cup (\alpha_0, \alpha_1)$ of interval (a, b) (or (b, a)) such that, for every ε small enough, two critical points of perturbed equation (3.3) are connected by a heteroclinic orbit $\Gamma_w(z)$ for $w \in W$.

Heteroclinic orbit $\Gamma_w(z)$ connects two one-dimensional stable/unstable manifolds of perturbed saddle points as $w \in \{w_1, \dots, w_N\}$, $N \rightarrow \infty$ as $\varepsilon \downarrow 0$, hence $\Gamma_w(z)$ corresponds to a transition solution (of eq. (2.1)) between two counter-rotating stable periodic patterns. For $w \in (\alpha_0, \alpha_1)$, $\Gamma_w(z)$ connects two centerpoints corresponding to unstable periodic patterns. The proofs of these

propositions rely heavily on the symmetry in system (3.3): $z \rightarrow -z$, $V \rightarrow -V$, $\Omega \rightarrow -\Omega$.

Remark. Eq. (3.1) has no quasi-periodic solutions which correspond to fixed points of Poincaré map P (proposition 3.1). However, numerical simulation of the flow induced by (3.3) strongly suggests the existence of periodic points of map P which also correspond to quasi-periodic solutions of (3.1). The numerical simulation exhibits also chaotical behaviour [4].

3.2. The non-classical case

In section 2 we assumed that the imaginary parts of all coefficients of non-classical modulation equation (1.3) are zero. This caused the disappearance of a special feature of eq. (1.3), the $\mathcal{O}(1)$ term $ib_i|\Phi|^2\Phi$. Since all other terms of (1.3) are $\mathcal{O}(\delta)$ it was possible to eliminate δ from (1.3) by rescaling τ . If one wants to study a perturbed non-classical modulation equation with real coefficients (see (2.2)) one first has to assume that $b_i = \mathcal{O}(\delta)$ to eliminate δ from (1.3). Then one assumes, as in the classical case, that the imaginary parts of all coefficients of the rescaled equation are $\mathcal{O}(\varepsilon)$ (thus, all in all, $b_i = \mathcal{O}(\varepsilon\delta)$). This yields, after rescaling,

$$\begin{aligned} \psi_t = & \psi + i\varepsilon b_i|\psi|^2\psi \\ & + (-1 + i\varepsilon h_i)|\psi|^4\psi - i(\hat{B} + i\varepsilon\hat{B}_i)|\psi|^2\psi_z \\ & + i(\check{B} + i\varepsilon\check{B}_i)\psi^2\psi_z^* + (1 + i\varepsilon c_i)\psi_{zz}, \end{aligned} \quad (3.4)$$

which is a “proper” perturbed version of (2.2). Like in the classical case [4] we consider slow time-periodic solutions

$$\psi(z, t) = \rho(z) e^{i[\theta(z) - \varepsilon w t]}, \quad (3.5)$$

w is (again) a free parameter. We obtain the

three-dimensional system

$$\begin{aligned} \rho_z = & V, \\ V_z = & \rho^5(H^2 - BH + 1) + \rho[-1 + \Omega(2H - B)] \\ & + \frac{\Omega^2}{\rho^3} - \varepsilon\sigma\rho^2V \\ & - \varepsilon^2c_i\rho[\rho^4(\delta H - \zeta) + \rho^2\eta + \delta\Omega + \xi], \\ \Omega_z = & -\varepsilon\rho^2[\rho^4(\delta H - \zeta) + \rho^2\eta + \delta\Omega + \xi] \\ & + \varepsilon^2c_i\sigma\rho^3V, \end{aligned} \quad (3.6)$$

with $\Omega(z) = \rho^2\theta_z - H\rho^4$, the integral of the unperturbed system, and parameters B and H as defined in section 2.2. The new parameters, δ , σ , η , ζ and ξ can be expressed in the complex perturbation coefficients b_i , h_i , $\hat{B}_i$, $\check{B}_i$ and c_i of eq. (3.4). Parameter ξ plays the rôle of free parameter; it depends on w , the free parameter introduced in (3.5):

$$\xi = -\frac{c_i + w}{1 + \varepsilon^2c_i^2}. \quad (3.7)$$

We remark that critical points of (3.6) (if they exist) are also critical points of the unperturbed system ($\varepsilon = 0$), and the position does not depend on ε . This is in agreement with one of the basic properties of the perturbed classical system (3.3). However, system (3.6) does not satisfy the other two basic properties of the classical system:

(i) There is no preservation of volume, unless $\sigma + \delta = 0$.

(ii) There is no symmetry $z \rightarrow -z$, $V \rightarrow -V$, $\Omega \rightarrow -\Omega$, unless $2H - B = 0$, $\sigma = \delta = 0$.

We divide this section into two parts: in the first part we do not impose any conditions on the parameters of eq. (3.6) (“real” parameters B and H and perturbation parameters δ , σ , η , ξ , ζ and c_i); in the second part we choose the parameters of (3.6) such that the flow induced by (3.6) exhibits the same features as the classical system.

