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Gavrylenko, P.

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Isomonodromic deformations and conformal field theory with W-symmetry

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Isomonodromic deformations and conformal field theory with W-symmetry
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Introduction

In my thesis I present a correspondence between isomonodromic deformations of higher-rank Fuchsian linear systems and conformal field theory with higher-spin symmetry, or W-symmetry. The correspondence that I describe is a generalization to higher rank of the one found by Gamayun, Iorgov and Lisovyy in [GIL12]. This generalization is first found numerically and then proved in the free-fermionic framework by an explicit construction of the twist-fields that are at the same time monodromy fields and W-primary fields. Next I use this construction to give the Fredholm-determinant representation of the general isomonodromic tau-function. The determinantal representation found in this way can also be proven without using any field theory by a careful analysis of derivatives of the determinant.

Another part of thesis deals with the special case that the monodromy group is given by quasi-permutation matrices I present a construction of the W-primary fields in terms of twisted bosons and give an expression of their conformal blocks in terms of algebro-geometric objects associated with branched covers of the complex sphere. Such correlation functions are related to exact isomonodromic tau-functions introduced by Korotkin. I also present the interpretation of such fields in terms of free fermions and a computation of the characters of related W-algebra representations. In this part of the investigations also W-algebras of the orthogonal series are considered.

Basic concepts

In this section I try to give a self-contained overview of basic objects considered in this thesis. My goal is to make this into an introduction for non-experts.

Conformal field theory

By a conformal field theory (CFT) is meant by default a two-dimensional quantum field theory with conformal symmetry, i.e., the symmetry that preserves angles and multiplies metrics by a scalar factor: $g_{\mu\nu} \mapsto \lambda g_{\mu\nu}$. A remarkable feature of the two-dimensional case is the fact that Lie algebra of local conformal transformations becomes infinite-dimensional and is generated by the holomorphic functions $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto f(z)$. In the infinitesimal form such transformations may be rewritten as

$$z \mapsto z + \epsilon(z) + O(\epsilon^2) \quad (1.1)$$
1. Introduction

The Lie algebra of the corresponding vector fields $\epsilon(z)\partial_z$ has as a natural basis the $\{\ell_n = -z^{n+1}\partial_z\}$. In this basis its commutations relations acquire the following form:

$$[\ell_n, \ell_m] = (n - m)\ell_{n+m} \tag{1.2}$$

This Lie algebra is called Witt algebra, or Virasoro algebra with zero central extension.

All local fields in a conformal theory transform under conformal transformations in some non-trivial way. It happens though that one may always choose a basis in the space of fields which is formed by elements that transform as differential forms of the kind $\phi_\Delta, z, \bar{z}$ $dz, d\bar{z}$. Such fields are called primary fields, and $(\Delta, \bar{\Delta})$ are called their dimensions. Though in actual physical models $\bar{\Delta}$ is important, we will always consider only the holomorphic part. Infinitesimal transformation of the primary field, or, in other words, the action of the Lie derivative, is given by the formula

$$\delta_{\epsilon(z)} \phi_\Delta(z) = (\epsilon'(z)\Delta + \epsilon\partial_z) \phi_\Delta(z) \tag{1.3}$$

By Noether’s theorem, any symmetry in a quantum theory gives rise to conserved charges. Conformal transformations in CFT give rise to charges that are encoded by a single energy-momentum tensor $T(z)$. The quantum version of Noether’s theorem is formed by the Ward identities that relate infinitesimal transformations of fields with the action of conserved charges. In CFT they read as

$$\delta_{\epsilon(z)} \phi_\Delta(z) = \oint d\frac{w}{2\pi i} R T(w) \phi(z) \tag{1.4}$$

In this formula the integral goes around a small circle around $w = z$, and the symbol $R$ means radial ordering of the operators

$$R\phi(z)\psi(w) = \phi(z)\psi(w), \quad |z| > |w|$$
$$R\phi(z)\psi(w) = (-1)^{p_\phi-p_\psi}\psi(w)\phi(z), \quad |z| < |w| \tag{1.5}$$

Here $p_\phi$ is fermionic parity. If we work in the path integral formulation we may just skip this notation: any product is already radially ordered.

A very important concept in CFT is the so-called operator product expansion, i.e., the expansion of the radially ordered product of two fields in the neighbouring points:

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{(AB)_n(w)}{(z-w)^{n+1}} \tag{1.6}$$

Radial ordering will be usually omitted in all OPE expansions due to historical reasons, though it is important.

The singular part of the OPE contains all information about the commutation relations between modes of the operators $^1$:

$$[A_n, B(w)] = \frac{1}{2\pi i} \oint dzz^{n+\Delta-1} A(z)B(w) \tag{1.7}$$

$^1$This can be easily deduced using properties of radial ordering and doing manipulations with contour integrals.
where $A_n = \frac{1}{2\pi i} \oint z^{n+\Delta-1} A(z) dz$.

For example, one can write down an OPE of the primary field with the energy-momentum tensor:

$$T(z)\phi_\Delta(w) = \frac{\Delta \phi_\Delta(w)}{(z-w)^2} + \frac{\partial \phi_\Delta(w)}{z-w} + \text{reg.}$$  \hspace{1cm} (1.8)

An analogous OPE for the energy-momentum tensor itself has the form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular}(= \text{reg.})$$  \hspace{1cm} (1.9)

One can introduce the components of the Laurent expansion of the energy-momentum tensor

$$T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}$$  \hspace{1cm} (1.10)

and then rewrite the above OPEs in terms of these components:

$$[L_n, \phi_\Delta(z)] = (n+1)\Delta z^n \phi_\Delta(z) + z^{n+1} \phi_\Delta(z)$$  \hspace{1cm} (1.11)

$$[L_n, L_m] = \frac{c}{12} (n^3 - n) + (n-m) L_{n+m}$$  \hspace{1cm} (1.12)

The Lie algebra generated by the operators $L_n$ is called the Virasoro algebra, and as we see, it is a central extension of the algebra of vector fields by the element $c$ called central charge. The value of the central charge is an important characteristic of CFT.

**Free bosonic CFT**

One of not the most elementary, but very important examples of CFT is a free bosonic theory with $N$ elementary fields $\phi_\alpha(z, \bar{z})$ – Gaussian fields with the following OPEs:

$$\phi_\alpha(z)\phi_\beta(w) = -\delta_{\alpha\beta} \log |z-w|^2 + \text{reg.}$$  \hspace{1cm} (1.13)

It is also useful to introduce derivatives of the $\phi_\alpha$, the so-called $U(1)$ currents $J_\alpha(z) = i\partial \phi_\alpha(z)$ with conformal dimension $(1, 0)$. The commutation relation of the modes of such currents are given by $[J_{\alpha,n}, J_{\beta,k}] = n\delta_{\alpha+k,0} \delta_{\alpha\beta}$. The Lie algebra with these commutation relations is called the Heisenberg algebra.

One can check that the energy-momentum tensor

$$T(z) = \sum_{\alpha=1}^{N} : J_\alpha(z)^2 :$$  \hspace{1cm} (1.14)

actually generates the Virasoro algebra with $c = N$. In this way we can get a realization of the complicated Virasoro algebra in terms of a simpler Heisenberg.

The simplest primary fields, or vertex operators of the constructed Virasoro, can be given explicitly by the exponents

$$V_{\vec{a}}(z) = : e^{i \sum_{\alpha} a_\alpha \phi_\alpha(z)} :$$  \hspace{1cm} (1.15)
One can check that the following OPE with the energy-momentum tensor holds for such fields:

\[
T(z)V_{\vec{a}}(w) = \frac{\Delta(\vec{a})V_{\vec{a}}(w)}{(z-w)^2} + \frac{\partial V_{\vec{a}}(w)}{z-w} + \text{reg.} \tag{1.16}
\]

In this formula the conformal dimension is given by the formula \(\Delta(\vec{a}) = \frac{1}{2} \sum_{\alpha} a_{\alpha}^2\).

In this way we can construct some examples of primary fields, but not an arbitrary one: in our case we have a serious problem, conservation of the \(U(1)\) charge. Namely, colliding two fields with charges \(\vec{a}\) and \(\vec{b}\) we get another field with charge \(\vec{a} + \vec{b}\):

\[
V_{\vec{a}}(z)V_{\vec{b}}(w) = (z-w)^{(\vec{a},\vec{b})}V_{\vec{a}+\vec{b}}(w) + \ldots, \tag{1.17}
\]

whereas in the general CFT any fields can appear in this OPE. However, we will present an almost free-field generalization of this construction in **Chapter 3**, which is not restricted by this charge-conservation condition.

The free-bosonic theory gives also an example of a theory with \(W\)-symmetry, the non-linear higher spin symmetry. Generators of this symmetry are expressed via initial bosonic fields as elementary symmetric polynomials (the energy-momentum tensor was a quadratic symmetric polynomial):

\[
W_{k}(z) = \sum_{\alpha_1 < \ldots < \alpha_k} a_{\alpha_1}(z) \ldots a_{\alpha_k}(z) : \tag{1.18}
\]

Clearly, there are only \(N\) such currents. It happens so that their commutators are actually non-linear functions of the initial generators. For example, in the \(N = 3\) case they look schematically like

\[
T \cdot T \sim T \\
T \cdot W_3 \sim W_3 \\
W_3 \cdot W_3 \sim T + (TT) \tag{1.19}
\]

This algebra is very complicated in the general case, but nevertheless it can be studied with the help of various free-field techniques.

The field \(V_{\vec{a}}(z)\) introduced above is also an example of a \(W\)-primary field since its OPE with \(W_{k}(z)\) is given by the following formula:

\[
W_{k}(z)V_{\vec{a}}(w) = \frac{e_k(\vec{a})V_{\vec{a}}(w)}{(z-w)^k} + \text{less singular}, \tag{1.20}
\]

where the \(\{e_k\}\) are elementary symmetric polynomials. The main difference with the usual conformal symmetry (1.8) is that, in general, coefficients near the lower orders of this expansion are not given in terms of \(V_{\vec{a}}(z)\). This causes one of the main problems of \(W\)-algebras: their vertex operators are not defined uniquely in the general situation. We propose some solution of this problem in **Chapter 3**.
Free fermionic CFT

Another very important concept in two-dimensional physics is the boson-fermion correspondence, which relates free bosonic and free fermionic theories. The transformation between these two theories can be given approximately (precise expressions are written in the main text) by the following formulas:

\[
\begin{align*}
\psi^*_\alpha(z) & \approx e^{i\phi_\alpha(z)} : \\
\psi_\alpha(z) & \approx e^{i\phi_\alpha(z)} : \\
J_\alpha(z) & = : \psi^*_\alpha(z) \psi_\alpha(z) :
\end{align*}
\]

(1.21)

Here \( \psi_\alpha(z) \) and \( \psi^*_\alpha(z) \) are \( N \)-component fermionic fields with the following OPEs and anticommutation relations:

\[
\begin{align*}
\psi^*_\alpha(z) \psi_\beta(w) &= \frac{\delta_{\alpha\beta}}{z - w} + \text{reg.} \\
\{ \psi^*_\alpha, \psi_\beta \} &= \delta_{\alpha\beta} \delta_{p+q,0}
\end{align*}
\]

(1.22)

The conformal dimensions of both \( \psi \) and \( \psi^* \) are the same and equal to \((\frac{1}{2}, 0)\).

From many points of view the fermionic description is much better. For example, instead of the complicated non-linear generators (1.20) the W-algebra has another set of nice fermionic operators which are just bilinear:

\[
\tilde{W}_k(z) = \sum_{\alpha=1}^{N} : \partial^{k-1} \psi^*_\alpha(z) \psi_\alpha(z) :
\]

(1.23)

Such a representation also gives us a better understanding of what is W-symmetry. Namely, its action on fermions is given by formula

\[
\tilde{W}_k(z) \psi^*_\alpha(w) = \frac{\partial^{k-1} \psi^*_\alpha(w)}{z - w} + \text{reg.}
\]

(1.24)

It may also be rewritten using (1.7) in terms of the modes \( \tilde{W}_{k,n} = \frac{1}{2\pi i} \oint \tilde{W}_k(z) z^{k+n-1} dz \):

\[
\left[ \tilde{W}_{k,n}, \psi^*_\alpha(w) \right] = w^{n+k-1} \partial^{k-1} \psi^*_\alpha(w)
\]

(1.25)

The above calculation demonstrates that the analogy between the vector fields \(-z^{n+1} \partial\) and the Virasoro generators \( L_n \) can be continued to an analogy between arbitrary differential operators \( z^{n+k-1} \partial^k \) and W-generators \( \tilde{W}_{k,n} \).

Another important concept in the free-fermionic theory are the group-like elements: such operators act on the generators of the Clifford algebra \( \psi_{\alpha,n}, \psi^*_{\alpha,n} \) in a linear way

\[
O^{-1} \psi_{\alpha,n} O = \sum_{\beta,q} C_{\alpha,n;\beta,q} \psi_{\beta,q}
\]

(1.26)

Such operators were widely used before in the literature to construct solutions of integrable hierarchies, like KP, Toda, and their multi-component generalizations. Here we show that they also appear in conformal theory: we find the general vertex operators for the W-algebra in such a form.
1. Introduction

A remarkable property of a group-like element is the fact that any of its matrix elements can be expressed as a determinant of just two-particle ones. As an immediate consequence of this property any correlation function of the group-like elements can be expressed as some Fredholm determinant.

The AGT relation

There is an important object in conformal field theory, called the conformal block. For simplicity we consider the 4-point one:

\[ F(\Delta_0, \Delta_t, \Delta_1, \Delta_\infty; \Delta_0 t; c|t) = \langle \Delta_\infty | \phi_{\Delta_1} (1) P_{\Delta_0 t} \phi_{\Delta_t} (t) | \Delta_0 \rangle \] (1.27)

To explain the meaning of this definition one has to recall that the symmetry algebra of the theory is the Virasoro algebra, and since it acts on the Hilbert space of the theory, this Hilbert space decomposes into the sum of its highest-weight irreducible representations. In the general position they are Verma modules, i.e. modules with highest weight \(|\Delta\rangle\) such that

\[
\begin{align*}
L_{k>0}\Delta\rangle &= 0 \\
L_0\Delta\rangle &= \Delta\Delta\rangle
\end{align*}
\] (1.28)

and the module itself is spanned by the vectors \(L_{-k_1} \ldots L_{-k_n} \Delta\rangle\) with \(k_1 \geq k_2 \geq \ldots \geq k_n\).

Now one can say that projector \(P_{\Delta_0 t}\) is a projector onto the Verma module with highest weight (dimension) \(\Delta\), and any 4-point correlation function in conformal field theory can be expanded over conformal blocks since its Hilbert space can be expanded into Verma modules.

Conformal blocks itself are purely algebraic universal objects that can be computed just from the commutation relations in the Virasoro algebra (1.12) and from the definition of the primary field (1.11). However, for the more complicated W-algebra case they can be computed algebraically only for the cases when two charges 2 \(\vec{a}_t\) and \(\vec{a}_1\) have a very special form: \(\vec{a}_1 = (a_1, b_1, \ldots, b_1), \vec{a}_t = (a_t, b_t, \ldots, b_t)\). Fields with such charges are called semi-degenerate.

Virasoro conformal block is in general a concrete, but very complicated special function, and until 2009 there were only two ways to compute it: by doing order-by-order computations in the Virasoro algebra or by using the Zamolodchikov recursion formula. The situation changed in 2009 when Alday, Gaiotto and Tachikawa proposed the correspondence between 2D CFT and 4D \(\mathcal{N} = 2\) supersymmetric gauge theories. In this approach the conformal blocks become equal to the so-called Nekrasov instantonic partition functions. For our purposes the most important fact is that any coefficient in the expansion of the instantonic partition function, and so of the conformal block, is given by an explicit combinatorial formula (we will write it for simplicity

---

2As we have seen in (1.20) for the free bosonic field, the action of the W-generators was expressed in terms of elementary symmetric polynomials \(e_k(\vec{a})\). It turns out that it is useful to use this parametrization not only for the free case.
only for \( c = 1 \) of the following kind:

\[
\mathcal{F}(a_0^2, a_1^2, a_2^2, \sigma_0^2; c = 1; |t|) = (1 - t)^{2a_0 a_1} \times \\
\times \sum_{Y_+, Y_-} t^{Y_+ - Y_-} \prod_{s, s' = \pm} Z_b(a_s + sa_0 - s' \sigma_0 |Y_s, Y_{s'}) Z_b(a_1 + s\sigma_0 - s' a_\infty |Y_s, Y_{s'})
\]

(1.29)

In this formula \( Y_+, Y_- \) are two Young diagrams, and \( Z_b(\nu|Y_1, Y_2) \) is some explicit factorized combinatorial expression depending on two Young diagrams and one complex number.

This formula for a conformal block was proved in 2010 by Alba, Fateev, Litvinov and Tarnopolsky. In their proof they presented such a basis that any matrix element of the Virasoro vertex operator can be expressed in terms of \( Z_b \). What matters for us is that for \( c = 1 \) their basis is exactly the free-fermionic one. This is one more hint that a fermionic description of conformal field theory is better than a bosonic one.

**Isomonodromic deformations**

There is a story from the beginning of 20th century when mathematicians started to study \( N \times N \) matrix linear systems with first-order singularities:

\[
\frac{d\Phi(z)}{dz} = \Phi(z) \sum_{k=1}^{n} A_k \frac{1}{z - z_k}
\]

(1.30)

where \( \sum_{k=1}^{n} A_k = 0 \). One may ask, at first, when such a system can be solved explicitly in terms of some known special function. I present below an important list of some examples which, however, does not cover everything:

- \( N = 2, n = 3 \). Always solvable in terms of hypergeometric function \( _2F_1 \).
- \( n = 3, N \) – arbitrary, but the spectral type of \( A_1 \) is special: \( A_1 \sim \text{diag}(a_1, b_1, \ldots, b_1) \). Always solvable in terms of \( _N F_{N-1} \). Here the analogy with the semi-degenerate fields is absolutely not accidental.
- \( n \) – arbitrary, but the monodromy group is a semidirect product of a permutation group and the diagonal matrices (quasi-permutation group). Always solvable in terms of higher-genus theta-functions. This case corresponds at the CFT side to the twist fields and is considered in Chapters 5, 6.

For \( n > 4 \) and for general \( A \)'s the Fuchsian system cannot be solved explicitly (though in Chapter 4 we give the formula that can give its explicit expansion in some region of parameters). Instead of this it is reasonable to ask about the monodromy of such a system. Namely, if we take some solution and continue it analytically around the loop \( \gamma \) encircling some singular point, we get another solution. Now any two solutions of the system are connected by linear transformations, so we have

\[
\gamma : \Phi(z) \mapsto M_\gamma \Phi(z)
\]

(1.31)
1. Introduction

The matrices $M_\gamma \in GL(N)$ generate the monodromy group of the system, and analytic continuation around closed loops generates a map from $\pi_1(\mathbb{C} \setminus \{z_1, \ldots, z_n\})$ to $GL(N)$ with the image coinciding with the monodromy group.

The problem of finding the monodromy group for a given system is also complicated. Instead of this one may look for such transformations of the systems that preserve the monodromy, the so-called isomonodromic transformations. It happens so that in general position we are able to move all singular points and to make some modifications of the matrices $A_k$ that preserve the monodromy: in this setting all matrices $A_k$ become functions of $\{z_1, \ldots, z_n\}$. Such a functional dependence is described by a non-linear system of matrix equations, the Schlesinger equations:

$$\frac{\partial A_j}{\partial z_k} = [A_j, A_k] \frac{z_j - z_k}{z_j - z_k}$$
$$\frac{\partial A_j}{\partial z_j} = -\sum_{k \neq j} [A_j, A_k] \frac{z_j - z_k}{z_j - z_k}$$

There is also a non-trivial statement that can be verified explicitly that any solution of the Schlesinger system gives some function of the $\{z_k\}$, the tau-function, defined by its derivatives:

$$\frac{\partial \log \tau(z_1, \ldots, z_n)}{\partial z_k} = \sum_{j \neq k} \frac{\text{tr} A_k A_j}{z_k - z_j}$$

This function is simpler than the fundamental solution itself. For example, for $n = 3$ the singular points it can be given explicitly by $\tau(z_1, z_2, z_3) = \prod_{i<j} (z_i - z_j)^{\text{tr} A_i A_j}$, while the fundamental solution is still unknown in general. One of the first interesting cases is $n = 4, N = 2$: this tau-function solves Painlevé VI equation and gives actually its general solution. This fact is one of the motivations to study isomonodromic deformations: they give a convenient framework to study the equations from the Painlevé family.

One of the achievements in the study of this tau-function for the Painlevé VI case was the work of Jimbo in 1982 where he obtained the first 3 terms of the tau-function in terms of the monodromy. The next breakthrough in this direction was done by Gamayun, Iorgov, Lisovyy and Teschner when they gave the general formula for the $N = 2$ tau-function, including arbitrary number of points, in terms of conformal blocks, which easily recovers the Jimbo formula. In the present thesis, see Chapters 2-4, I present the generalization of their result for arbitrary $N$. In particular, I give in Chapter 4 a rigorous proof of this result without using any field theory.

Isomonodromy-CFT correspondence

Various parts of the correspondences between isomonodromic deformations, Painlevé equations and quantum field theory (QFT) have been found in the late 70’s by Sato, Miwa and Jimbo. Archetypal formulas of such correspondence look like follows:

$$\tau(x_1, \ldots, x_n) = \langle O(x_1) \ldots O(x_n) \rangle, \quad \Phi(y, y_0) = \frac{y - y_0}{\tau} \langle \bar{\psi}^* (y) \psi(y_0) O(x_1) \ldots O(x_n) \rangle$$

(1.34)
Here the $O(x)$ are some disorder fields in some free field theory (like spin variable in the Ising model), $\psi(y), \psi^*(y_0)$ are initial free fields, and $\Phi(y,y_0)$ is a solution of some linear problem. Such a correspondence was found for various massive and massless bosonic and fermionic models. The only problem was that this correspondence was found 5 years before the creation of conformal field theory, otherwise this research could be related to CFT at that time.

There were several guesses that belong to Knizhnik and Moore that CFT is actually related to isomonodromic deformations, but they were not developed to get a final explicit answer. Such a development was done by Gamayun, Iorgov and Lisovyy in 2012, when they gave the general solution of the Painlevé VI equation as a linear combination of $c = 1$ conformal blocks:

$$\tau(t) = \sum_{n \in \mathbb{Z}} s_{0t} n(t^{\sigma_0 t+n})^2 - \theta^2_0 C_n(\sigma_{0t}, \{\theta\}) F(\theta^2_0, \theta^2_1, \theta^2_\infty; (\sigma_{0t} + n)^2 |t)$$  \hspace{1cm} (1.35)$$

Together with the AGT formula (1.29) this gave the general tau-function as an explicit series. To explain this formula I give below the short dictionary of the correspondence:

<table>
<thead>
<tr>
<th>Painlevé VI</th>
<th>CFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2} \text{tr} A_\nu = \theta_\nu^2$</td>
<td>$\Delta_\nu = \theta_\nu^2$</td>
</tr>
<tr>
<td>$\text{tr} M_0 M_t = 2 \cos \pi \sigma_{0t}$</td>
<td>$\Delta = (\sigma_{0t} + n)^2$</td>
</tr>
<tr>
<td>some function of $\text{tr} M_\mu M_\nu, \text{tr} M_\nu$</td>
<td>$s_{0t}$</td>
</tr>
<tr>
<td>$\tau(t)$</td>
<td>$\langle \Delta_{\infty}</td>
</tr>
<tr>
<td>$\frac{[\Phi(z) \Phi(z_0)]^{-1} \alpha \beta}{z-z_0}$</td>
<td>$\frac{1}{\tau(t)} \langle \Delta_{\infty}</td>
</tr>
<tr>
<td>$\text{tr} \left( \sum_{k=1}^n \frac{A_k}{z-z_k} \right)^2$</td>
<td>$\frac{1}{\tau(t)} \langle \Delta_{\infty}</td>
</tr>
</tbody>
</table>

So the main rule is the following: dimensions (or higher W-charges) are symmetric functions of the eigenvalues of logarithms of the monodromy matrices.

Formula (1.35) was proved in several different ways, it was also generalized to arbitrary number of points with $2 \times 2$ matrices.

In this thesis I present the same construction for the $N \times N$ case which relates isomonodromic tau-function to a linear combination of conformal blocks of the W-algebra. In Chapter 2 we solve Schlesinger system numerically and conjecture the general form of the tau-function, in Chapters 3, 4 we prove it using two different approaches. In Chapter 3 we construct explicitly W-primary fields as some fermionic group-like elements with given monodromy and then find the Fredholm-determinant formula for their correlator; in Chapter 4 we give the generalization of this formula to an arbitrary number of points and prove it.

**Twist fields**

The archetypal example of a twist field is Zamolodchikov’s construction of conformal field with dimension $\Delta = \frac{1}{16}$ in $c = 1$ CFT. The first ingredient is the expression of
1. Introduction

the Virasoro algebra in terms of the half-integer Heisenberg algebra:

\[
\left[ J_{n+\frac{1}{2}}, J_{-m-\frac{1}{2}} \right] = (n + \frac{1}{2}) \delta_{m,n}
\]

\[
L_n = \frac{\delta_{n,0}}{16} + \sum_{k \in \mathbb{Z} + \frac{1}{2}} : J_k J_{n-k} :
eq (1.36)
\]

The usual bosonic representation (Fock module) is reducible, and it is expanded over the infinite series of Verma modules with dimensions \((\frac{1}{4} + n)^2\). This statement can be obtained from the computation of characters with the help of the well-known Gauss formula:

\[
q^{\frac{1}{16}} \prod_{k=0}^{\infty} (1 - q^{k+\frac{1}{2}}) = \sum_{n \in \mathbb{Z}} q^{(\frac{1}{4} + n)^2}
\]

\[
\prod_{k=1}^{\infty} (1 - q^k) (1.37)
\]

The picture corresponding to this situation looks as follows: there is a bosonic field \(J(z)\) which has monodromy around the origin \(J(e^{2\pi i} z) = -J(z)\). This monodromy is actually related to the twist field \(\mathcal{O}(0)\) sitting in the point \(z = 0\). Its dimension equals to \(\frac{1}{16}\).

Another ingredient of the construction concerns the corresponding vertex operator: the field \(\mathcal{O}(x)\) sitting in the arbitrary point and changing the sign of \(J(z)\) when it goes around. The great discovery of Zamolodchikov was an exact formula for the conformal block of such fields. For example, the 4-point block is given by simple formula:

\[
\mathcal{F}(\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}; \Delta; c = 1|t) = \frac{(16t^{-1})^{\Delta} e^{\pi i \Delta |t|}}{(1 - t)^{\frac{5}{2}} \theta_3(0|\tau(t))}
\]

\[
(1.38)
\]

where \(\tau(t)\) is a period of the elliptic curve \(y^2 = z(z - t)(z - 1)\).

As far as we have the isomonodromy-CFT correspondence, we can use this conformal block in (1.35): this leads us to so-called Picard solution of Painlevé VI. From the point of view of the monodromy it corresponds to the quasi-permutations that were mentioned above: in this case one can find explicitly the general solution of the \(n\)-point system.

In this thesis I present the generalization of Zamolodchikov’s construction to the case of W-algebra. In contrast to the previous situation, here there is a richer collection of twist fields that are labelled by the elements of the permutation group. They permute the bosonic currents leaving the W-generators untouched:

\[
J_k(e^{2\pi i} z) \mathcal{O}_s(0) = J_{s(k)}(z) \mathcal{O}_s(0)
\]

\[
(1.39)
\]

In Chapter 5 we construct such fields and find the generalization of Zamolodchikov’s formula for their conformal blocks. We also show that using extended isomonodromy-CFT correspondence we can construct the tau-function from these conformal blocks and then identify it with the known tau-function found by Korotkin.

In Chapter 6 we find many generalizations of the character formula (1.37), we also find a very close relation between the construction of the W-algebra twist fields and the Lepowski-Wilson construction of the integrable representations of \(\hat{sl}(N)_{1}\). We also relate this construction to the free-fermionic approach from Chapter 2. In contrast to the previous considerations, here we also touch upon the W-algebras for the orthogonal series and generalize all results related to twist-fields to this case.
Outline

Here I list the most important results of the thesis and then explain how the different parts of the text are related to each other.

List of the key results

- **Formula 2.53:**
  \[
  \tau(t) = \sum_{w \in \mathcal{Q}} e^{(\beta, w)} c^{(0t)}_w(\theta_0, \theta_t, \sigma_{0t}, \mu_{0t}, \nu_{0t}) C^{(1\infty)}_w(\theta_1, \theta_\infty, \sigma_{01}, \mu_{1t}, \nu_{1t}) \\
  \times t^\frac{1}{2}(\sigma_{0t}+w, \sigma_{0t}+w) - \frac{1}{2}(\theta_0, \theta_0) - \frac{1}{2}(\theta_t, \theta_t) B_w(\{\theta_i\}, \sigma_{0t}, \mu_{0t}, \mu_{1t}, \nu_{1\infty}, \nu_{1t}; t)
  \]
  This formula describes the conjectural form of the general \(N = 3, n = 4\) tau-function.

- **Formula 2.58:**
  \[
  C^{(0t)}_w(\theta_0, a_1, \sigma) C^{(1\infty)}_w(\sigma, a_1, \theta_\infty) = \prod_{i,j} C[1-\frac{w_i}{w_j} + (e_i, \theta_0) - (e_j, \sigma+w)] C[1-\frac{w_i}{w_j} + (e_i, \sigma+w) + (e_j, \theta_\infty)]
  \]
  This formula gives the conjectural form of the structure constants for two semi-degenerate fields.

- **Theorem 3.2:** \(V_\nu(t)\) is a primary field of the conformal \(W_N \oplus H\) algebra with the highest weights \(u_k(\nu)\).

- **Theorem 3.5:**
  Solution of the linear problem with \(n\) marked points is given by
  \[
  (z - w) \mathcal{R}_{\alpha \beta}(z, w) = \frac{\langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1) \psi_{\alpha}^{\theta_0}(z) \psi_{\beta}^{\theta_0}(w) | \theta_0 \rangle}{\langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1) | \theta_0 \rangle}
  \]
  whereas its isomonodromic tau-function is defined by
  \[
  \tau(t_1, \ldots, t_{n-2}) = \langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1) | \theta_0 \rangle
  \]

- **Formula 3.136:** \(\tau(t) = \det (1 + \mathcal{R}_t)\)

  This formula expresses the 4-point isomonodromic tau-function as a Fredholm determinant with explicitly given kernel.

- **Theorem 4.22:** Fredholm determinant \(\tau(a)\) giving the isomonodromic tau function \(\tau_{\text{JMU}}(a)\) can be written as a combinatorial series
  \[
  \tau(a) = \sum_{\tilde{Q}_1, \ldots, \tilde{Q}_{n-3} \in \tilde{Q}_N} \sum_{\gamma_1, \ldots, \gamma_{n-3} \in \gamma_N} \prod_{k=1}^{n-2} Z_{\gamma_k}^{\tilde{Q}_k-1, \tilde{Q}_k-1}(T[k]),
  \]
  where \(Z_{\gamma_k}^{\tilde{Q}_k-1, \tilde{Q}_k-1}(T[k])\) are expressed by (4.66), (4.63) in terms of matrix elements of 3-point Plemelj operators in the Fourier basis.
1. Introduction

• **Theorem 4.32:** This theorem describes the relation of our general Fredholm determinant and the particular hypergeometric one found before by Borodin and Deift.

• **Theorem 5.1:** Function

\[
\log \tau_{SW} = \frac{1}{2} \sum_{I,J} a_I T_{IJ} a_J + \sum_I a_I U_I + \frac{1}{2} Q(r)
\]

solves the system (5.55), iff \(Q(r)\) solves the system \(\frac{\partial Q(r)}{\partial q_\alpha} = \sum_{\tau(q_\alpha)} (d^2) \frac{dz}{d\tau(q_\alpha)}\)

for \(\alpha = 1, \ldots, 2L\), \(d\Omega = \sum_\alpha d\Omega_{\tau_\alpha}\) and other ingredients in the r.h.s. are given by (5.16), (5.20) and the period matrix of \(C\).

This theorem gives the solution for a Seiberg-Witten system.

• **Formula 5.78:** \(Q(r) = \sum_{q_\alpha \neq q_\beta} \tau^{i_j} \delta^{i_j} \log \Theta_s (A(q_\alpha) - A(q_\beta)) - \sum_{q_\alpha} (\tau^{i_j} \delta^{i_j} \log \frac{d(\tau(q_\alpha))}{dz} \big|_{q_\alpha = q_\alpha})\)

This formula gives the “\(r\)-charge contribution” to the exact conformal block.

• **Formula 5.87:** \(G_0(q|a) = \tau_B(q) \exp \left( \frac{1}{2} \sum_{I,J} a_I T_{IJ}(q) a_J + \sum_I a_I U_I(q, r) + \frac{1}{2} Q(r) \right)\)

This is the general formula for the conformal block of twist fields (generalization of Zamolodchikov’s formula).

• **Theorem 6.2:** The characters of the twisted representations are given by the formulas (6.85), (6.88), (6.95), (6.97).

• **Theorem 6.3:** If \(g_1 \sim g_2 \text{ in } G\) for different \(g_1, g_2 \in \text{N}_G(h)\), then \(\chi_{g_1}(q) = \chi_{g_2}(q)\).

This theorem generalizes the Gauss identity from Zamolodchikov’s construction.

• **Theorem 6.4:** The conformal blocks (6.163) for generic \(W(o(2N))\) twist fields are given by

\[
G_0(a, r, q) = \tau_B(\Sigma |q) \tau_B^{-1}(\tilde{\Sigma}|q) \tau_{SW}(a, r, q)
\]

where

\[
\partial_q \log \tau_B(\Sigma |q) = \sum_{\pi_{2N}(\xi) = \eta_i} \text{Res} \tau(z) d\xi, \quad \partial_q \log \tau_B(\tilde{\Sigma}|q) = \sum_{\pi_{2N}(\xi) = \eta_i} \text{Res} \tilde{\tau}(\zeta) d\zeta
\]

for \(i = 1, \ldots, 2M\)

and

\[
\partial_q \log \tau_{SW}(a, r, q) = \frac{1}{4} \sum_{\pi_{2N}(\xi) = \eta_i} \text{Res} \left(\frac{dS}{dz} \right)^2, \quad i = 1, \ldots, 2M
\]

\[
\frac{\partial}{\partial a_I} \log \tau_{SW} = \oint_{B_I} dS, \quad A_I \circ B_J = \delta_{IJ}, \quad I, J = 1, \ldots, g-
\]
Organization of the thesis

All the parts of this thesis are self-contained papers with their own introductions, so they can be read independently. But nevertheless, there are some logical dependencies between different parts. I show them on the following diagram:

Chapter 2 is devoted to the numerical solution of the Schlesinger system of the rank 3 and to the computation of corresponding isomonodromic tau-function. Its main task was to formulate and check the main conjectures for the higher-rank isomonodromy-CFT correspondence, which are then proved in the next chapters.

Chapter 3 deals with the free-fermionic construction of monodromy fields. Axiomatically such fields are defined by:

1) it is a fermionic group-like element

2) its two-particle matrix elements are expressed through the solution of 3-point Fuchsian system.

Then we prove that such fields are W-primaries, and at the same time their correlation function can be given as some Fredholm determinant. In this way we give the free-fermionic proof of the conjectures from Chapter 2. Also we get some integrable hierarchies which are related to such fields.

Chapter 4 is written in a pure mathematical language and absolutely rigorously, so it does not require any field-theory background. In this chapter we develop the framework in which the Fredholm determinant formula can be proved rigorously. To do this first we cut the sphere with $n$ punctures into $n - 2$ three-punctured sphere, and then introduce the spaces of functions on the obtained boundaries. Then we construct two projectors onto the space of functions that can be continued between the different boundaries, $\mathcal{P}_\Sigma$ and $\mathcal{P}_\oplus$. After that we restrict these projectors on some other space $\mathcal{H}_+; \mathcal{P}_{\Sigma,+}, \mathcal{P}_{\oplus,+}$. Thanks to this procedure they become non-degenerate. Then we define an infinite-dimensional determinant $\tau = \det \mathcal{P}_{\Sigma,+}^{-1} \mathcal{P}_{\oplus,+}$. Next we prove that:

1) derivatives of this determinant coincide with the derivatives of isomonodromic tau-function,

2) it is a Fredholm determinant, whose kernel in the 4-point case reproduces the one obtained in Chapter 3,
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3) the minor expansion of the determinant reproduces a combinatorial formula which in the known cases can be obtained from the isomonodromy-CFT + AGT correspondences. In this sense it gives one more independent proof of AGT for $c = N$.

Chapter 5 is devoted to the study of W-twist fields. The main techniques here are the free-field conformal theory and the algebraic geometry of complex curves. From the field theory we get equations that are satisfied by the correlation functions of twist fields, and then solve them in terms of period matrices, Abel maps and the theta-function of some branched cover of the punctured sphere. This construction generalizes the conformal block of Zamolodchikov.

Chapter 6 is also devoted to W-twist fields, but from the algebraic point of view. Here we consider the W-algebras for the orthogonal series, too. We start from the free-fermionic definition of W-algebras, their vertex operators in the spirit of Chapter 3, and then show that for quasi-permutation monodromy a lot of other bosonic constructions of these algebra can be obtained with the help of various exotic bosonizations. We also find a sequence of character identities that come from equivalences between different representations. In addition, we give a simple generalization of the exact conformal block from Chapter 5 to the orthogonal series.

References

The content of Chapters 2-6 is based on the following papers \(^3\), in order. Almost no changes were done to avoid producing mistakes. Therefore some mathematical objects are introduced several times, but any time the only properties that are needed for a given chapter are introduced, so it would not confuse the reader.

- M. Bershtein, P. Gavrylenko, A. Marshakov, *Twist-field representations of W-algebras, exact conformal blocks and character identities*, [hep-th/1705.00957], Under review in Communications in Mathematical Physics
- P. Gavrylenko, O. Lisovyy, *Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions*, [math-ph/1608.00958], Submitted to Communications in Mathematical Physics

\(^3\)Formal remark on co-authorship, required by the *Promotiereglement 2014* of the University of Amsterdam: the authors of these papers have equally contributed to the obtained results.
Isomonodromic $\tau$-functions and $W_N$ conformal blocks

Abstract

We study the solution of the Schlesinger system for the 4-point $\mathfrak{sl}_N$ isomonodromy problem and conjecture an expression for the isomonodromic $\tau$-function in terms of 2d conformal field theory beyond the known $N = 2$ Painlevé VI case. We show that this relation can be used as an alternative definition of conformal blocks for the $W_N$ algebra and argue that the infinite number of arbitrary constants arising in the algebraic construction of $W_N$ conformal block can be expressed in terms of only a finite set of parameters of the monodromy data of rank $N$ Fuchsian system with three regular singular points. We check this definition explicitly for the known conformal blocks of the $W_3$ algebra and demonstrate its consistency with the conjectured form of the structure constants.