3.2.1. General parameter values

First we observe that, due to the character of the phase portrait of the unperturbed flow (see

section 2.2.2), bounded solutions of (3.6) have to remain $\mathcal{O}(\varepsilon)$ close to the periodic solutions of the unperturbed system. Hence we again study system (3.6) by a Poincaré map P (as in the classical case, see section 3.1): $P: E \rightarrow E$, with E the region in (K, Ω) space corresponding to periodic solutions of the unperturbed system (with integrals K and Ω , see section 2.2.2). Consulting section 2.2.2 we establish

I. $S < 0$

E is a bounded region, as in the classical case.

II. $S > 0$, $H^2 - BH + 1 > 0$

E is unbounded; however, there exist for every Ω_0 a $K_1(\Omega_0)$ and a $K_2(\Omega_0)$ such that periodic solutions of the unperturbed system correspond, for $\Omega = \Omega_0$, with points (K, Ω_0) with $K \in (K_1(\Omega_0), K_2(\Omega_0))$: a finite interval (which is empty for $\Omega_0 < \Omega_-$ or $\Omega_0 > \Omega_+$).

III. $S > 0$, $H^2 - BH + 1 \leq 0$

E is unbounded, moreover every intersection $\Omega = \Omega_0 \cap E$ is an unbounded (half) line (this situation is similar to one studied by Holmes [10]).

We define $\Delta K(K, \Omega)$ and $\Delta \Omega(K, \Omega)$ by

$$P: E \rightarrow E,$$

$$(K, \Omega) \mapsto (K + \Delta K(K, \Omega), \Omega + \Delta \Omega(K, \Omega)).$$

Using the unperturbed system we are able to compute ΔK and $\Delta \Omega$ accurate up to $\mathcal{O}(\varepsilon)$. These quantities are the main tools of our analysis. We shall establish the following results:

Theorem 3.3. Consider the parameters $B, H, \eta, \zeta, \delta, c_i$ and σ of eq. (3.6) with $(\delta H - \zeta, \eta) \neq (0, 0)$. There is for every choice of $B, H, \eta, \zeta, \delta, c_i$ at least one non-empty interval (Σ_1, Σ_2) , $\Sigma_1 = -\infty$ and $\Sigma_2 = \infty$, is possible, such that for $\sigma \in (\Sigma_1, \Sigma_2)$ system (3.6) has periodic solutions for a range of $\xi = \xi(w)$, w the free parameter (the w -interval may also be unbounded).

Thus, for a wide range of parameters, partial differential equation (3.4) has solutions which are quasi-periodic in space and slowly periodic in time: some structure of the stationary solutions of unperturbed equation (2.2) survives the perturbation.

Theorem 3.3 establishes the existence of a region D in the 7-dimensional parameter space: for parameter combinations inside D Poincaré map P has fixed points. Determining the boundaries of D can be considered as performing a bifurcation analysis in the parameter space. A way to annihilate (or create) a periodic orbit is to transform it, by adjusting the parameters, into a homoclinic loop (a global bifurcation). We shall prove the following theorem:

Theorem 3.4. There exists a bifurcation manifold in parameter space of codimension one; points on this manifold correspond to parameter combinations for which system (3.6) has a homoclinic solution (for a certain choice of w), i.e. a homoclinic solution of the unperturbed system has survived the perturbation.

Thus eq. (3.4) has slow time-periodic solutions which tend, for $z \rightarrow \pm\infty$, to the same stable periodic “wave” solution.

To prove theorem 3.3 we first have to derive the expressions for $\Delta K(K, \Omega)$ and $\Delta \Omega(K, \Omega)$. When we have proved the existence of region D we set up a more detailed analysis of D : we determine parts of its boundaries, search for subdomains (and interior boundaries) in which P has several fixed points, etc. Theorem 3.4 shall be proved by this bifurcation analysis.

The derivation of $\Delta K(K, \Omega)$ and $\Delta \Omega(K, \Omega)$ runs along the same lines as in the classical case [4].

Choose $(K_0, \Omega_0) \in E$ (or in an ε -neighbourhood of E). Define Γ_ε as the solution of (3.6) with initial data $V(0) = 0$, $\Omega(0) = \Omega_0$ and $\rho(0)$ such that $K(0) = K_0$, $(\bar{\rho}_0, 0, \bar{\Omega}_0)$ is defined as the next intersection point of Γ_ε with the $V = 0$ plane,

with $V_z < 0$ (if such a point exists). As in the classical case: $P(K_0, \Omega_0) = (\bar{K}_0, \bar{\Omega})$ with $\bar{K}_0$ defined by the value of the integral K of the unperturbed system in point $(\bar{\rho}_0, 0, \bar{\Omega}_0)$ (see (2.20)). Let $Z_\varepsilon = Z_\varepsilon(K_0, \Omega_0)$ be the return time ("time" = z) of Γ_ε . It is clear that

$$\Delta K(K_0, \Omega_0) = \int_0^{Z_\varepsilon} K_z(\Gamma_\varepsilon(z)) dz, \quad (3.8)$$

$$\Delta \Omega(K_0, \Omega_0) = \int_0^{Z_\varepsilon} \Omega_z(\Gamma_\varepsilon(z)) dz. \quad (3.9)$$