Introduction

There are two topics in the mathematical physics that remained independent for a long time: the theory of isomonodromic deformations, initiated by R. Fuchs, P. Painlevé and L. Schlesinger in the beginning of 20th century (see [IN] and references therein), and the 2d conformal field theory (CFT) founded by A. Belavin, A. Polyakov and A. Zamolodchikov in 1984 [BPZ]. Both theories have wide range of applications. Conformal field theory describes perturbative string theory and second order phase transitions in the 2d systems. The theory of isomonodromic deformations gives rise to non-linear special functions such as Painlevé transcendent, which appear in different problems of mathematical physics: for example, in the random matrix theory and general relativity.

First relations between the theory of isomonodromic deformations and 2d quantum field theory have been established in 1978-80 by M. Sato, M. Jimbo and T. Miwa [SMJ]. More recently, O. Gamayun, N. Iorgov and O. Lisovyy have discovered that the $\tau$-function of the Painlevé VI equation (related to the rank two Fuchsian system with four regular singular points on the Riemann sphere) can be expressed as a sum of $c = 1$ conformal blocks, multiplied by certain ratios of the Barnes functions – a typical expansion of the correlation function in CFT [GIL12]. Their formula gives the general
solution of Painlevé VI equation. This conjecture has already been proved in two ways: one proof is purely representation-theoretic and adapted initially for the 4-point \( \tau \)-function [BShch] but can provide us with a collection of nontrivial bilinear relations for the \( n \)-point conformal blocks, whereas another one is based on the computation of monodromies of conformal blocks with degenerate fields and allows to consider an arbitrary number of regular singular points on the Riemann sphere [ILTe]. The correspondence also extends to the irregular case: for instance, it gives exact solutions of the Painlevé V and III equations [GIL13], [ILT14], which are known to describe correlation functions in certain massive field theories.

The present chapter is concerned with the extension of the isomonodromy-CFT correspondence to higher rank. Already in [GIL12] there was a suggestion that the monodromy preserving deformations of Fuchsian systems of rank \( N \) should be related to 2d CFT with central charge \( c = N - 1 \). One obvious and natural candidate for such a theory is the Toda CFT with \( W_N \) algebra of extended conformal symmetry. We show that indeed the \( N \times N \) isomonodromic problem corresponds to the \( W_N \) algebra, whose Virasoro part has central charge \( c = N - 1 \). These algebras were first introduced by A. Zamolodchikov in [ZamW], and their study was substantially developed in [FZ] (for the first nontrivial \( W_3 \)-case) and [FL] (for generic \( W_N \)). Other developments in the theory of \( W \)-algebras are discussed in the review [BS].

The most condensed form of the commutation relations of \( W_3 \) is given by the operator product expansions (OPEs) of the energy-momentum tensor \( T(z) \) and the \( W \)-current \( W(z) \):

\[
\begin{align*}
T(z)T(w) &= \frac{c}{2(z-w)^4} + \frac{2T \left( \frac{z+w}{2} \right)}{(z-w)^2} + \text{reg.}, \\
T(z)W(w) &= \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + \text{reg.}, \\
W(z)W(w) &= \frac{c}{3(z-w)^6} + \frac{2T \left( \frac{z+w}{2} \right)}{(z-w)^4} + \\
&\quad + \frac{1}{(z-w)^2} \left( \frac{32}{22+5c} \Lambda \left( \frac{z+w}{2} \right) + \frac{1}{20} \partial^2 T \left( \frac{z+w}{2} \right) \right) + \text{reg.}
\end{align*}
\]

where \( \Lambda(z) = (TT)(z) - \frac{3}{10} \partial^2 T(z) \).

The representation theory of this algebra is very similar to that of the Virasoro algebra. In the generic case one has the Verma module with the highest vector \( |\Delta, w\rangle \) such that \( L_0|\Delta, w\rangle = \Delta|\Delta, w\rangle, \ W_0|\Delta, w\rangle = w|\Delta, w\rangle \). Hence the representation space is spanned by the vectors

\[
L_{-m_1}L_{-m_2} \cdots L_{-m_k}W_{-n_1}W_{-n_2} \cdots |\Delta, w\rangle, \quad m_1 \geq m_2 \geq \ldots \geq m_k, \ n_1 \geq n_2 \geq \ldots \geq n_k,
\]

while the set of the highest weight vectors themselves corresponds to primary fields (vertex operators) of the 2d CFT. As in the Virasoro case, these fields can be determined by their OPEs with higher-spin currents \( T(z) \) and \( W(z) \):
2.2. Isomonodromic deformations and moduli spaces of flat connections

\[ T(z)\phi(w) = \frac{\Delta \phi(w)}{(z-w)^2} + \frac{\partial \phi(w)}{z-w} + \text{reg.} \]

\[ W(z)\phi(w) = \frac{w\phi(w)}{(z-w)^3} + \frac{(W_{-1}\phi)(w)}{(z-w)^2} + \frac{(W_{-2}\phi)(w)}{z-w} + \text{reg.} \]

However, the W-descendants such as \((W_{-1}\phi)\) and \((W_{-2}\phi)\) are not defined in general (this is to be contrasted with the Virasoro case where one has e.g. \((L_{-1}\phi)(w) = \partial \phi(w))\), which means that the 3-point functions involving such fields are not really defined. As a consequence, one cannot express the matrix elements

\[ \langle \Delta_{\infty}, w_{\infty} | \phi(1) L_{-m_1} L_{-m_2} \ldots L_{-m_k} W_{-n_1} W_{-n_2} \ldots | \Delta_0, w_0 \rangle \]

in terms of \(\langle \Delta_{\infty}, w_{\infty} | \phi(1) | \Delta_0, w_0 \rangle\) only. It was shown in [BW] in an elegant way that all such 3-point functions can be expressed in terms of an infinite number of unknown constants

\[ C_k = \langle \Delta_{\infty}, w_{\infty} | \phi(1) W^{-k}_{-1} | \Delta_0, w_0 \rangle, \quad k = 1, 2, \ldots \]  

(2.4)

The problem is that having this infinite number of constants (which for the 4-point conformal block actually becomes doubly infinite) one can adjust them as to obtain any function as a result. In this chapter we show that the isomonodromic approach can fix this ambiguity in such a way that all these parameters become functions on the moduli space of the flat connections on the sphere with 3 punctures. In the \(\mathfrak{sl}_3\) case this space is 2-dimensional (we denote the corresponding coordinates by \(\mu\) and \(\nu\), so all \(C_k = C_k(\mu, \nu)\). Note that for the \(W_N\) algebra one would have the set of constants \(C_{k_1, \ldots, k_l}\) with \(l = \frac{1}{2}(N-1)(N-2)\) non-negative indices (e.g., this easily follows from analysis of [BW]), which is half of the dimension of the moduli space of flat \(\mathfrak{sl}_N\) connections on the 3-punctured sphere.

The chapter is organized as follows. In Section 2 we briefly discuss the origins of the Schlesinger system and the space of flat connections on the punctured Riemann sphere. Then we introduce a collection of convenient local coordinates on this space, which are related to pants decomposition of the sphere. In Section 3 an iterative algorithm of the solution of the Schlesinger system is proposed. We then present a set of non-trivial properties of this solution, discovered experimentally, and put forward a conjecture about isomonodromy-CFT correspondence in higher rank, which relates \(W_N\) conformal blocks to the isomonodromic tau function. In particular, for a collection of known \(W_3\) conformal blocks we present the 3-point functions that can be used to construct the \(\tau\)-function in the form of explicit expansion. In Section 4 we describe the problems of definition of the general \(W_3\) conformal block and discuss how they can be addressed using the global analytic structure induced by crossing symmetry. We conclude with a brief discussion of open questions.

Isomonodromic deformations and moduli spaces of flat connections

The main object of our study will be the Fuchsian linear system
2. Isomonodromic \( \tau \)-functions and \( W_N \) conformal blocks

\[
\frac{d}{dz} \Phi(z) = \sum_{\nu=1}^{n} \frac{A_{\nu}}{z - z_{\nu}} \Phi(z) = A(z)\Phi(z),
\]
\[
\sum_{\nu} A_{\nu} = 0.
\]  \( (2.5) \)

Here \( A_{\nu} \) are traceless matrices with distinct eigenvalues, \( \Phi(z) \) is the matrix of \( N \) independent solutions of the system normalized as \( \Phi(z_0) = 1 \). It is obvious that upon analytic continuation of the solutions along a contour \( \gamma_{\nu} \) encircling \( z_{\nu} \) they transform into some linear combination of themselves:

\[
\gamma_{\nu} : \Phi(z) \mapsto \Phi(z)M_{\nu},
\]  \( (2.6) \)

where \( M_{\nu} \in GL_N(\mathbb{C}) \). The relation \( \gamma_n \ldots \gamma_1 = 1 \) from \( \pi_1(\mathbb{C}P^1 \setminus \{z_1,\ldots,z_n\}, z_0) \) imposes the condition

\[
M_1 \ldots M_n = 1.
\]  \( (2.7) \)

The well-known Riemann-Hilbert problem is to find the correspondence

\[
\{M_1,\ldots,M_n\} \to \{A_1,\ldots,A_n\}.
\]  \( (2.8) \)

It is easy to see that the conjugacy classes of \( M_{\nu} \) are

\[
M_{\nu} \sim \exp(2\pi i A_{\nu}).
\]  \( (2.9) \)

The eigenvalues of \( A_{\nu} \) determine the asymptotics of the fundamental matrix solution near the singularities, so one can fix even this asymptotics and study the corresponding refined Riemann-Hilbert problem. We will work only with traceless matrices \( A_{\nu} \) since the scalar part trivially decouples.

**Schlesinger system**

Since it is difficult to solve the generic Riemann-Hilbert problem exactly, one can first ask a simpler question: how to deform simultaneously the positions of the singularities \( z_{\nu} \) and matrices \( A_{\nu} \) but preserve the monodromies \( M_{\nu} \). The answer follows from the infinitesimal gauge transformation

\[
\Phi(z) \mapsto \left(1 + \epsilon \frac{A_{\nu}}{z - z_{\nu}}\right)\Phi(z),
\]
\[
A(z) \mapsto A(z) + \epsilon \frac{A_{\nu}}{(z - z_{\nu})^2} - \epsilon \left[ \frac{A_{\nu}}{z - z_{\nu}}, A(z) \right],
\]  \( (2.10) \)

that is

\[
z_{\nu} \mapsto z_{\nu} + \epsilon,
\]
\[
A_{\mu \neq \nu} \mapsto A_{\mu} + \epsilon \frac{[A_{\nu}, A_{\mu}]}{z_{\nu} - z_{\mu}},
\]
\[
A_{\nu} \mapsto A_{\nu} - \epsilon \sum_{\mu \neq \nu} \frac{[A_{\nu}, A_{\mu}]}{z_{\nu} - z_{\mu}},
\]  \( (2.11) \)

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leading to the Schlesinger system of non-linear equations
\[
\begin{align*}
\frac{\partial A_\mu}{\partial z_\nu} &= [A_\mu, A_\nu] z_\mu - z_\nu, \\
\frac{\partial A_\nu}{\partial z_\nu} &= -\sum_{\mu \neq \nu} [A_\mu, A_\nu] z_\mu - z_\nu.
\end{align*}
\] (2.12)

Note that one can fix \( z_n = \infty \), then the corresponding matrix \( A_\infty = -\sum A_\nu \) will be constant. A non-trivial statement is that the relations
\[
\frac{\partial}{\partial z_\mu} \log \tau = \sum_{\nu \neq \mu} \frac{\text{tr} A_\mu A_\nu}{z_\mu - z_\nu}
\] (2.13)are compatible and define the \( \tau \)-function \( \tau(z_1, \ldots, z_n) \) of the Schlesinger system. It is easy to see that the 3-point \( \tau \)-function is given by a simple expression:
\[
\tau(z_1, z_2, z_3) = \text{const} \cdot (z_1 - z_2)^{\Delta_3 - \Delta_1 - \Delta_2} (z_2 - z_3)^{\Delta_1 - \Delta_2 - \Delta_3} (z_1 - z_3)^{\Delta_2 - \Delta_1 - \Delta_3},
\] where \( \Delta_\nu = \frac{1}{2} \text{tr} A_\nu^2 \). Let us now attempt to solve the Schlesinger system for the 4-point case and compute the corresponding \( \tau \)-function in the form of certain expansion.

**Moduli spaces of flat connections**

The main object of our interest is the \( \tau \)-function. It depends on monodromy data which provide the full set of integrals of motion for the Schlesinger system. It will be useful to start by introducing a convenient parametrization of this space.

One starts with \( n \) matrices \( M_\nu \in SL_N \), with fixed nondegenerate eigenvalues, i.e. there are \( n(N^2 - N) \) parameters. These matrices are constrained by one equation (2.7) and are considered up to an overall \( SL_N \) conjugation, which decreases the number of parameters by \( 2(N^2 - 1) \). So the resulting number of parameters is
\[
\dim \mathcal{M}^{SL_N}(\theta_1, \ldots, \theta_n) = (n - 2)N^2 - nN + 2. \] (2.14)

Here \( \theta_\nu \in \mathfrak{h} \) (\( \mathfrak{h} \) is the Cartan subalgebra) define the conjugacy classes: \( M_\nu \sim e^{2\pi i \theta_\nu} \). It is obvious that \( \theta_\nu \) is equivalent to \( \theta_\nu + \mathfrak{h}_\nu \), such that for all weights of the first fundamental representation \( e_i \) one has \( (e_i, \mathfrak{h}_\nu) \in \mathbb{Z} \). It means that \( \mathfrak{h}_\nu \in \oplus_{i=1}^r \mathbb{Z} \alpha^\vee_i \), where \( \alpha^\vee_i \in \mathfrak{h} \) are simple coroots (for the simply-laced case they coincide with the roots).

For the general Lie algebra this formula can be written as
\[
\dim \mathcal{M}^{\mathfrak{g}}(\theta_1, \ldots, \theta_n) = (n - 2) \dim \mathfrak{g} - n \cdot \text{rank} \mathfrak{g}. \] (2.15)

In particular, for \( n = 3 \) punctures on the sphere
\[
\dim \mathcal{M}^{\mathfrak{sl}_N}(\theta_1, \theta_2, \theta_3) = \dim \mathfrak{g} - 3 \cdot \text{rank} \mathfrak{g}. \]
This formula gives the number of non-simple roots of \( \mathfrak{g} \). In the \( \mathfrak{sl}_N \) case it specializes to
\[
\dim \mathcal{M}^{\mathfrak{sl}_N}(\theta_1, \theta_2, \theta_3) = (N - 1)(N - 2). \] (2.16)
This expression vanishes for $\mathfrak{sl}_2$, which drastically simplifies the study of the corresponding isomonodromic problem. However already for $\mathfrak{sl}_3$ this dimension is equal to 2, i.e. it is nonvanishing. One way to simplify the problem is to set $\theta_2 = a e_1$: in this case the orbit of the adjoint action

$$e^{2\pi i a e_1} \mapsto g^{-1} e^{2\pi i a e_1} g$$

has the dimension $\dim \mathcal{O}_{a e_1} = \dim \mathfrak{g} - \dim \mathrm{stab}(e_1) = N^2 - 1 - (N - 1)^2 = 2N - 2$. The total dimension is $2(N^2 - N) + (2N - 2) - 2(N^2 - 1) = 0$. In this calculation the first two terms correspond to the dimensions of orbits: two generic and one with a large stabilizer. The last term corresponds to one equation and one factorization. Hence

$$\dim \mathcal{M}^\mathfrak{sl}_N(\theta_1, a e_1, \theta_3) = 0.$$  \hspace{1cm} (2.17)

This case is the best known on the side of $W$-algebras [FLitv07], [FLitv09], [FLitv12]. In the mathematical framework, this situation corresponds to rigid local systems.

**Pants decomposition of $\mathcal{M}^\mathfrak{g}_4$**

We begin our consideration with an arbitrary Lie group $G$ containing a Cartan torus $H \subset G$. The corresponding Lie algebras are $\mathfrak{g}$ and $\mathfrak{h}$, respectively. At some point we will switch to $G = SL_N(\mathbb{C})$ case.

The moduli space $\mathcal{M}^\mathfrak{g}_4$ is described by 4 matrices satisfying $M_1 M_2 M_3 M_4 = 1$, defined up to conjugation:

$$\mathcal{M}^\mathfrak{g}_4 = \{(M_1, M_2, M_3, M_4)\}/G.$$ \hspace{1cm} (2.18)

Let us introduce $S = M_1 M_2$ and consider two triples

$$\{(M_1, M_2, S^{-1}), (S, M_3, M_4)\}.$$ \hspace{1cm} (2.19)

Note that the products inside each of these triples are equal to the identity. Let us now choose the submanifold with fixed eigenvalues of $M_1, \ldots, M_4, S$. One may also use the freedom of the adjoint action to diagonalize $S$

$$S = e^{2\pi i \sigma},$$

where $\sigma \in \mathfrak{h}$. We thereby obtain a submanifold

$$\mathcal{M}^\mathfrak{g}_3(\theta_1, \theta_2; \sigma; \theta_3, \theta_4) = \{(M_1, M_2, e^{-2\pi i \sigma}), (e^{2\pi i \sigma}, M_3, M_4)\}/H \subset \mathcal{M}^\mathfrak{g}_4(\theta_1, \theta_2, \theta_3, \theta_4),$$ \hspace{1cm} (2.20)

where the remaining factorization is performed over the Cartan torus $H \subset G$. It is very similar to what happens for $\mathcal{M}^\mathfrak{g}_3$:

$$\mathcal{M}^\mathfrak{g}_3 = \{(M_1, M_2, M_3)\}/G = \{(M_1, M_2, e^{2\pi i \theta_1})\}/H,$$ \hspace{1cm} (2.21)

except that the conjugation is simultaneous for both triples. To relax this condition, let us define an extra Cartan torus acting on $\mathcal{M}^\mathfrak{g}_3$:

$$h : \{(M_1, M_2, e^{-2\pi i \sigma}), (e^{2\pi i \sigma}, M_3, M_4)\} \mapsto \{(M_1, M_2, e^{-2\pi i \sigma}), h^{-1}(e^{2\pi i \sigma}, M_3, M_4)h\},$$ \hspace{1cm} (2.22)
which looks like a relative twist of one part of the sphere with respect to another (in the \(\mathfrak{sl}_2\) case it will be exactly the geodesic flow). Therefore one can say that

\[
\mathcal{M}_4^\theta(\theta_1, \theta_2; \sigma, \theta_3, \theta_4)/H = \mathcal{M}_3^\theta(\theta_1, \theta_2, -\sigma) \times \mathcal{M}_3^\theta(\sigma, \theta_3, \theta_4). \tag{2.23}
\]

The torus action is free, so locally it looks as a product (actually it is true even globally because the fibration \((M_1, M_2, M_3) \mapsto (M_1, M_2, M_3)/G\) is trivial: we can give an algebraic parametrization for one representative from each conjugacy class). Therefore we have the equality for the open subsets (denoted by \(\approx\)):

\[
\mathcal{M}_4^\theta(\theta_1, \theta_2; \sigma, \theta_3, \theta_4) \approx \mathcal{M}_3^\theta(\theta_1, \theta_2, -\sigma) \times H \times \mathcal{M}_3^\theta(\sigma, \theta_3, \theta_4). \tag{2.24}
\]

The above considerations suggest the following choice of coordinates on \(\mathcal{M}_4^\theta\):

- **Gluing parameters** \(\sigma\): rank \(g\) items.
- **Invariant functions on** \(\mathcal{M}_3^\theta \times \mathcal{M}_3^\theta\) (for example, \(\text{tr} M_1 M_2^{-1}\), \(\text{tr} M_3^{-1} M_4\)). They are invariant with respect to the action of “relative twists”: we have \(2 \dim \mathcal{M}_3^\theta\) such functions.
- **Relative twist parameters**, which change under the twist (for example, \(\text{tr} M_2 M_3^{-1}\), \(\text{tr} M_2^{-1} M_3\)), rank \(g\) items. These coordinates will be denoted by \(\beta \in \mathfrak{h}\).

This procedure is schematically depicted in Fig.4.6 for the \(\mathfrak{sl}_3\) case, where \(\dim \mathcal{M}_3^{\text{sl}_3} = 2, \dim \mathcal{M}_4^{\text{sl}_3} = 8\). The coordinates on each copy of \(\mathcal{M}_3^{\text{sl}_3}\) are denoted by \(\mu, \nu\). The indices \(\{1, 2, 3, 4\}\) of the matrices are replaced by \(\{0, t, 1, \infty\}\).

![Figure 2.1](image)

**Figure 2.1:** Coordinates on \(\mathcal{M}_4^{\text{sl}_3}\): eight = two \(\sigma\)’s + two \(\beta\)’s + \(\mu_0 t + \nu_0 t + \mu_1 \infty + \nu_1 \infty\)

### Pants decomposition for \(\mathcal{M}_n^g\)

Suppose that the coordinates on \(\mathcal{M}_n^g\) are chosen via the pants decomposition. Split the matrices into two groups and define

\[
S_{n-3} = M_1 \ldots M_{n-2},
\]

\[
\mathcal{M}_n^g = \{(M_1, \ldots, M_{n-2}, S_{n-3}^{-1}), (S_{n-3}, M_{n-1}, M_n)\}/G =
\]

\[
= \{(M_1, \ldots, M_{n-2}, e^{-2\pi i \sigma}), (e^{2\pi i \sigma}, M_{n-1}, M_n)\}/H \approx \mathcal{M}_{n-1}^g \times H \times \mathcal{M}_3^g. \tag{2.25}
\]

Iteratively repeating this procedure, one is led to the following choice of coordinates on \(\mathcal{M}_n^g\):

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- $(n - 3)$ rank $g$ gluing parameters $\sigma_i$,
- $(n - 3)$ rank $g$ relative twist parameters $\beta_i$,
- $\sum_{i=1}^{n-2} \dim \mathcal{M}_3^g(\sigma_{i-1}, \theta_{i+1}, -\sigma_i)$ 3-point moduli of flat connections (here we identify $\sigma_0 = \theta_1$ and $\sigma_{n-2} = -\theta_n$).

Anticipating the result, let us mention that these coordinates are convenient from the CFT point of view: $\sigma_i$ will parametrize intermediate charges in the conformal block and $\beta_i$ will be the Fourier transformation parameters. This description was shown to be valid in the $\mathfrak{sl}_2$ case [GIL12], [ILTe] and was recently demonstrated to hold for $\mathfrak{sl}_N$ case with $\dim \mathcal{M}_3^g = 0$ [GavIL]. From a more conceptual point of view, this decomposition illustrates that all extra parameters in the $\tau$-function expansion come from the 3-point functions.

Iterative solution of the Schlesinger system

In order to study the generic Schlesinger system, let us follow the approach proposed in the original paper of M. Jimbo [Jimbo] and in [SMJ, part 2].

Let us take the 4-point Schlesinger system and fix the singularities to be 0, $t$, 1, $\infty$. The system becomes

\[
\begin{align*}
\theta_t A_0 &= [A_t, A_0], \\
\theta_t A_1 &= \frac{t}{t-1} [A_t, A_1], \\
\partial_t A_0 &= -\frac{1}{t} [A_t, A_0] - \frac{1}{t-1} [A_t, A_1].
\end{align*}
\] (2.26)

Fixing the integral of motion $A_\infty = -A_0 - A_t - A_1$, one obtains

\[
\begin{align*}
\theta_t A_0 &= [A_0, A_1 + A_\infty], \\
\theta_t A_1 &= t(1-t)^{-1} [A_0 + A_\infty, A_1].
\end{align*}
\] (2.27)

The isomonodromic $\tau$-function is defined by

\[
\partial_t \log \tau = \frac{1}{t} \text{tr} A_t A_0 + \frac{1}{t-1} \text{tr} A_t A_1.
\] (2.28)

Let us study the solution of the system (2.27) for the case when $A_1(t)$ is finite in the limit $t \to 0$: $A_1(t) = A_1(0) + O(t^{\epsilon>0})$. Under this assumption we have

\[
\theta_t A_0(t) = [A_0, A_\infty + A_1(0) + O(t^{\epsilon>0})].
\]

If the last term were absent, then the solution would be $A_0 = t^{-A_\infty - A_1(0)} \tilde{A}_0 t^{A_\infty + A_1(0)}$. Therefore it is natural to introduce

\[
B = -A_1(0) - A_\infty = \lim_{t \to 0} (A_0(t) + A_t(t)), \quad \tilde{A}_0(t) = t^{-B} A_0(t) t^B,
\] (2.29)
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where \( \tilde{A}_0(t) \) has a well-defined limit as \( t \to 0 \). We see that in view of its definition \( B \) describes the total monodromy around 0 and \( t \) in the limit \( t \to 0 \). Since the deformation is isomonodromic, this monodromy is constant and is given by \( M_0 M_t = M_{0t} \sim e^{2\pi i B} \). This allows to make the identification

\[
B = \sigma . \tag{2.30}
\]

Our system then becomes

\[
t \partial_t \tilde{A}_0(t) = [\tilde{A}_0(t), t^{-\sigma} (A_1(t) - A_1(0)) t^\sigma],
t \partial_t A_1 = t(1 - t)^{-1} [t^\sigma \tilde{A}_0(t) t^{-\sigma} + A_\infty, A_1(t)]. \tag{2.31}
\]

Here we see an operator \( t^{ad_\sigma} \), which produces some fractional powers of \( t \). It is convenient to impose the condition that \( (\sigma, \sigma) \ll 1 \), or at least that for all roots \( \alpha \) one has \( |(\sigma, \alpha)| < \frac{1}{2} \). This allows to organize the terms of the expansion in powers of \( t \) according to their order of magnitude in the asymptotic behavior. If necessary, one can perform an analytic continuation of the solution from the region with small \( \sigma \).

We know that in the Lie algebra the operator \( t^{ad_\sigma} \) acts by

\[
t^\sigma E_{\alpha} t^{-\sigma} = t^{(\sigma, \alpha)} E_{\alpha},
t^\sigma H_{\alpha} t^{-\sigma} = H_{\alpha}, \tag{2.32}
\]

where \( \alpha \in \mathfrak{g}^* \) is a root and \( E_{\alpha}, H_{\alpha} \) are the elements of the Cartan-Weyl basis. Let us define a grading on the space of monomials

\[
\text{deg}[t^{k+(\sigma, w)}] = (k, w),
\]

where \( w \in Q_\mathfrak{g} \) is an element of the root lattice \( Q_\mathfrak{g} = \bigoplus_{i=1}^{\text{rank} \mathfrak{g}} \mathbb{Z} \alpha_i \) of \( \mathfrak{g} \). It is useful to define a filtration

\[
Q_\mathfrak{g}^0 \subset Q_\mathfrak{g}^1 \subset Q_\mathfrak{g}^2 \subset \ldots \subset Q_\mathfrak{g} \tag{2.33}
\]

on this root lattice, which is recursively constructed as follows: \( Q_\mathfrak{g}^0 = \{0\} \), \( Q_\mathfrak{g}^1 \) is the set of all roots of \( \mathfrak{g} \) and 0, and

\[
Q_\mathfrak{g}^{i+1} = \{ x + y | x \in Q_\mathfrak{g}^i, y \in Q_\mathfrak{g}^1 \} = Q_\mathfrak{g}^1 + \ldots + Q_\mathfrak{g}^1 .
\]

Also define the double filtration \( V_{n,m} \) on the space of all fractional-power series:

\[
t^{k+(\sigma, w)} \in V_{n,m} \Leftrightarrow (k \geq n) \land (w \in Q_\mathfrak{g}^m),
V_{n+1,m} \subset V_{n,m}, \quad V_{n,m} \subset V_{n,m+1} . \tag{2.34}
\]

Each term of the filtration is generated by these monomials. This definition turns out to be useful because of the properties

\[
t \cdot : V_{n,m} \to V_{n+1,m},
t^{ad_\sigma} : V_{n,m} \to V_{n,m+1}, \tag{2.35}
V_{n_1,m_1} \cdot V_{n_2,m_2} \to V_{n_1+n_2,m_1+m_2} .
\]
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One can also see that the degrees present in $V_{n+1,m+k}$ are larger then in $V_{n,m}$ if $\sigma$ is sufficiently small ($\forall \alpha \in Q^1_0 : |(\sigma, \alpha)| < \frac{1}{\epsilon}$). We also define a slightly ambiguous notation $V_{n,w}$ by

$$tk^+(\sigma,w) \in V_{n,w} \Leftrightarrow (k \geq n). \quad (2.36)$$

Now we have all the ingredients that are necessary for the construction of an iterative solution of the system (2.31). Our initial data will be given by the triple of matrices $\sigma$, $\tilde{A}_0(0)$ and $A_1(0)$. Symbolically, the system (2.31) can be written as

$$\tilde{A}_0(t) = F_0(\tilde{A}_0(t), A_1(t)), \quad A_1(t) = F_1(\tilde{A}_0(t), A_1(t)), \quad (2.37)$$

where “affine” bilinear (in the sense $f(x,y) = axy + bx + cy + d$) functions $F_0$, $F_1$ have the following properties:

- $F_0 : V_{n_0,m_0} \times V_{0,0} \rightarrow 0$,
- $F_0 : V_{n_0,m_0} \times V_{n_1,m_1} \rightarrow V_{n_0+n_1,m_0+m_1+1} \subset V_{n_0+n_1,\infty}$,
- $F_1 : V_{n_0,m_0} \times V_{n_1,m_1} \rightarrow V_{n_0+n_1+1,m_0+m_1+1} \cup V_{n_1+1,m_1} \subset V_{n_1+1,\infty}$.

Let us substitute into (2.37) the expressions

$$\tilde{A}_0(t) = \tilde{A}_0(0) + \sum_{k=1}^{\infty} t^k \tilde{A}_0^k(t),$$
$$A_1(t) = A_1(0) + \sum_{k=1}^{\infty} t^k A_1^k(t),$$
$$t^k \tilde{A}_0^k(t), t^k A_1^k(t) \in V_{k,\infty}. \quad (2.39)$$

From (2.38) we immediately see that (2.31) takes the form

$$\tilde{A}_0^k(t) = f_0^k(\tilde{A}_0^{<k}(t), A_1^{<k}(t)), \quad A_1^k(t) = f_1^k(\tilde{A}_0^{<k}(t), A_1^{<k}(t)). \quad (2.40)$$

Because of the $\leq$ sign our strategy of solving will be to compute first $A_1^k(t)$, and then subsequently determine $\tilde{A}_0^k(t)$. One can also write down explicit formulas for bilinears $f_1^k$ and $f_0^k$, which are immediate (though cumbersome) consequences of the system (2.31).

Now let us determine which powers $(k, w)$ will be actually present in the solution. This will be done again iteratively, using only the properties (2.38):

- Taking $\tilde{A}_0(0) \in V_{0,0}$ and $A_1(0) \in V_{0,0}$, and computing $F_1$, we get an element of $V_{1,1}$, therefore $A_1 \in V_{0,0} + V_{1,1} + \ldots$
- Take $\tilde{A}_0(0) \in V_{0,0}$ and $A_1 \in V_{0,0} + V_{1,1} + \ldots$, then $\tilde{A}_0 \in V_{0,0} + V_{1,2} + \ldots$
- For $\tilde{A}_0 \in V_{0,0} + V_{1,2} + \ldots$ and $A_1 \in V_{0,0} + V_{1,1} + \ldots$ one finds $A_1 \in V_{0,0} + V_{1,1} + V_{2,3} + \ldots$
• Setting \( \hat{A}_0 \in V_{0,0} + V_{1,2} + \ldots \) and \( A_1 \in V_{0,0} + V_{1,1} + V_{2,3} + \ldots \) yields \( \hat{A}_0 \in V_{0,0} + V_{1,2} + V_{2,4} \ldots \)

• ...

Continuing this procedure one finds the following structure

\[
\hat{A}_0(t) \in \sum_{k=0}^{\infty} V_{k,2k}, \tag{2.41}
\]

\[
A_1(t) \in V_{0,0} + \sum_{k=1}^{\infty} V_{k,2k-1}.
\]

It is easy to check that these spaces are stable under the action of \((F_0, F_1)\) described by the rules (2.38). This is somewhat similar to the statement that the cone is stable under the addition operation.

Indeed, let us try to find an element of \( \hat{A}_0(t) \) lying in \( V_{k,2k+1} \). For this one would need \( n_0 + n_1 \leq k \), \( m_0 + m_1 \geq 2k \), so \( m_0 + m_1 \geq 2(n_0 + n_1) \). Since \( m_1 \leq 2n_1 - 1 \) for \( n_1 \neq 0 \) (when \( F_0 \) vanishes) and \( m_0 \leq 2n_0 \), such an element cannot exist. Similarly, for \( A_1 \), let us take an element lying in \( V_{k,2k} \). One then needs to satisfy the constraints \( n_1 \leq k - 1 \), \( m_1 \geq 2k \) (impossible) or \( n_0 + n_1 + 1 \leq k \) and \( m_0 + m_1 + 1 \geq 2k \), which implies \( m_0 + m_1 \geq 2n_0 + 2n_1 + 1 \). But \( m_1 \leq 2n_1 \) and \( m_0 \leq 2n_0 \), therefore one cannot get such an element neither.

Now let us compute the \( \tau \)-function and try to understand in which elements of the filtration does it lie. Since we have

\[
t \partial_t \log \tau(t) = - \text{tr} [t^{-\sigma}(A_1 + A_\infty)t^\sigma \hat{A}_0 + \hat{A}_0^2] + t(1 - t)^{-1} \text{tr} [(A_1 + A_\infty + t^\sigma \hat{A}_0 t^{-\sigma})A_1],
\]

naively it could be a term in \( V_{0,1} \). However, computing the constant part one finds

\[
t \partial_t \log \tau(t) = \text{tr} (B \hat{A}_0 - \hat{A}_0^2) + \ldots = \text{tr} (A_t A_0) + \ldots = \frac{1}{2} \text{tr} (A_t + A_0)^2 - \frac{1}{2} \text{tr} A_0^2 - \frac{1}{2} \text{tr} A_t^2 + \ldots = \frac{1}{2} (\sigma, \sigma) - \frac{1}{2} (\theta_0, \theta_0) - \frac{1}{2} (\theta_t, \theta_t) + \ldots, \tag{2.43}
\]

where \( A_\nu \sim \theta_\nu \). For convenience, let us introduce the notation

\[
\chi = \frac{1}{2} (\sigma, \sigma) - \frac{1}{2} (\theta_0, \theta_0) - \frac{1}{2} (\theta_t, \theta_t) \tag{2.44}
\]

The terms present in \( \text{tr} (t^{-\sigma} A_1(t) t^\sigma \hat{A}_0(t)) \) that are closest to the boundary originate from the constant part of \( A_1(t) \). These terms belong to \( \sum_{k=0}^{\infty} V_{k,2k} \), therefore

\[
\log \tau(t) \in \sum_{k=0}^{\infty} V_{k,2k} \tag{2.45}
\]
2. Isomonodromic $\tau$-functions and $W_N$ conformal blocks

Note that these estimates are too rough, since we have not taken into account that a number of the commutators actually vanish. The actual result turns out to be the same for all three functions

$$\log \tau, \tilde{A}_0, A_1 = \sum_{k=0}^{\infty} V_{k,k},$$

and it can be checked numerically. Moreover, it turns out that the expansion of the $\tau$-function itself is even more restricted:

$$t^{-\chi} \tau(t) = \sum_{w \in \mathbb{Q}_{\pm \alpha}} V_{1/2(w,w),w} f_n(t),$$ (2.46)

which in fact provides an evidence for the 2d CFT description: different fractional powers come from $t^\Delta$ for the different $\Delta$’s, but the conformal dimension $\Delta = \frac{1}{2}(\sigma + w, \sigma + w)$ is a quadratic function of $w$ leading to the structure (2.46).

$\mathfrak{sl}_2$ case

In this case we illustrate all procedures, definitions and statements using the exact solution of [GIL12].

The Lie algebra $\mathfrak{sl}_2$ is given by 3 generators $E_\alpha, E_{-\alpha}, H_\alpha$, such that

$$[E_\alpha, E_{-\alpha}] = H_\alpha,$$
$$[H_\alpha, E_{\pm \alpha}] = \pm 2E_{\pm \alpha}. \quad (2.47)$$

The root lattice $Q_{\mathfrak{sl}_2}$ is shown in Fig.2.2. It is spanned by one root $\alpha$. $Q_{\mathfrak{sl}_2}^0$ is the empty square, $Q_{\mathfrak{sl}_2}^1$ is the red rectangle, $Q_{\mathfrak{sl}_2}^2$ is green and $Q_{\mathfrak{sl}_2}^3$ is blue.

All monomials have the form $t^\sigma f_n(t)$, and therefore can be depicted by the points of a two-dimensional lattice. Note that in our normalization $(\alpha, \alpha) = 2$. Several examples of the elements of this filtration are presented in Fig.2.3.

Here the blue region represents $V_{0,0}$, red corresponds to $V_{1,1}$ and green is $V_{3,4}$. We can also show the “true” and “naive” lattice supports of the quantities $\tilde{A}_0(t)$, $A_1(t)$, $\log \tau(t)$ and $t^{-\chi} \tau(t)$. See Fig.2.4: green region is the “naive” support of $A_1(t)$, blue region is the true support of $\tilde{A}_0(t), A_1(t), \log \tau(t)$, which can be derived experimentally. Now one can use an exact formula for the tau function expansion [GIL12] (cf (2.50) below) to see that

$$\tau(t) = t^{\sigma^2 - \theta_0^2 - \theta_1^2} \sum_{n \in \mathbb{Z}} t^{2\sigma n} f_n(t),$$ (2.48)

which in turn implies

$$t^{\theta_0^2 + \theta_1^2 - \sigma^2} \tau(t) = \sum_{k=0}^{\infty} V_{k^2,k}. \quad (2.49)$$
It looks like a miracle and means that a huge number of terms cancel out when we exponentiate, but this answer confirms the conjecture (2.46). This phenomenon is illustrated in Fig.2.5 in two ways. Upper bold numbers account for the degree in $\tau(t)$ (blue region), lower numbers correspond to the degree in $\log \tau(t)$ (green region). Horizontal coordinate corresponds to the position in the $\mathfrak{sl}_2$ root lattice.

Figure 2.5: Supports of $\tau(t)$ and $\log \tau(t)$: circles correspond to the integral points of the $x$-axis, numbers inside show the $y$-coordinates of the cone and parabola.
2. Isomonodromic $\tau$-functions and $W_N$ conformal blocks

Let us take the main formula from [GIL12]:

$$\tau(t) = \sum_{n \in \mathbb{Z}} s^n C_n^{(0t)}(\theta_0, \theta_t, \sigma_{0t}) C_n^{(1\infty)}(\theta_1, \theta_\infty, \sigma_{0t}) t^{(\sigma_{0t}+n)^2 - \theta_0^2 - \theta_\infty^2} B(\{\theta_i\}, \sigma_{0t} + n; t),$$

(2.50)

where $B(\ldots; t)$ is the $c = 1$ Virasoro conformal block and

$$C_n^{(0t)}(\theta_0, \theta_t, \sigma_{0t}) C_n^{(1\infty)}(\theta_1, \theta_\infty, \sigma_{0t}) = \prod_{\epsilon = \pm, \epsilon' = \pm} \frac{G(1 + \theta_t + \epsilon \theta_0 + \epsilon' (\sigma_{0t} + n)) G(1 + \theta_1 + \epsilon \theta_\infty + \epsilon' (\sigma_{0t} + n))}{G(1 - 2\sigma_{0t}) G(1 + 2\sigma_{0t})}.$$  (2.51)

Here $(\theta_\nu, -\theta_\nu)$ are the eigenvalues of the matrices $A_\nu$ in the linear system (2.5), $(e^{2\pi i \sigma_{\mu\nu}}, e^{-2\pi i \sigma_{\mu\nu}})$ are the eigenvalues of $M_\mu M_\nu$, $s$ is the only variable depending on $\sigma_1 t$ (in a complicated way). The main properties of (2.50) and (2.51) can be summarized as follows:

1. The support of $\tau(t)$ is as indicated in (2.49).
2. Relative twist parameter enters only via the factor $s^n$ in the structure constants.
3. The 3-point coefficients $C_n$ factorize with respect to the pants decomposition parametrization.

We are now going to check these important properties in the $\mathfrak{sl}_3$ case.

$\mathfrak{sl}_3$ case

Fig.2.6 illustrates the filtration on the $\mathfrak{sl}_3$ root lattice. The red hexagon corresponds to $Q^1_{\mathfrak{sl}_3}$, $Q^2_{\mathfrak{sl}_3}$ is shown in green and $Q^3_{\mathfrak{sl}_3}$ is blue. It is difficult to visualize $V_{m,n}$. 

![Figure 2.6: Filtration $Q^*_{\mathfrak{sl}_3}$](image-url)
since one would then need a 3d picture. One can however think of \( \sum_{k=0}^{\infty} V_{k,k} \) as being a cone with hexagonal section.

Let us perform the numerical study of the 3\times3 Schlesinger system. We first determine which degrees \((k, w)\) are present in \(\log \tau(t)\) and in \(\tau(t)\) (Fig. 2.7).

![Figure 2.7: Degrees present in \(t^{-x}\tau(t)\) and in \(\log \tau(t)\). Number \(\chi\) is given by (2.44).](image)

As above, the upper bold numbers correspond to degrees in \(t^{-x}\tau(t)\) and the lower ones to \(\log \tau(t)\). We mark with "?" sign those values which are obtained at the limit of machine precision or which are greater than 7 (so that they are not seen in the solution up to the 7th order). Carefully analyzing this picture, one deduces that

\[
\begin{align*}
\log \tau(t) & \in \sum_{k=0}^{\infty} V_{k,k}, \\
t^{-x}\tau(t) & \in \sum_{w \in Q_{\nu_{ij}}} V_{k,k}^{(w,w,w)}.
\end{align*}
\]

It means that nonzero monomials of \(\tau(t)\) fill a paraboloid, and not the naively expected cone. In other words, a lot of nontrivial cancellations take place, which provides further evidence for the conjecture (2.46). We now list other nontrivial properties of \(\tau(t)\) revealed by our experimental study.

1. The expansion has the form

\[
\tau(t) = \sum_{w \in Q} e^{(\beta, w)} C_{w}^{(1)}(\omega_{0}, \theta_{t}, \sigma_{0t}, \mu_{0t}, \nu_{0t}) C_{w}^{(1,\infty)}(\omega_{1}, \sigma_{0t}, \mu_{1t}, \nu_{1t}) \times
\]

\[
\times t^{\frac{1}{2}}(\sigma_{0t} + w, \sigma_{0t} + w) - \frac{1}{2}(\theta_{0}, \theta_{t}) - \frac{1}{2}(\theta_{0}, \theta_{t}) B_{w}(\{\theta_{1}\}, \sigma_{0t}, \mu_{0t}, \nu_{0t}; \mu_{1t}, \nu_{1t}; t).
\]

(2.53)
2. Isomonodromic $\tau$-functions and $W_N$ conformal blocks

2. The non-zero coefficients of the expansion start from $t^{\frac{1}{2}(w,w)}$.

3. All the dependence on the relative twist parameters is hidden in $\beta \in \mathfrak{h}$, which enters in a trivial way.

4. The dependence of structure constants on the 3-point monodromy parameters is factorized.

5. The first term in the expansion of conformal block has the form

$$B_0 = 1 + [\alpha + \beta C_1(\mu_0t, \nu_0t) + \gamma \tilde{C}_1(\mu_{1\infty}, \nu_{1\infty}) + \delta C_1(\mu_0t, \nu_0t) \tilde{C}_1(\mu_{1\infty}, \nu_{1\infty})]t + \ldots$$

This property is new, as compared to the $N = 2$ case, and we will see later that it is very important.

All these facts tell us that almost all properties of $\mathfrak{sl}_2$ case hold in the $\mathfrak{sl}_3$ case. This leads us to

**Main conjecture:**

$$B_0(\{\theta_i\}, \sigma_{0t}, \mu_{0t}, \nu_{0t}, \mu_{1\infty}, \nu_{1\infty}; t)$$

is a conformal block of $W_3$ algebra

The corresponding dimensions and $W$-charges are given by

$$\Delta_\nu = \frac{1}{2}(\theta_\nu, \theta_\nu)$$

$$w_\nu = \sqrt{\frac{3}{2}} \prod_i (\theta_\nu, e_i). \quad (2.54)$$

The main advantage of the above definition of conformal block is that it depends only on 4 extra variables instead of a doubly-infinite set.

It is easy to check this definition for the case when $W_3$-block can be defined algebraically. This becomes possible when $\theta_t = a_te_1$ and $\theta_1 = a_1e_1$, where $e_1$ is the weight of the first fundamental representation. The best way to present this conformal block is to use Nekrasov formulas [Nek] which can be applied to conformal field theory in view of the extended AGT [AGT] correspondence, first established in [Wyll], [MirMor]. The most convenient (for $c = 2$) expression for the conformal block can be found in [FLitv12]:

$$B_w(\theta_\infty, a_1, \sigma, a_t, \theta_0; t) = B(\theta_\infty, a_1, \sigma + w, a_t, \theta_0; t)$$

$$B(\theta_\infty, a_1, \sigma, a_t, \theta_0; t) = (1 - t)^{\frac{1}{2}(a_1a_1)} \sum_{Y} \delta_{\nu|\bar{Y}, Y} Z_{bf}(-\theta_\infty, a_1, \sigma|0, \bar{Y})$$

$$\times \frac{1}{Z_{bf}(\sigma, 0, \sigma|\bar{Y}, Y)} Z_{bf}(\sigma, a_t, \theta_0|\bar{Y}, \bar{Y}),$$

where

$$Z_{bf}(\theta, a, \theta'|i\bar{\nu}, \bar{\nu}) = \prod_{i,j=1}^3 \prod_{s \in \nu_i} (-E_{\nu_i, \nu_j}(i(\theta, e_j) - i(\theta', e_i)|s) - i\frac{a}{3}) \times$$

$$\times \prod_{t \in \nu_j} (E_{\nu_j, \nu_i}(i(\theta', e_i) - i(\theta, e_j)|t) - i\frac{a}{3}), \quad (2.56)$$
and the quantities $E$ are defined by

$$E_{\lambda,\mu}(x|s) = x - il_{\mu}(s) - ia_{\lambda}(s) - i.$$  \hspace{1cm} (2.57)

It yields exactly the same result as our computations using iterative solution of the Schlesinger system.

We have also conjectured in this case and checked experimentally a formula for the structure constants, which is a straightforward generalization of (2.51):

$$C_w(0, \theta_0, a_t, \sigma)C_w(1, \infty, \sigma, a_1, \theta_\infty) = \prod_{ij} G[1 - \frac{a_t}{N} + (e_i, \theta_0) - (e_j, \sigma + w)] G[1 - \frac{a_1}{N} + (e_i, \sigma + w) + (e_j, \theta_\infty)].$$  \hspace{1cm} (2.58)

Here $e_i$ denote the weights of the first fundamental representation and $\alpha_i$ are all roots of $\mathfrak{sl}_N$ (in our case $N = 3$). This formula was recently proved [GavIL] for general $N$. One can also observe a similarity between this formula and Toda 3-point function computed in [FLitv07].

**Remarks on $W_3$ conformal blocks**

**General conformal block**

Here we consider for simplicity the $c = 2$ case, but the generalization to arbitrary $c$ is straightforward. First we explain how the $W_N$ conformal block is defined algebraically. For that let us compute the following expression:

$$B(\theta_\infty, \theta_1, \sigma, \theta_t, \theta_0; t) = \langle -\theta_\infty|\phi_{\theta_1}(1)P_{\sigma}\phi_{\theta_t}(t)|\theta_0\rangle,$$  \hspace{1cm} (2.59)

where $|\theta_0\rangle$ and $\langle -\theta_\infty|$ are the highest-weight vectors with the charges given by (2.54), $P_{\sigma}$ is the projector onto the whole Verma module (2.2) with given highest weight. This conformal block can be computed by inserting the resolution of the identity in the Verma module. One can take, for instance, the naive basis (2.2), or (if we do not necessarily want to preserve the $L_0$ grading) the basis from [BW], or (if we wish to add the Heisenberg algebra) the AGT basis from [FLitv12], [BBFLT]. Let us call the vectors of this basis $|\sigma, \vec{Y}\rangle$ and suppose that

$$L_0|\sigma\rangle = (\Delta(\sigma) + |\vec{Y}|)|\sigma, \vec{Y}\rangle.$$

Their scalar products will be $K_{\sigma}(\vec{Y}, \vec{Y}') = \langle \sigma, \vec{Y}|\sigma, \vec{Y}'\rangle$. This allows to express conformal block as

$$B(\theta_\infty, \theta_1, \sigma, \theta_t, \theta_0; t) = t^x \sum_{\vec{Y}, \vec{Y}'} t^{\vec{Y}}|\langle -\theta_\infty|\phi_{\theta_1}(1)|\sigma, \vec{Y}\rangle K^{-1}(\vec{Y}, \vec{Y}')\langle \sigma, \vec{Y}'|\phi_{\theta_t}(1)|\theta_0\rangle =$$

$$= t^x \sum_{\vec{Y}} t^{\vec{Y}}Q(\vec{Y})\tilde{Q}(\vec{Y}),$$  \hspace{1cm} (2.60)
where \( \tilde{Q}(\vec{Y}) = \sum_{\vec{Y}'} K^{-1}(\vec{Y}, \vec{Y}') \langle \sigma, \vec{Y}' | \phi_{\theta_1}(1) | \theta_0 \rangle \) and \( Q(\vec{Y}) = \langle \theta_\infty | \phi_{\theta_1}(1) | \sigma, \vec{Y} \rangle \) and \( \chi \) is given by (2.44). The claim of [BW] is that \( Q(\vec{Y}) \) and \( \tilde{Q}(\vec{Y}) \)

\[
Q(\vec{Y}) = Q(\vec{Y} | C_1, \ldots, C_{|\vec{Y}|}) = \gamma_0(\vec{Y}) + \sum_{k=1}^{||\vec{Y}||} \gamma_k(\vec{Y}) C_k, \\
\tilde{Q}(\vec{Y}) = Q(\vec{Y} | \tilde{C}_1, \ldots, \tilde{C}_{|\vec{Y}|}) = \tilde{\gamma}_0(\vec{Y}) + \sum_{k=1}^{||\vec{Y}||} \tilde{\gamma}_k(\vec{Y}) \tilde{C}_k, 
\]

are “triangular” “affine” linear functions of infinitely many arbitrary parameters \( C_k, \tilde{C}_k \) defined by

\[
C_k = \langle -\theta_\infty | \phi_{\theta_1}(1) W_{-1}^{k} | \sigma \rangle, \\
\tilde{C}_k = \langle \sigma | W_{1}^{k} \phi_{\theta_1}(1) | \theta_0 \rangle. 
\]

### Degenerate field

Let us consider the case \( \theta_1 = e_1 \) (the weight of the first fundamental representation). The fusion rules for such fields are known to be given by

\[
[e_1] \bigotimes [\theta] = \bigoplus_k [\theta + e_k]. 
\]

Let us also shift \( \theta_0 \mapsto \theta_0 - e_n \), multiply the conformal block by \( t^2 = t^{(e_1, e_1)} \), and define the quantity

\[
\Phi_{nk}(t) = t^{(e_1, e_1)} B(\theta_\infty, \theta_1, \theta_0 + e_k - e_n, e_1, \theta_0 - e_n; t) = t^{(\theta_0, e_k) + (1-\delta_{kn})} \sum_{\vec{Y}} Q(\vec{Y}, C_1, \ldots, C_{|\vec{Y}|}) \tilde{q}(\vec{Y}),
\]

where \( \tilde{q}(\vec{Y}) \) do not contain any free parameters [BW].

Now denote the degenerate field \( \phi_{e_1}(t) \) by \( \psi(t) \) and consider the correlation function

\[
l^{(e_1, e_1)} \langle -\theta_\infty | \phi_{\theta_1}(1) \psi(t) | \theta_0 - e_k \rangle.
\]

In the region \( t \to 0 \) (s-channel) it can be expanded in the basis of conformal blocks written above. But if we set \( t \to 1 \) or \( t \to \infty \) (t- and u-channel), then we will have the following OPEs

\[
\psi(t) \phi_{\theta_1}(1) = \sum_k C^{\theta_1, e_k}_{\psi_1, \theta_1} \cdot (t - 1)^{(\theta_1, e_k)} (\phi_{\theta_1 + e_k}(1) + \text{ descendants}), \\
l^{(e_1, e_1)} \langle -\theta_\infty | \psi(t) \rangle = \sum_k C^{\theta_\infty + e_k}_{\psi_1, e_1} t^{-(\theta_\infty, e_k)} (\langle -\theta_\infty - e_k \rangle + \text{ descendants}).
\]

These formulas suggest that the space of conformal blocks involving \( \psi(t) \) is 3-dimensional and near each point we have a basis with asymptotics prescribed by \( \theta_\mu \). It is clear that
2.4. Remarks on \( W_3 \) conformal blocks

Upon analytic continuation of \( \Psi_{1k}(t) \) around 0, 1, \( \infty \) one gets some linear combinations of the basis elements

\[
\gamma_0 : \Phi_{1k}(t) \mapsto \sum_{k'} \Phi_{1k'}(t)(M_0)_{k'k}, \\
\gamma_1 : \Phi_{1k}(t) \mapsto \sum_{k'} \Phi_{1k'}(t)(M_1)_{k'k}, \\
\gamma_{\infty} : \Phi_{1k}(t) \mapsto \sum_{k'} \Phi_{1k'}(t)(M_{\infty})_{k'k}. \tag{2.66}
\]

In our case \( M_0 = \text{diag}(e^{2\pi i \theta_0}, e^{2\pi i \theta_1}, e^{2\pi i \theta_2}) \). That these formulas must hold can be expected on general grounds (crossing symmetry) and from the fact that the space is 3-dimensional. However, looking at the formula (2.64), the freedom in choice of \( C_k \) can give us \( \Phi_{1k}(t) = \sum_{k'} \Phi_{1k'}(t)(M_0)_{k'k} \). It means that \( W \)-algebra itself does not account for the global structure of conformal blocks and this information should be introduced as an extra input.

Now suppose that we have some globally-defined multivalued functions \( \Phi_{1k} \). Then we have three monodromies \( M_0, M_1, M_{\infty} \) and one can solve the refined 3-point Riemann-Hilbert problem. Suppose that its solution is given by the matrix \( F(t) \) such that

\[
\frac{d}{dt} F(t) = \left( A_0 + A_1 \right) F(t), \tag{2.67}
\]

\( A_0 = \text{diag}((\theta_0, e_1), (\theta_0, e_2), (\theta_0, e_3)) \) and \( F(t) \) is normalized in such a way that \( F(t) = t^{A_0}(1 + O(t)) \). Next let us compute \( R_i(t) = \sum_k \Phi_{1k}(t)(F(t)^{-1})_{ki} \). This vector has the trivial monodromies around all singular points, it is regular there and \( R(0) = (1, 0, 0) \), so that \( R(t) = (1, 0, 0) \). It means that

\[
\Phi_{1k}(t) = F_{1k}(t). \tag{2.68}
\]

This formula allows us to fix all constants \( C_k \). This is done in the following way: we solve the 3-point Riemann-Hilbert problem, take \( F_{11}(t) \) and read the coefficients of conformal block from its series. These coefficients are triangular linear combinations (2.61) of \( C_k \) (i.e., \( k \)th term of the conformal block expansion involves only \( C_{j \leq k} \)). This construction thus expresses \( C_k \) via the moduli \( (\mu, \nu) \) of flat connections on the 3-punctured sphere.

\[
C_k = C_k(\mu, \nu). \tag{2.69}
\]

All constants are expressed in terms of only two parameters. If we now recall the 5th experimental property of the \( \tau \)-function, its origin can be easily understood: the first term of the conformal block (with the structure constants fixed above) depends only on \( C_1(\mu, \nu) \) and \( \tilde{C}_1(\mu, \nu) \) and this dependence is at most bilinear.

**Verlinde loop operators**

Here we can slightly modify our point of view: now all possible vertex operators defined by (2.3) and (2.4) have to be considered simultaneously. They form some \( \infty \)-dimensional vector space, which can be identified with the space of 3-point conformal
blocks (and which was one-dimensional in the Virasoro case). One can define the action of the Verlinde loop operators on this space in the same way as it was done in [CGTe]. This action is given by some operators $\hat{V}(\gamma)$ depending on the loop $\gamma$.

If we now look at the results of [ILTe] then we realize that (2.50) can be defined alternatively as the common eigenvector of all possible Verlinde loops. One can act in the same way for the case of 3-point conformal blocks

$$\hat{V}(\gamma) \cdot \langle Y|\phi_{\theta_1,\mu,\nu}(1)|Y'\rangle = M_{\gamma}(\mu, \nu) \cdot \langle Y|\phi_{\theta_1,\mu,\nu}(1)|Y'\rangle$$  \hspace{1cm} (2.70)

This procedure defines the basis of the “right” vertex operators $\phi_{\theta_1,\mu,\nu}(1)$ characterized by some $C_k(\mu, \nu)$. It looks more natural in this approach that $\tau$-function constructed from such operators should solve the Riemann-Hilbert problem.

The question about interpretation in $c \neq N - 1$ case is still open: the problem is caused by non-commutativity of the algebra of $\hat{V}(\gamma)$. Moreover, even in the minimal model-like cases $c = N - 1 - N(N^2 - 1)^{(k-1)/k}$ for integer $k$, when the algebra is commutative again, the relation to the isomonodromic deformations becomes unclear (see “concluding remarks” in [BShch] for discussion of the Virasoro case).

**Conclusions**

We have discovered several important properties of the isomonodromic $\tau$-functions in higher rank, which can be interpreted as signatures of the isomonodromy-CFT correspondence for the $W_N$ case. This allows to give a definition of the general $W_N$ conformal block, depending only on a finite number of parameters. It is also possible to prove [GavIL] that the algebraic way to define well-known conformal blocks for semi-degenerate fields agrees with the above definition.

We have also considered a particular conformal block with degenerate field and shown that its global structure is not fixed algebraically. The requirement of the correct global behavior of this object yields an expression for the whole infinite series of constants in the $W_3$ conformal block in terms of the solution of the 3-point Riemann-Hilbert problem.

These expressions can be written in terms of coordinates on the moduli space of flat connections on sphere with 3 punctures. This is expected to be universal and work for any conformal block (not only for those with degenerate fields). We have checked experimentally some properties, which support this conjecture.

Finally, let us list some remaining open problems:

- One needs to check that the procedure of fixing $C_k$ is self-consistent.
- If the constants $C_k$ can be fixed in such a way, we may try to prove that the $\tau$-function can be given as a sum of the general $W_N$ conformal blocks.
- A constructive solution of the 3-point Riemann-Hilbert problem is still missing.
- It would be interesting to understand the meaning of $Z\delta f(\theta_t, \sigma, \theta_0; \mu, \nu|\tilde{Y}, \tilde{Y}')$ in the context of isomonodromy-CFT correspondence. It can be done for the case $\tilde{Y}' = 0$ and it is interesting what happens for the arbitrary Young diagrams.
• There as an approach to the definition of conformal blocks of the light fields in the limit $c \to \infty$ [FR]. In that case explicit integral expression for the conformal block was derived. All the information about the 3-point functions enters this definition via several functions of one variable. The open problem is to obtain the monodromy properties of such conformal blocks and to identify the choice of 3-point functions that gives the conformal blocks arising in our approach.

• It is also important to understand the meaning of the results [BMPTY] about partition functions of $T_N$ theories without lagrangian description (which are believed to be the counterparts of the general $W_N$ 3-point functions) from the isomonodromic point of view.
Free fermions, W-algebras and isomonodromic deformations

Abstract

We consider the theory of multicomponent free massless fermions in two dimensions and use it for construction of representations of W-algebras at integer Virasoro central charges. We define the vertex operators in this theory in terms of solutions of the corresponding isomonodromy problem. We use this construction to get some new insights on tau-functions of the multicomponent Toda type hierarchies for the class of solutions, given by the isomonodromy vertex operators and get useful representation for the tau-function of isomonodromic deformations.

Introduction

The aim of the chapter is to present briefly the main free-fermionic constructions that appear in the study of correspondence between the problem of isomonodromic deformations and two-dimensional conformal field theories – for some class of the theories with extended conformal symmetry. An interest to the two-dimensional conformal field theories (CFT) with extended nonlinear symmetries, generated by the higher spin holomorphic currents, has been initiated by pioneering work [ZamW]. These theories with so called W-symmetry possess many features of ordinary CFT, including the free field representation [FZ, FL], which becomes especially simple for the case of integer Virasoro central charges. However, even in this relatively simple case it turns already to be impossible to construct in generic situation the W-conformal blocks [BW], which are the main ingredients of the conformal bootstrap definition of the physical correlation functions [BPZ].

This interest has been seriously supported already in our century by rather nontrivial correspondence between two-dimensional CFT and four-dimensional supersymmetric gauge theory [LMN, NO, AGT], where the conformal blocks have to be compared with the Nekrasov instanton partition functions [Nek, NP] producing in the quasiclassical limit the Seiberg-Witten prepotentials [SW]. This correspondence also meets serious problems beyond SU(2)/Virasoro level: both on four-dimensional gauge theory and two-dimensional CFT sides. These difficulties can be attacked using different approaches, for example in [GMtw] we have demonstrated how the
exact conformal blocks for the twist fields [ZamAT87, ZamAT86, ApiZam] in theories with W-symmetry can be computed, using the technique developed previously in [KriW, Mtau, GMqui].

Here we present another approach to the study of the CFT vertex operators in the theories with extended conformal symmetry, based on their free-fermionic construction. It is clear, that it should work (at least) in the cases of integral central charges, where it is intimately related with the recently discovered there CFT/isomonodromy correspondence [GIL12, Gav]. We are going to discuss the operator content of these theories with nontrivial monodromy properties, and then turn to the problem of computation of the matrix elements of generic monodromy operators. Finally, we are going to relate these matrix elements with the tau-functions of two different classes of problems – the tau-functions of the multicomponent classical integrable hierarchies of Toda type, and the tau-functions of the isomonodromic deformations.

Abelian \(U(1)\) theory

Fermions and vertex operators

Introduce the standard two-dimensional holomorphic fermionic fields with the action
\[
S = \frac{1}{\pi} \int_{\Sigma} d^2 z \bar{\psi} \partial \psi ,
\]
so that
\[
\tilde{\psi}(z) \psi(z') = \frac{1}{z - z'} + \ldots \tag{3.1}
\]
or
\[
\{ \psi_r, \psi_s \} = \delta_{r+s,0}, \quad r, s \in \mathbb{Z} + \frac{1}{2},
\]
\[
\psi(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{\psi_r}{z^{r+1/2}}, \quad \tilde{\psi}(z) = \sum_{s \in \mathbb{Z} + \frac{1}{2}} \frac{\tilde{\psi}_s}{z^{s+1/2}} \tag{3.2}
\]
with the half-integer mode expansion. The bosonization formulas read
\[
\tilde{\psi}(z) =: e^{i\phi(z)} := e^{\sum_{n<0} J_n z^n} e^{-\sum_{n>0} J_n z^n} e^{Q z J_0},
\]
\[
\psi(z) =: e^{-i\phi(z)} := \sum_{n<0} \frac{J_n}{n} z^n \sum_{n>0} \frac{J_n}{n} z^{-n} e^{-Q z} e^{-J_0}, \tag{3.3}
\]
where
\[
J(z) =: \tilde{\psi}(z) \psi(z) := i \partial \phi(z) = \sum_{n \in \mathbb{Z}} \frac{J_n}{z^{n+1}},
\]
\[
[J_n, J_m] = n \delta_{n+m,0}, \quad n, m \in \mathbb{Z}, \quad [J_n, Q] = \delta_{n0}, \tag{3.4}
\]
where normal ordering means, that all negative modes stand to the left of all positive, and all \(Q\) to the left of \(J_0\).

Consider now generic vertex operators for the bosonic fields
\[
V_\nu(z) =: e^{i \nu \phi(z)} := e^{-\nu \sum_{n<0} J_n z^n} e^{-\nu \sum_{n>0} J_n z^{-n}} e^{\nu Q z} e^{\nu J_0} \equiv V_\nu^-(z) V_\nu^+(z) e^{\nu Q z} e^{\nu J_0} \tag{3.5}
\]
which satisfy the obvious exchange relations, following from the Campbell-Hausdorff formula

\[ V_\alpha^+(z)V_\beta^-(w) = \left(1 - \frac{w}{z}\right)^{\alpha\beta} V_\beta^-(w)V_\alpha^+(z), \tag{3.6} \]
\[ V_\alpha(z)V_\beta(w) = \left(\frac{z}{w}\right)^{\alpha\beta} \left(1 - \frac{w}{z}\right)^{\alpha\beta} \left(1 - \frac{z}{w}\right)^{-\alpha\beta} V_\beta(w)V_\alpha(z). \]

One can also write

\[ V_\alpha(z)V_\beta(w) = (z - w)^{\alpha\beta} : V_\alpha(z)V_\beta(w) :. \tag{3.7} \]

Since vertex operators contain the factor \( e^{\nu Q} \), they shift the vacuum charge

\[ V_\nu(z) : \mathcal{H}^\sigma \to \mathcal{H}^{\sigma + \nu} \tag{3.8} \]

when acting onto a sector in full Hilbert space

\[ \mathcal{H} = \bigoplus_{\sigma} \mathcal{H}^\sigma \tag{3.9} \]

corresponding to the definite value of this charge. Notice that we do not impose any special constraints to the (real) values of the vacuum charges \( \sigma \in \mathbb{R} \).

The Hilbert space \( \mathcal{H}^\sigma \) is constructed by the action of the negative bosonic generators

\[ J_{-n_1} \ldots J_{-n_k} |\sigma\rangle \tag{3.10} \]
on the vacuum vector \( J_0 |\sigma\rangle = \sigma |\sigma\rangle \), and these states can be labeled by the Young diagrams with the row lengths \( n_1, \ldots, n_k \).

One can also construct the action of the fermionic operators on this vector space. Then the bosonization formulas (3.3) will generally produce the fractional powers in holomorphic coordinate \( z \) due to the factors \( z^0 \), while \( e^{\pm Q} \) just shift the vacuum charge by \( \pm 1 \). It means that one can define the (multiple) action of the modes of the operators

\[ \psi^\sigma(z) = \sum_r \frac{\psi_r^\sigma}{z^{r+1/2+\sigma}}, \quad \tilde{\psi}^\sigma(z) = \sum_r \frac{\tilde{\psi}_r^\sigma}{z^{r+1/2-\sigma}} \tag{3.11} \]
in the direct sum of the Hilbert spaces

\[ \mathcal{H}^\sigma = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n^\sigma \tag{3.12} \]
naturally labeled by some fractional \( \sigma \in \mathbb{R}/\mathbb{Z} \).

Basis in each space \( \mathcal{H}_n^\sigma \) can be given by the vectors generated by the zero-charge expressions of the fermionic modes. As in bosonic representation, these vectors can be labeled by the Young diagrams

\[ |Y,\sigma\rangle = \prod_i \tilde{\psi}_{-p_i}^\sigma \psi_{-q_i}^\sigma |\sigma\rangle \tag{3.13} \]

where now \( p_i \) and \( q_i \) are the Frobenius coordinates of the Young diagram. In our convention they are half-integer, and can be easily read of the following picture:
3. Free fermions, W-algebras and isomonodromic deformations

i.e. one has to cut the diagram by the main diagonal and just take the areas of the rows and columns starting from the diagonal cells. For example, the Young diagram from the picture has \( \{p_i\} = \{1, \frac{5}{2}, \frac{3}{2}\} \) and \( \{q_i\} = \{\frac{1}{2}, \frac{5}{2}, \frac{3}{2}\} \).

The states in the dual to \( \mathcal{H}^\sigma \) module can be obtained by the Hermitian conjugation

\[
\langle \sigma, Y \rangle = \langle \sigma | \prod_i \tilde{\psi}_{q_i}^* \psi_{p_i}^\sigma .
\]

\[(3.14)\]

Our main aim in what follows is to compute the matrix elements of the operator \( V_{\nu}(1) = V_\nu \) between the arbitrary fermionic states

\[
Z(\nu|Y', Y) = \langle \theta + \nu, Y'|V_{\nu}(1)|Y, \theta \rangle .
\]

\[(3.15)\]

The most straightforward way is to use explicit bosonic representation (3.5) of the vertex operator

\[
Z(\nu|Y', Y) = \langle \sigma + \nu | \prod_j \tilde{\psi}_{q_j}^* \psi_{p_j}^\nu V_{\nu}^+ e^{\nu q} \prod_i \tilde{\psi}_{-p_i}^\nu \psi_{-q_i} | \sigma \rangle =
\]

\[
= \langle 0 | \prod_j V_{-\nu}^+ \tilde{\psi}_{q_j}^* V_{-\nu}^- \cdot V_{-\nu}^- \psi_{p_j} \prod_i V_{\nu}^+ \tilde{\psi}_{-p_i}^\nu V_{\nu}^- \cdot V_{\nu}^+ \psi_{-q_i} V_{\nu}^- | 0 \rangle =
\]

\[
= \langle 0 | \prod_j (V_{\nu}^-)^{-1} \tilde{\psi}_{q_j}^* V_{\nu}^- \cdot (V_{\nu}^-)^{-1} \psi_{p_j} V_{\nu}^- \prod_i V_{\nu}^+ \tilde{\psi}_{-p_i} (V_{\nu}^+)^{-1} \cdot V_{\nu}^+ \psi_{-q_i} (V_{\nu}^+)^{-1} | 0 \rangle .
\]

\[(3.16)\]

It is easy to understand from (3.3) and (3.5) that the consequent triple products of operators in this formula can be considered as certain adjoint action, or just conjugations of the fermions, which turn under such action just into the linear combinations of themselves. At the level of generating functions it looks like

\[
V_{\nu}^+ \tilde{\psi}(z)(V_{\nu}^-)^{-1} = (1 - z)^\nu \tilde{\psi}(z) , \quad V_{\nu}^+ \psi(z)(V_{\nu}^+)^{-1} = (1 - z)^{-\nu} \psi(z) ,
\]

\[
(V_{\nu}^-)^{-1} \tilde{\psi}(z)V_{\nu}^- = \left(1 - \frac{1}{z}\right)^\nu \tilde{\psi}(z) , \quad (V_{\nu}^-)^{-1} \psi(z)V_{\nu}^- = \left(1 - \frac{1}{z}\right)^{-\nu} \psi(z) .
\]

\[(3.17)\]

or, more generally

\[
V_{\nu}(w)^{-1} \tilde{\psi}^{\sigma + \nu}(z)V_{\nu}(w) = \left(\frac{z}{w}\right)^\nu \exp \left(\nu \sum_{n \in \mathbb{Z}} \frac{1}{n^\nu} \right) \tilde{\psi}^\sigma(z) ,
\]

\[
V_{\nu}(w)^{-1} \psi^{\sigma + \nu}(z)V_{\nu}(w) = \left(\frac{z}{w}\right)^{-\nu} \exp \left(-\nu \sum_{n \in \mathbb{Z}} \frac{1}{n^\nu} \right) \psi^\sigma(z) ,
\]

\[(3.18)\]

where the formal series in the r.h.s. can be rewritten with the help of the Fourier transformation as

\[
\exp \left(\nu \sum_{n \in \mathbb{Z}} \frac{z^n}{n}\right) = \frac{\sin \pi \nu}{\pi} \sum_{k \in \mathbb{Z}} \frac{z^k}{k + \nu} .
\]

\[(3.19)\]
This is a particular case of transformations from \( GL(\infty) \), realized by

\[
\sum a_{rs} : \tilde{\psi}_r \psi_s \in \mathfrak{gl}(\infty), \quad a_{rs} \to \infty, \quad |r - s| \to \infty, \quad (3.20)
\]

moreover, corresponding to the situation, when \( a_{rs} = a_{r-s} \) (a well known example of such transformation is generated by the currents \( J_n = \sum_{r} : \tilde{\psi}_r \psi_{n-r} : \) from (3.4)). It is true in the most general case: if one computes any matrix elements of such operator, they always can be expressed in terms of those with only two extra fermion insertions, i.e. we do not need an explicit form of the operator \( V_{\nu} = V_{\nu}^{-} V_{\nu}^{+} \) – just the only fact of the adjoint action, and we are going to use this property in more complicated non Abelian situation below.

In particular, one can compute (3.16) first using the Wick theorem

\[
Z(\nu|Y', Y) = \det \left( \begin{array}{cc} \langle \sigma + \nu | \tilde{\psi}_{q'} \tilde{\psi}_{p'} V_{\nu} | \sigma \rangle & \langle \sigma + \nu | \tilde{\psi}_{q'} V_{\nu} \tilde{\psi}_{-q} | \sigma \rangle \\ -\langle \sigma + \nu | \tilde{\psi}_{q'} V_{\nu} \tilde{\psi}_{-p} | \sigma \rangle & \langle \sigma + \nu | V_{\nu} \tilde{\psi}_{-p} \tilde{\psi}_{-q} | \sigma \rangle \end{array} \right) = \det G_{\nu} \quad (3.21)
\]

and then to apply (3.17) to the matrix elements in (3.21).