Expressions K_z and Ω_z are slow ($= \mathcal{O}(\varepsilon)$): K and Ω are integrals of the unperturbed system. Let $\Gamma_0(z) = (\rho_0(z), V_0(z), \Omega_0(z))$ be the periodic solution of the unperturbed system with the same initial data as Γ_ε , and period $Z_0(K_0, \Omega_0)$. The initial data have to be adapted a little ($= \mathcal{O}(\varepsilon)$) for choices of (K_0, Ω_0) which are an $\mathcal{O}(\varepsilon)$ distance outside E . Since $|\Gamma_\varepsilon - \Gamma_0| = \mathcal{O}(\varepsilon)$ on $\mathcal{O}(1)$ "time" scale, we obtain from (3.8) and (3.9), using $K_z, \Omega_z = \mathcal{O}(\varepsilon)$,

$$\Delta(K_0, \Omega_0) = \int_0^{Z_0} K_z(\Gamma_0(z)) dz + \mathcal{O}(\varepsilon^2), \quad (3.10)$$

$$\Delta \Omega(K_0, \Omega_0) = \int_0^{Z_0} \Omega_z(\Gamma_0(z)) dz + \mathcal{O}(\varepsilon^2). \quad (3.11)$$

Solution $\Gamma_0(z)$ is the orbit $(\rho_0(z), V_0(z))$ in the $\Omega = \Omega_0$ plane, described by the K integral (2.20). It crosses the ρ axis ($V = \rho_z = 0$) in two points $0 < \hat{\rho}(K_0, \Omega_0) < \check{\rho}(K_0, \Omega_0)$, see fig. 4. Defining

$$\begin{aligned} F_{K, \Omega}(\rho^2) &= \rho_z^2 = V^2 \\ &= \frac{1}{3}\rho^6(H^2 - BH + 1) \\ &\quad + \rho^2[-1 + \Omega(2H - B)] - \frac{\Omega^2}{\rho^2} + K, \end{aligned} \quad (3.12)$$

see again (2.20), and calculating $K_z(\rho, V, \Omega)$ and $\Omega_z(\rho, V, \Omega)$ we establish

$$\begin{aligned} \Delta \Omega(K, \Omega) &= -2\varepsilon \int_{\hat{\rho}(K, \Omega)}^{\check{\rho}(K, \Omega)} \frac{\rho^2 G_\Omega(\rho^2)}{\sqrt{F_{K, \Omega}(\rho^2)}} d\rho \\ &\quad + \mathcal{O}(\varepsilon^2), \end{aligned} \quad (3.13)$$

$$\begin{aligned} \Delta K(K, \Omega) &= -2\varepsilon \sigma \int_{\hat{\rho}(K, \Omega)}^{\check{\rho}(K, \Omega)} \rho^2 \sqrt{F_{K, \Omega}(\rho^2)} d\rho \\ &\quad + 2\varepsilon \int_{\hat{\rho}(K, \Omega)}^{\check{\rho}(K, \Omega)} \frac{[\rho^4(2H - B) - 2\Omega] G_\Omega(\rho^2)}{\sqrt{F_{K, \Omega}(\rho^2)}} d\rho \\ &\quad + \mathcal{O}(\varepsilon^2) \end{aligned} \quad (3.14)$$

with

$$G_\Omega(\rho^2) = \rho^4(\delta H - \zeta) + \rho^2\eta + (\delta\Omega + \xi). \quad (3.15)$$

It is natural to substitute $R = \rho^2$ in the expressions of ΔK and $\Delta \Omega$. Since parameter σ does not appear in the expression for $\Delta \Omega$ one easily deduces:

if one can find $K_c, \Omega_c \in E$ such that $\Delta \Omega(K_c, \Omega_c) = 0$ then there exists (exactly) one value of σ such that $\Delta K(K_c, \Omega_c) = 0$.

Let us fix $\delta, \eta, \zeta, B, H$ and choose $\Omega_0 \neq 0$ such that

$\triangleright H^2 - BH + 1 > 0$;
 $\triangleright \Omega_0 \in (\Omega_-(H, B), \Omega_+(H, B))$, for a definition of Ω_-, Ω_+ see section 2.2.2; $\Omega_- = -\infty$, or $\Omega_+ = +\infty$ for $S > 0$.

Hence the phase portrait of the unperturbed system is as in fig. 4. Define $\rho_1(\Omega_0)$ and $\rho_2(\Omega_0)$ as the center and the saddle point of the unperturbed system, with corresponding values of K : $K \in [K_1(\Omega_0), K_2(\Omega_0)]$. $\Delta \Omega$ has to be zero for some K_0 if

$$\begin{aligned} &\text{sgn } \Delta \Omega(K_1(\Omega_0), \Omega_0) \\ &\neq \text{sgn} \left(\lim_{K \uparrow K_2(\Omega_0)} \Delta \Omega(K, \Omega_0) \right). \end{aligned} \quad (3.16)$$

The limit expression is needed since $\Delta\Omega(K, \Omega_0)$ tends to infinite as $K \uparrow K_2(\Omega_0)$. This is due to the fact that $F_{K, \Omega_0}(\rho^2)$ has a double zero at $K = K_2(\Omega_0)$: the integral diverges as $K \uparrow K_2(\Omega_0)$. It should also be remarked that the periodic solution of the unperturbed equation with integrals K and Ω_0 tends to the homoclinic orbit (see fig. 4), hence the return time of the map becomes ∞ . This interferes with the $\mathcal{O}(\varepsilon)$ approximation of solutions of the perturbed equation (3.6) on $\mathcal{O}(1)$ time scale. Remark that the situation is essentially different when $G_\Omega(\rho_2^2(\Omega_0)) = 0$: $(\rho_2(\Omega_0), 0, \Omega_0)$ is now a critical point of (3.6), due to this zero in the numerator of (3.13) $\Delta\Omega$ does not diverge as $K \uparrow K_2(\Omega_0)$ but tends to a finite number. We will come back to this situation in the bifurcation analysis (and in the proof of theorem 3.4).

One observes that (3.16) is satisfied if

$$\text{sgn}[G_\Omega(\rho_1^2(\Omega_0))] \neq \text{sgn}[G_\Omega(\rho_2^2(\Omega_0))]. \quad (3.17)$$

Since $\xi = \xi(w)$, w the free parameter, ξ can be varied at will. This reduces (3.17) to the following non-degeneracy condition (see (3.15)):

$$[\rho_1^2(\Omega_0) + \rho_2^2(\Omega_0)](\delta H - \zeta) + \eta \neq 0, \quad (3.18)$$

i.e. if (3.18) is satisfied there exists $w_1(\Omega_0) < w_2(\Omega_0)$ ($w_{1,2} = \pm\infty$ is possible) such that for $w \in (w_1, w_2)$ we can find $K_0(w)$ such that $\Delta\Omega(K_0, \Omega_0) = 0$. Remark that if (3.16) is not satisfied it is still possible that there is a K_0 such that $\Delta\Omega(K_0, \Omega_0) = 0$. With the observation above we know that we can now find a $\sigma = \sigma(w)$ such that $\Delta K(K_0, \Omega_0) = \Delta\Omega(K_0, \Omega_0) = 0$.

Define

$$\Sigma_1 = \inf_{\Omega_0 \in (\Omega_1, \Omega_2)} \{ \sigma(w) : w \in (w_1(\Omega_0), w_2(\Omega_0)) \},$$

$$\Sigma_2 = \sup_{\Omega_0 \in (\Omega_1, \Omega_2)} \{ \sigma(w) : w \in (w_1(\Omega_0), w_2(\Omega_0)) \}.$$

Remark that $\Sigma_1 = -\infty$ and $\Sigma_2 = \infty$ are possible.

It is not necessary to assume that the unperturbed flow has a center and a saddle point in a $\Omega = \Omega_0$ plane (fig. 4). The same reasoning as above can be applied to the case $H^2 - BH + 1 \leq 0$ (section 2.2.2, III): $\Delta\Omega(K, \Omega)$ has a zero on the line $\Omega = \Omega_0$ for some K_0 if

$$\text{sgn}[\Delta\Omega(K_1(\Omega_0), \Omega_0)] \neq \text{sgn}\left(\lim_{K \rightarrow \infty} \Delta\Omega(K, \Omega_0)\right)$$

(in this situation the Poincaré map is defined for every K, Ω with $K > K_1(\Omega)$). This condition can be satisfied by varying $\xi = \xi(w)$ for every combination of parameters H, δ, η, ζ and δ (except if $\Omega_z \equiv \text{constant}$: $\delta H - \zeta = \eta = 0$). One can now proceed as above defining the quantities $\Sigma_j = \Sigma_j(B, H, \eta, \zeta, \delta) = \Sigma_j(\hat{B}, \hat{\delta}, b_i, h_i, \hat{b}_i, \hat{B}_i, c_i)$ in terms of the coefficients of the original equation ($j = 1, 2$). Remark that only six parameters of (3.6) are significant: c_i can be varied at will, this is due to the fact that c_i only appears directly in $\mathcal{O}(\varepsilon^2)$ terms of (3.6). However, adjusting c_i changes the values of η, ζ, δ and σ (which are functions of the parameters of eq. (2.2), see for instance (3.7)).