**Matrix elements and Nekrasov functions**

The two-fermion matrix elements of the matrix \( G = G_{\nu} \) (its rows are labeled by \( \{x_{\nu}\} = \{q_{\nu}\} \cup \{-p_{\nu}\} \), whereas columns are labeled by \( \{y_{\nu}\} = \{p'_{\nu}\} \cup \{-q_{\nu}\} \), here we denote by \( p \) and \( q \) some positive half-integer numbers) are expressed as

\[
G(q', p') = \langle 0 | \psi_{q'} \tilde{\psi}_{p'} V_{\nu}^{-} | 0 \rangle = \sum_{m=0}^{q' - \frac{1}{2}} \frac{(\nu)^{m} (-\nu)_{p' + q' - m}}{m! (p' + q' - m)!},
\]

\[
G(-p, -q) = \langle 0 | V_{\nu}^{+} \tilde{\psi}_{-p} \tilde{\psi}_{-q} | 0 \rangle = \sum_{n=0}^{q - \frac{1}{2}} \frac{(-\nu)^{n} (\nu)_{p + q - n}}{n! (p + q - n)!}, \quad (3.22)
\]

\[
G(-p, p') = -\langle 0 | \tilde{\psi}_{p'} V_{\nu}^{+} V_{\nu}^{-} | 0 \rangle = -\sum_{m=0}^{p' - \frac{1}{2}} \frac{(-\nu)^{m} (\nu)_{m + p - p'}}{m! (m + p - p')!},
\]

\[
G(q', -q) = \langle 0 | \psi_{q'} V_{\nu}^{+} V_{\nu}^{-} \tilde{\psi}_{-q} | 0 \rangle = \sum_{n=0}^{q - \frac{1}{2}} \frac{(-\nu)^{n} (\nu)_{n + q' - q}}{n! (n + q' - q)!}.
\]

These expressions are easily computed, using adjoint action (3.17) for the components

\[
V_{\nu}^{+} \tilde{\psi}_{-p} (V_{\nu}^{+})^{-1} = \sum_{m=0}^{\infty} \frac{(-\nu)^{m}}{m!} \tilde{\psi}_{-p + m}, \quad V_{\nu}^{+} \tilde{\psi}_{-q} (V_{\nu}^{+})^{-1} = \sum_{m=0}^{\infty} \frac{(-\nu)^{m}}{m!} \tilde{\psi}_{-q + m}, \quad (3.23)
\]

\[
(V_{\nu}^{-})^{-1} \psi_{q} V_{\nu}^{-} = \sum_{m=0}^{\infty} \frac{(\nu)^{m}}{m!} \psi_{q - m}, \quad (V_{\nu}^{-})^{-1} \tilde{\psi}_{p} V_{\nu}^{-} = \sum_{m=0}^{\infty} \frac{(-\nu)^{m}}{m!} \tilde{\psi}_{p - m}.
\]
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with \((\nu)_m = \nu(\nu + 1) \ldots (\nu + m - 1)\), \((\nu)_0 = 1\), and there are explicit formulas for the sums in the r.h.s. of (3.22)

\[
\sum_{m=0}^{b} \frac{(\nu)_m}{m!} \frac{(-\nu)_{a-m}}{(a-m)!} = \frac{(\nu)_{b+1}(-\nu)_{a-b}}{\nu b!(a-b-1)!}
\]

and

\[
\sum_{m=0}^{b} \frac{(-\nu)_m}{m!} \frac{(\nu)_{a+m}}{(a+m)!} = \frac{(-\nu)_{b+1}(\nu)_{a+b+1}}{\nu(a+\nu)b!(a+b)!}
\]

which can be easily proven by induction. It allows to rewrite matrix elements (3.22) in the factorized form

\[
G(q', p') = \frac{1}{\nu(p' + q')(q' - \frac{1}{2})!(p' - \frac{1}{2})!},
\]

\[
G(-p, -q) = -\frac{1}{\nu(p + q)(p - \frac{1}{2})!(q - \frac{1}{2})!},
\]

\[
G(-p, p') = \frac{1}{\nu(p - p' + q')(p - \frac{1}{2})!(p' - \frac{1}{2})!},
\]

\[
G(q', -q) = -\frac{1}{\nu(q' - q + \nu)(q' - \frac{1}{2})!(q - \frac{1}{2})!}.
\]

The determinant from (3.21) can be therefore written as

\[
\det_{a,b} G(x_a, y_b) = \prod_{j} \frac{(-\nu)_{p'_j + \frac{1}{2}}(\nu)_{q'_j + \frac{1}{2}}}{\nu(p'_j - \frac{1}{2})!(q'_j - \frac{1}{2})!} \prod_{i} \frac{(\nu)_{p_i + \frac{1}{2}}(-\nu)_{q_i + \frac{1}{2}}}{\nu(p_i - \frac{1}{2})!(q_i - \frac{1}{2})!} \cdot \det_{a,b} \tilde{G}(\tilde{x}_a, \tilde{y}_b)
\]

where now for two new sets \(\{\tilde{x}_a\} = \{q'_j\} \cup \{-p_i - \nu\}, \{\tilde{y}_b\} = \{-p'_j\} \cup \{q_i - \nu\}\)

\[
\tilde{G}(\tilde{x}_a, \tilde{y}_b) = \frac{\text{sgn}(\tilde{x}_a \tilde{y}_b)}{\tilde{x}_a - \tilde{y}_b},
\]

and the corresponding determinant can be computed using the Cauchy determinant formula

\[
\det_{a,b} \frac{1}{\tilde{x}_a - \tilde{y}_b} = \prod_{a < b}(\tilde{x}_a - \tilde{x}_b) \prod_{a > b}(\tilde{y}_a - \tilde{y}_b),
\]

so one gets finally

\[
Z(\nu|Y', Y) = \pm \prod_{j} \frac{(-\nu)_{p'_j + \frac{1}{2}}(\nu)_{q'_j + \frac{1}{2}}}{\nu(p'_j - \frac{1}{2})!(q'_j - \frac{1}{2})!} \prod_{i} \frac{(\nu)_{p_i + \frac{1}{2}}(-\nu)_{q_i + \frac{1}{2}}}{\nu(p_i - \frac{1}{2})!(q_i - \frac{1}{2})!} \times \prod_{i > j} (p'_i - p'_j) \prod_{i < j} (p_i - p_j) \prod_{i > j} (q'_i - q'_j) \prod_{i < j} (q_i - q_j) \prod_{i,j} (q'_i + p_j + \nu) \prod_{i,j} (p'_i + q_j + \nu)
\]

\[
\prod_{i,j} (p'_i + q'_j) \prod_{i,j} (p_i + q_j) \prod_{i,j} (q'_i + q_j + \nu) \prod_{i,j} (p_i - p'_j + \nu)
\]

\[
(3.28)
\]

It is easy to see that this expression has the structure

\[
Z(\nu|Y', Y) = \pm \frac{Z_b(\nu|Y', Y)}{Z_0^3(Y') Z_0^3(Y)}
\]

(3.29)
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where

\[ Z_0^{\frac{1}{2}}(Y) = \prod_i (p_i - \frac{1}{2})! (q_i - \frac{1}{2})! \prod_{i<j} (p_i + q_j) \prod_{i<j} (p_i - p_j) \]  

(3.30)

while

\[ Z_b(\nu|Y', Y) = \prod_i \nu^{-1}(-\nu)_{\nu_i + \frac{1}{2}}(\nu)_{\nu_i + \frac{1}{2}} \prod_j \nu^{-1}(-\nu)_{\nu_j + \frac{1}{2}}(\nu)_{\nu_j + \frac{1}{2}} \times \]

\[ \times \prod_{i<j} (p_i' + p_j + \nu) \prod_{i<j} (p_i' - q_j + \nu) \prod_{i<j} (p_i' - p_j - \nu) \]  

(3.31)

In this normalization one can check that

\[ Z_b(\nu|Y', Y) = \pm \prod_{t \in Y} (1 + a_Y(t) + l_Y(t) + \nu) \prod_{s \in Y'} (1 + a_{Y'}(s) + l_Y(s) - \nu) \]  

(3.32)

is exactly the Nekrasov bi-fundamental function of the $U(1)$ gauge theory at $c = 1$ or $\epsilon_1 + \epsilon_2 = 0$. Notice also that

\[ Z_b(0|Y, Y) = Z_0^{\frac{1}{2}}(Y)Z_0^{\frac{1}{2}}(Y) = \prod_{s \in Y} (1 + a_Y(s) + l_Y(s))^2 = Z_{V}(Y)^{-1} \]  

(3.33)

is Nekrasov function for the pure $U(1)$ gauge theory, which corresponds to the Plancherel measure on partitions [LMN].

Riemann-Hilbert problem

The following simple observation is extremely important for our generalizations below. Consider the correlator

\[ \langle \theta|V_{\nu}(1)\tilde{\psi}^{\sigma}(z)\psi^{\sigma}(w)|\sigma \rangle = \delta_{\theta, \sigma + \nu} z^\sigma w^{-\sigma} (1 - z)^{\nu}(1 - w)^{-\nu} \]  

(3.34)

which is easily computed using bosonization rules (3.3). One finds then, that

\[ (z - w)\langle \theta|V_{\nu}(1)\tilde{\psi}^{\sigma}(z)\psi^{\sigma}(w)|\sigma \rangle = \phi(z)\phi(w)^{-1} \]  

(3.35)

is expressed actually through the solutions of a simple linear system

\[ \frac{d\phi(z)}{dz} = \phi(z) \left( \frac{\sigma}{z} + \frac{\nu}{z - 1} \right) \]  

(3.36)

It means that this linear system can be used to define all two-fermion matrix elements, e.g. in the region $1 > |z| > |w|$}

\[ \langle \theta|V_{\nu}(1)\tilde{\psi}^{\sigma}(z)\psi^{\sigma}(w)|\sigma \rangle = \sum_{p,q} \frac{1}{z^{p+\frac{1}{2}-\sigma}w^{q+\frac{1}{2}+\sigma}} \langle \theta|V_{\nu}(1)\tilde{\psi}^{\sigma}_{p}\psi^{\sigma}_{q}|\sigma \rangle \]  

(3.37)

and together with the Wick theorem it defines all matrix elements, or just the vertex operator $V_{\nu}$, uniquely – up to a numeric factor. In its turn the linear system itself is determined by the monodromy properties (here very simple) of $\phi(z)$ at $z = 0$ and $z = 1$ (and related to them monodromy at $z = \infty$). Hence, the problem of computation of the two-fermion correlation functions can be reformulated in terms of a Riemann-Hilbert problem.
3. Free fermions, W-algebras and isomonodromic deformations

Remarks

- Formulas (3.28), (3.31) give a very explicit representation for the matrix element and bi-fundamental Nekrasov function in terms of the Frobenius coordinates of the corresponding Young diagrams (this representation, for example, is far more adapted for practical computation, than the formulas (3.32)). However, it is sometimes not easy to see directly, that these formulas possess some nice properties: for example satisfy the “sum rules” like

\[
\sum_{Y} t^{|Y|} Z_a(\alpha_1 | \emptyset, Y) Z_a(\alpha_2 | Y, \emptyset) = \sum_{Y} t^{|Y|} Z_a(\alpha_1 | \emptyset, Y) Z_a(\alpha_2 | Y, \emptyset) = (1-t)^{\alpha_1 \alpha_2}
\]  

(3.38)

where the r.h.s. immediately follows from resolution of unity and the correlator of two exponentials

\[
\langle 0 | e^{i \alpha_1 \phi(1)} | Y, 0 \rangle \langle Y, 0 | e^{i \alpha_2 \phi(1)} | 0 \rangle = (1-t)^{\alpha_1 \alpha_2}
\]  

(3.39)

which is instructive to compare with the computation from [KKMST, Koz].

- One can also easily extract some useful information from particular cases of (3.34), which include a nice identity (cf. with [NO, BAW])

\[
\frac{(1-z)^\nu (1-w)^{-\nu}}{z-w} = \frac{1}{z-w} + \sum_{a,b=0}^{\infty} \frac{(-\nu)_{a+1}}{(a+b+1)\nu a! b!} z^a w^b
\]  

(3.40)

containing some part of the matrix elements from (3.25).

- According to (3.7)

\[
\tilde{\psi}(z + t/2) \psi(z - t/2) = \frac{1}{t} : \exp \left( \int_{z-t/2}^{z+t/2} J(\xi) d\xi \right) : \]  

(3.41)

Expansion into the powers of \(t\) gives the infinite series of the currents of \(W_{1+\infty}\) algebra

\[
: \tilde{\psi}(z + t/2) \psi(z - t/2) := \frac{1}{t} : \left( \exp \left( \int_{z-t/2}^{z+t/2} J(\xi) d\xi \right) - 1 \right) :=
\]  

(3.42)

where explicitly

\[
U_1(z) = J(z), \quad U_2 = \frac{1}{2} : J(z)^2 :, \quad U_3 = \frac{1}{3} \left( : J(z)^3 : + \frac{1}{4} \partial^2 J(z) \right), \quad \ldots
\]  

(3.43)

and one implies bosonic normal ordering for the bosons and fermionic for the fermions. These formulas have been used many times (see e.g. [Pogr, OP, LMN, NO, Mint]) to relate the generators of the \(W_{1+\infty}\) algebra with the fermionic bilinear operators, and we just recall them in order to generalize below to much less trivial non Abelian case.
Non-Abelian $U(N)$ theory

Nekrasov functions

Consider now more general case of Nekrasov functions, corresponding to the $U(N)$ non-Abelian theory. They can be expressed in terms of $U(1)$ functions (3.31), (3.32) by the following product formula

$$
\hat{Z}_b(\theta', \nu, \theta|Y', Y) = \prod_{\alpha,\beta=1}^{N} Z_b(\nu - \theta'_\alpha + \theta_\beta|Y'_\alpha, Y_\beta)
$$

(3.44)

For the diagonal elements $\hat{Z}_0(\theta|Y) = \hat{Z}_b(\theta, 0, \theta|Y, Y)$ one gets

$$
\hat{Z}_0(\theta|Y) = \prod_{\alpha,\beta=1}^{N} Z_b(-\theta_\alpha + \theta_\beta|Y'_\alpha, Y_\beta) = \pm \prod_{i<j} Z_b^2(-\theta_\alpha + \theta_\beta|Y'_\alpha, Y_\beta) \cdot \prod_{\alpha} Z_0(Y_\alpha)
$$

(3.45)

or, after taking the square root, just

$$
\hat{Z}_0^{\frac{1}{2}}(\theta|Y) = \prod_{\alpha<\beta} Z_b(-\theta_\alpha + \theta_\beta|Y'_\alpha, Y_\beta) \cdot \prod_{\alpha} Z_0^{\frac{1}{2}}(Y_\alpha)
$$

(3.46)

Now for simplicity it is better to replace $\theta'_\alpha - \nu \mapsto \theta'_\alpha$ or $\theta_\alpha + \nu \mapsto \theta_\alpha$, then $\nu$ simply disappears from (3.44). Consider now the normalized matrix element

$$
\hat{Z}(\theta', \theta|Y', Y) = \frac{\hat{Z}_b(\theta', \theta|Y', Y)}{\hat{Z}_0^{\frac{1}{2}}(\theta'|Y') \hat{Z}_0^{\frac{1}{2}}(\theta|Y)} = \\
= \prod_{\alpha,\beta=1}^{N} Z_b(-\theta'_\alpha + \theta_\beta|Y'_\alpha, Y_\beta) \prod_{\alpha<\beta} Z_b(-\theta'_\alpha + \theta'_\beta|Y'_\alpha, Y'_\beta) \cdot \prod_{\alpha} Z_0^{\frac{1}{2}}(Y_\alpha)
$$

(3.47)

Using representation (3.28), (3.31) for the $U(1)$ functions in terms of the Frobenius coordinates, one finds that the ratio of products of the elementary Cauchy determinants from there is actually combined into more sophisticated unique Cauchy determinant

$$
\hat{Z}(\theta', \theta|Y', Y) = \frac{1}{\prod_{I,J} x_I - y_J} \times \\
\prod_{I \alpha} f_{I,\alpha}(\theta', \theta, p_{\alpha,i}) f_{2,\alpha}(\theta', \theta, q_{\alpha,i}) \prod_{I \alpha} f_{I,\alpha}(\theta, \theta', p_{\alpha,i}) f_{2,\alpha}(\theta, \theta', q_{\alpha,i})
$$

(3.48)

with two multi-sets of variables entering the determinant of the form

$$
\{x_I\} = \{-q'_{\alpha,i} - \theta'_\alpha\} \cup \{p_{\alpha,i} - \theta_\alpha\} \\
\{y_I\} = \{p'_{\alpha,i} - \theta'_\alpha\} \cup \{-q_{\alpha,i} - \theta_\alpha\}
$$

(3.49)
up to quite nontrivial diagonal part, which can be still read from (3.28) and (3.47),
giving the following factors for (3.48)

\[
f_1,\alpha(\theta, \theta', p_{\alpha,i}) = \frac{1}{(p_{\alpha,i} - \frac{1}{2})!} \prod_\beta \frac{(\theta'_\beta - \theta_\alpha)_{p_{\alpha,i} + \frac{1}{2}}}{\sqrt{\theta'_\beta - \theta_\alpha}} \prod_{\beta \neq \alpha} \frac{\sqrt{\theta_\beta - \theta_\alpha}}{(\theta_\beta - \theta_\alpha)_{p_{\alpha,i} + \frac{1}{2}}}
\]

\[
f_2,\alpha(\theta, \theta', q_{\alpha,i}) = \frac{1}{(q_{\alpha,i} - \frac{1}{2})!} \prod_\beta \frac{(\theta_\alpha - \theta'_\beta)_{q_{\alpha,i} + \frac{1}{2}}}{\sqrt{\theta_\alpha - \theta'_\beta}} \prod_{\beta \neq \alpha} \frac{\sqrt{\theta_\alpha - \theta_\beta}}{(\theta_\alpha - \theta_\beta)_{q_{\alpha,i} + \frac{1}{2}}}
\]

Existence of the determinant formula (3.48) is very important, since it actually implies
that Nekrasov functions \( \hat{Z}(\theta', \theta|Y', Y) \) can be identified with the matrix elements of
some vertex operator, characterized as in the Abelian \( U(1) \) case by its adjoint action,
which is still a linear transformation but now of the \( N \)-component fermions. We
are going indeed to introduce this vertex operator below using the theory of \( (N- \)component) free fermions, generalizing the Abelian case considered above. In general
situation this operator is characterized by solution to auxiliary linear problem on
sphere with three marked points, while explicit formulas of this section just correspond
to particular case of the hypergeometric-type solutions.

**N-component free fermions**

Hence, consider the generalization of the free-fermionic construction from \( U(1) \) to the
non-Abelian \( U(N) \) case. First, introduce the algebra

\[
\{ \psi_{\alpha,r}, \psi_{\beta,s} \} = 0, \quad \{ \tilde{\psi}_{\alpha,r}, \tilde{\psi}_{\beta,s} \} = 0,
\]

\[
\{ \tilde{\psi}_{\alpha,r}, \tilde{\psi}_{\beta,s} \} = \delta_{\alpha,\beta} \delta_{r+s,0}
\]

\[
r, s \in \mathbb{Z} + \frac{1}{2}, \quad \alpha, \beta = 1, \ldots, N
\]

of the canonical anticommutation relations for the components of the fermionic fields
with free first-order action \( S = \frac{1}{\pi} \sum_{\alpha=1}^{N} \int_{\Sigma} d^2z \tilde{\psi}_{\alpha} \partial \psi_{\alpha} \), so that (3.51) are equivalent to the operator product expansions

\[
\tilde{\psi}_{\alpha}(z)\psi_{\beta}(w) = \frac{\delta_{\alpha\beta}}{z-w} + J_{\alpha\beta}(w) + O(z-w)
\]

\[
\psi_{\alpha}(z)\psi_{\beta}(w) = \text{reg.} \quad \tilde{\psi}_{\alpha}(z)\tilde{\psi}_{\beta}(w) = \text{reg.}
\]

Similarly to (3.3) it is also possible and useful to introduce the bosonization formulas
for these fermionic fields

\[
\tilde{\psi}_{\alpha}(z) = \exp \left( -\sum_{n<0} J_{\alpha,n} \frac{z^n}{n!} \right) \exp \left( -\sum_{n>0} J_{\alpha,n} \frac{z^{-n}}{n!} \right) e^{Q_{\alpha} z^{J_{0,0}\epsilon_\alpha}(J_0)}
\]

\[
\psi_{\alpha}(z) = \exp \left( \sum_{n<0} J_{\alpha,n} \frac{z^n}{n!} \right) \exp \left( \sum_{n>0} J_{\alpha,n} \frac{z^{-n}}{n!} \right) e^{-Q_{\alpha} z^{J_{0,0}\epsilon_\alpha}(J_0)}
\]

Here \( J_{\alpha,n} \) form the Heisenberg algebra

\[
[J_{\alpha,n}, J_{\beta,m}] = n\delta_{\alpha\beta} \delta_{m+n,0}, \quad [J_{0,\alpha}, Q_{\beta}] = \delta_{\alpha\beta}
\]
and $\epsilon_{\alpha}(J_0) = \prod_{\beta=1}^{\alpha-1} (-1)^{J_0,\beta}$, we may also note that $\epsilon_{\alpha}(x + y) = \epsilon_{\alpha}(x)\epsilon_{\beta}(y)$. These extra sign factors do the same as the Jordan-Wigner transformation: they convert commuting objects into the anticommuting ones.

A standard representation of this algebra $\mathcal{H}^\sigma$ is constructed from the vacuum vector $|\sigma\rangle$, with the charges $J_0|\sigma\rangle = \sigma|\sigma\rangle$ and killed by all positive modes

$$\psi_{\alpha,r}^\sigma \neq 0, \quad \tilde{\psi}_{\alpha,-r}^\sigma \neq 0.$$ (3.55)

Basis vectors of this representation can be given by

$$|\{p_{\alpha,i}\}, \{q_{\alpha,i}\}, \sigma\rangle = \prod_{\alpha=1}^{N} \left( \prod_{i=1}^{\#p_{\alpha,i}} \psi_{\alpha,-p_{\alpha,i}} \prod_{j=1}^{\#q_{\alpha,i}} \psi_{\alpha,-q_{\alpha,i}} \right) |\sigma\rangle.$$ (3.56)

The letters $p_{\alpha,i}$ and $q_{\alpha,i}$, at least in the case when $\#p_{\alpha} = \#q_{\alpha}$, should be interpreted as Frobenius coordinates of the $N$-tuple of the Young diagrams. It will be also convenient in what follows to use the vacuum-shifting operators $P_n^\alpha$

$$P_0^\alpha = 1, \quad P_n^{\alpha<0} = \psi_{\alpha,n+\frac{1}{2}}^\sigma \psi_{\alpha,n+\frac{3}{2}}^\sigma \cdots \psi_{\alpha,-\frac{1}{2}}^\sigma |\sigma\rangle,$$

and the corresponding states

$$|n, \sigma\rangle = \prod_{\alpha=1}^{N} P_n^\alpha |\sigma\rangle.$$ (3.58)

in particular for the vectors $n = \pm 1$ with components $n_\alpha = \pm \delta_{\alpha\beta}$.

### Level one Kac-Moody and W-algebras

Consider the W-algebras for $\mathfrak{g} = \mathfrak{sl}(N)$ series, possibly extended to $\mathfrak{gl}(N)$ where we shall call it $W_N \oplus H$. Their generators in current representation can be identified with the symmetric functions of the normally ordered currents $J(z) \in \mathfrak{h} \subset \mathfrak{g}$ with the values in Cartan subalgebra, or equivalently, up to a coefficient, as certain “Casimir elements” in the universal enveloping $U(\mathfrak{sl}(N)_1)$. The Virasoro central charge at level $k = 1$ is

$$c = \frac{k \dim \mathfrak{g}}{k + C_V} = \frac{N^2 - 1}{1 + N} = N - 1.$$ (3.59)

When embedded to $U(\mathfrak{gl}(N)_1)$ this current algebra has nice representation in terms of the multi-component free holomorphic fermionic fields

$$J_{\alpha\beta}(z) = :\bar{\psi}_{\alpha}(z)\psi_{\beta}(z):, \quad \alpha, \beta = 1, \ldots, N.$$ (3.60)

The $W_N$-algebra can be defined in terms of invariant Casimir polynomials of the currents, commuting with the screening charges $Q_{\alpha\beta} = \not\!J_{\alpha\beta}(z)$ (it is enough to require commutativity only with those, corresponding to the positive simple roots).
Then the W-generators turn to be just the symmetric polynomials of the diagonal Cartan currents \( J_\alpha = J_{\alpha\alpha}(z) \in \mathfrak{h} \), i.e.

\[
W_n(z) = \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_n} : J_{\alpha_1}(z) J_{\alpha_2}(z) \ldots J_{\alpha_n}(z) : , \quad n = 1, \ldots, N \tag{3.61}
\]

One can consider the representations of \( U(\widehat{\mathfrak{gl}(N)}_1) \) and \( W_N \oplus H \) in \( \mathcal{H}^\sigma \). For this purpose it is convenient to introduce the generating functions

\[
\varphi_\sigma(z) = \sum_{r \in \mathbb{Z}^+} \frac{\varphi_{\sigma,r} z^r}{r!} , \quad \tilde{\varphi}_\sigma(z) = \sum_{r \in \mathbb{Z}^+} \frac{\tilde{\varphi}_{\sigma,r} z^r}{r!}. \tag{3.62}
\]

where shifts of the powers of the coordinate \( z \) come naturally, e.g. from the bosonization formulas (3.53). For these fields instead of (3.52) one gets

\[
\tilde{\varphi}_\sigma(z) \varphi_\beta(w) = \delta_{\alpha \beta} \frac{z^{\sigma_\alpha} w^{-\sigma_\alpha}}{z - w} + : \tilde{\varphi}_\sigma(z) \varphi_\beta(w) : = \delta_{\alpha \beta} \frac{\sigma_\alpha}{w} + : \varphi_\alpha(w) \varphi_\beta(w) : + O(z - w), \tag{3.63}
\]

then it is clear that the modes of \( J_{\alpha \beta}(w) \) from (3.52) acquire in this representation the form

\[
J_{\alpha \beta,n} = \delta_{\alpha \beta} \delta_{n,0} \sigma_\alpha + \sum_{p \in \mathbb{Z}^+} : \varphi_{\alpha,n-p} \varphi_{\beta,p} :. \tag{3.64}
\]

As in the \( U(1) \) case (see (3.42)) in the fermionic realization of \( W_N \oplus H \), then one can choose the set of generators in a form of the fermionic bilinears:

\[
\sum_\alpha \tilde{\varphi}_\sigma(z + \frac{t}{2}) \varphi_\alpha(z - \frac{t}{2}) = \frac{N}{t} \sum_{k=1}^\infty \frac{t^{k-1}}{(k-1)!} U_k(z). \tag{3.65}
\]

The l.h.s. of this formula gives

\[
\tilde{\varphi}_\sigma(z + \frac{t}{2}) \varphi_\alpha(z - \frac{t}{2}) = \frac{1}{t} \left( \frac{1 + \frac{t}{2z}}{1 - \frac{t}{2z}} \right)^{\sigma_\alpha} + \\
+ \frac{1}{t} \sum_{m \in \mathbb{Z}} \frac{1}{z^m (1 + \frac{t}{2z})^{m+1}} \sum_{p \in \mathbb{Z}^+} \left( \frac{1 + \frac{t}{2z}}{1 - \frac{t}{2z}} \right)^{p+\frac{1}{2} + \sigma_\alpha} : \tilde{\varphi}_{m-p,\alpha} \varphi_{\alpha,p} :. \tag{3.66}
\]
Introducing two collections of polynomials

\[ \left( \frac{1 + \frac{x}{2}}{1 - \frac{x}{2}} \right)^p = 1 + \sum_{k=0}^{\infty} u_k(p) \frac{x^k}{(k-1)!}, \]

\[ \frac{x}{(1 + \frac{x}{2})^{m+1}} \left( \frac{1 + \frac{x}{2}}{1 - \frac{x}{2}} \right)^{p+\frac{1}{2}} = \sum_{k=1}^{\infty} v_{k,m}(p) \frac{x^k}{(k-1)!}, \]

the generators in the r.h.s. of (3.65) explicitly become

\[ U_{k,m} = \sum_{\alpha} \left( \delta_{m,0} u_k(\sigma_\alpha) + \sum_{p \in \mathbb{Z}} v_{k,m}(p + \sigma_\alpha) : \tilde{\psi}_{\alpha,m-p}^{\sigma} \psi_{\alpha}^{\sigma} : \right). \]

This set of generators of \( W_N \oplus H \) contains commuting zero modes \( U_{k,0}^{\sigma} \) which were shown to play an important role in the study of the extended Seiberg-Witten theory and AGT correspondence [LMN, MN, Mint, FLitv12]. It is also important to notice that commutation relations between these generators are linear, the only place when the non-linearity appears are the relations between these generators.

Using the bosonization rules (3.53) one can rewrite these generators in the conventional form. To perform explicit splitting of this algebra into \( W_N \oplus H \) it is convenient to redefine \( J_{\alpha}(z) \mapsto J_{\alpha}(z) + j(z) \), where the new currents already satisfy the condition \( \sum J_{\alpha} = 0 \) and the operator product expansions (OPE)

\[ j(z)j(w) = \frac{1}{(z-w)^2} + \text{reg.} \quad J_{\alpha}(z)J_{\beta}(w) = \frac{\delta_{\alpha\beta} - \frac{1}{N}}{(z-w)^2} + \text{reg.} \]

Now we take the bilinear expression

\[ \sum_{\alpha} \tilde{\psi}_{\alpha}(z + \frac{t}{2}) \psi_{\alpha}(z - \frac{t}{2}) = \sum_{\alpha} : e^{i\varphi(z + \frac{t}{2}) + i\phi_\alpha(z + \frac{t}{2})} : e^{-i\varphi(z - \frac{t}{2}) - i\phi_\alpha(z - \frac{t}{2})} : =
\]

\[ = \frac{1}{t} : e^{i\varphi(z + \frac{t}{2}) - i\varphi(z - \frac{t}{2})} : \sum_{\alpha} : e^{i\phi_\alpha(z + \frac{t}{2}) - i\phi_\alpha(z - \frac{t}{2})} : \]
3. Free fermions, W-algebras and isomonodromic deformations

with (3.65) we get the following formulas:

\[ U_1(z) = N j(z), \quad U_2(z) = T(z) + \frac{N}{2} : j^2(z) :, \]
\[ U_3(z) = W_3(z) + 2N T(z) j(z) + \frac{N}{3} \left( : j^3(z) : + \frac{1}{4} \partial^2 j(z) \right), \]
\[ U_4(z) = -W_4(z) + \frac{1}{2} (TT)(z) + 3W_3(z) j(z) + 3 : j^2(z) : T(z) + \]
\[ + \frac{N}{4} \left( : j^4(z) : + : j(z) \partial^2 j(z) : \right), \quad U_5(z) = \ldots \]

where \( T(z) = -W_2(z) \) is the stress-energy tensor, and \( (AB)(z) \) is the “interacting” normal ordering

\[ (AB)(z) = \oint_z \frac{dw}{w-z} A(w) B(z) \]

One find therefore, that one basis is related with the other by some complicated, though explicit and triangular transformation. Here we can see that generators \( U_k(z) \) are actually dependent, namely, if \( N = 3 \), then \( W_4(z) = 0 \) and \( U_4(z) \) becomes some non-linear expression of the lower generators.

It is also easy to see that for the states (3.58)

\[ J_{\alpha,0}^\sigma |n, \sigma\rangle = (\sigma_\alpha + n_\alpha) |n, \sigma\rangle, \quad U_{k,0}^\sigma |n, \sigma\rangle = u_k(\sigma + n) |n, \sigma\rangle, \]
\[ U_{k,m>0}^\sigma |n, \sigma\rangle = 0. \tag{3.73} \]

It is sometimes useful to decompose the whole Hilbert space into the sectors \( \mathcal{H}_n^\sigma = \bigoplus_{\sum n = l} \mathcal{H}_n^\sigma \) with fixed \( h \in \mathfrak{gl}(N) \) charges and also into the sectors \( \mathcal{H}_{\mathfrak{gl}}^\sigma = \bigoplus_{\sum n = l} \mathcal{H}_{\mathfrak{gl}}^\sigma \) with fixed overall \( u(1) = \mathfrak{gl}(1) \) charge. Summarizing all these facts we can formulate the following

**Theorem 3.1.** Spaces \( \mathcal{H}_{\mathfrak{gl}}^\sigma \) are representations of \( \mathfrak{gl}(N)_1 \), and for general \( \sigma \) spaces \( \mathcal{H}_n^\sigma \) are the Verma modules of \( W_N \oplus H \) algebra with the highest weight vectors \( |\sigma, n\rangle \) and with basis vectors \( |Y, n, \sigma\rangle, \forall Y \).

**Proof** is extremely simple: \( \mathfrak{gl}(N)_1 \) generators have zero fermionic \( \mathfrak{gl}_1 \)-charge, \( W_N \oplus H \) generators have zero charges with respect to the whole Cartan subalgebra \( \mathfrak{h} \), so the spaces \( \mathcal{H}_{\mathfrak{gl}}^\sigma \) and \( \mathcal{H}_n^\sigma \) are closed under the action of these algebras. We also know from (3.73) that \( |\sigma, n\rangle \) are the highest weight vectors of \( W_N \oplus H \), so we have a non-zero map from the Verma module to \( \mathcal{H}_n^\sigma \), but this Verma module is generally irreducible and has the same character \( \text{tr} q^{L_0} \), so we actually have an isomorphism. \( \square \)

**Free fermions and representations of W-algebras**

Let us now illustrate how can free fermions appear in the theory with \( W_N \)-symmetry at integer central charges after inclusion of extra Heisenberg algebra. Construction below is a straightforward generalization of the bosonization procedure from [ILTe].

It is well-known [FZ, FL] that conformal theory with \( W_N \)-symmetry contains two degenerate fields \( V_{\mu_1}(z) \) and \( V_{\mu_{N-1}}(z) \), such that their \( W \)-charges are determined by
the highest weights of the fundamental ($N$) and antifundamental ($\bar{N}$) representations, respectively. Their dimensions are

$$\Delta(\mu_i) = \frac{1}{2} \mu_i^2 = \frac{N - 1}{2N}, \quad i = 1, N - 1$$

and they have the following fusion rules with arbitrary primary field

$$[\mu_1] \otimes [\sigma] = \bigoplus_{\alpha=1}^{N} [\sigma + e_\alpha]$$
$$[\mu_{N-1}] \otimes [\sigma] = \bigoplus_{\alpha=1}^{N} [\sigma - e_\alpha]$$

(3.75)

where $\{\pm e_\beta\}$ is the set of all weights of $N$ and $\bar{N}$. One can define now the vertex operators

$$\Psi_\alpha(z) = \sum_{\sigma} P_{\sigma + e_\alpha} V_{\mu_1}(z) P_\sigma, \quad \tilde{\Psi}_\alpha(z) = \sum_{\sigma} P_{\sigma - e_\alpha} V_{\mu_{N-1}}(z) P_\sigma$$

(3.76)

which, due to extra projector operators, act only from one Verma module to another, just extracting the corresponding term from the fusion rules (3.75). Using the general structure of the OPE of two initial degenerate fields

$$V_{\mu_1}(z)V_{\mu_{N-1}}(w) = \left( 1 \cdot (z-w)^{1/N} + \#(z-w)^{1+2/N} T(w) \right) + \sum_{\beta} c_{\alpha \beta} V_\beta(w) + \ldots$$

(3.77)

one finds, that $\tilde{\Psi}_\alpha(z) \Psi_\beta(w) = \delta_{\alpha \beta} 1 \cdot (z-w)^{1/N} + \text{reg.}$, i.e. these fields look almost like fermions, except for the wrong power in the OPE. To fix this let us add an extra scalar field $\phi(z)$, such that

$$\phi(z)\phi(w) = -\frac{1}{N} \log(z-w) + \ldots$$

(3.78)

and define the new, the true fermionic, vertex operators

$$\psi_\alpha(z) = e^{-i\phi(z)} \Psi_\alpha(z), \quad \tilde{\psi}_\alpha(z) = e^{i\phi(z)} \tilde{\Psi}_\alpha(z), \quad \alpha = 1, \ldots, N$$

(3.79)

which have the canonical OPE (cf. with (3.52))

$$\psi_\alpha(z) \psi_\beta(w) = \frac{\delta_{\alpha \beta}}{z-w} + \text{reg.}$$
$$\psi_\alpha(z) \tilde{\psi}_\beta(w) = \text{reg.}$$
$$\tilde{\psi}_\alpha(z) \tilde{\psi}_\beta(w) = \text{reg.}$$

(3.80)

The rest is to understand, how to express the W-algebra generators in terms of these free fermions. One can easily write for the structure of the sum

$$\sum_{\alpha} \tilde{\Psi}_\alpha(z) \Psi_\alpha(w) = \frac{1}{z-w} + \#(z-w)(L_{-2}1)(w) + \#(z-w)^2 \cdot (L_{-1}L_{-2}1) + \#(z-w)^2 \cdot (W_{-3}1)(w) + \ldots$$

(3.81)

which have the canonical OPE (cf. with (3.52))

$$\psi_\alpha(z) \psi_\beta(w) = \frac{\delta_{\alpha \beta}}{z-w} + \text{reg.}$$
$$\psi_\alpha(z) \tilde{\psi}_\beta(w) = \text{reg.}$$
$$\tilde{\psi}_\alpha(z) \tilde{\psi}_\beta(w) = \text{reg.}$$

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(3.81)

which have the canonical OPE (cf. with (3.52))

$$\psi_\alpha(z) \psi_\beta(w) = \frac{\delta_{\alpha \beta}}{z-w} + \text{reg.}$$
$$\psi_\alpha(z) \tilde{\psi}_\beta(w) = \text{reg.}$$
$$\tilde{\psi}_\alpha(z) \tilde{\psi}_\beta(w) = \text{reg.}$$

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$$\sum_{\alpha} \tilde{\Psi}_\alpha(z) \Psi_\alpha(w) = \frac{1}{z-w} + \#(z-w)(L_{-2}1)(w) + \#(z-w)^2 \cdot (L_{-1}L_{-2}1) + \#(z-w)^2 \cdot (W_{-3}1)(w) + \ldots$$

(3.81)
with some coefficients (and where we have used obvious notations for the descendants).
We do not need their exact numeric values at the moment, just the very fact that only the unit operator $1$ enters the r.h.s. of this OPE together with its descendants. Using additionally the OPE of the $U(1)$ factors

$$(z - w)^{1/N} e^{i\phi(z)} e^{-i\phi(w)} =$$

$$=: \exp \left( i(z - w) \partial \phi(w) + \frac{1}{2}(z - w)^2 \partial^2 \phi(w) + \frac{1}{6}(z - w)^3 \partial^3 \phi(w) \right) =:$$

$$= 1 + (z - w) j(w) + \frac{1}{2}(z - w)^2 \partial j(w) + \frac{1}{6}(z - w)^3 \partial^2 j(w) +$$

$$+ \frac{1}{2}(z - w)^2 : j(w)^2 : + \frac{1}{2}(z - w)^3 : j(w) \partial j(w) : + \frac{1}{6}(z - w)^3 j(w)^3 + \ldots$$

one can get

$$\sum_{\alpha} \tilde{\psi}_\alpha(z) \psi_\alpha(w) = \frac{1}{z - w} + j(w) + (z - w) \left( \# \cdot T(w) + \frac{1}{2} j(w) + \frac{1}{2} : j(w)^2 : \right) +$$

$$+ (z - w)^2 \left( \# \cdot W(w) + \# \cdot j(w) T(w) + \frac{1}{6} \partial^2 j(w) + \frac{1}{2} : j(w) \partial j(w) : + \frac{1}{6} : j(w)^3 : \right) + \ldots$$

(3.83)

This formula states, how the standard $W$-generators can be expressed via the fermionic bilinears by some triangular transformation, and its symmetric form is equivalent to (3.71), (3.72).

**Vertex operators and Riemann-Hilbert problem**

**Vertex operators and monodromies**

Let us now turn to general construction of the monodromy vertex operator $^2$

$$V_\nu(t) : \mathcal{H}^\sigma \rightarrow \mathcal{H}^\theta \quad \text{(3.84)}$$

Actually one can define only the operator $V_\nu(1)$ due to conformal Ward identity

$$V_\nu(t) = t^{-\Delta_\nu} t^{L_0} V_\nu(1) t^{-L_0} \quad \text{(3.85)}$$

and the operator $V_\nu(1)$ is defined by the following three properties:

- $V_\nu(1)$ is a (quasi)-group element, i.e.

$$V_\nu(1) \mathcal{H}^\sigma (V_\nu(1))^{-1} \subseteq \mathcal{H}^\theta, \quad (V_\nu(1))^{-1} \mathcal{H}^\theta V_\nu(1) \subseteq \mathcal{H}^\sigma$$

As we discussed already in sect. 3.2 this fact actually implies that all correlators of fermions in the presence of such an operator can be computed using the Wick theorem.

$^2$Notice, that we have here only the conservation of the “total charge” $\sum_\alpha \sigma_\alpha + \sum_\alpha \nu_\alpha = \sum_\alpha \theta_\alpha$, and apart of that their values are arbitrary.
• \(\langle \theta | V_\nu(1) | \sigma \rangle = 1\), which is a kind of convenient normalization. Notice, however, that vertex operator is defined by the adjoint action only up to some diagonal factor \(S = \exp(\beta)\), \(\beta \in \mathfrak{h} \subset \mathfrak{gl}(N)\). In what follows we shall restore these diagonal factors when necessary.

• All two-fermionic correlators give the solution for the 3-point Riemann-Hilbert problem in the different regions

\[
\begin{align*}
\langle \theta | V_\nu(1) \tilde{\psi}_\alpha^\sigma(z) \psi_\beta^\sigma(w) | \sigma \rangle &= K_{\alpha\beta}(z, w), \quad |z| \leq 1, |w| \leq 1 \\
\langle \theta | \tilde{\psi}_\alpha^\sigma(z) \psi_\beta^\sigma(w) | V_\nu(1) | \sigma \rangle &= K_{\alpha\beta}(z, w), \quad |z| \geq 1, |w| \geq 1 \\
\langle \theta | \tilde{\psi}_\alpha^\sigma(z) V_\nu(1) \psi_\beta^\sigma(w) | \sigma \rangle &= K_{\alpha\beta}(z, w), \quad |z| \geq 1, |w| \leq 1 \\
-\langle \theta | \psi_\beta^\sigma(w) V_\nu(1) \tilde{\psi}_\alpha^\sigma(z) | \sigma \rangle &= K_{\alpha\beta}(z, w), \quad |z| \leq 1, |w| \geq 1
\end{align*}
\] (3.86)

In terms of some matrix kernels \(K(z, w) = K^\nu(z, w)\), where we have used \(\{\alpha, \beta\}\) and \(\{\tilde{\alpha}, \tilde{\beta}\}\) to denote matrix indices, corresponding to different bases, associated with the points \(z = 0\) and \(z = \infty\) respectively.

By this moment the only claim is that this operator is uniquely defined by th properties listed above, and this follows from the fact, that all matrix elements of the quasi-group \(V_\nu(1)\) element are given by certain determinants of the matrices with the entries, constructed from \(K(z, w)\). Existence of this operator is therefore obvious, since one can compute all its matrix elements using the Wick theorem.

Now, we would like to specify the kernels \(K(z, w)\) first by their monodromy properties. We associate the basis at \(z = 0\) with the eigenvectors of \(M_0 \sim e^{2\pi i \sigma}\), while the basis at \(z = \infty\) with the eigenvectors of \(M_\infty \sim e^{2\pi i \theta}\) (only the conjugacy classes of these two matrices are fixed, and certainly in general \([M_0, M_\infty] \neq 0\) ). We propose an explicit form of the kernel

\[
K_{\alpha\beta}(z, w) = \frac{[\phi(z)\phi(w)^{-1}]_{\alpha\beta}}{z - w}
\] (3.87)
given in terms of the solution to the linear system

\[
\frac{d}{dz} \phi(z) = \phi(z) \left( \frac{A_0}{z} + \frac{A_1}{z - 1} \right) = \phi(z) A(z)
\] (3.88)

with \(A_0 \sim \sigma\), \(A_1 \sim \nu\), \(A_\infty \sim \theta\) and prescribed monodromies

\[
\gamma_0 : \phi_{\alpha i}(z) \mapsto \sum_\beta (M_0)_{\alpha\beta} \phi(z)_{\beta i}
\]

\[
\gamma_\infty : \phi_{\alpha i}(z) \mapsto \sum_\beta (M_\infty)_{\alpha\beta} \phi(z)_{\beta i}
\] (3.89)

also implying monodromy around \(z = 1\), i.e. \(\gamma_1 : \phi_{\alpha i}(z) \mapsto \sum_\beta (M_1)_{\alpha\beta} \phi(z)_{\beta i}\), with \(M_1 \sim e^{2\pi i \nu}\) and \(M_0 M_1 M_\infty = 1\). Solutions for a linear system (3.88) can be expressed themselves in terms of a fermionic correlators, namely

\[
\begin{align*}
\phi_{\alpha \gamma}(z) &= z \cdot \langle \theta | V_\nu(1) \tilde{\psi}_\alpha(z) | 1, \gamma, \sigma \rangle \\
\phi^{-1}_{\gamma \beta}(z) &= z \cdot \langle \theta | V_\nu(1) \psi_\beta(z) | 1, \gamma, \sigma \rangle
\end{align*}
\] (3.90)
3. Free fermions, W-algebras and isomonodromic deformations

for some fixed normalization at \( z \to 0 \). We are going to prove in next section, that definitions (3.87) are indeed self-consistent and also consistent with (3.90), which follows from the generalized Hirota bilinear relations, satisfied by the monodromy vertex operators.

Actually, we have four different matrix kernels (3.87) with the indices \( \alpha \beta, \alpha \dot{\beta}, \dot{\alpha} \beta, \dot{\alpha} \dot{\beta} \), corresponding to all possible combinations of different regions. When we change from one region to another one, then we have to change the basis of solutions, and this transition can be given by some matrix \( C_{\alpha}^{\alpha} \).

The expansion of these kernels, e.g. for \( 0 < z, w < 1 \)

\[
K_{\alpha\beta}(z, w) = \frac{\delta_{\alpha\beta}}{z - w} + \sum_{p,q>0} \langle \theta | V_{\nu}(1) \tilde{\psi}_p^\alpha \tilde{\psi}_q^\beta | \sigma \rangle z^{-\frac{1}{2} + \sigma_\alpha} w^{-\frac{1}{2} - \sigma_\beta} = \\
= \frac{\delta_{\alpha\beta}}{z - w} + \sum_{p,q>0} K_{pq}^{\alpha\beta} z^{-\frac{1}{2} + \sigma_\alpha} w^{-\frac{1}{2} - \sigma_\beta}
\]

(3.91)

or at \( z, w > 1 \)

\[
K_{\dot{\alpha}\dot{\beta}}(z, w) = \frac{\delta_{\dot{\alpha}\dot{\beta}}}{z - w} + \sum_{p,q>0} \langle \theta | \tilde{\psi}_p^\dot{\alpha} \tilde{\psi}_q^\dot{\beta} V_{\nu}(1) | \sigma \rangle z^{-\frac{1}{2} + \theta_\alpha} w^{-\frac{1}{2} - \theta_\beta} = \\
= \frac{\delta_{\dot{\alpha}\dot{\beta}}}{z - w} + \sum_{p,q>0} K_{pq}^{\dot{\alpha}\dot{\beta}} z^{-\frac{1}{2} + \theta_\alpha} w^{-\frac{1}{2} - \theta_\beta}
\]

(3.92)

give the corresponding matrix elements for the fermionic modes. The corresponding matrix elements (\( K_{pq}^{\alpha\beta} \) and \( \tilde{K}_{pq}^{\alpha\beta} \)) are in fact defined up to the factors \( s_\alpha s_\beta^{-1} \) which comes from the ambiguity in normalization of the vertex operator. For any three monodromy matrices \( M_0, M_1, M_\infty = 1 \) one can fix all their invariant functions (e.g. traces) and diagonalize \( M_\infty \), but then one possible transformation survives: a simultaneous conjugation

\[
M_i \mapsto S^{-1} M_i S
\]

(3.93)

by diagonal \( S = \text{diag}(s_1, \ldots, s_N) \). This gives vertex operators, actually different by \( s^{J_0} \) factor with corresponding multiplicative renormalization of their matrix elements.

For special vertex operators with \( \nu = \nu N e_j \) these matrix elements can be expressed in terms of the products (3.50), for example

\[
K_{pq}^{\alpha\beta} = \langle \theta | V_{\nu}(1) | p_\alpha, q_\beta; \sigma \rangle = \langle \theta | V_{\nu}(1) \tilde{\psi}_p^\alpha \tilde{\psi}_q^\beta | \sigma \rangle = \\
= \frac{1}{p_\alpha + q_\beta - \sigma_\alpha + \sigma_\beta} f_{1,\alpha}(\sigma, \theta + \nu, p_\alpha) f_{2,\beta}(\sigma, \theta + \nu, q_\beta)
\]

(3.94)

\[
\tilde{K}_{pq}^{\alpha\beta} = \langle p_\alpha, q_\beta; \theta | V_{\nu}(1) | \sigma \rangle = \langle \theta | \tilde{\psi}_p^\alpha \tilde{\psi}_q^\beta V_{\nu}(1) | \sigma \rangle = \\
= -\frac{1}{p_\alpha + q_\beta - \sigma_\alpha + \sigma_\beta} f_{1,\alpha}(\theta + \nu, \sigma, p_\alpha) f_{2,\beta}(\theta + \nu, \sigma, q_\beta)
\]

We shall return to discussion of special case below in sect. 3.4.3.
The general formula for $2n$-point fermionic correlator is given by the Wick formula

$$
\langle \theta | \prod_{\alpha=1}^{N} \tilde{\psi}_i^\sigma (z_{\alpha,i}) \psi_i^\sigma (w_{\alpha,i}) V_\nu (1) \prod_{\alpha=1}^{N} \tilde{\psi}_i^\theta (z_{\alpha,i}) \psi_i^\theta (w_{\alpha,i}) | \sigma \rangle =
$$

$$
= \langle \theta | V_\nu (1) | \sigma \rangle \cdot \det \left( \begin{array}{cc}
K_{\alpha \beta} (z_{\alpha,i}, w_{\beta,j}) & K_{\alpha \beta} (z_{\alpha,i}, w_{\beta,j}) \\
K_{\alpha \beta} (z_{\alpha,i}, w_{\beta,j}) & K_{\alpha \beta} (z_{\alpha,i}, w_{\beta,j})
\end{array} \right)
$$

On the punctured unit circle $|z| = |w| = 1$, $z \neq 1$, $w \neq 1$ one has

$$
K_{\alpha \beta} (z, w) = \sum_{\alpha} C_{\alpha}^\alpha K_{\alpha \beta} (z, w)
$$

$$
K_{\alpha \beta} (z, w) = \sum_{\beta} K_{\alpha \beta} (z, w) (C^{-1})_{\beta}^\beta
$$

It follows then from (3.95), that there are two operator identities

$$
\tilde{\psi}_i^\theta (z) V_\nu (1) = V_\nu (1) \sum_{\alpha} C_{\alpha}^\alpha \tilde{\psi}_i^\sigma (z)
$$

$$
\psi_i^\theta (z) V_\nu (1) = V_\nu (1) \sum_{\alpha} \psi_i^\sigma (z) (C^{-1})_{\alpha}^\alpha
$$

Actually, these identities are enough to define the operator $V_\nu (1)$. The simplest quantity to compute is

$$
V_\nu (1) \psi_{\alpha, r}^\sigma V_\nu (1)^{-1} = \oint_{|z|=1} \frac{dz}{2\pi i} z^{r-\frac{1}{2}+\sigma_{\alpha}} V_\nu (1) \psi_i^\sigma (z) V_\nu (1)^{-1}
$$

Using (3.97) one can rewrite this equivalently as

$$
V_\nu (1) \psi_{\alpha, r}^\sigma V_\nu (1)^{-1} = \sum_{\beta} C_{\alpha}^\beta \oint_{|z|=1} \frac{dz}{2\pi i} z^{r-\frac{1}{2}+\sigma_{\alpha}} \psi_i^\theta (z) =
$$

$$
= \sum_{\beta, s} C_{\alpha}^\beta \oint_{|z|=1} \frac{dz}{2\pi i} z^{r-s-1+\sigma_{\alpha}-\theta_{\beta,s}} \psi_i^\theta (z) = \sum_{\beta, s} C_{\alpha}^\beta \int \frac{d\phi}{2\pi} e^{2\pi i (r-s+\sigma_{\alpha}-\theta_{\beta,s}) \phi} \psi_i^\theta (z) =
$$

$$
= \sum_{\beta, s} C_{\alpha}^\beta \frac{-i (e^{2\pi i (\sigma_{\alpha}-\theta_{\beta,s})} - 1)}{r-s+\sigma_{\alpha}-\theta_{\beta,s}} \psi_i^\theta (z)
$$

In principle, this formula includes all possible information about $V_\nu (t)$. Now it is easy to prove

**Theorem 3.2.** $V_\nu (t)$ is a primary field of the conformal $W_N \oplus H$ algebra with the highest weights $u_k (\nu)$.

**Proof:** First we notice that due to (3.97) and to the definitions (3.85), (3.65) one has

$$
U^\theta_k (z) V_\nu (t) = V_\nu (t) U^\sigma_k (z)
$$

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in the region $|z| = t, z \neq t$. This means that $U_k(z)$ are actually single-valued operators (with trivial monodromies). Actually, we have already proved in Theorem 3.1 that states $|\sigma\rangle$ are highest weight vectors, so

$$\langle \theta | \ldots U_k(z) | \sigma \rangle = \left( \frac{u_k(\sigma)}{z^t} \right) \langle \theta | \ldots | \sigma \rangle$$  \hspace{1cm} (3.101)$$

and, since (3.88) is symmetric under the permutation of the singular points, one can also conclude, that for a different point

$$\langle \theta | \ldots U_k(z)V_{\nu}(t) \ldots | \sigma \rangle = \left( \frac{u_k(\nu)}{(z-t)^k} \right) \langle \theta | \ldots V_{\nu}(t) \ldots | \sigma \rangle$$  \hspace{1cm} (3.102)$$

so $(U_{k,n>0})_{\nu}(t) = 0$, and it means, that $V_{\nu}(t)$ is just a primary field.

\vspace{0.5cm}

**Generalized Hirota relations**

Now consider any operator $O$ with linear adjoint action on fermions

$$O^{-1}\psi_{\alpha,r}O = \sum_{s,\beta} R_{\alpha,s\beta} \psi_{\beta,s}, \quad O^{-1}\tilde{\psi}_{\alpha,-r}O = \sum_{s,\beta} \tilde{\psi}_{\beta,-s}(R^O)^{-1}_{s\beta,r\alpha}$$  \hspace{1cm} (3.103)$$

which is generally a relabeling of a $GL(\infty)$ transformation for a single fermion. It leads to a standard statement of commutativity of two operators in $\mathcal{H} \otimes \mathcal{H}$

$$O \otimes O \sum_{r,\alpha} \psi_{\alpha,-r} \otimes \tilde{\psi}_{\alpha,r} = \sum_{r,\alpha} \psi_{\alpha,-r} \otimes \tilde{\psi}_{\alpha,r}O \otimes O$$  \hspace{1cm} (3.104)$$

which is an operator form of the bilinear Hirota relation [MJD/KvdL, AZ].

Let us now point out, that we have already introduced by (3.97) a particular subclass of general transformations (3.103)

$$V^{-1}\psi_{\alpha}(z)V = \sum_{\alpha} (C^{-1})_{\alpha,\beta} \psi_{\beta}(z), \quad V^{-1}\tilde{\psi}_{\alpha}(z)V = \sum_{\alpha} C^{\alpha}_{\beta} \tilde{\psi}_{\beta}(z)$$  \hspace{1cm} (3.105)$$

where $C$ and $C^{-1}$ can be now interpreted as monodromy matrices: one can consider (3.105) as a linear relation between two analytic continuations of the fermionic fields at $|z| = 1$ towards $z \to \infty$ and $z \to 0$, preserving the OPE $\tilde{\psi}(z)_{\alpha}(z)\psi_{\beta}(z') = \frac{\delta_{\alpha\beta}}{z-z'} + \ldots$

An immediate consequence of (3.105) is

**Theorem 3.3.** The Fourier modes of the bilinear operators

$$\mathcal{I}(z) = \sum_{\alpha} \psi_{\alpha}(z) \otimes \tilde{\psi}_{\alpha}(z) = \sum_{k \in \mathbb{Z}} \frac{\mathcal{I}_k}{z^{k+1}} \quad \mathcal{I}^\dagger(z) = \sum_{\alpha} \tilde{\psi}_{\alpha}(z) \otimes \psi_{\alpha}(z) = \sum_{k \in \mathbb{Z}} \frac{\mathcal{I}_k^\dagger}{z^{k+1}}$$  \hspace{1cm} (3.106)$$

commute with $V_{\nu}(t) \otimes V_{\nu}(t)$ in the sense

$$\mathcal{I}_k \cdot V_{\nu}(t) \otimes V_{\nu}(t) = V_{\nu}(t) \otimes V_{\nu}(t) \cdot \mathcal{I}_k^\dagger \quad \mathcal{I}_k^\dagger \cdot V_{\nu}(t) \otimes V_{\nu}(t) = V_{\nu}(t) \otimes V_{\nu}(t) \cdot \mathcal{I}_k$$  \hspace{1cm} (3.107)$$

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Proof: First we notice that
\[ \mathcal{I}^\theta(z) \cdot V_{\nu}(t) \otimes V_{\nu}(t) = V_{\nu}(t) \otimes V_{\nu}(t) \cdot \mathcal{I}^\theta(z) \]  
(3.108)
holds at \(|z| = t, z \neq t\), due to (3.97)
\[ \sum_{\alpha} \psi^\theta_{\alpha}(z) \otimes \tilde{\psi}^\theta_{\alpha}(z) \cdot V_{\nu}(t) \otimes V_{\nu}(t) = V_{\nu}(t) \otimes V_{\nu}(t) \sum_{\alpha, \beta, \gamma} (C^{-1})_{\alpha}^{\beta} C_{\gamma}^{\beta} \psi^\sigma_{\beta}(z) \otimes \tilde{\psi}^\sigma_{\gamma}(z) = \]
\[ = V_{\nu}(t) \otimes V_{\nu}(t) \sum_{\beta} \psi^\beta_{\beta}(z) \otimes \tilde{\psi}^\beta_{\beta}(z) \]
(3.109)
To continue this equality to \(z = t\) one has just to check that \(\mathcal{I}^\theta(z) \cdot V_{\nu}(t) \otimes V_{\nu}(t)\) is regular. Due to the symmetry of (3.88) this is the same as to check that \(\mathcal{I}^\sigma(z) \cdot |\sigma\rangle \otimes |\sigma\rangle\) is regular. Since,
\[ \mathcal{I}^\sigma(z) \cdot |\sigma\rangle \otimes |\sigma\rangle = \sum_{\alpha} \sum_{n < 0} \psi^\sigma_{\alpha,n} |\sigma\rangle \otimes \sum_{m < 0} \tilde{\psi}^\sigma_{\alpha,m} |\sigma\rangle \]
\[ = \sum_{\alpha} \psi^\sigma_{\alpha,-1/2} |\sigma\rangle \otimes \tilde{\psi}^\sigma_{\alpha,-1/2} |\sigma\rangle + O(z) \]
(3.110)
this expression is regular, this completes the proof. \(\square\)

Let us notice that we have also got the equalities
\[ \mathcal{I}^\sigma_{k \geq 0} \cdot |\sigma\rangle \otimes |\sigma\rangle = 0, \quad \mathcal{I}^\sigma_{k \geq 0} \cdot |\sigma\rangle \otimes |\sigma\rangle = 0 \]
(3.111)
while, for example
\[ \mathcal{I}^\dagger_{-1} |\theta\rangle \otimes |\theta\rangle = \sum_{\alpha} \tilde{\psi}_{\alpha,-1/2} \otimes \tilde{\psi}_{\alpha,-1/2} |\theta\rangle \otimes |\theta\rangle = \sum_{\alpha} |1_{\alpha}, \theta\rangle \otimes |1_{\alpha}, \theta\rangle \]
(3.112)
but
\[ \langle \theta | \otimes \langle \theta | \cdot \mathcal{I}^\dagger_{-1} = \langle \theta | \otimes \langle \theta | \cdot \mathcal{I}_{-1} = 0 \]
(3.113)
We shall see below, that existence of extra bilinear operator relations lead actually to the infinite number of Hirota-like equations for the \(\tau\)-function.

Let us also notice that operator \(t^{L_0}\) belongs to the quasigroup, but it does not commute with \(\mathcal{I}_k\):
\[ t^{L_0} \mathcal{I}(z) t^{-L_0} = t \mathcal{I}(tz) \]
(3.114)
which means that \(t^{L_0} \mathcal{I}_k t^{-L_0} = t^{-k}\). So, in principle, vertex operator can contain some factors \(t^{L_0}_i\), but in such a combination with \(\prod t_i = 1\).

Now we are ready to prove, that the correlation functions (3.86) (and in fact any correlation function \(\langle \theta_\infty | O_{\nu}(z) \psi_\beta(w) | \theta_0 \rangle = \langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1) \psi_\alpha(z) \psi_\beta(w) | \theta_0 \rangle\) with two fermions) can be decomposed into two correlation functions with a single
fermion insertion. In addition to (3.112), (3.113) one has to compute commutator of this operator with \( \tilde{\psi} \otimes \psi \) using the contour integral representation

\[
\left[ I_{-1}, \tilde{\psi}_\alpha(z) \otimes \psi_\beta(w) \right] = \left( \oint \frac{dx}{z} \sum_\gamma \psi_\gamma(x) \otimes \tilde{\psi}_\gamma(x) \cdot \tilde{\psi}_\alpha(z) \otimes \psi_\beta(w) = \right.
\]

\[
= \oint \frac{dx}{z} \sum_\gamma \delta_{\gamma \alpha} \otimes \tilde{\psi}_\gamma(x) \psi_\beta(w) + \oint \frac{dx}{w} \sum_\gamma \psi_\gamma(x) \tilde{\psi}_\alpha(z) \otimes \delta_{\gamma \beta} = \]

\[
= \frac{1}{z} \cdot 1 \otimes \tilde{\psi}_\alpha(z) \psi_\beta(w) + \frac{1}{w} \cdot \psi_\beta(w) \tilde{\psi}_\alpha(z) \otimes 1
\]

Inserting this operator identity inside the correlation functions, and using (3.112), (3.113) we get

\[
0 = \langle \theta_\infty | \otimes \langle \theta_\infty | \cdot I_{-1} \cdot \mathcal{O} \otimes \mathcal{O} \cdot \tilde{\psi}_\alpha(z) \otimes \psi_\beta(w) \cdot | \theta_0 \rangle \otimes | \theta_0 \rangle = \]

\[
= \langle \theta_\infty | \otimes \langle \theta_\infty | \cdot \mathcal{O} \otimes \mathcal{O} \cdot \tilde{\psi}_\alpha(z) \otimes \psi_\beta(w) \sum_\gamma | -1_\gamma, \theta_0 \rangle \otimes | 1_\gamma, \theta_0 \rangle + \]

\[
+ \left( \frac{1}{z} - \frac{1}{w} \right) \langle \theta_\infty | \mathcal{O} | \theta_0 \rangle \cdot \langle \theta_\infty | \mathcal{O} \tilde{\psi}_\alpha(z) \psi_\beta(w) | \theta_0 \rangle
\]

The first term in the r.h.s. is equal to the bilinear combination of the correlation functions with a single fermion insertion, so one gets finally

\[
\langle \theta_\infty | \mathcal{O} \tilde{\psi}_\alpha(z) \psi_\beta(w) | \theta_0 \rangle \langle \theta_\infty | \mathcal{O} | \theta_0 \rangle = \]

\[
= \frac{zw}{z - w} \sum_\gamma \langle \theta_\infty | \mathcal{O} \tilde{\psi}_\alpha(z) \rangle \cdot 1_\gamma, \theta_0 \rangle \langle \theta_\infty | \mathcal{O} \psi_\beta(w) | 1_\gamma, \theta_0 \rangle
\]

which for \( \mathcal{O} = V_{\nu}(1) \) gives the relation between (3.90) and (3.87). Substituting here the OPE \( \psi_\alpha(z) \psi_\beta(w) = \frac{\delta_{\alpha \beta}}{z - w} + \text{reg.} \) and taking residue at \( z \to w \) one also proves that matrices in (3.90) are indeed inverse to each other.

**Riemann-Hilbert problem: hypergeometric example**

A hypergeometric solution to the Riemann-Hilbert problem with three singular points at \( z = 0, 1, \infty \) can be given by the following formulas

\[
\phi(z) = \begin{pmatrix}
z^\beta \mathcal{F}(\alpha, \beta, \nu|z) \\
-1^{1+\beta} C(\alpha, -\beta, \nu) \mathcal{F}(\alpha, 1 - \beta, \nu|z)
\end{pmatrix},
\]

\[
\phi^{-1}(z) = \begin{pmatrix}
z^{-\beta} \mathcal{F}(\alpha, -\beta, -\nu|z) \\
z^{1+\beta} C(\alpha, -\beta, -\nu) \mathcal{F}(\alpha, 1 - \beta, -\nu|z)
\end{pmatrix}
\]

where we have introduced \( \mathcal{F}(\alpha, \beta, \nu|z) = _2F_1 \left[ -\frac{\alpha - \beta + \nu}{2\beta}, \frac{\alpha + \beta - \nu}{2\beta}; \frac{\alpha + \beta + \nu}{2\beta}; z \right] \) for a standard hypergeometric function and the constant \( C(\alpha, \beta, \nu) = \frac{(-\alpha - \beta + \nu)(\alpha - \beta + \nu)}{2\beta(2\beta + 1)} \).

These formulas give solution to the linear system (3.88) with the residues in the following conjugacy classes:

\[
A_0 \sim \theta_0 = \sigma = \text{diag}(\beta, -\beta), \quad A_\infty \sim \theta_\infty = \theta = \text{diag}(\alpha, -\alpha)
\]

\[
A_1 \sim \theta_1 = \nu = \text{diag}(2\nu, 0)
\]

(3.118)
According to (3.86), (3.87)

\[ \langle \theta | V_\nu \tilde{\psi}_\alpha(z) \psi_\beta(w) | \sigma \rangle = \frac{[\phi(z)\phi(w)^{-1}]_{\alpha \beta}}{z - w} \]  

(3.119)

It means, for example, that in order to study the matrix elements with \( \psi_1, \tilde{\psi}_1 \) one needs to consider the function

\[ \widehat{K}_{11}(z, w) = z^{-\beta}w^{\beta}[\phi(z)\phi(w)^{-1}]_{11} = \mathcal{F}(\alpha, -\beta, -\nu|z)\mathcal{F}(\alpha, \beta, \nu|w) - \prod_{c, c' = \pm 1} (c\alpha + c'\beta + \nu) \]

\[ \times \frac{4\beta^2(4\beta^2 - 1)}{z w} \mathcal{F}(\alpha, 1 - \beta, -\nu|z)\mathcal{F}(\alpha, 1 + \beta, \nu|w) \]  

(3.120)

Already the simplest fact, that \( \widehat{K}_{11}(z, z) = 1 \) becomes a non-trivial bilinear relation for the hypergeometric function. However, our claim is much stronger: this function is almost as nice as (3.40) since its expansion (3.91) is given by

\[ \mathcal{K}(z, w)_{11} = \frac{1}{z - w} - \sum_{a, b = 0}^{\infty} 2\beta(\alpha - \beta + \nu)_{a+1}(\alpha + \beta + \nu)_{a+1}(-\alpha + \beta - \nu)_{b+1}(\alpha + \beta - \nu)_{b+1} \]

\[ \times \frac{z^b w^a}{(a + b + 1)(\alpha + \beta - \nu)(\alpha + \beta - \nu)_{a+1-b}b!(2\beta)_{b+1}(-2\beta)_{a+1}} \]  

(3.121)

and it is indeed a generation function of the matrix elements we are interested in. One can substitute here \( a = q - \frac{1}{2}, b = p - \frac{1}{2} \)

\[ \langle (\alpha, -\alpha) | V_{(2q, 0)}(1)\psi_1, -p \tilde{\psi}_1, -q | (\beta, -\beta) \rangle = \frac{2\beta(\alpha + \beta - \nu)_{q+\frac{1}{2}}(\alpha - \beta + \nu)_{p+\frac{1}{2}}}{(p + q)(\alpha + \beta - \nu)(\alpha + \beta - \nu)_{p+\frac{1}{2}}(q - \frac{1}{2})!}(2\beta)_{q+\frac{1}{2}}(-2\beta)_{p+\frac{1}{2}} \]  

(3.122)

and compare this formula with (3.48)

\[ \frac{f_{1,1}(\theta, \theta', p) f_{2,1}(\theta, \theta', q)}{p + q} = \frac{1}{(p - \frac{1}{2})!} \prod_{\beta} \frac{\theta' - \theta}{\theta' - \theta_1} \prod_{\beta_1 \neq \beta} \frac{\sqrt{\theta - \theta_1}}{\sqrt{\theta - \theta_1}} \]

\[ \times \frac{1}{(q - \frac{1}{2})!} \prod_{\beta} \frac{\theta - \theta'}{\theta - \theta'_{q+\frac{1}{2}}} \prod_{\beta_1 \neq \beta} \frac{\sqrt{\theta' - \theta}}{\sqrt{\theta' - \theta_1}} \times \frac{1}{p + q} = \]  

\[ \frac{(\theta_1 - \theta_2)(\theta_1 + \theta_2')_{p+\frac{1}{2}}(\theta_1 + \theta_2')_{q+\frac{1}{2}}(\theta_1 - \theta_2'_{q+\frac{1}{2}})}{(p + q)(\theta_1 + \theta_2')(\theta_1 - \theta_2')(p - \frac{1}{2})!(q - \frac{1}{2})!(\theta_2 - \theta_1)_{p+\frac{1}{2}}(\theta_2 - \theta_1)_{q+\frac{1}{2}}} \]  

(3.123)

It is easy to see, that after the appropriate identification

\[ \theta_1 = \beta, \theta_2 = -\beta, \theta'_1 = \alpha + \nu, \theta'_2 = -\alpha + \nu \]  

(3.124)

the r.h.s.’s in two last formulas coincide exactly.

In addition to the hypergeometric case another explicit example can be provided by the exact conformal blocks, considered in [GMtw]. We are planning to consider it in detail elsewhere.
3. Free fermions, W-algebras and isomonodromic deformations

Isomonodromic tau-functions and Fredholm determinants

Isomonodromic tau-function

First we need to prove the simple

**Lemma 3.4.** Monodromies of $\psi_\beta(w)$ and $\tilde{\psi}_\alpha(z)$ in the matrix elements

$$\langle Y', n', \theta | V_\nu(1) \tilde{\psi}_\alpha(z) \psi_\beta^\sigma(w) | Y, n, \sigma \rangle$$

(3.125)

do not depend on $n, Y, n', Y'$.  

**Proof:** All these matrix elements can be obtained from (3.95) by certain contour integration, producing fermionic modes from the fermionic fields. However, in (3.95) due to the Wick theorem factorization, all contributions have the factorized form $K_{\alpha\beta}(z, \bullet) \times \ldots$, where all other factors do not depend at all on $z$, so that all monodromies comes from a single kernel $K$.  

Now it is easy to prove

**Theorem 3.5.** Solution of the linear problem with $n$ marked points is given by

$$(z - w) \mathcal{R}_{\alpha\beta}(z, w)$$

with

$$\mathcal{R}_{\alpha\beta}(z, w) = \frac{\langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1) \tilde{\psi}_0^\alpha(z) \psi_\beta^0(w) | \theta_0 \rangle}{\langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1) | \theta_0 \rangle}$$

(3.126)

whereas its isomonodromic tau-function is defined by

$$\tau(t_1, \ldots, t_{n-2}) = \langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1) | \theta_0 \rangle$$

(3.127)

**Proof:** First, insert resolutions of unity between each two (radially-ordered) vertex operators, e.g.

$$\tau \cdot \mathcal{R}_{\alpha\beta}(z, w) = \sum_{\{Y_1, m_1\}} \langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) | Y_{n-3}, m_{n-3}, \sigma_{n-3} \rangle \langle Y_{n-3}, m_{n-3}, \sigma_{n-3} | \times$$

$$\times \ldots \times \langle Y_2, m_2, \sigma_2 | V_{\theta_2}(t_2) | Y_1, m_1, \sigma_1 \rangle \langle Y_1, m_1, \sigma_1 | V_{\theta_1}(t_1) \tilde{\psi}_0^\alpha(z) \psi_\beta^0(w) | \theta_0 \rangle$$

(3.128)

for $0 < |z|, |w| < |t_1|$ and similarly in the other regions. Due to Lemma 3.4 the monodromies of the fermionic fields do not depend on the intermediate states, but only on the vertex operators and the set of charges $\sigma$'s, therefore it is enough to reduce the problem of computation of all monodromies to the collection of corresponding three-point problems with different vertex operators $V_{\theta_j}(t_j)$ inserted. So we have proven that

---

3In addition to $(n-3)$ time parameters $\{t_1, \ldots, t_n\}$ modulo Möbius transformation, which always allow to fix three of them to 0, 1, $\infty$ and $n$ sets of W-charges $\{\theta_j\}$ the isomonodromic tau-function depends upon the charges $\{\sigma_k\} \in (\mathbb{R}/\mathbb{Z})^{n-1}$, $k = 1, \ldots, n-3$ in the intermediate channels and their duals $\{\beta_k\}$, which we had already discussed in the context of ambiguity in normalization of the vertex operators and their matrix elements.
(z - w)\mathcal{R}_{\alpha\beta}(z, w) = [\Phi(z)\Phi^{-1}(w)]_{\alpha\beta}$ (to cancel extra singularity in (3.126)), actually gives a solution to the multi-point Riemann-Hilbert problem.

In order to prove (3.127) consider
\[
\sum_{\alpha} \tilde{\psi}_\alpha(z + \frac{t}{2})\psi_\alpha(z - \frac{t}{2}) = \frac{N}{t} + J(z) + tU_2(z) + \ldots
\]
so that
\[
t \text{Tr} \mathcal{R}(z + \frac{t}{2}, z - \frac{t}{2}) = \text{Tr} \Phi(z + \frac{t}{2})\Phi(z - \frac{t}{2})^{-1} = \nonumber
\]
\[
= N + t \frac{\langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1)U_2(z)\rangle_0}{\langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1)\rangle_0} + t^2 \frac{\langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1)U_2(z)\rangle_0}{\langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1)\rangle_0} + \ldots
\]
where from (3.72) and the conformal Ward identities
\[
\frac{\langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1)U_2(z)\rangle_0}{\langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1)\rangle_0} = \sum_{i=1}^n \left( \frac{1}{2} \frac{\partial_i^2}{z - t_i} + \frac{\partial_i}{z - t_i} \log \langle \theta_\infty | V_{\theta_{n-2}}(t_{n-2}) \ldots V_{\theta_1}(t_1)\rangle_0 \right)
\]
where we have extended this formula to include $t_1 = 0$ and $t_n = \infty$.

Now solving the linear system (3.88) with $A(z) = \sum_i \frac{A_i}{z - t_i}$ we get
\[
\Phi(z + \frac{t}{2})\Phi(z - \frac{t}{2})^{-1} = \Phi(z) \left( 1 + \frac{t}{2} A(z) + \frac{t^2}{8} (\partial A(z) + A(z)^2) + \ldots \right) \times \nonumber
\]
\[
\times \left( 1 + \frac{t}{2} A(z) + \frac{t^2}{8} (-\partial A(z) + A(z)^2) + \ldots \right) \Phi(z)^{-1} = \nonumber
\]
\[
= \Phi(z) \left( 1 + tA(z) + \frac{t^2}{2} A(z) + \ldots \right) \Phi(z)^{-1}
\]
Therefore, due to the definition of the tau-function
\[
\text{Tr} \Phi(z + \frac{t}{2})\Phi(z - \frac{t}{2})^{-1} = \frac{t^2}{2} \sum_{i=1}^n \left( \frac{1}{2} \frac{\partial_i^2}{(z - t_i)^2} + \frac{\partial_i}{z - t_i} \log \tau(t_1, \ldots, t_n) \right) + \ldots
\]
Comparing this formula with (3.131) completes the proof. \ \Box

**Fredholm determinant**

Consider now the isomonodromic tau-function $\tau(t) = \langle \theta_\infty | V_{\nu_1}(1)V_{\nu_1}(t)\rangle_0$, corresponding to the problem on sphere with four marked points at $z = 0, t, 1, \infty$. Inserting the resolution of unity one can write
\[
\tau(t) = \langle \theta_\infty | V_{\nu_1}(1)V_{\nu_1}(t)\rangle_0 = \sum_{Y, m} \langle \theta_\infty | V_{\nu_1}(1)|Y, m; \sigma\rangle \langle Y, m; \sigma| V_{\nu_1}(t)\rangle_0 = \nonumber
\]
\[
= \sum_{\{p_{\alpha,i}\}, \{q_{\alpha,i}\}} \langle \theta_\infty | V_{\nu_1}(1)|\{p_{\alpha,i}\}, \{q_{\alpha,i}\}; \sigma\rangle \langle \{q_{\alpha,i}\}, \{p_{\alpha,i}\}; \sigma| V_{\nu_1}(t)\rangle_0
\]
Here we have used first just a particular case of the expansion (3.128), applying it to the simplest nontrivial isomonodromic tau-function. However, now it is useful to notice, that summation over the basis in total space $\mathcal{H}_m = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_m^\sigma$ can be performed in Frobenius coordinates just forgetting restriction $#p_\alpha = #q_\alpha$ for the states (3.56) in $\mathcal{H}_m^\sigma$, hence there is no restriction in summation range in the r.h.s. of (3.134).

Now, one can still apply formulas (3.91), (3.92) for the matrix elements in (3.134). It gives

$$\langle \theta_\infty | V_{\nu_1}(1) | \{p_{\alpha,i}\}, \{q_{\alpha,i}\}; \sigma \rangle = \det K_{x,y} \quad \langle \{p_{\alpha,i}\}, \{q_{\alpha,i}\}; \sigma | V_{\nu_1}(t) | \theta_0 \rangle = \det \tilde{K}_{x,y} (t) \quad (3.135)$$

where we have used again the multi-indices $\cup_\alpha \{(\alpha, p_{\alpha,i})\} = \{x_I\}$ and $\cup_\alpha \{(\alpha, q_{\alpha,i})\} = \{y_J\}$. It means, that the tau-function (3.134) can be summed up into a single Fredholm determinant

$$\tau(t) = \sum_{\{p_{\alpha,i}\}, \{q_{\alpha,i}\}} \langle \theta_\infty | V_{\nu_1}(1) | \{p_{\alpha,i}\}, \{q_{\alpha,i}\}; \sigma \rangle \langle \{q_{\alpha,i}\}, \{p_{\alpha,i}\}; \sigma | V_{\nu_1}(t) | \theta_0 \rangle =$$

$$= \sum_{\{x\}, \{y\}} \det K_{x,y} \cdot \det \tilde{K}_{y,x} (t) = \sum_{n=0}^{\infty} \sum_{\{x\} = n} \det K_{x,y} \cdot \det \tilde{K}_{y,x} (t) =$$

$$= \sum_{n=0}^{\infty} \text{Tr} \wedge^n (K \tilde{K}(t)) = \det (1 + K \tilde{K}(t)) = \det (1 + \mathcal{R}_t) \quad (3.136)$$

where basically only the Wick theorem has been used. One can also present the kernel of this operator $\mathcal{R}_t = K \tilde{K}(t)$ explicitly by the formula

$$\mathcal{R}(x, z) = \oint_{|y| = r} \frac{(\phi(x) \phi(y)^{-1} - x^\sigma y^{-\sigma})(S^{-1} \phi(y/t) \tilde{\phi}(z/t)^{-1} S - y^\sigma z^{-\sigma})}{t^{-1} (x - y)(y - z)} dy \quad (3.137)$$

(where $S$ is the diagonal matrix introduced before), so that this integral operator acts from the space of vector-valued functions $f(z) = (f_1(z), \ldots, f_N(z))$ on the circle $|z| = r, t < r < 1$. These functions have the fractional Laurent expansion

$$f_\alpha(z) = z^{\sigma_\alpha} \sum_{n \in \mathbb{Z}} f_{\alpha,n} z^n \quad (3.138)$$

otherwise their convolution with our kernel will be ill-defined.