Combining the cases $H^2 - BH + 1 > 0$ and $H^2 - BH + 1 \leq 0$ proves theorem 3.3.

Thus there is a region D in parameter space in which Poincaré map P has fixed points. Remark that the periodic orbits corresponding to these fixed points may be attracting or repelling if $\sigma + \delta \neq 0$. It is natural that one wants to know more about D:

- Is it possible to determine its boundaries?
- Are there subdomains of D for which eq. (3.2.3) has several periodic orbits?

The boundaries of D and of its subdomains can be found by projecting the bifurcation manifolds for the flow of (3.6) in the 8-dimensional $(B, H, \eta, \zeta, \delta, \sigma, c_i, \xi)$ space along the axis of free parameter ξ into the 7-dimensional parameter space. Some of these manifolds can be computed, although the computational effort should not be underestimated.

(a) A critical point of (3.6), which is a perturbation of a centerpoint of the unperturbed system, can undergo a Hopf bifurcation. One can compute the 7-dimensional manifold at which a Hopf bifurcation can occur. Crossing this manifold means the creation or annihilation of a periodic orbit (a fixed point of map P). Projection of this manifold into $(H, B, \eta, \zeta, \delta, \sigma, c_i)$ space results in a subdomain of D in which Hopf bifurcations appear as free parameter ξ is varied. One can compute explicitly the non-degeneracy conditions (by determining the center-manifold, etc., see, for instance, ref. [8]). These conditions also define manifolds in parameter space (which are submanifolds of the Hopf-bifurcation manifold).

(b) A periodic orbit can also “disappear” into a homoclinic orbit (or vice versa): a global bifurcation. At the bifurcation point there exists a homoclinic solution in system (3.6), which is a $\mathcal{O}(\varepsilon)$ perturbation of a homoclinic orbit of the unperturbed system. It is possible to compute a manifold in parameter space at which the (unperturbed) homoclinic orbit of saddle point $(\rho_*, 0)$ in a $\Omega = \Omega_*$ plane (see fig. 4) survives the perturbation. It is, of course, necessary that $(\rho_*, 0, \Omega_*)$ is a critical point of perturbed system (3.2.3), i.e.

$$\rho_*^4 (\delta H - \zeta) + \rho_*^2 \eta + (\delta \Omega_* + \xi) = 0.$$

Remark that this can be achieved by choosing $\xi = \xi(w)$ correctly. Define $K_* = K_2(\Omega_*)$: the value of unperturbed integral K (2.20) at saddle point $(\rho_*, 0, \Omega_*)$. As was remarked above: $\Delta K(K_*, \Omega_*)$ and $\Delta \Omega(K_*, \Omega_*)$ do not diverge. Due to the results of the theory of invariant manifolds one can show that $\Delta K(K_*, \Omega_*)$ and $\Delta \Omega(K_*, \Omega_*)$ really do measure the changes of perturbed integrals K and Ω along the perturbed homoclinic loop, although the approximations are not on an $\mathcal{O}(1)$ time scale (see, for instance refs. [9, 19]). Thus one may conclude that the homoclinic orbit of $(\rho_*, 0, \Omega_*)$ survives the perturbation if

$$\Delta K(K_*, \Omega_*) = \Delta \Omega(K_*, \Omega_*) = 0. \quad (3.19)$$

Furthermore one remarks that expressions (3.13) and (3.14) can be integrated explicitly for $K = K_*$, $\Omega = \Omega_*$: F_{K_*, Ω_*} has a double zero at a saddle point (see (3.12)), hence there remain only quadratic polynomials (in $R = \rho^2$) under the root expressions of ΔK and $\Delta \Omega$. One can show that identities (3.19) can be satisfied, by the same reasoning as above: find a zero of $\Delta \Omega(K_*, \Omega_*)$ using parameters $B, H, \eta, \zeta, \delta$ and free parameter ξ and construct a zero of $\Delta K(K_*, \Omega_*)$ by choosing σ . Remark that $F_{K_*, \Omega_*}(\rho^2)$ can be written as (see (3.12))

$$R \cdot F_{K_*, \Omega_*}(R) = \frac{1}{3}(H^2 - BH + 1)(R - R_*)^2 \times (R - R_u)(R + R_-),$$

with $R = \rho^2$, $R_* = \rho_*^2$, $R_u = \rho_u^2$ the intersection of the homoclinic loop with the ρ -axis and $R_- > 0$, the third zero. The numerator of the integrand of $\Delta \Omega(K_*, \Omega_*)$ can be factorized into $(\delta H - \zeta)R(R - R_*)(R - R_s)$. Hence $\Delta \Omega(K_*, \Omega_*)$ becomes, see (3.13),

$$\Delta \Omega(K_*, \Omega_*) = -\varepsilon \frac{(\delta H - \zeta)}{\sqrt{\frac{1}{3}(H^2 - BH + 1)}} \times \int_{R_*}^{R_u} \frac{R(R - R_s)}{\sqrt{(R - R_u)(R + R_-)}} dR + \mathcal{O}(\varepsilon^2).$$