The representation in terms of the Fredholm determinant definitely requires further careful investigation, and it could appear to be useful for practical computations with isomonodromic tau-functions, which basically have no explicit representations.

**Conclusion**

We have considered in this chapter the free fermion formalism, which allows to study representations of the W-algebras at least at integer values of the central charges. The
vertex operators are defined by their two-fermion matrix elements, which are fixed by monodromies of auxiliary linear system, and can be obtained from solution of the corresponding Riemann-Hilbert problem.

This chapter is just the first step of studying this relation (apart of the well-known and effectively used for different applications Abelian case). A natural development of the above ideas is only outlined in sect. 3.5. We are going to return elsewhere to the problem of rewriting the isomonodromic tau-functions in terms of the Fredholm determinants, which can be quite useful representations (though still not an explicit form) for these complicated objects. Another point, which has to be understood better is the relation of class of the isomonodromic solutions to the Toda lattices, which have been defined above using the generalized Hirota bilinear relations, to the class of solutions, obeying the Virasoro-W constraints.
Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

Abstract

We derive Fredholm determinant representation for isomonodromic tau functions of Fuchsian systems with $n$ regular singular points on the Riemann sphere and generic monodromy in $\text{GL}(N, \mathbb{C})$. The corresponding operator acts in the direct sum of $N(n - 3)$ copies of $L^2(S^1)$. Its kernel has a block integrable form and is expressed in terms of fundamental solutions of $n - 2$ elementary 3-point Fuchsian systems whose monodromy is determined by monodromy of the relevant $n$-point system via a decomposition of the punctured sphere into pairs of pants. For $N = 2$ these building blocks have hypergeometric representations, the kernel becomes completely explicit and has Cauchy type. In this case Fredholm determinant expansion yields multivariate series representation for the tau function of the Garnier system, obtained earlier via its identification with Fourier transform of Liouville conformal block (or a dual Nekrasov-Okounkov partition function). Further specialization to $n = 4$ gives a series representation of the general solution to Painlevé VI equation.

Introduction

Motivation and some results

The theory of monodromy preserving deformations plays a prominent role in many areas of modern nonlinear mathematical physics. The classical works [WMTB, JMMS, TW1] relate, for instance, various correlation and distribution functions of statistical mechanics and random matrix theory models to special solutions of Painlevé equations. The relevant Painlevé functions are usually written in terms of Fredholm or Toeplitz determinants. Further study of these relations has culminated in the development by Tracy and Widom [TW2] of an algorithmic procedure of derivation of systems of PDEs satisfied by Fredholm determinants with integrable kernels [IJKS] restricted to a union of intervals; the isomonodromic origin of Tracy-Widom equations has been elucidated in [Pal94] and further studied in [HI]. This raises a natural
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

question:

Can the general solution of isomonodromy equations be expressed in terms of a Fredholm determinant?

One of the goals of the present chapter is to provide a constructive answer to this question in the Fuchsian setting. Let us consider a Fuchsian system with \( n \) regular singular points \( a := \{ a_0, \ldots, a_{n-2}, a_{n-1} \equiv \infty \} \) on \( \mathbb{P}^1 \equiv \mathbb{P}^1 (\mathbb{C}) \):

\[
\partial_z \Phi = \Phi A(z), \quad A(z) = \sum_{k=0}^{n-2} \frac{A_k}{z - a_k},
\]

where \( A_0, \ldots, A_{n-2} \) are \( N \times N \) matrices independent of \( z \) and \( \Phi(z) \) is a fundamental matrix solution, multivalued on \( \mathbb{P}^1 \setminus a \). The monodromy of \( \Phi(z) \) realizes a representation of the fundamental group \( \pi_1 (\mathbb{P}^1 \setminus a) \) in \( \text{GL}(N, \mathbb{C}) \). When the residue matrices \( A_0, \ldots, A_{n-2} \) and \( A_{n-1} := -\sum_{k=0}^{n-2} A_k \) are non-resonant, the isomonodromy equations are given by the Schlesinger system,

\[
\begin{align*}
\partial_{a_i} A_k &= [A_i, A_k], \\
\partial_{a_i} A_i &= \sum_{k \neq i} [A_i, A_k].
\end{align*}
\]

Integrating the flows associated to affine transformations, we may set without loss of generality \( a_0 = 0 \) and \( a_{n-2} = 1 \), so that there remains \( n - 3 \) nontrivial time variables \( a_1, \ldots, a_{n-3} \). In the case \( N = 2 \), Schlesinger equations reduce to the Garnier system \( G_{n-3} \), see for example [IKSY, Chapter 3] for the details. Setting further \( n = 4 \), we are left with only one time \( t \equiv a_1 \) and the latter system becomes equivalent to a nonlinear 2nd order ODE — the Painlevé VI equation.

The main object of our interest is the isomonodromic tau function of Jimbo-Miwa-Ueno [JMU]. It is defined as an exponentiated primitive of the 1-form

\[
d_q \ln \tau_{\text{JMU}} := \frac{1}{2} \sum_{k=0}^{n-2} \text{res}_{z=a_k} \text{Tr} A^2(z) \, da_k.
\]

The definition is consistent since the 1-form on the right is closed on solutions of the deformation equations (4.2). It generates the hamiltonians of the Schlesinger system. Dealing with the Garnier system, we will assume the standard gauge where \( \text{Tr} A(z) = 0 \) and denote the eigenvalues of \( A_k \) by \( \pm \theta_k \) with \( k = 0, \ldots, n-1 \). In the Painlevé VI case, it is convenient to modify this notation as \( (\theta_0, \theta_1, \theta_2, \theta_3) \mapsto (\theta_0, \theta_t, \theta_1, \theta_\infty) \). The logarithmic derivative \( \zeta(t) := t (t-1) \frac{d}{dt} \ln \tau_{\text{VI}}(t) \) then satisfies the \( \sigma \)-form of Painlevé VI,

\[
\left( t(t-1) \zeta'' \right)^2 = -2 \det \begin{pmatrix}
\frac{2\theta_0^2}{t\zeta' - \zeta} & \frac{t\zeta' - \zeta}{\theta_1^2} & \frac{\zeta' + \theta_0^2 + \theta_1^2 + \theta_\infty^2 - t^2}{(t-1)\zeta' - \zeta} \\
\frac{t\zeta' - \zeta}{\theta_1^2} & \frac{2\theta_1^2}{\zeta' + \theta_0^2 + \theta_1^2 + \theta_\infty^2 - t^2} & \frac{2\theta_1^2}{(t-1)\zeta' - \zeta} \\
\frac{\zeta' + \theta_0^2 + \theta_1^2 + \theta_\infty^2 - t^2}{(t-1)\zeta' - \zeta} & \frac{2\theta_1^2}{(t-1)\zeta' - \zeta} & \frac{2\theta_1^2}{(t-1)\zeta' - \zeta}
\end{pmatrix}.
\]
Monodromy of the associated linear problems provides a complete set of conserved quantities for Painlevé VI, the Garnier system and Schlesinger equations. By the general solution of deformation equations we mean the solution corresponding to generic monodromy data. The precise genericity conditions will be specified in the main body of the text.

In [Pal90], Palmer (developing earlier results of Malgrange [Mal]) interpreted the Jimbo-Miwa-Ueno tau function (4.3) as a determinant of a singular Cauchy-Riemann operator acting on functions with prescribed monodromy. The main idea of [Pal90] is to isolate the singular points \( a_0, \ldots, a_{n-1} \) inside a circle \( C \subset \mathbb{P}^1 \) and represent the Fuchsian system (4.1) by a boundary space of functions on \( C \) that can be analytically continued inside with specified branching. The variation of positions of singularities gives rise to a trajectory of this space in an infinite Grassmannian. The tau function is obtained by comparing two sections of an associated determinant bundle.

The construction suggested in the present chapter is essentially a refinement of Palmer’s approach, translated into the Riemann-Hilbert framework. A single circle \( C \) is replaced by the boundaries of \( n \) annuli which cut the \( n \)-punctured sphere \( \mathbb{P}^1 \setminus a \) into trinions (pairs of pants), see e.g. Fig. 4.2a below. To each trinion is assigned a Fuchsian system with 3 regular singular points whose monodromy is determined by monodromy of the original system. We show that the isomonodromic tau function is proportional to a Fredholm determinant:

\[
\tau_{JMU}(a) = \Upsilon(a) \cdot \det (1 - K),
\]

where the prefactor \( \Upsilon(a) \) is a known elementary function. The integral operator \( K \) acts on holomorphic vector functions on the union of annuli and involves projections on certain boundary spaces.

The pay-off of a more complicated Grassmannian model is that the kernel of \( K \) may be written explicitly in terms of 3-point solutions\(^1\). In particular, for \( N = 2 \) (i.e. for the Garnier system) the latter have hypergeometric expressions. The \( n = 4 \) specialization of our result is as follows.

**Theorem 4.1.** Let the independent variable \( t \) of Painlevé VI equation vary inside the real interval \( ]0, 1[ \) and let \( C = \{ z \in \mathbb{C} : |z| = R, t < R < 1 \} \) be a counter-clockwise oriented circle. Let \( \sigma, \eta \) be a pair of complex parameters satisfying the conditions

\[
|\Re \sigma| \leq \frac{1}{2}, \quad \sigma \neq 0, \pm \frac{1}{2}, \\
\theta_0 \pm \theta_t + \sigma \notin \mathbb{Z}, \quad \theta_0 \pm \theta_t - \sigma \notin \mathbb{Z}, \quad \theta_1 \pm \theta_\infty + \sigma \notin \mathbb{Z}, \quad \theta_1 \pm \theta_\infty - \sigma \notin \mathbb{Z}.
\]

General solution of the Painlevé VI equation (4.4) admits the following Fredholm determinant:

\[
\tau_{JMU}(a) = \Upsilon(a) \cdot \det (1 - K),
\]

\(^1\)We would like to note that somewhat similar refined construction emerged in the analysis of massive Dirac equation with \( U(1) \) branching on the Euclidean plane [Pal93]. Every branch point was isolated there in a separate strip, which ultimately allowed to derive an explicit Fredholm determinant representation for the tau function of appropriate Dirac operator [SMJ]. In physical terms, the determinant corresponds to a resummed form factor expansion of a correlation function of \( U(1) \) twist fields in the massive Dirac theory. The paper [Pal93] was an important source of inspiration for the present work, although it took us more than 10 years to realize that the strips should be replaced by pairs of pants in the chiral problem.
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

determinant representation:

\[
\tau_{VI}(t) = \text{const} \cdot t^{\sigma_{+}^{2} - \sigma_{-}^{2}} (1 - t)^{-2\theta_{1} \theta_{2}} \det (1 - U), \quad U = \begin{pmatrix} 0 & a \\ d & 0 \end{pmatrix}, \quad (4.6)
\]

where the operators \( a, d \in \text{End} (\mathbb{C}^{2} \otimes L^{2} (\mathcal{C})) \) act on \( g = \begin{pmatrix} g_{+} \\ g_{-} \end{pmatrix} \) with \( g_{\pm} \in L^{2} (\mathcal{C}) \) as

\[
(ag)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} a (z, z') g (z') dz', \quad (dg)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} d (z, z') g (z') dz', \quad (4.7)
\]

and their kernels are explicitly given by

\[
a(z, z') = \frac{(1 - z')^{2\theta_{1}} \begin{pmatrix} K_{++} (z) & K_{+-} (z) \\ K_{-+} (z) & K_{--} (z) \end{pmatrix} \begin{pmatrix} K_{--} (z') & -K_{+-} (z') \\ -K_{-+} (z') & K_{++} (z') \end{pmatrix} - 1}{z - z'},
\]

\[
d(z, z') = \frac{1 - (1 - \frac{t}{z})^{2\theta_{1}} \begin{pmatrix} \bar{K}_{++} (z) & \bar{K}_{+-} (z) \\ \bar{K}_{-+} (z) & \bar{K}_{--} (z) \end{pmatrix} \begin{pmatrix} \bar{K}_{--} (z') & -\bar{K}_{+-} (z') \\ -\bar{K}_{-+} (z') & \bar{K}_{++} (z') \end{pmatrix}}{z - z'}, \quad (4.8)
\]

with

\[
K_{\pm \pm} (z) = {}_{2}F_{1} \left[ \begin{array}{c} \theta_{1} + \theta_{\infty} \pm \sigma, \theta_{1} - \theta_{\infty} \pm \sigma \\ 2\sigma \end{array} ; z \right],
\]

\[
K_{\pm \mp} (z) = \pm \frac{\theta_{\infty}^{2} - (\theta_{1} \pm \sigma)^{2}}{2\sigma (1 \pm 2\sigma)} z {}_{2}F_{1} \left[ \begin{array}{c} 1 + \theta_{1} + \theta_{\infty} \pm \sigma, 1 + \theta_{1} - \theta_{\infty} \pm \sigma \\ 2 \pm 2\sigma \end{array} ; z \right],
\]

\[
\bar{K}_{\pm \mp} (z) = {}_{2}F_{1} \left[ \begin{array}{c} \theta_{1} + \theta_{0} \mp \sigma, \theta_{1} - \theta_{0} \mp \sigma \\ \mp 2\sigma \end{array} ; t/z \right],
\]

\[
\bar{K}_{\pm \pm} (z) = \mp \frac{t^{2\sigma} e^{\mp i\sigma \theta_{0}^{2} - (\theta_{1} \mp \sigma)^{2} t}}{2\sigma (1 \mp 2\sigma)} z {}_{2}F_{1} \left[ \begin{array}{c} 1 + \theta_{1} + \theta_{0} \mp \sigma, 1 + \theta_{1} - \theta_{0} \mp \sigma \\ 2 \mp 2\sigma \end{array} ; t/z \right]. \quad (4.9)
\]

Moreover, we demonstrate that for a special choice of monodromy in the Painlevé VI case, \( U \) becomes equivalent to the hypergeometric kernel of [BO05] and thereby reproduces previously known family of Fredholm determinant solutions [BD]. The hypergeometric kernel is known to produce other random matrix integrable kernels in confluent limits.

Another part of our motivation comes from isomonodromy/CFT/gauge theory correspondence. It was conjectured in [GL12] that the tau function associated to the general Painlevé VI solution coincides with a Fourier transform of 4-point \( c = 1 \) Virasoro conformal block with respect to its intermediate momentum. Two independent derivations of this conjecture have been already proposed in [ILTe] and [BShch]. The first approach [ILTe] also extends the initial statement to the Garnier system. Its main idea is to consider the operator-valued monodromy of conformal blocks with additional level 2 degenerate insertions. At \( c = 1 \), Fourier transform of such conformal blocks reduces their “quantum” monodromy to ordinary \( 2 \times 2 \) matrices. It can therefore be used to construct the fundamental matrix solution of a Fuchsian system.
4.1. Introduction

with prescribed SL\((2,\mathbb{C})\) monodromy. The second approach [BSch] uses an embedding of two copies of the Virasoro algebra into super-Virasoro algebra extended by Majorana fermions to prove certain bilinear differential-difference relations for 4-point conformal blocks, equivalent to Painlevé VI equation. An interesting feature of this method is that bilinear relations admit a deformation to generic values of Virasoro central charge.

Among other developments, let us mention the papers [GIL13, ILT14, Nag] where asymptotic expansions of Painlevé V, IV and III tau functions were identified with Fourier transforms of irregular conformal blocks of different types. The study of relations between isomonodromy problems in higher rank and conformal blocks of \(W_N\) algebras has been initiated in [Gav, GMTw, GMfer].

The AGT conjecture [AGT] (proved in [AFLT]) identifies Virasoro conformal blocks with partition functions of \(\mathcal{N} = 2\) 4D supersymmetric gauge theories. There exist combinatorial representations of the latter objects [Nek], expressing them as sums over tuples of Young diagrams. This fact is of crucial importance for isomonodromy theory, since it gives (contradicting to an established folklore) explicit series representations for the Painlevé VI and Garnier tau functions. Since the very first paper [GIL12] on the subject, there has been a puzzle to understand combinatorial tau function expansions directly within the isomonodromic framework. There have also been attempts to sum up these series to determinant expressions; for example, in [Bal] truncated infinite series for \(c = 1\) conformal blocks were shown to coincide with partition functions of certain discrete matrix models.

In this work, we show that combinatorial series correspond to the principal minor expansion of the Fredholm determinant (4.5), written in the Fourier basis of the space of functions on annuli of the pants decomposition. Fourier modes which label the choice of rows for the principal minor are related to Frobenius coordinates of Young diagrams. It should be emphasized that this combinatorial structure is valid also for \(N > 2\) where CFT/gauge theory counterparts of the tau functions have yet to be defined and understood.

We prove in particular the following result, originally conjectured in [GIL12] (the details of notation concerning Young diagrams are explained in the next subsection):

**Theorem 4.2.** General solution of the Painlevé VI equation (4.4) can be written as

\[
\tau_{VI}(t) = \text{const} \cdot \sum_{n \in \mathbb{Z}} e^{i \eta n} B\left(\bar{\theta}; \sigma + n; t\right),
\]

where \(B(\bar{\theta}, \sigma; t)\) is a double sum over Young diagrams,

\[
B\left(\bar{\theta}, \sigma; t\right) = \mathcal{N}_{\bar{\theta}_{-\infty}; \sigma} \mathcal{N}_{\bar{\theta}_{\infty}; \sigma} t^{2\sigma - \theta_0^2} (1 - t)^{2\theta_1 \theta_0} \sum_{\lambda, \mu} B_{\lambda, \mu}\left(\bar{\theta}, \sigma\right) t^{b_{\lambda} + b_{\mu}},
\]

\[
B_{\lambda, \mu}\left(\bar{\theta}, \sigma\right) = \prod_{(i,j) \in \lambda} \frac{(\theta_i + \sigma + i - j)^2 - \theta_0^2}{h_{\bar{\lambda}}^2(i, j) (\lambda'_i - i + \mu_i - j + 1 + 2\sigma)^2} \times \prod_{(i,j) \in \mu} \frac{(\theta_i - \sigma + i - j)^2 - \theta_0^2}{h_{\bar{\mu}}^2(i, j) (\mu'_i - i + \lambda_i - j + 1 - 2\sigma)^2},
\]

where \(b_{\lambda} + b_{\mu}\) is the number of boxes in the Young diagrams \(\lambda\) and \(\mu\).
Here \( \sigma \notin \mathbb{Z}/2, \eta' \) are two arbitrary complex parameters, and \( G(z) \) denotes the Barnes \( G \)-function.

The parameters \( \sigma \) play exactly the same role in the Fredholm determinant (4.6) and the series representation (4.10), whereas \( \eta \) and \( \eta' \) are related by a simple transformation. An obvious quasiperiodicity of the second representation with respect to integer shifts of \( \sigma \) is by no means manifest in the Fredholm determinant.

\[
N_{\theta_1, \theta_2}^{\eta_2} = \prod_{\epsilon = \pm} \frac{G(1 + \theta_3 + \epsilon(\theta_1 + \theta_2)) G(1 - \theta_3 + \epsilon(\theta_1 - \theta_2))}{G(1 - 2\theta_1) G(1 - 2\theta_2) G(1 + 2\theta_3)}.
\]

### Notation

The monodromy matrices of Fuchsian systems and the jumps of associated Riemann-Hilbert problems appear on the left of solutions. These somewhat unusual conventions are adopted to avoid even more confusing right action of integral and infinite matrix operators. The indices corresponding to the matrix structure of rank \( N \) Riemann-Hilbert problem are referred to as color indices and are denoted by Greek letters, such as \( \alpha, \beta \in \{1, \ldots, N\} \). Upper indices in square brackets, e.g. \([k]\) in \( T^{[k]} \), label different trinions in the pants decomposition of a punctured Riemann sphere. We denote by \( \mathbb{Z}' := \mathbb{Z} + \frac{1}{2} \) the half-integer lattice, and by \( \mathbb{Z}'_{\pm} = \{ p \in \mathbb{Z}' | p \gtrless 0 \} \) its positive and negative parts. The elements of \( \mathbb{Z}', \mathbb{Z}'_{\pm} \) will be generally denoted by the letters \( p \) and \( q \).

**Figure 4.1**: Young diagram associated to the partition \( \lambda = \{6, 5, 4, 2\} \).

The set of all partitions identified with Young diagrams is denoted by \( \mathbb{Y} \). For \( \lambda \in \mathbb{Y} \), we write \( \lambda' \) for the transposed diagram, \( \lambda_i \) and \( \lambda'_j \) for the number of boxes in the \( i \)th row and \( j \)th column of \( \lambda \), and \( |\lambda| \) for the total number of boxes in \( \lambda \). Let \( \square = (i, j) \) be the box in the \( i \)th row and \( j \)th column of \( \lambda \in \mathbb{Y} \) (see Fig. 4.1). Its arm-length \( a_\lambda(\square) \) and leg-length \( l_\lambda(\square) \) denote the number of boxes on the right and below. This definition is extended to the case where the box lies outside \( \lambda \) by the formulae \( a_\lambda(\square) = \lambda_i - j \) and \( l_\lambda(\square) = \lambda'_j - i \). The hook length of the box \( \square \in \lambda \) is defined as \( h_\lambda(\square) = a_\lambda(\square) + l_\lambda(\square) + 1 \).

### Outline of the chapter

The chapter is organized as follows. Section 4.2 is devoted to the derivation of Fredholm determinant representation of the Jimbo-Miwa-Ueno isomonodromic tau function. It starts from a recast of the original rank \( N \) Fuchsian system with \( n \) regular
singular points on \( \mathbb{P}^1 \) in terms of a Riemann-Hilbert problem. In Subsection 4.2.2 we associate to it, via a decomposition of \( n \)-punctured Riemann sphere into pairs of pants, \( n - 2 \) auxiliary Riemann-Hilbert problems of Fuchsian type having only 3 regular singular points. Section 4.2.3 introduces Plemelj operators acting on functions holomorphic on the annuli of the pants decomposition, and deals with their basic properties. The main result of the section is formulated in Theorem 4.11 of Subsection 4.2.4, which relates the tau function of a Fuchsian system with prescribed generic monodromy to a Fredholm determinant whose blocks are expressed in terms of 3-point Plemelj operators. In Subsection 4.2.5, we consider in more detail the example of \( n = 4 \) points and show that the Fredholm determinant representation can be efficiently used for asymptotic analysis of the tau function. In particular, Theorem 4.13 provides a generalization of the Jimbo asymptotic formula for Painlevé VI valid in any rank and up to any asymptotic order.

In Section 4.3 we explain how the principal minor expansion of the Fredholm determinant leads to a combinatorial structure of the series representations for isomonodromic tau functions. Theorem 4.15 of Subsection 4.3.1 shows that 3-point Plemelj operators written in the Fourier basis are given by sums of a finite number of infinite Cauchy type matrices twisted by diagonal factors. Combinatorial labeling of the minors by \( N \)-tuples of charged Maya diagrams and partitions is described in Subsection 4.3.2.

Section 4.4 deals with rank \( N = 2 \). Hypergeometric representations of the appropriate 3-point Plemelj operators are listed in Lemma 4.28 of Subsection 4.4.1. Theorem 4.30 provides an explicit combinatorial series representation for the tau function of the Garnier system. In the final subsection, we explain how Fredholm determinant of the Borodin-Olshanski hypergeometric kernel arises as a special case of our construction. Appendix contains a proof of a combinatorial identity expressing Nekrasov functions in terms of Maya diagrams instead of partitions.

**Perspectives**

In an effort to keep the chapter of reasonable length, we decided to defer the study of several straightforward generalizations of our approach to separate publications. These extensions are outlined below together with a few more directions for future research:

1. In higher rank \( N > 2 \), it is an open problem to find integral/series representations for general solutions of 3-point Fuchsian systems and to obtain an explicit description of the Riemann-Hilbert map. There is however an important exception of rigid systems having two generic regular singularities and one singularity of spectral multiplicity \( (N - 1, 1) \); these can be solved in terms of generalized hypergeometric functions of type \( _N F_{N-1} \). The spectral condition is exactly what is needed to achieve factorization in Lemma 4.26. The results of Section 4.4 can therefore be extended to Fuchsian systems with two generic singular points at 0 and \( \infty \), and \( n - 2 \) special ones. The corresponding isomonodromy equations (dubbed \( \mathcal{G}_{N,n-3} \) system in [Tsu]) are the closest higher rank relatives of Painlevé VI and Garnier system. It is natural to expect their tau functions to be related on the 2D CFT and gauge theory side, respectively, to \( W_N \) conformal blocks.
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

with semi-degenerate fields [FLitv12, Bul] and Nekrasov partition functions of 4D linear quiver gauge theories with the gauge group $U(N)^{\otimes(n-3)}$.

In the generic non-rigid case the 3-point solutions depend on $(N-1)(N-2)$ accessory parameters and may be interpreted as matrix elements of a general vertex operator for the $W_N$ algebra. They should also be related to the so-called $T_N$ gauge theory without lagrangian description [BMPTY].

2. Fredholm determinants and series expansions considered in the present work are associated to linear pants decompositions of $\mathbb{P}^1 \setminus \{n \text{ points}\}$, which means that every pair of pants has at least one external boundary component (see Fig. 4.2a). Plemelj operators assigned to each trinion act on spaces of functions on internal boundary circles only. To be able to deal with arbitrary decompositions, in addition to 4 operators $a^{[k]}, b^{[k]}, c^{[k]}, d^{[k]}$ appearing in (4.20) one has to introduce 5 more similar operators associated to other possible choices of ordered pairs of boundary components.

![Figure 4.2: (a) Linear and (b) Sicilian pants decomposition of $\mathbb{P}^1 \setminus \{6 \text{ points}\}$; (c) gluing 1-punctured torus from a pair of pants.](image)

A (tri)fundamental example where this construction becomes important is known in the gauge theory literature under the name of Sicilian quiver (Fig. 4.2b). Already for $N = 2$ the monodromies along the triple of internal circles of this pants decomposition cannot be simultaneously reduced to the form “1 + rank 1 matrix” by factoring out a suitable scalar piece. The analog of expansion (4.87) in Theorem 4.30 will therefore be more intricate yet explicitly computable. Since the identification [ILTe] of the tau function of the Garnier system with a Fourier transform of $c=1$ Virasoro conformal block does not put any constraint on the employed pants decomposition, Sicilian expansion of the Garnier tau function may be used to produce an analog of Nekrasov representation for the corresponding conformal blocks. It might be interesting to compare the results obtained in this way against instanton counting [HKS].

Extension of the procedure to higher genus requires introducing additional simple (diagonal in the Fourier basis) operators acting on some of the internal annuli. They give rise to a part of moduli of complex structure of the Riemann surface and correspond to gluing a handle out of two boundary components. Fig. 4.2c shows how a 1-punctured torus may be obtained by gluing two boundary circles of a pair of pants. The gluing operator encodes the elliptic modulus, which plays a role of the time variable in the corresponding isomonodromic
problem. Elliptic isomonodromic deformations have been studied e.g. in [K00], where the interested reader can find further references.

3. It is natural to wonder to what extent the approach proposed in the present work may be followed in the presence of irregular singularities, in particular, for Painlevé I–V equations. The contours of appropriate isomonodromic RHPs become more complicated: in addition to circles of formal monodromy, they include anti-Stokes rays, exponential jumps on which account for Stokes phenomenon [FIKN]. We will sketch here a partial answer in rank $N = 2$. For this it is useful to recall a geometric representation of the confluence diagram for Painlevé equations recently proposed by Chekhov, Mazzocco and Rubtsov [CM, CMR], see Fig. 4.3. To each of the equations (or rather associated linear problems) is assigned a Riemann surface with a number of cusped boundary components. They are obtained from Painlevé VI 4-holed sphere using two surgery operations: i) a “chewing-gum” move creating from two holes with $k$ and $l$ cusps one hole with $k + l + 2$ cusps and ii) a cusp removal reducing the number of cusps at one hole by 1. The cusps may be thought of as representing the anti-Stokes rays of the Riemann-Hilbert contour.

An extension of our approach is straightforward for equations from the upper part of the CMR diagram and, more generally, when the Poincaré ranks of all irregular singular points are either $\frac{1}{2}$ or 1. The associated surfaces may be decomposed into irregular pants of three types corresponding to solvable RHPs: Gauss hypergeometric, Whittaker and Bessel systems (Fig. 4.4). They serve to construct local Riemann-Hilbert parametrices which in turn produce the relevant Plemelj operators.

The study of higher Poincaré rank seems to require new ideas. Moreover, even for Painlevé V and Painlevé III Fredholm determinant expansions naturally give series representations of the corresponding tau functions of regular type, first
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

![Diagram of Gauss, Whittaker, and Bessel]

Figure 4.4: Some solvable RHPs in rank $N = 2$: Gauss hypergeometric (3 regular punctures), Whittaker (1 regular + 1 of Poincaré rank 1) and Bessel (1 regular + 1 of rank $\frac{1}{2}$).

proposed in [GIL13] and expressed in terms of irregular conformal blocks of [G, BMT, GT]. It is not clear to us how to extract from them irregular (long-distance) asymptotic expansions. Let us mention a recent work [Nag] which relates such expansions to irregular conformal blocks of a different type.

4. Given a matrix $K \in \mathbb{C}^{X \times X}$ indexed by elements of a discrete set $X$, it is almost a tautology to say that the principal minors $\det K_{\mathbb{N}^* X}$ define a determinantal point process on $X$ and a probability measure on $2^{\mathbb{N}^* X}$. Fredholm determinant representations and combinatorial expansions of tau functions thus generalize in a natural way various families of measures of random matrix or representation-theoretic origin, such as $Z$- and $ZW$-measures [BO05, BO01] (the former correspond to the scalar case $N = 1$ with $n = 4$ regular singular points, and the latter are related to hypergeometric kernel considered in the last subsection). We believe that novel probabilistic models coming from isomonodromy deserve further investigation.

5. Perhaps the most intriguing perspective is to extend our setup to $q$-isomonodromy problems, in particular $q$-difference Painlevé equations, presumably related to the deformed Virasoro algebra [SKAO] and 5D gauge theories. Among the results pointing in this direction, let us mention a study of the connection problem for $q$-Painlevé VI [Ma] based on asymptotic factorization of the associated linear problem into two systems solved by the Heine basic hypergeometric series $\varphi_1$, and critical expansions for solutions of $q$-$P(A_1)$ equation recently obtained in [JR].

**Tau functions as Fredholm determinants**

**Riemann-Hilbert setup**

The classical setting of the Riemann-Hilbert problem (RHP) involves two basic ingredients:
4.2. Tau functions as Fredholm determinants

- A contour \( \Gamma \) on a Riemann surface \( \Sigma \) of genus \( g \) consisting of a finite set of smooth oriented arcs that can intersect transversally. Orientation of the arcs defines positive and negative side \( \Gamma^\pm \) of the contour in the usual way, see Fig. 4.5.

- A jump matrix \( J : \Gamma \to \text{GL}(N, \mathbb{C}) \) that satisfies suitable smoothness requirements.

The RHP defined by the pair \((\Gamma, J)\) consists in finding an analytic invertible matrix function \( \Psi : \Sigma \setminus \Gamma \to \text{GL}(N, \mathbb{C}) \) whose boundary values \( \Psi^\pm \) on \( \Gamma^\pm \) are related by \( \Psi^+ = J \Psi^- \). Uniqueness of the solution is ensured by adding an appropriate normalization condition.

In the present work we are mainly interested in the genus 0 case: \( \Sigma = \mathbb{P}^1 \). Let us fix a collection

\[
a := (a_0 = 0, a_1, \ldots, a_{n-3}, a_{n-2} = 1, a_{n-1} = \infty)
\]

of \( n \) distinct points on \( \mathbb{P}^1 \) satisfying the condition of radial ordering \( 0 < |a_1| < \ldots < |a_{n-3}| < 1 \). To reduce the amount of fuss below, it is convenient to assume that \( a_1, \ldots, a_{n-3} \in \mathbb{R}^>0 \). The contour \( \Gamma \) will then be chosen as a collection

\[
\Gamma = \left( \bigcup_{k=0}^{n-1} \gamma_k \right) \cup \left( \bigcup_{k=0}^{n-2} \ell_k \right)
\]

of counter-clockwise oriented circles \( \gamma_k \) of sufficiently small radii centered at \( a_k \), and the segments \( \ell_k \subset \mathbb{R} \) joining the circles \( \gamma_k \) and \( \gamma_{k+1} \), see Fig. 4.6.

The jumps will be defined by the following data:
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

- An $n$-tuple of diagonal $N \times N$ matrices $\Theta_k = \text{diag} \{ \theta_{k,1}, \ldots, \theta_{k,N} \} \in \mathbb{C}^{N \times N}$ (with $k = 0, \ldots, n - 1$) satisfying Fuchs consistency relation $\sum_{k=0}^{n-1} \text{Tr} \Theta_k = 0$ and having non-resonant spectra. The latter condition means that $\theta_{k,\alpha} - \theta_{k,\beta} \notin \mathbb{Z} \setminus \{0\}$.

- A collection of $2n$ matrices $C_{k,\pm} \in \text{GL} (N, \mathbb{C})$ subject to the constraints

$$
M_{0\rightarrow k} := C_{k,-} e^{2\pi i \Theta_k} C_{k,+}^{-1} = C_{k+1,-} C_{k+1,+}, \quad k = 0, \ldots, n - 3,
$$

$$
M_{0\rightarrow n-2} := C_{n-2,-} e^{2\pi i \Theta_{n-2}} C_{n-2,+}^{-1} = C_{n-1,-} e^{-2\pi i \Theta_{n-1}} C_{n-1,+},
$$

$$
M_{0\rightarrow n-1} := I = C_{n-1,-} C_{n-1,+} = C_{0,-} C_{0,+},
$$

which are simultaneously viewed as the definition of $M_{0\rightarrow k} \in \text{GL} (N, \mathbb{C})$. Only $n$ of the initial matrices (for example, $C_{k,+}$) are therefore independent.

The jump matrix $J$ that we are going to consider is then given by

$$
J (z) \Big|_{\gamma_k} = M_{0\rightarrow k}^{-1}, \quad k = 0, \ldots, n - 2,
$$

$$
J (z) \Big|_{\gamma_k} = (a_k - z)^{-\Theta_k} C_{k,\pm}^{-1}, \quad \Im z \gtrless 0, \quad k = 0, \ldots, n - 2,
$$

$$
J (z) \Big|_{\gamma_{n-1}} = (-z)^{\Theta_{n-1}} C_{n-1,\pm}, \quad \Im z \gtrless 0.
$$

Throughout this chapter, complex powers will always be understood as $z^\theta = e^{\theta \ln z}$, the logarithm being defined on the principal branch. The subscripts $\pm$ of $C_{k,\pm}$ are sometimes omitted to lighten the notation.

A major incentive to study the above RHP comes from its direct connection to systems of linear ODEs with rational coefficients. Indeed, define a new matrix $\Phi$ by

$$
\Phi (z) = \begin{cases} 
\Psi (z), & z \text{ outside } \gamma_{0\ldots n-1}, \\
C_k (a_k - z)^{\Theta_k} \Psi (z), & z \text{ inside } \gamma_k, \quad k = 0, \ldots, n - 2, \\
C_{n-1} (-z)^{-\Theta_{n-1}} \Psi (z), & z \text{ inside } \gamma_{n-1}.
\end{cases}
$$

It has only piecewise constant jumps $J_k (z) \big|_{a_k, a_{k+1}} = M_{0 \rightarrow k}^{-1}$ on the positive real axis. The matrix $A (z) := \Phi^{-1} \partial_z \Phi$ is therefore meromorphic on $\mathbb{P}^1$ with poles only possible at $a_0, \ldots, a_{n-1}$. It follows immediately that

$$
\partial_z \Phi = \Phi A (z), \quad A (z) = \sum_{k=0}^{n-2} \frac{A_k}{z - a_k},
$$

with $A_k = \Psi (a_k)^{-1} \Theta_k \Psi (a_k)$. Thus $\Phi (z)$ is a fundamental matrix solution for a class of Fuchsian systems related by constant gauge transformations. It has prescribed monodromy and singular behavior that are encoded in the connection matrices $C_k$ and local monodromy exponents $\Theta_k$. The freedom in the choice of the gauge reflects the dependence on the normalization of $\Psi$.

The monodromy representation $\rho : \pi_1 (\mathbb{P}^1 \setminus a) \rightarrow \text{GL} (N, \mathbb{C})$ associated to $\Phi$ is uniquely determined by the jumps. It is generated by the matrices $M_k = \rho (\xi_k)$.
4.2. Tau functions as Fredholm determinants

\[ a_0 \ a_1 \ a_2 \ a_{n-2} \]

Figure 4.7: Generators of $\pi_1(\mathbb{P}^1 \setminus a)$

assigned to counter-clockwise loops $\xi_0, \ldots, \xi_{n-1}$ represented in Fig. 4.7. They may be expressed as

\[ M_0 = M_{0 \rightarrow 0}, \quad M_{k+1} = M_{0 \rightarrow -k}^{-1} M_{0 \rightarrow k+1}, \]

which means simply that $M_{0 \rightarrow k} = M_0 \ldots M_{k-1} M_k$. It is a direct consequence of the definition (4.11) that the spectra of $M_k$ coincide with those of $e^{2\pi i \Theta_k}$.

**Assumption 4.3.** The matrices $M_{0 \rightarrow k}$ with $k = 1, \ldots, n - 3$ are assumed to be diagonalizable:

\[ M_{0 \rightarrow k} = S_k e^{2\pi i S_k} S_k^{-1}, \quad \Theta_k = \text{diag} \{ \sigma_{k,1}, \ldots, \sigma_{k,N} \}. \]

It can then be assumed without loss in generality that $\text{Tr} \Theta_k = \sum_{j=0}^k \text{Tr} \Theta_j$ and $|\Re (\sigma_{k,\alpha} - \sigma_{k,\beta})| \leq 1$. We further impose a non-resonancy condition $\sigma_{k,\alpha} - \sigma_{k,\beta} \neq \pm 1$.

In order to have uniform notation, we may also identify $\Theta_0 \equiv \Theta_0$, $\Theta_{n-2} \equiv -\Theta_{n-1}$. Note that any sufficiently generic monodromy representation can be realized as described above.