This expression can be made zero by adapting the parameters: there are enough parameters to adjust R_s , while R_* , R_- and R_u remain fixed. Remark that it is necessary that $0 < R_* < R_s < R_u$ and thus, for instance,

$$\frac{\eta}{\delta H - \zeta} < 0$$

(see (3.15), we assume here that $\delta H - \zeta \neq 0$). Thus, by choosing σ correctly it is possible to satisfy (3.19). Eliminating ξ from identities (3.19) yields one expression $P(B, H, \eta, \zeta, \delta, \sigma) = 0$, a manifold of codimension 1 in parameter space determining a boundary, or an interior boundary, of D : crossing this manifold in parameter space

means the creation or annihilation of a periodic solution by a global bifurcation.

This proves theorem 3.4.

(c) It is possible to study Poincaré map P close to a fixed point: one has to linearize $\Delta\Omega$ and ΔK in the neighbourhood of a zero (remark that it is difficult to determine $d\Delta\Omega/dK$, etc.). One can now look for local bifurcations of map P :

- a saddle-node bifurcation: two periodic solutions melt together and disappear;
- A Hopf bifurcation: this causes the creation of a invariant torus for the flow of (3.6), corresponding to a quasi-periodic solution of original equation (3.4) with three different frequencies;
- a period doubling bifurcation; etc.

There are of course more complex bifurcations possible.

We may conclude that one can determine (parts of) the boundary of domain D and its subdomains. However, the computations are very extensive: we have (for the greater part) computed the bifurcation domains in $(B, H, \eta, \zeta, \delta, \sigma, c_i)$ space in the case of Hopf bifurcation and the homoclinic bifurcation. We did not analyze the bifurcations of map P (mentioned in part (c)).

3.2.2. $2H - B = \delta = \sigma = 0$

In this subsection we study the case in which the non-classical system behaves like the classical one. The conditions $2H - B = \delta = \sigma = 0$ relate quantities which are essentially independent. Therefore they are non-generic and will only by accident occur in applications. Nevertheless it is worthwhile to work out briefly the implications.

In terms of coefficients of eq. (3.4) $2H - B = \delta = \sigma = 0$ means:

$$\check{B} + 3\hat{B} = 0, \quad \check{B}_i + 3\check{B}_i = 0 \quad \text{and} \quad \check{B}_i - c_i \check{B} = 0,$$

or, in terms of the coefficients f and g of the original equation (1.3), with $f = -i(\check{B} + i\varepsilon\check{B}_i)$, and $g = i(\hat{B} + i\varepsilon\hat{B}_i)$:

$$f = 3g \quad \text{and} \quad f_i + c_i f_i = 0.$$

With this choice of the parameters, system (3.6) is symmetrical and volume-preserving:

$$\begin{aligned} \rho_z &= V, \\ V_z &= -\rho + \rho^5(-S) + \frac{\Omega^2}{\rho^3} \\ &\quad - \varepsilon^2 c_i \rho [\xi - \eta \rho^2 - \zeta \rho^4], \\ \Omega_z &= -\varepsilon \rho^2 [\xi - \eta \rho^2 - \zeta \rho^4], \end{aligned} \quad (3.20)$$

with S as defined in (2.6). In order to apply the method used in ref. [4] we have to impose some further conditions:

- $S < 0$ (see section 2: the curve of periodic solutions of (2.2) is an ellipse).
- Parameters ζ and η are such that, for an arbitrary value of the free parameter $\xi = \xi(w)$, system (3.20) either has no critical points or only two critical points (with again a degeneration in the $\Omega = 0$ plane): for arbitrary values of ζ and η , 4 critical points may exist; this disturbs the analysis of the important Poincaré map P (see 3.1). Thus we choose

$$\zeta = 0 \quad \text{or} \quad \frac{\eta}{\zeta} \leq 0 \quad \text{or} \quad \frac{\eta}{\zeta} \geq \frac{2}{\sqrt{-S}}. \quad (3.21)$$

Conditions (3.21) become in terms of the parameters of the original equation (3.4):

$$\begin{aligned} h_i + c_i = 0 \quad \text{or} \quad \frac{b_i}{h_i + c_i} \geq 0 \\ \text{or} \quad \frac{b_i}{h_i + c_i} \leq \frac{2}{\sqrt{-S}}. \end{aligned}$$

If the coefficients satisfy the conditions above we can use the methods of ref. [4]: one can define a Poincaré map P and compute it up to $\mathcal{O}(\varepsilon^2)$. System (3.20) exhibits now the necessary properties (divergence free, symmetries, etc.) which are essential for the analysis. Hence we establish:

- The “regular” periodic orbits of the unperturbed system (fig. 4) break open as a consequence of the perturbation. There is, however, one singular periodic solution in the $\Omega = 0$ plane

(fig. 5) which survives the perturbation. The situation is not exactly similar to the classical case: the degenerated periodic solution of (3.20) does not correspond to a periodic solution of (3.4), but to a *degenerate quasi-periodic* solution. This is due to the fact that, unlike in the classical case, $\theta_z \neq 0$ in the $\Omega = 0$ plane of the unperturbed non-classical system: $\theta_z = H\rho^2 \neq 0$ if $H \neq 0$ (see (2.19)). Since this quasi-periodic solution oscillates through the origin of the complex plane it differs essentially from the non-degenerated quasi-periodic solutions of the unperturbed equation.

– There is a set X such that there exists a heteroclinic orbit connecting two critical points of (3.20) for every $\xi \in X$ (set X is a transformation of a set W since $\xi = \xi(w)$, with w the free parameter). These heteroclinic orbits correspond to solutions of (3.4) connecting counterrotating periodic patterns.

Remark. As we can check in section 2: for $S < 0$ and $2H - B = 0$ the unperturbed system has a phase portrait in a $\Omega = \Omega_0$ plane as in fig. 4 for $0 < |\Omega_0| < 1/2\sqrt{-S}$. Thus, as in the critical case: $\Omega_+ = -\Omega_-$. The critical separation points $P_- = (k_-, R_-^2)$ and $P_+ = (k_+, R_+^2)$ satisfy $R_- = R_+$; however, $k_+ \neq -k_-$ in general, which is the case in the classical situation (to satisfy also $k_+ = -k_-$ in the non-classical situation we have to demand $B = 0$ (i.e. $\hat{B} = \check{B} = 0$), conic (2.5) is now, as in the classical case, a semicircle). Consulting the results of the stability analysis of section 2.1 and applying the extra condition $2H - B = 0$, we observe

- $0 \leq B \leq \sqrt{2}$: all saddle points of the unperturbed system correspond to stable periodic orbits.
- $\sqrt{2} < B < 2$: some saddle points are unstable.

Appendix

In this appendix we give a short sketch of the derivation of the degenerate modulation equation (1.3). This derivation is based on the non-degenerate non-truncated modulation equation (1.2).

Let us consider a periodic solution of (1.2) of the form $\Phi(\xi, \tau) = R e^{i(k\xi + w\tau)}$, with $R, k, w \in \mathbb{R}$. Substituting this expression into eq. (1.2) yields, for the real parts,

$$\begin{aligned} 0 = & a_r \lambda + b_r R^2 - c_r k^2 \\ & + \hat{\varepsilon}(-d_i \lambda k + e_i k^3 - f_i R^2 k + g_i R^2 k) \\ & + \hat{\varepsilon}^2(\dots + h_r R^4) + \dots \end{aligned} \quad (\text{A.1})$$

We can consider (4.1) as an equation for R^2 . This equation has at most one solution R^2 of $\mathcal{O}(1)$ and solutions which are $\mathcal{O}(\hat{\varepsilon}^{-2})$. If $R^2 = \mathcal{O}(\hat{\varepsilon}^{-2})$ then $\Phi(\xi, \tau) = \mathcal{O}(\hat{\varepsilon}^{-1})$, which violates the important condition that $\Phi = \mathcal{O}(1)$ (this is essential in the derivation of (1.3): Φ is a (part of a) perturbation of a basic, linear unstable, solution, see refs. [2, 5, 15, 17]). The $\mathcal{O}(1)$ solution is the untruncated version of the classical periodic solutions of section 2.1 (compare (4.1) with $\hat{\varepsilon} = 0$ to (2.4)). In the degenerate case, i.e. $b_r = \delta \hat{b}_r$ with $0 < \delta \ll 1$, we observe that the $\mathcal{O}(1)$ solution has become $\mathcal{O}(1/\delta)$. In order to solve this new “blow-up” we introduce the rescalings: $\lambda = \delta \lambda'$ and $\xi' = \sqrt{\delta} \xi$, which yields $K = \sqrt{\delta} K'$. Eq. (A.1) then reads

$$\begin{aligned} & (\delta a_1 + \hat{\varepsilon} \delta \sqrt{\delta} a_2) + R^2 (\delta \hat{b}_r + \hat{\varepsilon} \sqrt{\delta} a_3) \\ & + R^4 (\hat{\varepsilon}^2 h_r) + \text{h.o.t.} = 0 \end{aligned} \quad (\text{A.2})$$