**Auxiliary 3-point RHPs**

Consider a decomposition of the original $n$-punctured sphere into $n - 2$ pairs of pants $T^{[1]}, \ldots, T^{[n-2]}$ by $n - 3$ annuli $A_1, \ldots, A_{n-3}$ represented in Fig. 4.8. The labeling is designed so that two boundary components of the annulus $A_k$ that belong to trinions $T^{[k]}$ and $T^{[k+1]}$ are denoted by $C^{[k]}_{\text{out}}$ and $C^{[k+1]}_{\text{in}}$. We are now going to associate to the $n$-point RHP described above $n - 2$ simpler 3-point RHPs assigned to different trinions and defined by the pairs $(\Gamma^{[k]}, J^{[k]})$ with $k = 1, \ldots, n - 2$.

The curves $C^{[k]}_{\text{in}}$ and $C^{[k]}_{\text{out}}$ are represented by circles of positive and negative orientation as shown in Fig. 4.9. For $k = 2, \ldots, n - 3$, the contour $\Gamma^{[k]}$ of the RHP assigned to trinion $T^{[k]}$ consists of three circles $C^{[k]}_{\text{in}}, C^{[k]}_{\text{out}}, \gamma_k$ associated to boundary components, and two segments of the real axis. For leftmost and rightmost trinions $T^{[1]}$ and $T^{[n-2]}$, the role of $C^{[1]}_{\text{in}}$ and $C^{[n-2]}_{\text{out}}$ is played respectively by the circles $\gamma_0$ and $\gamma_{n-1}$ around 0 and $\infty$.

The jump matrix $J^{[k]}$ is constructed according to two basic rules:
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

![Diagram](image)

Figure 4.8: Labeling of trinions, annuli and boundary curves

![Diagram](image)

Figure 4.9: Contour $\Gamma^k$ (left) and $\hat{\Gamma}$ for $n = 5$ (right)

- The arcs that belong to original contour give rise to the same jumps:
  \[(J^k - J) \big|_{\Gamma^k \cap \Gamma} = 0.\]

- The jumps on the boundary circles $C_{\text{out}}^k, C_{\text{in}}^{k+1}$ mimic regular singularities characterized by counter-clockwise monodromy matrices $M_0 \to k - 1, M_k, M_{-1}^0 \to k$:
  \[
  J^k \big|_{C_{\text{out}}^k} = (-z)^{-\Theta_k} S_{k}^{-1}, \quad J^{k+1} \big|_{C_{\text{in}}^{k+1}} = (-z)^{-\Theta_k} S_{k}^{-1}, \quad k = 1, \ldots, n - 3.
  \]

The solution $\Psi^k$ of the RHP defined by the pair $(\Gamma^k, J^k)$ is thus related in a way analogous to (4.13) to the fundamental matrix solution $\Phi^k$ of a Fuchsian system with 3 regular singular points at 0, $a_k$ and $\infty$ characterized by monodromies $M_0 \to k - 1, M_k, M_{-1}^0 \to k$:

\[
\partial_z \Phi^k = \Phi^k A^k (z), \quad A^k (z) = \frac{A_0^k}{z} + \frac{A_1^k}{z - a_k}.
\]

We note in passing that the spectra of $A_0^k, A_1^k$ and $A_{\infty}^k := -A_0^k - A_1^k$ coincide with the spectra of $\Theta_{k-1}, \Theta_k$ and $-\Theta_k$. The non-resonancy constraint in Assumption 4.3
ensures the existence of solution $\Phi^{[k]}$ with local behavior leading to the jumps (4.15) in $\Psi^{[k]}$.

It will be convenient to replace the $n$-point RHP described in the previous subsection by a slightly modified one. It is defined by a pair $\left(\hat{\Gamma}, \hat{J}\right)$ such that (cf right part of Fig. 4.9)

$$\hat{\Gamma} = \bigcup_{k=1}^{n-2} \Gamma^{[k]}, \quad \hat{J}\big|_{\Gamma^{[k]}} = J^{[k]}.$$  (4.17)

Constructing the solution $\hat{\Psi}$ of this RHP is equivalent to finding $\Psi$: it is plain that

$$\hat{\Psi} (z) = \begin{cases} 
(-z)^{-\Theta_k} S_k^{-1} \Psi (z), & z \in A_k, \\
\Psi (z), & z \in \mathbb{P} \setminus \bigcup_{k=1}^{n-3} A_k.
\end{cases}$$  (4.18)

Our aim in the next subsections is to construct the isomonodromic tau function in terms of $3$-point solutions $\Phi^{[k]}$. This construction employs in a crucial way integral Plemelj operators acting on spaces of holomorphic functions on $A := \bigcup_{k=1}^{n-3} A_k$.

**Plemelj operators**

Given a positively oriented circle $C \subset \mathbb{C}$ centered at the origin, let us denote by $V (C)$ the space of functions holomorphic in an annulus containing $C$. Any $f \in V (C)$ is canonically decomposed as $f = f_+ + f_-$, where $f_+$ and $f_-$ denote the analytic and principal part of $f$. Let us accordingly write $V (C) = V_+ (C) \oplus V_- (C)$ and denote by $\Pi_{\pm} (C)$ the projectors on the corresponding subspaces. Their explicit form is

$$\Pi_{\pm} (C) f (z) = \frac{1}{2\pi i} \oint_{C_{\pm}, |z|=|z|\pm 0} f (z') \frac{dz'}{z' - z},$$

where the subscript of $C_\pm$ indicates the orientation of $C$. Projectors $\Pi_{\pm} (C)$ are simple instances of Plemelj operators to be extensively used below.

Let us next associate to every trinion $T^{[k]}$ with $k = 2, \ldots, n - 3$ the spaces of vector-valued functions

$$\mathcal{H}^{[k]} = \bigoplus_{\epsilon = \text{in, out}} \left( \mathcal{H}_{\epsilon, +}^{[k]} \oplus \mathcal{H}_{\epsilon, -}^{[k]} \right), \quad \mathcal{H}_{\epsilon, \pm}^{[k]} = \mathbb{C}^N \otimes V_{\pm} (C_{\epsilon}^{[k]}).$$

With respect to the first decomposition, it is convenient to write the elements $f^{[k]} \in \mathcal{H}^{[k]}$ as

$$f^{[k]} = \begin{pmatrix} f_{\text{in}, -}^{[k]} \\ f_{\text{in}, +}^{[k]} \end{pmatrix} \oplus \begin{pmatrix} f_{\text{out}, +}^{[k]} \\ f_{\text{out}, -}^{[k]} \end{pmatrix}.$$  (4.19)

Here $f_{\epsilon, \pm}^{[k]}$ denote $N$-column vectors which represent the restrictions of analytic and principal part of $f^{[k]}$ to boundary circle $C_{\epsilon}^{[k]}$. Now define an operator $\mathcal{P}^{[k]} : \mathcal{H}^{[k]} \to \mathcal{H}^{[k]}$ by

$$\mathcal{P}^{[k]} f^{[k]} (z) = \frac{1}{2\pi i} \oint_{C_{\text{in}}^{[k]} \cup C_{\text{out}}^{[k]}} \frac{\Psi_+^{[k]} (z) \Psi_+^{[k]} (z')^{-1} f^{[k]} (z') dz'}{z - z'},$$  (4.19)
The singular factors $1/(z-z')$ for $z, z' \in C_{\text{in,out}}^{k}$ are interpreted with the following prescription: the contour of integration is deformed to appropriate annulus (e.g. $A_{k-1}$ for $C_{\text{in}}^{k}$ and $A_{k}$ for $C_{\text{out}}^{k}$) as to avoid the pole at $z' = z$. Matrix function $\Psi^{[k]}(z)$ is a solution of the 3-point RHP described in the previous subsection. Its normalization is irrelevant as the corresponding factor cancels out in (4.19).

**Lemma 4.4.** We have $(\mathcal{P}^{[k]})^{2} = \mathcal{P}^{[k]}$ and $\ker \mathcal{P}^{[k]} = \mathcal{H}_{\text{in,+}}^{[k]} \oplus \mathcal{H}_{\text{out,-}}^{[k]}$. Moreover, $\mathcal{P}^{[k]}$ can be explicitly written as

$$\mathcal{P}^{[k]} : \left( f_{\text{in,-}}^{[k]} \atop f_{\text{out,+}}^{[k]} \right) \oplus \left( f_{\text{in,+}}^{[k]} \atop f_{\text{out,-}}^{[k]} \right) \mapsto \left( f_{\text{in,-}}^{[k]} \atop f_{\text{out,+}}^{[k]} \right) \oplus \left( a^{[k]} b^{[k]} \atop c^{[k]} d^{[k]} \right) \left( f_{\text{in,-}}^{[k]} \atop f_{\text{out,+}}^{[k]} \right),$$

where the operators $a^{[k]}, b^{[k]}, c^{[k]}, d^{[k]}$ are defined by

$$(a^{[k]} g)(z) = \frac{1}{2\pi i} \oint_{C_{\text{in}}^{k}} \frac{\left[ \Psi^{[k]}(z) \Psi^{[k]}(z')^{-1} - 1 \right] g(z') \, dz'}{z-z'}, \quad z \in C_{\text{in}}^{k}, \quad (4.20a)$$

$$(b^{[k]} g)(z) = \frac{1}{2\pi i} \oint_{C_{\text{out}}^{k}} \frac{\Psi^{[k]}(z) \Psi^{[k]}(z')^{-1} g(z') \, dz'}{z-z'}, \quad z \in C_{\text{in}}^{k}, \quad (4.20b)$$

$$(c^{[k]} g)(z) = \frac{1}{2\pi i} \oint_{C_{\text{in}}^{k}} \frac{\Psi^{[k]}(z) \Psi^{[k]}(z')^{-1} g(z') \, dz'}{z-z'}, \quad z \in C_{\text{out}}^{k}, \quad (4.20c)$$

$$(d^{[k]} g)(z) = \frac{1}{2\pi i} \oint_{C_{\text{out}}^{k}} \frac{\left[ \Psi^{[k]}(z) \Psi^{[k]}(z')^{-1} - 1 \right] g(z') \, dz'}{z-z'}, \quad z \in C_{\text{out}}^{k}, \quad (4.20d)$$

**Proof.** Let us first prove that $\mathcal{H}_{\text{in,+}}^{[k]}, \mathcal{H}_{\text{out,-}}^{[k]} \subset \ker \mathcal{P}^{[k]}$. This statement follows from the fact that $\Psi^{[k]}$ holomorphically extends inside $C_{\text{in}}^{k}$ and outside $C_{\text{out}}^{k}$, so that the integration contours can be shrunk to 0 and $\infty$. To prove the projection property, decompose for example

$$\left( \mathcal{P}^{[k]} f_{\text{out,+}}^{[k]} \right)_{\text{out}}(z) = \frac{1}{2\pi i} \oint_{C_{\text{out}}^{k}} \frac{\left[ \Psi^{[k]}(z) \Psi^{[k]}(z')^{-1} - 1 \right] f_{\text{out,+}}^{[k]}(z') \, dz'}{z-z'} + \frac{1}{2\pi i} \oint_{C_{\text{out}}^{k}} f_{\text{out,+}}^{[k]}(z') \, dz'. \quad (4.20d)$$

The first integral admits holomorphic continuation in $z$ outside $C_{\text{out}}^{k}$ thanks to nonsingular integral kernel, and leads to (4.20d), whereas the second term is obviously equal to $f_{\text{out,+}}^{[k]}$. The action of $\mathcal{P}^{[k]}$ on $f_{\text{in,-}}^{[k]}$ is computed in a similar fashion. □

The leftmost and rightmost trinions $\mathcal{T}^{[1]}$ and $\mathcal{T}^{[n-2]}$ play somewhat distinguished role. Let us assign to them boundary spaces

$$\mathcal{H}^{[1]} := \mathcal{H}_{\text{out,+}}^{[k]} \oplus \mathcal{H}_{\text{out,-}}^{[k]}, \quad \mathcal{H}^{[n-2]} := \mathcal{H}_{\text{in,+}}^{[k]} \oplus \mathcal{H}_{\text{in,-}}^{[k]}.$$
and the operators $\mathcal{P}^{[k]} : \mathcal{H}^{[k]} \to \mathcal{H}^{[k]}$ with $k = 1, n - 2$ defined by

\[
\mathcal{P}^{[1]} f^{[1]} (z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{[1]}} \frac{\Psi^{[1]}_+ (z) \Psi^{[1]}_+ (z')^{-1} f^{[1]} (z') \, dz'}{z - z'},
\]
\[
\mathcal{P}^{[n-2]} f^{[n-2]} (z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{[n-2]}} \frac{\Psi^{[n-2]}_+ (z) \Psi^{[n-2]}_+ (z')^{-1} f^{[n-2]} (z') \, dz'}{z - z'}.
\]

Analogously to the above, one can show that

\[
\mathcal{P}^{[1]} : f^{[1]}_{\text{out},+} \oplus f^{[1]}_{\text{out},-} \mapsto f^{[1]}_{\text{out},+} \oplus d^{[1]} f^{[1]}_{\text{out},+},
\]
\[
\mathcal{P}^{[n-2]} : f^{[n-2]}_{\text{in},-} \oplus f^{[n-2]}_{\text{in},+} \mapsto f^{[n-2]}_{\text{in},-} \oplus a^{[n-2]} f^{[n-2]}_{\text{in},-},
\]

where the operators $d^{[1]}, a^{[n-2]}$ are given by the same formulae (4.20a), (4.20d). Note in particular that $\mathcal{P}^{[1]}$ and $\mathcal{P}^{[n-2]}$ are projections along their kernels $\mathcal{H}^{[1]}_{\text{out},-}$ and $\mathcal{H}^{[n-2]}_{\text{in},+}$.

Let us next introduce the total space

\[
\mathcal{H} := \bigoplus_{k=1}^{n-2} \mathcal{H}^{[k]}.
\]

It admits a splitting that will play an important role below. Namely,

\[
\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,
\]
\[
\mathcal{H}_\pm := \mathcal{H}^{[1]}_{\text{out},\pm} \oplus \left( \mathcal{H}^{[2]}_{\text{in},\mp} \oplus \mathcal{H}^{[2]}_{\text{out},\mp} \right) \oplus \ldots \oplus \left( \mathcal{H}^{[n-3]}_{\text{in},\mp} \oplus \mathcal{H}^{[n-3]}_{\text{out},\mp} \right) \oplus \mathcal{H}^{[n-2]}_{\text{in},\mp}.
\]

Combine the 3-point projections $\mathcal{P}^{[k]}$ into an operator $\mathcal{P}_\oplus : \mathcal{H} \to \mathcal{H}$ given by the direct sum

\[
\mathcal{P}_\oplus = \mathcal{P}^{[1]} \oplus \ldots \oplus \mathcal{P}^{[n-2]}.
\]

Clearly, we have

**Lemma 4.5.** $\mathcal{P}_\oplus^2 = \mathcal{P}_\oplus$ and $\ker \mathcal{P}_\oplus = \mathcal{H}_-.$

Another important operator $\mathcal{P}_\Sigma : \mathcal{H} \to \mathcal{H}$ is defined using the solution $\hat{\Psi} (z)$ (defined by (4.17)) of the $n$-point RHP in a way similar to construction of the projection (4.19):

\[
\mathcal{P}_\Sigma f (z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_\Sigma} \frac{\hat{\Psi}^+ (z) \hat{\Psi}^+ (z')^{-1} f (z') \, dz'}{z - z'}, \quad \mathcal{C}_\Sigma := \bigcup_{k=1}^{n-3} \mathcal{C}^{[k]}_{\text{out}} \cup \mathcal{C}^{[k+1]}_{\text{in}}.
\]

We use the same prescription for the contours: whenever it is necessary to interpret the singular factor $1/(z - z')$, the contour of integration goes clockwise around the pole.

Let $\mathcal{H}_\mathcal{A}$ be the space of boundary values on $\mathcal{C}_\Sigma$ of functions holomorphic on $\mathcal{A} = \bigcup_{k=1}^{n-3} \mathcal{A}_k$.

**Lemma 4.6.** $\mathcal{P}_\Sigma^2 = \mathcal{P}_\Sigma$ and $\mathcal{H}_\mathcal{A} \subseteq \ker \mathcal{P}_\Sigma.$
Proof. Given \( f \in \mathcal{H}_A \), the integration contours \( C^{[k]}_\text{in} \) and \( C^{[k+1]}_\text{in} \) in (4.22) can be merged thanks to the absence of singularities inside \( A_k \), which proves the second statement. To show the projection property, it suffices to notice that

\[
P^{[k]}_\Sigma f^{[k]}(z) = \frac{1}{(2\pi i)^2} \oint_{C_{\Sigma}} \hat{\Psi}_+(z) \hat{\Psi}_+(z'')^{-1} f^{[k]}(z'')\ dz'\ dz''.
\]

Because of the ordering of contours prescribed above, the only obstacle to merging \( C^{[k]}_\text{out} \) and \( C^{[k]}_\text{in} \) in the integral with respect to \( z' \) is the pole at \( z' = z \). The result follows by residue computation. \( \Box \)

Lemma 4.7. \( P_\Sigma \circ P_\oplus = P_\oplus \) and \( P_\oplus \circ P_\Sigma = P_\Sigma \).

Proof. Similar to the proof of Lemma 4.6. Use that \( \hat{\Psi}^{-1} \Psi^{[k]} \) has no jumps on \( \Gamma^{[k]} \) to compute by residues the intermediate integrals in \( P_\Sigma \circ P_\oplus \) and \( P_\oplus \circ P_\Sigma \). \( \Box \)

The above suggests to introduce the notation

\[
\mathcal{H}_T := \text{im} \ circ \circ P_\oplus = \text{im} P_\Sigma.
\]

(4.23)

The space \( \mathcal{H}_T \subset \mathcal{H} \) can be thought of as the subspace of functions on the union of boundary circles \( C^{[k]}_\text{in} \), \( C^{[k]}_\text{out} \) that can be continued inside \( \bigcup_{k=1}^{n-2} \mathcal{T}^{[k]} \) with monodromy and singular behavior of the \( n \)-point fundamental matrix solution \( \Phi(z) \). The only exception is the regular singularity at \( \infty \) where the growth is slower.

The structure of elements of \( \mathcal{H}_T \) is described by Lemma 4.4. Varying the positions of singular points, one obtains a trajectory of \( \mathcal{H}_T \) in the infinite-dimensional Grassmannian \( \text{Gr}(\mathcal{H}) \) defined with respect to the splitting \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \). Note that each of the subspaces \( \mathcal{H}_+ \) may be identified with \( N(n-3) \) copies of the space \( L^2(S^1) \) of functions on a circle; the factor \( n-3 \) corresponds to the number of annuli and \( N \) is the rank of the appropriate RHP.

We can also write

\[
\mathcal{H} = \mathcal{H}_T \oplus \mathcal{H}_-.
\]

(4.24)

The operator \( P_\oplus \) introduced above gives the projection on \( \mathcal{H}_T \) along \( \mathcal{H}_- \). Similarly, the operator \( P_\Sigma \) is a projection on \( \mathcal{H}_T \) along \( \text{ker} P_\Sigma \supseteq \mathcal{H}_A \). We would like to express it in terms of 3-point projectors. To this end let us regard \( f^{[k]}_\text{in}, f^{[k]}_\text{out} \) as coordinates on \( \mathcal{H}_T \). Suppose that \( f \in \mathcal{H} \) can be decomposed as \( f = g + h \) with \( g \in \mathcal{H}_T \) and \( h \in \mathcal{H}_A \). The latter condition means that

\[
h^{[k]}_\text{out,\pm} = h^{[k+1]}_\text{in,\pm}, \quad k = 1, \ldots, n-3,
\]

which can be equivalently written as a system of equations for components of \( g \):

\[
\begin{align*}
g^{[k]}_\text{in,\pm} - c^{[k-1]}_\text{in,\pm} g^{[k-1]}_\text{in,\pm} - d^{[k-1]}_\text{out,\pm} g^{[k-1]}_\text{out,\pm} &= f^{[k]}_\text{in,\pm} - f^{[k-1]}_\text{in,\pm}, \\
g^{[k]}_\text{out,\pm} - a^{[k+1]}_\text{in,\pm} g^{[k+1]}_\text{in,\pm} - b^{[k+1]}_\text{out,\pm} g^{[k+1]}_\text{out,\pm} &= f^{[k]}_\text{out,\pm} - f^{[k+1]}_\text{in,\pm},
\end{align*}
\]

(4.25)

where \( g^{[1]}_\text{in,\pm} = 0, g^{[n-2]}_\text{out,\pm} = 0 \). The first and second equations are valid in sufficiently
narrow annuli containing \( C_{in}^{[k]} \) and \( C_{out}^{[k]} \), respectively. Define

\[
\tilde{g}_k = \begin{pmatrix} g_{out,+}^{[k]} \\ g_{in,-}^{[k]+1} \end{pmatrix}, \quad \tilde{f}_k = \begin{pmatrix} f_{out,+}^{[k]} - f_{in,+}^{[k]+1} \\ f_{in,-}^{[k]} - f_{out,-}^{[k]+1} \end{pmatrix},
\]

\[
U_k = \begin{pmatrix} 0 & a^{[k+1]} \\ d^{[k]} & 0 \end{pmatrix}, \quad k = 1, \ldots, n - 3,
\]

\[
V_k = \begin{pmatrix} b^{[k+1]} \\ 0 \end{pmatrix}, \quad W_k = \begin{pmatrix} 0 & 0 & c^{[k+1]} \end{pmatrix}, \quad k = 1, \ldots, n - 4,
\]

\[
K = \begin{pmatrix} U_1 & V_1 & 0 & \ldots & 0 \\ W_1 & U_2 & V_2 & \ldots & 0 \\ 0 & W_2 & U_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & W_{n-4} & U_{n-3} \end{pmatrix}, \quad \bar{\tilde{g}} = \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \vdots \\ \tilde{g}_{n-3} \end{pmatrix}, \quad \bar{\tilde{f}} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_{n-3} \end{pmatrix}.
\]

(4.26)

The system (4.25) can then be rewritten in a block-tridiagonal form

\[
(1 - K) \bar{\tilde{g}} = \bar{\tilde{f}}.
\]

(4.27)

The decomposition \( \mathcal{H} = \mathcal{H}_T \oplus \mathcal{H}_A \) thus uniquely exists provided that \( 1 - K \) is invertible.

Let us prove a converse result and interpret \( K \) in a more invariant way. Consider the operators \( P_{\oplus,+} : \mathcal{H}_+ \to \mathcal{H}_T \) and \( P_{\Sigma,+} : \mathcal{H}_+ \to \mathcal{H}_T \) defined as restrictions of \( P_{\oplus} \) and \( P_{\Sigma} \) to \( \mathcal{H}_+ \). The first of them is invertible, with the inverse given by the projection on \( \mathcal{H}_+ \) along \( \mathcal{H}_- \). Hence one can consider the composition \( L = \text{End} (\mathcal{H}_+) \) defined by

\[
L := P_{\oplus,+}^{-1} P_{\Sigma,+}.
\]

(4.28)

We are now going to make an important assumption which is expected to hold generically (more precisely, outside the Malgrange divisor). It will soon become clear that it is satisfied at least in a sufficiently small finite polydisk \( \mathbb{D} \subset \mathbb{C}^{n-3} \) in the variables \( a_1, \ldots, a_{n-3} \), centered at the origin.

**Assumption 4.8.** \( P_{\Sigma,+} \) is invertible.

**Proposition 4.9.** For \( g \in \mathcal{H}_+ \), let \( \tilde{g}_k \) and \( \bar{\tilde{g}} \) be defined by (4.26). In these coordinates, \( L^{-1} = 1 - K \).

**Proof.** Rewrite the equation \( L^{-1} f' = f \) as \( P_{\oplus,+} f' = P_{\Sigma,+} f \). Setting \( f = P_{\oplus,+} f' + h \), the latter equation becomes equivalent to \( P_{\Sigma} h = 0 \). The solution thus reduces to constructing \( h \in \mathcal{H}_A \) such that \( (h + P_{\oplus,+} f')_+ = 0 \), where the projection is taken with respect to the splitting \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \). This can be achieved by setting

\[
h_{out,+}^{[k]} = h_{in,+}^{[k]+1} = - (P_{\oplus,+} f')_{in,+}^{[k]+1},
\]

\[
h_{out,-}^{[k]} = h_{in,-}^{[k]+1} = - (P_{\oplus,+} f')_{out,-}^{[k]+1}.
\]

It then follows that \( f = f' + h_+ = (1 - K) f' \). \( \Box \)
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

**Tau function**

**Definition 4.10.** Let \( L \in \text{End} (\mathcal{H}_+) \) be the operator defined by (4.28). We define the tau function associated to the Riemann–Hilbert problem for \( \Psi \) as

\[
\tau (a) := \det \left( L^{-1} \right).
\]  

(4.29)

In order to demonstrate the relation of (4.29) to conventional definition [JMU] of the isomonodromic tau function and its extension [ILP], let us compute the logarithmic derivatives of \( \tau \) with respect to isomonodromic times \( a_1, \ldots, a_{n-3} \). At this point it is convenient to introduce the notation

\[
\Delta_k = \frac{1}{2} \text{Tr} \Theta_k^2, \quad \bar{\Delta}_k = \frac{1}{2} \text{Tr} \bar{\Theta}_k^2.
\]

(4.30)

Recall that \( \bar{\Delta}_0 \equiv \Delta_0 \) and \( \bar{\Delta}_{n-2} \equiv \Delta_{n-1} \).

**Theorem 4.11.** We have

\[
\tau (a) = \Upsilon (a) \tau_{\text{JMU}} (a),
\]

(4.31)

where \( \tau_{\text{JMU}} (a) \) is defined up to a constant independent of \( a \) by

\[
d_a \ln \tau_{\text{JMU}} = \sum_{0 \leq k < l \leq n-2} \text{Tr} A_k A_l \, d \ln (a_k - a_l),
\]

(4.32)

and the prefactor \( \Upsilon (a) \) is given by

\[
\Upsilon (a) = \prod_{k=1}^{n-3} a_k^{\bar{\Delta}_k - \Delta_{k-1} - \Delta_k}.
\]

(4.33)

**Proof.** We will proceed in several steps.

**Step 1.** Choose a collection of points \( a^0 \) close to \( a \) in the sense that the same annuli can be used to define the tau function \( \tau (a^0) \). The collection \( a^0 \) will be considered fixed whereas \( a \) varies. Let us compute the logarithmic derivatives of the ratio \( \tau (a) / \tau (a^0) \).

First of all we can write

\[
\frac{\tau (a)}{\tau (a^0)} = \det \left( P_{\oplus,+} (a^0)^{-1} P_{\Sigma,+} (a^0) P_{\Sigma,+} (a)^{-1} P_{\oplus,+} (a) \right)
\]

(4.34)

Note that since \( P_{\Sigma,+} (a) : \mathcal{H}_+ \to \mathcal{H}_T (a) \) can be viewed as a projection of elements of \( \mathcal{H} \) along \( \mathcal{H}_A \), the composition

\[
P_{a^0 \to a} := P_{\Sigma,+} (a) P_{\Sigma,+} (a^0)^{-1} : \mathcal{H}_T (a^0) \to \mathcal{H}_T (a)
\]

is also a projection along \( \mathcal{H}_A \). It therefore coincides with the restriction \( P_{\Sigma} \big|_{\mathcal{H}_T (a^0)} \).

One similarly shows that

\[
F_{a^0 \to a} := P_{\oplus,+} (a) P_{\oplus,+} (a^0)^{-1} = P_{\oplus} \big|_{\mathcal{H}_T (a^0)}.
\]
The exterior logarithmic derivative of (4.34) can now be written as

\[
\begin{align*}
\frac{d_a \ln \tau(a)}{\tau(a^0)} & = - \text{Tr}_{H(a^0)} \left\{ d_a \left( \mathcal{F}_{a \rightarrow a^0} \mathcal{P}_{a^0 \rightarrow a} \right) \cdot \mathcal{P}_{a \rightarrow a^0} \mathcal{F}_{a^0 \rightarrow a} \right\} \\
& = - \text{Tr}_{H(a^0)} \left\{ \mathcal{F}_{a \rightarrow a^0} \cdot d_a \mathcal{P}_{a^0 \rightarrow a} \cdot \mathcal{P}_{a \rightarrow a^0} \mathcal{F}_{a^0 \rightarrow a} \right\} = (4.35) \\
& = - \text{Tr}_{H} \left\{ \mathcal{P}_{\otimes} (a^0) \cdot d_a \mathcal{P}_{\Sigma} (a) \cdot \mathcal{P}_{\Sigma} (a^0) \right\} \mathcal{P}_{\otimes} (a).
\end{align*}
\]

The possibility to extend operator domains as to have the second equality is a consequence of (4.23). Furthermore, using once again the projection properties, one shows that

\[
\mathcal{P}_{\Sigma} (a) \left( 1 - \mathcal{P}_{\Sigma} (a^0) \right) = 0, \quad \mathcal{P}_{\otimes} (a) \left( 1 - \mathcal{P}_{\otimes} (a^0) \right) = 0.
\]

which reduces the equation (4.35) to

\[
\frac{d_a \ln \tau(a)}{\tau(a^0)} = - \text{Tr}_{H} \left\{ \mathcal{P}_{\otimes} d_a \mathcal{P}_{\Sigma} \right\} = - \sum_{k=1}^{n-2} \text{Tr}_{H^{[k]}} \left\{ \mathcal{P}^{[k]} d_a \mathcal{P}_{\Sigma} \right\}. \quad (4.36)
\]

**Step 2.** Let us now proceed to calculation of the right side of (4.36). Computations of the same type have already been used in the proofs of Lemmata 4.6 and 4.7. The idea is that \( \Psi^{[k]} \) and \( \hat{\Psi} \) have the same jumps on the contour \( \Gamma^{[k]} \) which reduces the integrals in (4.19), (4.22) to residue computation. In particular, for \( f^{[k]} \in \mathcal{H}^{[k]} \) with \( k = 2, \ldots, n-3 \) we have

\[
\mathcal{P}^{[k]} d_a \mathcal{P}_{\Sigma} f^{[k]}(z) = \frac{1}{(2 \pi i)^2} \oint_{\mathcal{C}^{[k]}_{\text{in}}} \oint_{\mathcal{C}^{[k]}_{\text{out}}} \Psi^{[k]}_{\pm}(z) \Psi^{[k]}_{\pm}(z')^{-1} d_a \left( \hat{\Psi}_{\pm}(z') \hat{\Psi}_{\pm}(z'')^{-1} \right) f^{[k]}(z'') dz' dz''
\]

(4.37)

The integrals are computed with the prescription that \( z \) is located inside the contour of \( z' \), itself located inside the contour of \( z'' \), and then passing to boundary values. But since the function \( (z' - z'')^{-1} d_a \left( \hat{\Psi}_{\pm}(z') \hat{\Psi}_{\pm}(z'')^{-1} \right) \) has no singularity at \( z'' = z' \), the contours of \( z' \) and \( z'' \) can be moved through each other. This identifies the trace of the integral operator on the right of (4.37) with

\[
\text{Tr} \left\{ \mathcal{P}_{\otimes} d_a \mathcal{P}_{\Sigma} \right\} = \\
= - \frac{1}{(2 \pi i)^2} \oint_{\mathcal{C}^{[k]}_{\text{in}}} \oint_{\mathcal{C}^{[k]}_{\text{out}}} \text{Tr} \left\{ \Psi^{[k]}_{\pm}(z) \Psi^{[k]}_{\pm}(z')^{-1} d_a \left( \hat{\Psi}_{\pm}(z') \hat{\Psi}_{\pm}(z')^{-1} \right) dz dz' \right\} = \\
= - \frac{1}{(2 \pi i)^2} \oint_{\mathcal{C}^{[k]}_{\text{in}}} \oint_{\mathcal{C}^{[k]}_{\text{out}}} \text{Tr} \left\{ d_a \left( \Psi_{\pm}(z)^{-1} \right) \Psi^{[k]}_{\pm}(z) \Psi^{[k]}_{\pm}(z')^{-1} \hat{\Psi}_{\pm}(z') \right\} dz dz' = \\
- \frac{1}{(2 \pi i)^2} \oint_{\mathcal{C}^{[k]}_{\text{in}}} \oint_{\mathcal{C}^{[k]}_{\text{out}}} \text{Tr} \left\{ d_a \left( \hat{\Psi}_{\pm}(z') \hat{\Psi}_{\pm}(z')^{-1} \right) \Psi^{[k]}_{\pm}(z) \Psi^{[k]}_{\pm}(z')^{-1} \hat{\Psi}_{\pm}(z') \right\} dz dz',
\]

where \( z \) is considered to be inside the contour of \( z' \). The first term vanishes since the contours \( \mathcal{C}^{[k]}_{\text{in}} \) and \( \mathcal{C}^{[k]}_{\text{out}} \) in the integral with respect to \( z \) can be merged. In the second
term the integral with respect to $z'$ is determined by the residue at $z' = z$, which yields
\[
\text{Tr} \left( \mathcal{P}_\Sigma^{[k]} d_a \mathcal{P}_\Sigma \right) = \frac{1}{2\pi i} \oint_{\mathcal{C}_\text{in}^{[k]} \cup \mathcal{C}_\text{out}^{[k]}} \text{Tr} \left\{ d_a \left( \hat{\Psi}^+ (z)^{-1} \right) \cdot \Psi^+_1 (z) \cdot \partial_z \left( \Psi^+_1 (z)^{-1} \hat{\Psi}^+ (z) \right) \right\} dz.
\]

Recall that $\hat{\Psi}^+_1$, $\Psi^+_1^{[k]}$ are related to fundamental matrix solutions $\Phi$, $\Phi^{[k]}$ of $n$-point and 3-point Fuchsian systems by
\[
\begin{align*}
\hat{\Psi}^+_1 (z) \big|_{\mathcal{C}_\text{in}^{[k]}} &= S_{k-1}^{-1} (-z)^{-\Theta_k} \Phi (z), &\Psi^+_1 (z) \big|_{\mathcal{C}_\text{in}^{[k]}} &= S_{k-1}^{-1} (-z)^{-\Theta_k} \Phi^{[k]} (z), \\
\Psi^+_1 (z) \big|_{\mathcal{C}_\text{out}^{[k]}} &= S_{k-1}^{-1} (-z)^{-\Theta_k} \Phi^{[k]} (z), &\Psi^+_1 (z) \big|_{\mathcal{C}_\text{out}^{[k]}} &= S_{k}^{-1} (-z)^{-\Theta_k} \Phi^{[k]} (z).
\end{align*}
\]

This leads to
\[
\text{Tr} \left( \mathcal{P}_\Sigma^{[k]} d_a \mathcal{P}_\Sigma \right) = \frac{1}{2\pi i} \oint_{\mathcal{C}_\text{in}^{[k]} \cup \mathcal{C}_\text{out}^{[k]}} \text{Tr} \left\{ d_a \left( \Phi^{-1} \right) \cdot \Phi^{[k]} \cdot \partial_z \left( \Phi^{-1} \Phi^{[k]} \right) \right\} dz = \text{res}_{z=\alpha_k} \text{Tr} \left\{ d_a \Phi \cdot \Phi^{-1} \left( \partial_z \Phi \cdot \Phi^{-1} - \partial_z \Phi^{[k]} \cdot \Phi^{-1} \Phi^{[k]} \right) \right\}. \tag{4.38}
\]

The contributions of the subspaces $\mathcal{H}^{[1]}$ and $\mathcal{H}^{[n-2]}$ to the trace (4.36) can be computed in a similar fashion. The only difference is that instead of merging $\mathcal{C}_\text{in}^{[k]}$ with $\mathcal{C}_\text{out}^{[k]}$ one should now shrink the contour $\mathcal{C}_\text{out}^{[1]}$ to 0 and $\mathcal{C}_\text{in}^{[n-2]}$ to $\infty$. The result is given by the same formula (4.38).

**Step 3.** To complete the proof, it now remains to compute the residues in (4.38). Note that near the regular singularity $z = \alpha_k$ the fundamental matrices $\Phi$, $\Phi^{[k]}$ are characterized by the behavior
\[
\begin{align*}
\Phi (z \to \alpha_k) &= C_k (\alpha_k - z)^{\Theta_k} \left( 1 + \sum_{l=1}^{\infty} g_{k,l} (z - \alpha_k)^l \right) G_k, \tag{4.39a} \\
\Phi^{[k]} (z \to \alpha_k) &= C_k (\alpha_k - z)^{\Theta_k} \left( 1 + \sum_{l=1}^{\infty} g_{1,l}^{[k]} (z - \alpha_k)^l \right) G_1^{[k]}.
\end{align*}
\]

The coinciding leftmost factors ensure the same local monodromy properties. The rightmost coefficients appear in the $n$-point and 3-point RHPs as $G_k = \hat{\Psi} (\alpha_k)$, $G_1^{[k]} = \Psi^{[k]} (\alpha_k)$. It becomes straightforward to verify that as $z \to \alpha_k$, one has
\[
\partial_z \Phi \cdot \Phi^{-1} - \partial_z \Phi^{[k]} \cdot \Phi^{-1} = C_k (\alpha_k - z)^{\Theta_k} \left[ g_{k,1} - g_{1,1}^{[k]} + O (z - \alpha_k) \right] (z - \alpha_k)^{-\Theta_k} C_k^{-1},
\]
\[
d_a \Phi \cdot \Phi^{-1} = C_k (\alpha_k - z)^{\Theta_k} \left[ -\Theta_k da_k \frac{1}{z - \alpha_k} + O (1) \right] (z - \alpha_k)^{-\Theta_k} C_k^{-1}.
\]

In combination with (4.36), (4.38), this in turn implies that
\[
d_a \ln \tau (a) = \sum_{k=1}^{n-3} \text{Tr} \Theta_k \left( g_{k,1} - g_{1,1}^{[k]} \right) da_k. \tag{4.40}
\]
Substituting local expansion (4.39a) into the Fuchsian system (4.14), we may recursively determine the coefficients $g_{k,l}$. In particular, the first coefficient $g_{k,1}$ satisfies
\[ g_{k,1} + [\Theta_k, g_{k,1}] = G_k^{-1} \left( \sum_{l=0, l \neq k}^{n-2} \frac{A_l}{a_k - a_l} \right) G_k, \] (4.41)
so that
\[ \sum_{k=1}^{n-3} \text{Tr} (\Theta_k g_{k,1}) da_k = \sum_{k=1}^{n-3} \sum_{l=0, l \neq k}^{n-2} \frac{\text{Tr} A_k A_l}{a_k - a_l} da_k = d_a \ln \tau_{\text{JMU}}. \] (4.42)
The 3-point analog of the relation (4.41) is
\[ g_{1,1}^{[k]} + [\Theta_k, g_{1,1}^{[k]}] = G_1^{[k]} \frac{A_0^{[k]}}{a_k} G_1^{[k]-1}, \] which gives
\[ \text{Tr} \left( \Theta_k g_{1,1}^{[k]} \right) = \frac{\text{Tr} A_0^{[k]} A_1^{[k]}}{a_k} = \frac{\text{Tr} \left( A_1^{[k]} - A_0^{[k]} - A_1^{[k]} \right)}{2a_k} = \frac{\Delta_k - \Delta_{k-1} - \Delta_k}{a_k}. \] (4.43)
Combining (4.40) with (4.42) and (4.43) finally yields the statement of the theorem.
\[ \square \]
**Corollary 4.12.** Jimbo-Miwa-Ueno isomonodromic tau function $\tau_{\text{JMU}} (a)$ admits a block Fredholm determinant representation
\[ \tau_{\text{JMU}} (a) = \Upsilon (a) \cdot \det (I - K), \] (4.44)
where the operator $K$ is defined by (4.26). Its $N \times N$ subblocks (4.20) are expressed in terms of solutions $\Psi^{[k]}$ of RHPs associated to 3-point Fuchsian systems with prescribed monodromy.