(coefficients $a_i = a_i(\lambda', K')$, $i = 1, 2, 3$, can be derived from (A.1)). This equation has two solutions R_1^2 and R_2^2 with $R_1^2 \cdot R_2^2 = \mathcal{O}(\delta/\hat{\varepsilon}^2)$. These solutions are both $\mathcal{O}(1)$ if we choose $\hat{\varepsilon} = \sqrt{\delta}$ (of course there are more solutions R^2 but these are again too large). Note that we are allowed to choose parameter $\hat{\varepsilon}$ to be equal to the “new” small parameter $\sqrt{\delta}$: $\hat{\varepsilon}$ has been introduced by setting $\Lambda = \Lambda_c + \lambda \hat{\varepsilon}^2$, see refs. [2, 5, 15, 17], thus its magnitude can be considered as free as long as it is small. Summarizing one can say: the fact that the real part of the coefficient of the cubic non-linearity, b_r , is small makes it necessary to perform the above rescalings in the classical Ginzburg–Landau modulation equation. These

rescalings yield a modulation equation which governs the nonlinear evolution of "patterns" for $\Lambda_c - \Lambda = \mathcal{O}(\delta^2)$, with δ the magnitude of b_r :

$$\begin{aligned} \Phi_\tau = & i b_i \Phi |\Phi|^2 \\ & + \delta \left(a \lambda' \Phi + \hat{b}_r \Phi |\Phi| + c \Phi_{\xi' \xi'} + f |\Phi|^2 \Phi_{\xi'} \right. \\ & \left. + g \Phi^2 \Phi_{\xi'}^* + h |\Phi|^4 \Phi \right) + \mathcal{O}(\delta^2). \end{aligned} \quad (\text{A.3})$$

This is the degenerate modulation equation (1.3) (with $\xi' = z$ and $\lambda' = \lambda$). Note that this equation again degenerates if (also) h_r turns out to be small (see (A.2): $R_{1,2}^2$ are $\mathcal{O}(1/h_r)$). This second degeneration appears in the Blasius boundary layer [7] and can be treated similar to the first one.

For more details on this derivation process we refer to refs. [5, 7].

References

- [1] H.R. Brand, P.S. Lomdahl and A.C. Newell, Benjamin-Feir turbulence in convective binary fluid mixtures, *Physica D* 23 (1986) 345.
- [2] R.C. DiPrima, W. Eckhaus and L.A. Segel, Non-linear wave-number interaction in near-critical two-dimensional flows, *J. Fluid Mech.* 49 (1971) 705.
- [3] R. DiPrima and R.N. Grannick, A non-linear investigation of the stability of flow between counter-rotating cylinders, in: *Proc. IUTAM Symp. on Instability of Continuous Systems*, Herrenalb (Germany, 1969), ed. H. Leipholz (Springer, Berlin, 1971), p. 55.
- [4] A. Doelman, Slow time-periodic solutions of the Ginz-Landau equation, *Physica D* 40 (1989) 156.
- [5] A. Doelman, On the nonlinear evolution of patterns; modulation equations and their solutions, Thesis, Rijks-universiteit Utrecht (1990).
- [6] M.P. Eagles, Supercritical flow in a diverging channel, *J. Fluid Mech.* 57 (1973) 149.
- [7] W. Eckhaus and G. Iooss, Strong selection or rejection of spatially periodic patterns in degenerate bifurcations, *Physica D* 39 (1989) 124.
- [8] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer, Berlin, 1983).
- [9] M.W. Hirsch, C.C. Pugh and M. Shub, *Invariant Manifolds*, Springer Lecture Notes in Mathematics, Vol. 583 (Springer, Berlin, 1977).
- [10] P. Holmes, Spatial structure of time-periodic solutions of the Ginzburg-Landau equation, *Physica D* 23 (1986) 84.
- [11] L.R. Keefe, Dynamics of perturbed wavetrain solutions to the Ginzburg-Landau equation, *Stud. Appl. Math.* 73 (1985) 91.
- [12] L. Kramer and W. Zimmerman, On the Eckhaus instability for spatially periodic patterns, *Physica D* 16 (1985) 221.
- [13] Y. Kuramoto and S. Koga, Anomalous period-doubling bifurcations leading to chemical turbulence, *Phys. Lett. A* 92 (1982) 1.
- [14] H.T. Moon, P. Huerre and L.G. Redekopp, Transitions to chaos in the Ginzburg-Landau equation, *Physica D* 7 (1983) 135.
- [15] A.C. Newell and J.A. Whitehead, Finite bandwidth, finite amplitude convection, *J. Fluid Mech.* 38 (1969) 279.
- [16] W. Schöpf and W. Zimmerman, Multicritical behaviour in binary fluid convection, *Europhys. Lett.* 8 (1989) 41.
- [17] K. Stewartson and J.T. Stuart, A non-linear instability theory for a wave system in plane Poiseuille flow, *J. Fluid Mech.* 48 (1971) 529.
- [18] J.T. Stuart and R.C. DiPrima, The Eckhaus and Benjamin-Feir resonance mechanisms, *Proc. R. Soc. London A* 362 (1978) 27.
- [19] S. Wiggins, *Global Bifurcations and Chaos* (Springer, Berlin, 1988).