**Example: 4-point tau function**

In order to illustrate the developments of the previous subsection, let us consider the simplest nontrivial case of Fuchsian systems with $n = 4$ regular singular points. Three of them have already been fixed at $a_0 = 0$, $a_2 = 1$, $a_3 = \infty$. There remains a single time variable $a_1 \equiv t$. To be able to apply previous results, it is assumed that $0 < t < 1$.

The monodromy data are given by 4 diagonal matrices $\Theta_{0,t,1,\infty}$ of local monodromy exponents and connection matrices $C_0$, $C_{t,\pm}$, $C_{1,\pm}$, $C_{\infty}$ satisfying the relations
\[ M_0 \equiv C_0 e^{2\pi i \Theta_0} C_0^{-1} = C_{t,-} C_{t,+}^{-1}, \quad e^{2\pi i \Theta} = C_{t,-} e^{2\pi i \Theta} C_{t,+}^{-1} = C_{1,-} C_{1,+}^{-1} \]
Observe that, in the hope to make the notation more intuitive, it has been slightly changed as compared to the general case. The indices 0, 1, 2, 3 are replaced by 0, t, 1, \infty. Also, for $n = 4$ there is only one nontrivial matrix $M_{0 \rightarrow k}$ (namely, with $k = 1$). Therefore it becomes convenient to work from the very beginning in a
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

\[ \tau_{JMU}(t) = t^{\frac{1}{2}} \text{Tr}(e^{z^2 - e^{-2i} - e^{2i}}) \det (I - U), \quad U = \begin{pmatrix} 0 & a \\ d & 0 \end{pmatrix} \in \text{End}(\mathcal{H}_C), \]
4.2. Tau functions as Fredholm determinants

where the operators $a \equiv a^{[R]} \equiv a^{[2]} : \Pi_- \mathcal{H}_C \to \Pi_+ \mathcal{H}_C$ and $d \equiv d^{[L]} \equiv d^{[1]} : \Pi_+ \mathcal{H}_C \to \Pi_- \mathcal{H}_C$ are given by

\begin{align}
(a g) (z) &= \frac{1}{2\pi i} \oint_C a (z, z') g (z') \, dz', \\
(a) (z, z') &= \frac{\Psi^{[R]} (z) \Psi^{[R]} (z')^{-1} - 1}{z - z'},
\end{align}

(4.49a)

\begin{align}
(d g) (z) &= \frac{1}{2\pi i} \oint_C d (z, z') g (z') \, dz', \\
(d) (z, z') &= \frac{1 - \Psi^{[L]} (z) \Psi^{[L]} (z')^{-1}}{z - z'}.
\end{align}

(4.49b)

The contour $C$ is oriented counterclockwise, which is the origin of sign difference in the expression for $d$ as compared to (4.20d). In the Fourier basis (4.47), the operators $a$ and $d$ are given by semi-infinite matrices whose $N \times N$ blocks $a_{-p}^{-q}$, $d_{-p}^{-q}$ are determined by

\begin{align}
a (z, z') &= \sum_{p, q \in \mathbb{Z}_+^*} a_{-p}^{-q} z^{-q + p} z'^{-q - p}, \\
d (z, z') &= \sum_{p, q \in \mathbb{Z}_+^*} d_{-p}^{-q} z^{-q + p} z'^{-q - p}.
\end{align}

(4.50)

It should be emphasized that the indices of $a_{-p}^{-q}$ and $d_{-p}^{-q}$ belong to different ranges, since in both cases $p, q$ are positive half-integers.

The matrix functions $\Psi^{[L]} (z)$, $\Psi^{[R]} (z)$ appearing in the integral kernels of $a$ and $d$ solve the 3-point RHPs associated to Fuchsian systems with regular singularities at $0, t, \infty$ and $0, 1, \infty$, respectively. In order to understand the dependence of the 4-point tau function on the time variable $t$, let us rescale the fundamental solution of the first system by setting

\begin{equation}
\Phi^{[L]} (z) = \tilde{\Phi}^{[L]} \left( \frac{z}{t} \right).
\end{equation}

(4.51)

The rescaled matrix $\tilde{\Phi}^{[L]} (z)$ solves a Fuchsian system characterized by the same monodromy as $\Phi^{[L]} (z)$ but the corresponding singular points are located at $0, 1, \infty$. Denote by $\tilde{\Psi}^{[L]} (z)$ the solution of the RHP associated to $\tilde{\Phi}^{[L]} (z)$. To avoid possible confusion of the reader, we explicitly indicate the contours and jump matrices for RHPs for $\tilde{\Psi}^{[L]}$.
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

and $Ψ^{[R]}$ in Fig. 4.11; note the independence of jumps on $t$. In particular, inside the disk around $∞$ we have $\tilde{Φ}^{[L]} (z) = (−z)^Ω \tilde{Ψ}^{[L]} (z)$. Since the annulus $A$ belongs to the disk around $∞$ in the RHP for $Ψ^{[L]}$, the formula (4.51) yields the following expression for $Ψ^{[L]}$ inside $A$:

$$Ψ^{[L]} (z) \bigg|_A = (−z)^Ω \tilde{Φ}^{[L]} (z) = t^{-Ω} \tilde{Ψ}^{[L]} \left( \frac{z}{t} \right) = t^{-Ω} \left( 1 + \sum_{k=1}^{∞} g^L_k t^k z^{-k} \right) G^{[L]}_0, \quad (4.52a)$$

where the $N \times N$ matrix coefficients $g^L_k$ are independent of $t$. Analogous expression for $Ψ^{[R]} (z)$ inside $A$ does not contain $t$ at all:

$$Ψ^{[R]} (z) \bigg|_A = \left( 1 + \sum_{k=1}^{∞} g^R_k z^k \right) G^{[R]}_0. \quad (4.52b)$$

The formulae (4.52) allow to extract from the determinant representation (4.48) the asymptotics of 4-point Jimbo-Miwa-Ueno tau function $τ_{JMU} (t)$ as $t \to 0$ to any desired order. We are now going to explain the details of this procedure.

Rewrite the integral kernel $d (z, z')$ as

$$d (z, z') = t^{-Ω} \frac{1 - \tilde{Ψ}^{[L]} \left( \frac{z}{t} \right) \tilde{Ψ}^{[L]} \left( \frac{z'}{t} \right)^{-1}}{z - z'} t^Ω. \quad \text{The block matrix elements of } d \text{ in the Fourier basis are therefore given by}$$

$$d^{-p}_{−q} = t^{-Ω} \tilde{d}^{-p}_{−q} t^Ω, \quad p, q \in \mathbb{Z}_+, \quad (4.53)$$

where $N \times N$ matrix coefficients $\tilde{d}^{-p}_{−q}$ are independent of $t$. They can be extracted from the Fourier series

$$\frac{1 - \tilde{Ψ}^{[L]} (z) \tilde{Ψ}^{[L]} (z')^{-1}}{z - z'} = \sum_{p, q \in \mathbb{Z}_+} \tilde{d}^{-p}_{−q} z^{-1/2−p} z'^{-1/2−q}, \quad (4.54)$$

and are therefore expressed in terms of the coefficients of local expansion of the 3-point solution $\tilde{Φ}^{[L]} (z)$ around $z = ∞$ by straightforward algebra. For instance, the first few coefficients are given by

$$\tilde{d}^{-1/2}_{−1} = g^L_1, \quad \tilde{d}^{-3/2}_{−2} = g^L_2, \quad \tilde{d}^{-5/2}_{−3} = g^L_3,$$

$$\tilde{d}^{-1/2}_{−1} = g^L_2 - g^L_1, \quad \tilde{d}^{-3/2}_{−2} = g^L_3 - g^L_2,$$

$$\tilde{d}^{-1/2}_{−1} = g^L_3 - g^L_2 g^L_1 + g^L_1 g^L_2, \quad \tilde{d}^{-3/2}_{−2} = g^L_3 - g^L_2 g^L_1.$$ 

Different lines above contain the coefficients of fixed degree $p + q \in \mathbb{Z}_{>0}$ which appears in the power of $t$ in (4.53). Very similar formulas are also valid for matrix elements of $a$:

$$a^{-1/2}_{−1} = g^R_1, \quad a^{-1/2}_{−1} = g^R_2 - g^R_1,$$ 

$$a^{-3/2}_{−2} = g^R_3, \quad a^{-3/2}_{−2} = g^R_2 - g^R_1.$$
The crucial point for the asymptotic analysis of $\tau(t)$ is that for small $t$ the operator $d$ becomes effectively finite rank. Indeed, fix a positive integer $Q$. To obtain a uniform approximation of $d(z,z')$ up to order $O(t^Q)$, it suffices to take into account its Fourier coefficients $d_{-p}^{-q}$ with $p+q \leq Q$; recall that the eigenvalues of $\Theta$ are chosen as to satisfy (4.45). Since here $p,q \in \mathbb{Z}'_+$, the total number of relevant coefficients is finite and equal to $Q(Q-1)/2$. It follows that the only terms in the Fourier expansion of $a(z,z')$ that contribute to the determinant (4.48) to order $O(t^Q)$ correspond to monomials $z^{p-\frac{1}{2}}z'^{q-\frac{1}{2}}$ with $p+q \leq Q$. This is summarized in

**Theorem 4.13.** Let $Q \in \mathbb{Z}_{>0}$. The 4-point tau function $\tau_{JMU}(t)$ has the following asymptotics as $t \to 0$:

$$
\tau_{JMU}(t) \simeq t^{-\frac{1}{2}} \text{Tr}(e^{\sigma_2-\sigma_0^2-\sigma_2^2}) \left[ \det (1 - U_Q) + O(t^Q) \right], \quad U_Q = \begin{pmatrix} 0 & a_Q \\ d_Q & 0 \end{pmatrix}. \quad (4.55)
$$

Here $U_Q$ denotes a $2NQ \times 2NQ$ finite matrix whose $NQ \times NQ$-dimensional blocks $a_Q$ and $d_Q$ are themselves block lower and block upper triangular matrices of the form

$$
a_Q = \begin{pmatrix}
\begin{array}{cccc}
a_{-\frac{1}{2}}^{Q-\frac{1}{2}} & 0 & \cdots & 0 \\
\vdots & a_{-\frac{1}{2}}^{Q-\frac{1}{2}} & \ddots & \vdots \\
-a_{-\frac{1}{2}}^{\frac{1}{2}} & \cdots & \ddots & 0 \\
a_{-\frac{1}{2}}^{\frac{1}{2}} & a_{-\frac{1}{2}}^{\frac{1}{2}} & \cdots & a_{-\frac{1}{2}}^{\frac{1}{2}-Q}
\end{array}
\end{pmatrix}, \quad d_Q = t^{-\Theta}
$$

where $a_{-q}^{-p}$, $d_{-p}^{-q}$ are determined by (4.49a), (4.50), (4.54), and the conjugation by $t^{\Theta}$ in the expression for $d_Q$ is understood to act on each $N \times N$ block of the interior matrix. Moreover, strengthening the condition (4.45) to strict inequality $|\Re(\sigma_0 - \sigma_0)| < 1$ improves the error estimate in (4.55) to $o(t^Q)$.

**Remark 4.14.** The above theorem gives the asymptotics of $\tau_{JMU}(t)$ to arbitrary finite order $Q$ in terms of solutions $\Phi^{[R]}(z)$, $\Phi^{[L]}(z)$ of two 3-point Fuchsian systems with prescribed monodromy around regular singular points 0, 1, $\infty$. For $Q = 1$ and under assumption $|\Re(\sigma_0 - \sigma_0)| < 1$, its statement may be rewritten as

$$
\tau_{JMU}(t) \simeq t^{-\frac{1}{2}} \text{Tr}(e^{\sigma_2-\sigma_0^2-\sigma_2^2}) \left[ \det \left( 1 - g_1^{[R]} t^{1-\Theta} g_1^{[L]} t^{\Theta} \right) + o(t) \right]. \quad (4.56)
$$

A result equivalent to this last formula has been recently obtained in [ILP, Proposition 3.9] by a rather involved asymptotic analysis based on the conventional Riemann-Hilbert approach. For $N = 2$, the leading term in the expansion of the determinant appearing in (4.56) gives Jimbo asymptotic formula [Jimbo] for Painlevé VI.

**4.3. Fourier basis and combinatorics**

**Structure of matrix elements**

Let us return to the general case of $n$ regular singular points on $\mathbb{P}^1$. We have already seen in the previous subsection certain advantages of writing the operators which
appear in the Fredholm determinant representation (4.44) of the tau function in the Fourier basis. This motivates us to introduce the following notation for the integral kernels of the 3-point projection operators $a^{[k]}$, $b^{[k]}$, $c^{[k]}$, $d^{[k]}$ from (4.20):

\[
\begin{align*}
\Psi_+^{[k]}(z, z') := &\frac{-\Psi_+^{[k]}(z) \Psi_+^{[k]}(z')^{-1} - 1}{z - z'} = \sum_{p, q \in Z_+} a^{[k]}_{-q} z^{-\frac{1}{2} + p} z^{-\frac{1}{2} + q}, \quad z, z' \in C_{in}^{[k]}, \\
b^{[k]}(z, z') := &\frac{-\Psi_+^{[k]}(z) \Psi_+^{[k]}(z')^{-1}}{z - z'} = \sum_{p, q \in Z_+} b^{[k]}_{-q} z^{-\frac{1}{2} + p} z^{-\frac{1}{2} - q}, \quad z \in C_{in}^{[k]}, z' \in C_{out}^{[k]}, \\
c^{[k]}(z, z') := &\frac{\Psi_+^{[k]}(z) \Psi_+^{[k]}(z')^{-1}}{z - z'} = \sum_{p, q \in Z_+} c^{[k]}_{-q} z^{-\frac{1}{2} + p} z^{-\frac{1}{2} + q}, \quad z \in C_{out}^{[k]}, z' \in C_{in}^{[k]}, \\
d^{[k]}(z, z') := &\frac{1 - \Psi_+^{[k]}(z) \Psi_+^{[k]}(z')^{-1}}{z - z'} = \sum_{p, q \in Z_+} d^{[k]}_{-q} z^{-\frac{1}{2} + p} z^{-\frac{1}{2} - q}, \quad z, z' \in C_{out}^{[k]}. 
\end{align*}
\]

Just as before in (4.49b), the overall minus signs in the expressions for $b^{[k]}(z, z')$ and $d^{[k]}(z, z')$ are introduced to absorb the negative orientation of $C_{out}^{[k]}$.

Our task in this subsection is to understand the dependence of matrix elements $a^{[k]}_{p}, b^{[k]}_{p}, c^{[k]}_{-p}, d^{[k]}_{-p}$ on their indices $p, q \in Z_+$. To this end recall that (cf (4.15))

\[
\Psi_+^{[k]}(z) = \begin{cases} 
-\xi_{k-1} S_{k-1}^{-1} \Phi^{[k]}(z), & z \in C_{in}^{[k]}, \\
-\xi_{k} S_{k}^{-1} \Phi^{[k]}(z), & z \in C_{out}^{[k]}. 
\end{cases}
\]

where $\Phi^{[k]}(z)$ denotes the fundamental solution of the 3-point Fuchsian system (4.16).

**Theorem 4.15.** Denote by $\psi^{[k]}_r$ the rank of the matrix $A^{[k]}_1$ which appears in the Fuchsian system (4.16). Let $u^{[k]}_r, v^{[k]}_r \in \mathbb{C}^N$ with $r = 1, \ldots, \psi^{[k]}_r$ be the column and row vectors giving the decomposition

\[
a^{[k]}_k A^{[k]}_1 = - \sum_{r=1}^{\psi^{[k]}_r} u^{[k]}_r \otimes v^{[k]}_r. 
\]

Let $(\psi^{[k]}_r)^{-p}, (\bar{\psi}^{[k]}_r)^{p}, (\phi^{[k]}_r)^{-p}, (\bar{\phi}^{[k]}_r)^{-p} \in \mathbb{C}^N$ be the coefficients of the Fourier expansions
Then the operators $a^{[k]}_p, b^{[k]}_q, c^{[k]}_r, d^{[k]}_s$ can be represented as sums of a finite number of infinite-dimensional Cauchy matrices with respect to the indices $p,q$, explicitly given by

\begin{align}
  a^{[k]}_{-q} &= \sum_{r=1}^{e^{[k]}} (\bar{\psi}^{[k]}_r)^{p\alpha} (\bar{\psi}^{[k]}_r)^{q\beta}, \\
  b^{[k]}_{p\alpha} &= \sum_{r=1}^{e^{[k]}} (\bar{\psi}^{[k]}_r)^{p\alpha} (\bar{\psi}^{[k]}_r)^{-q\beta}, \\
  c^{[k]}_{-p\alpha} &= \sum_{r=1}^{e^{[k]}} (\bar{\psi}^{[k]}_r)^{-p\alpha} (\bar{\psi}^{[k]}_r)^{q\beta}, \\
  d^{[k]}_{-p\alpha} &= \sum_{r=1}^{e^{[k]}} (\bar{\psi}^{[k]}_r)^{-p\alpha} (\bar{\psi}^{[k]}_r)^{-q\beta},
\end{align}

where the color indices $\alpha, \beta = 1, \ldots, N$ correspond to internal structure of the blocks $a^{[k]}_p, b^{[k]}_q, c^{[k]}_r, d^{[k]}_s$.

**Proof.** The Fuchsian system (4.16) can be used to differentiate the integral kernels (4.57) with respect to $z$ and $z'$. Consider, for instance, the operator

\[ L_0 = z \partial_z + z' \partial_{z'} + 1. \]

It is easy to check that \( L_0 \frac{1}{z-z'} = 0 \). Combining this with (4.57a), (4.58) and (4.16), one obtains e.g. that

\[ L_0 a^{[k]}_{z} (z, z') = \frac{(z \partial_z + z' \partial_{z'}) \Psi^+[z] (z) \Psi^+[z] (z')^{-1}}{z - z'} = \left[ a^{[k]}_{z} (z, z'), \mathcal{S}_k \right] - \frac{\Psi^+[z] (z)}{z - a_k} a_k A^{[k]}_{z} \frac{\Psi^+[z] (z')^{-1}}{z' - a_k}, \]

where $z, z' \in C^{[k]}$. The crucial point here is that the dependence of the second term on $z$ and $z'$ is completely factorized. Indeed, it follows from the last identity, the

\[2\text{The reader with acquaintance with two-dimensional conformal field theory will recognize in this equation the dilatation Ward identity for the 2-point correlator of Dirac fermions.} \]
form of $\mathcal{L}_0$ and the notation (4.60a) that $N \times N$ matrix $a^{[k]}_{q}p$ from (4.57a) satisfies the equation

$$
(p + q + \text{ad}_{\mathfrak{g}_{k-1}}) a^{[k]}_{q}p = \sum_{r=1}^{\mathfrak{g}[k]} (\psi_{r}^{[k]})^{p} \otimes (\bar{\psi}_{r}^{[k]})^{q},
$$

The formula (4.61a) is nothing but a rewrite of this identity. The proof of Cauchy type representations (4.61b)–(4.61d) for the other three operators is completely analogous. □

**Combinatorics of determinant expansion**

This subsection develops a systematic approach to the computation of multivariate series expansion of the Fredholm determinant $\tau(a) = \det(\mathcal{I} - K)$. Recall that, according to Theorem 4.11, the isomonodromic tau function $\tau_{\text{IMU}}(a)$ coincides with $\tau(a)$ up to an elementary explicit prefactor.

Let $A \in \mathbb{C}^{X \times X}$ be a matrix indexed by a discrete and possibly infinite set $X$. Our basic tool for expanding $\tau(a)$ is the von Koch's formula:

$$
\det(\mathcal{I} + A) = \sum_{\emptyset \in \mathcal{X}} \det A_{\emptyset},
$$

where $\det A_{\emptyset}$ denotes the $|\emptyset| \times |\emptyset|$ principal minor obtained by restriction of $A$ to a subset $\emptyset \subseteq X$. Of course, the series in (4.62) terminates when $X$ is finite.

In our case, the role of the matrix $A$ is played by the operator $K$ written in the Fourier basis. The elements of $X$ are multi-indices which encode the following data:

- the positions of the blocks $a^{[k]}, b^{[k]}, c^{[k]}, d^{[k]}$ in $K$ defined by (4.26);
- a half-integer Fourier index of the appropriate block;
- a color index taking its values in the set $\{1, \ldots, N\}$.

It is useful to combine Fourier and color indices into one multi-index $\iota = (p, \alpha) \in \mathfrak{N} := \mathbb{Z}^{l} \times \{1, \ldots, N\}$. Unordered sets $\{i_1, \ldots, i_m\} \in 2^\mathfrak{N}$ of such multi-indices are denoted by capital Roman letters $I$ or $J$. Given a matrix $M \in \mathbb{C}^{\mathfrak{N} \times \mathfrak{N}}$, we denote by $M_I^J$ its $|I| \times |J|$ restriction to rows $I$ and columns $J$.

Principal submatrices of $K$ may be labeled by pairs $(\bar{I}, \bar{J})$, where $\bar{I} = (I_1, \ldots, I_{n-3})$, $\bar{J} = (J_1, \ldots, J_{n-3})$ and $J_{1..n-3}, J_{1..n-3} \in 2^\mathfrak{N}$. Namely, define

$$
K_{\bar{I}, \bar{J}} :=
\begin{pmatrix}
0 & (a^{[\mathfrak{N}]}_{I_1})_{J_1}^{r_1} & (a^{[\mathfrak{N}]}_{I_1})_{J_2}^{r_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & (a^{[\mathfrak{N}]}_{I_{n-3}})_{J_1}^{r_{n-3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
$$
For reasons that will become apparent below, the pairs \((\vec{I}, \vec{J})\) will be referred to as configurations. It is useful to keep in mind that the lower index in \(I_k, J_k\) corresponds to the annulus \(A_k\), and the blocks of \(K\) are acting between spaces of holomorphic functions on the appropriate annuli.

**Definition 4.16.** A configuration \((\vec{I}, \vec{J}) \in (2^n) \times 2^{(n-3)}\) is called

- balanced if \(|I_k| = |J_k|\) for \(k = 1, \ldots, n - 3\);
- proper if all elements of \(I_k\) (and \(J_k\)) have positive (resp. negative) Fourier indices for \(k = 1, \ldots, n - 3\).

The sets of all balanced and proper balanced configurations will be denoted by \(\text{Conf}\) and \(\text{Conf}_+\), respectively.

**Definition 4.17.** For \((\vec{I}, \vec{J}) \in \text{Conf}\), define

\[
Z^{I_{k-1}, J_{k-1}}_{I_k, J_k} (\mathcal{T}^{[k]}) := (-1)^{|I_k|} \det \begin{pmatrix} (a^{[k]})^{I_{k-1}}_{J_{k-1}} & (b^{[k]})^{I_{k-1}}_{J_k} \\ (c^{[k]})^{J_k}_{J_{k-1}} & (d^{[k]})^{J_k}_{I_k} \end{pmatrix}, \quad k = 1, \ldots, n - 2. \tag{4.63a}
\]

In order to have uniform notation, here we set \(I_0 = J_0 = I_{n-2} = J_{n-2} \equiv \emptyset\), so that

\[
Z^{0,0}_{I_1, J_1} (\mathcal{T}^{[1]}) = (-1)^{|I_1|} \det (d^{[1]})_{I_1}, \quad Z^{I_{n-3}, J_{n-3}}_{\emptyset, \emptyset} (\mathcal{T}^{[n-2]}) = \det (a^{[n-2]})^{I_{n-3}}_{J_{n-3}}. \tag{4.63b}
\]

**Proposition 4.18.** The principal minor \(D_{\vec{I}, \vec{J}} := \det K_{\vec{I}, \vec{J}}\) vanishes unless \((\vec{I}, \vec{J}) \in \text{Conf}_+\), in which case it factorizes into a product of \(n - 2\) finite \((|I_{k-1}| + |I_k|) \times (|J_{k-1}| + |J_k|)\) determinants as

\[
D_{\vec{I}, \vec{J}} = \prod_{k=1}^{n-2} Z^{I_{k-1}, J_{k-1}}_{I_k, J_k} (\mathcal{T}^{[k]}). \tag{4.64}
\]

**Proof.** For \(k = 1, \ldots, n - 3\), exchange the \((2k - 1)\)-th and \(2k\)-th block row of the matrix \(K_{\vec{I}, \vec{J}}\). As such permutation can only change the sign of the determinant, the proposition for balanced configurations follows immediately from the block structure of the resulting matrix. The sign change is taken into account by the factor \((-1)^{|I_k|}\) in (4.63a).

The only non-zero Fourier coefficients of \(a^{[k]}, b^{[k]}, c^{[k]}, d^{[k]}\) are given by (4.57). Therefore, if a configuration \((\vec{I}, \vec{J}) \in \text{Conf}\) is not proper, then at least one of the factors on the right of (4.64) vanishes due to the presence of zero rows or columns in the relevant matrices. \(\square\)

**Corollary 4.19.** Fredholm determinant \(\tau (a)\) is given by

\[
\tau (a) = \sum_{(\vec{I}, \vec{J}) \in \text{Conf}_+} \prod_{k=1}^{n-2} Z^{I_{k-1}, J_{k-1}}_{I_k, J_k} (\mathcal{T}^{[k]}). \tag{4.65}
\]
Proof. Another useful consequence of the block structure of the operator $K$ is that $\text{Tr } K^{2m+1} = 0$ for $m \in \mathbb{Z}_{\geq 0}$. This implies that $\text{det } (1 - K) = \text{det } (1 + K)$. It now suffices to combine this symmetry with von Koch's formula (4.62) and Proposition 4.18. □

Let us now give a combinatorial description of the set $\text{Conf}_+$ of proper balanced configurations in terms of Maya diagrams and charged partitions.

**Definition 4.20.** A Maya diagram is a map $m : \mathbb{Z}' \to \{-1, 1\}$ subject to the condition that $m(p) = \pm 1$ for all but finitely many $p \in \mathbb{Z}'_{\pm}$. The set of all Maya diagrams will be denoted by $M$.

A convenient graphical representation of $m \in M$ is obtained by replacing $-1$’s and 1’s by white and black circles located at the sites of half-integer lattice, see bottom part of Fig. 4.12 for an example. The white circles in $\mathbb{Z}'_+$ and black circles in $\mathbb{Z}'_-$ are referred to as particles and holes in the Dirac sea, which itself corresponds to the diagram $m_0$ defined by $m_0(p) = \pm 1$. An arbitrary diagram is completely determined by a sequence $p(m) = (p_1, \ldots, p_r)$ of strictly decreasing positive half-integers $p_1 > \ldots > p_r$, giving the positions of particles, and a sequence $h(m) = (-q_1, \ldots, -q_s)$ of strictly increasing negative half-integers $-q_1 < \ldots < -q_s$ corresponding to the positions of holes. The integer $Q(m) := |p(m)| - |h(m)|$ is called the charge of $m$.

Given a configuration $(\vec{I}, \vec{J}) \in \text{Conf}_+$, consider a pair of its multi-indices $(I_k, J_k)$ associated to the annulus $A_k$. Recall that the Fourier indices of elements of $I_k$ (and $J_k$) are positive (resp. negative). They can therefore be interpreted as positions of particles and holes of $N$ different colors. This yields a bijection between the set of pairs $(I_k, J_k)$ verifying the balance condition $|I_k| = |J_k|$ and the set

$$M_0^N = \left\{ (m^{(1)}, \ldots, m^{(N)}) \in M^N \bigg| \sum_{\alpha=1}^{N} Q(m^{(\alpha)}) = 0 \right\}$$

of $N$-tuples of Maya diagrams with vanishing total charge. We thereby obtain a one-to-one correspondence

$$\text{Conf}_+ \cong \underbrace{M_0^N \times \ldots \times M_0^N}_{n-3 \text{ factors}}.$$

**Definition 4.21.** A charged partition is a pair $\hat{Y} = (Y, Q) \in \mathbb{Y} \times \mathbb{Z}$. The integer $Q$ is called the charge of $\hat{Y}$.

There is a well-known bijection between Maya diagrams and charged partitions, whose construction is illustrated in Fig. 4.12. Given a Maya diagram $m \in M$, we start far on the north-west axis and draw a segment directed above each white circle and a segment directed north-east above each black circle. The resulting polygonal line defines the outer boundary of the Young diagram $Y$ corresponding to $m$. The charge $Q = Q(m)$ of $\hat{Y}$ is the signed distance between $Y$ and the north-east axis. In the case $Q(m) = 0$, the sequences $p(m)$ and $-h(m)$ give the Frobenius coordinates of $Y$.  

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4.3. Fourier basis and combinatorics

Figure 4.12: The correspondence between Maya diagrams and charged partitions; here the charge \( Q(m) = 2 \) and the positions of particles and holes are given by \( p(m) = \left( \frac{13}{2}, \frac{7}{2}, \frac{3}{2}, \frac{1}{2} \right) \) and \( h(m) = \left( -\frac{5}{2}, -\frac{1}{2} \right) \).

Let us write \( N \)-tuples \( \left( \vec{Y}^{(1)}, \ldots, \vec{Y}^{(N)} \right) \) of charged partitions as \( \left( \vec{Y}, \vec{Q} \right) \), with \( \vec{Y} = (Y^{(1)}, \ldots, Y^{(N)}) \in \mathbb{Y}^N \) and \( \vec{Q} = (Q^{(1)}, \ldots, Q^{(N)}) \in \mathbb{Z}^N \). The set of such \( N \)-tuples with zero total charge can be identified with \( \mathbb{M}_0^N \cong \mathbb{Y}^N \times \Omega_N \), where \( \Omega_N \) denotes the \( A_{N-1} \) root lattice:

\[
\Omega_N := \left\{ \vec{Q} \in \mathbb{Z}^N \mid \sum_{\alpha=1}^N Q^{(\alpha)} = 0 \right\}.
\]

This suggests to introduce an alternative notation for elementary finite determinant factors in (4.65). For \( |I_{k-1}| = |J_{k-1}| \) and \( |I_k| = |J_k| \), we define

\[
Z_{\vec{Y}_{k-1}, \vec{Q}_{k-1}}^{\vec{Y}_k, \vec{Q}_k} \left( \mathcal{T}^{[k]} \right) := Z_{I_{k-1}, J_{k-1}}^{I_k, J_k} \left( \mathcal{T}^{[k]} \right),
\]

where \( \left( \vec{Y}_{k-1}, \vec{Q}_{k-1} \right), \left( \vec{Y}_k, \vec{Q}_k \right) \in \mathbb{Y}^N \times \Omega_N \) are associated to \( N \)-tuples of Maya diagrams describing subconfigurations \( (I_{k-1}, J_{k-1}), (I_k, J_k) \). In what follows, the two notations are used interchangeably.

The structure of the expansion of \( \tau(a) \) may now be summarized as follows.

**Theorem 4.22.** Fredholm determinant \( \tau(a) \) giving the isomonodromic tau function \( \tau_{JMU}(a) \) can be written as a combinatorial series

\[
\tau(a) = \sum_{\vec{Q}_1, \ldots, \vec{Q}_n \in \Omega_N} \sum_{\vec{Y}_1, \ldots, \vec{Y}_n \in \mathbb{Y}^N} \prod_{k=1}^{n-2} Z_{\vec{Y}_k, \vec{Q}_k}^{\vec{Y}_{k-1}, \vec{Q}_{k-1}} \left( \mathcal{T}^{[k]} \right),
\]

where \( Z_{\vec{Y}_k, \vec{Q}_k}^{\vec{Y}_{k-1}, \vec{Q}_{k-1}} \left( \mathcal{T}^{[k]} \right) \) are expressed by (4.66), (4.63) in terms of matrix elements of 3-point Plemelj operators in the Fourier basis.

**Example 4.23.** Let us outline simplifications to the above scheme in the case \( N = 2 \), \( n = 4 \) corresponding to the Painlevé VI equation. Here a configuration \( \left( \vec{I}, \vec{J} \right) \in \text{Conf}_+ \) is given by a single pair \( (I, J) \) of multi-indices whose structure may be described as follows: \( I \) (and \( J \)) encode the positions of particles (resp. holes) of two colors \( \{+, -\} \),
and the total number of particles in $I$ coincides with the total number of holes in $J$. Relative positions of particles and holes of each color are described by two Young diagrams $Y_+, Y_- \in Y$. The vectors $(Q_+, Q_-) \in \Omega_2$ of the charge lattice are labeled by a single integer $n = Q_+ = -Q_- \in \mathbb{Z}$. In the notation of Subsection 4.2.5, the series (4.65) can be rewritten as

$$
\tau (t) = \sum_{n \in \mathbb{Z}} \sum_{P^+, P^- \in \mathbb{Z}^+; h^+, h_- \in \mathbb{Z}^-} (-1)^{|P^+| + |P^-|} \det a_{h^+, h_-}^{P^+, P^-} \det d_{h^+, h_-}^{P^+, P^-} =
\sum_{n \in \mathbb{Z}} \sum_{Y_+, Y_- \in Y} Z_{Y_+, Y_-} (T^{[L]}) Z_{Y_+, Y_-} (T^{[R]}) ，
$$

where $Z_{Y_+, Y_-} (T^{[L]}) = (-1)^{|P^+| + |P^-|} \det d_{h^+, h_-}^{P^+, P^-}$ and $Z_{Y_+, Y_-} (T^{[R]}) = \det a_{h^+, h_-}^{P^+, P^-}$. In these equations, the particle/hole positions $(P^+, h^+)$ and $(P^-, h_-)$ for the 1st and 2nd color are identified with a pair of Maya diagrams, subsequently interpreted as charged partitions $(Y_+, n)$ and $(Y_-, -n)$.

**Remark 4.24.** Describing the elements of $\text{Conf}_+$ in terms of $N$-tuples of Young diagrams and vectors of the $A_{N-1}$ root lattice is inspired by their appearance in the four-dimensional $\mathcal{N} = 2$ supersymmetric linear quiver gauge theories. Combinatorial structure of the dual partition functions of such theories [Nek, NO] coincides with that of (4.67). These partition functions can in fact be obtained from our construction or its higher genus/irregular extensions by imposing additional spectral constraints on monodromy. It will shortly become clear that the multiple sum over $\Omega_N$ is responsible for a Fourier transform structure of the isomonodromic tau functions. This structure was discovered in [GIL12, ILT13] for Painlevé VI, understood for $\mathcal{N} = 2$ and arbitrary number of punctures within the framework of Liouville conformal field theory [ILTc], and conjectured to appear in higher rank in [Gav]. It might be interesting to mention the appearance of a possibly related structure in the study of topological string partition functions [GHM, BGT].

**Rank two case**

For $N = 2$, the elementary 3-point RHPs can be solved in terms of Gauss hypergeometric functions so that Fredholm determinant representation (4.44) becomes completely explicit. Being rewritten in Fourier components, the blocks of $K$ may be reduced to single infinite Cauchy matrices acting in $\ell^2 (\mathbb{Z})$. We are going to use this observation to calculate the building blocks $Z_{Y_k, Q_k} (T^{[k]})$ of principal minors of $K$ in terms of monodromy data, and derive thereby a multivariate series representation for the isomonodromic tau function of the Garnier system.
Gauss and Cauchy in rank 2

The form of the Fuchsian system (4.14) is preserved by the following non-constant scalar gauge transformation of the fundamental solution and coefficient matrices:

\[ \Phi(z) \mapsto \hat{\Phi}(z) \prod_{l=0}^{n-2} (z - a_l)^{\kappa_l}, \]

\[ A_l \mapsto \hat{A}_l + \kappa_l \mathbb{1}, \quad l = 0, \ldots, n - 2. \]

Under this transformation, the monodromy matrices \( M_l \) are multiplied by \( e^{-2\pi i \kappa_l} \), and the associated Jimbo-Miwa-Ueno tau function transforms as

\[ \tau_{\text{JMU}}(a) \mapsto \hat{\tau}_{\text{JMU}}(a) \prod_{0 \leq k < l \leq n-2} (a_l - a_k)^{-N_{\kappa_l \kappa_l + \kappa_k \Theta_l + \kappa_l \Theta_k}}. \]

The freedom in the choice of \( \kappa_0, \ldots, \kappa_{n-2} \) allows to make the following assumption.

**Assumption 4.25.** One of the eigenvalues of each of the matrices \( \Theta_0, \ldots, \Theta_{n-2} \) is equal to 0.

This involves no loss in generality and means in particular that the ranks \( r \) of the coefficient matrices \( A_{1[l]} \) in the auxiliary 3-point Fuchsian systems (4.16) are at most \( N - 1 \).

For \( r^{[k]} = 1 \), the factor \( Z_{I_{k-1}J_k}^{I_{k-1}J_k} (\mathcal{T}^{[k]}) \) in (4.67) can be computed in explicit form. In this case the sums such as (4.59) or (4.61) contain only one term, and the index \( r \) can therefore be omitted. The matrix \( A_{1[l]}^{[k]} \in \mathbb{C}^{N \times N} \) may be written as

\[ a_k A_{1[l]}^{[k]} = -u^{[k]} \otimes v^{[l]}. \]

The crucial observation is that the blocks (4.61) are now given by single Cauchy matrices conjugated by diagonal factors (instead of being a sum of such matrices). In order to put this to a good use, let us introduce two complex sequences \( (x_i^{[k]}), (y_j^{[l]})_{j \in J_{k-1} \cup J_k} \) of the same finite length \( |I_{k-1}| + |I_k| \). Their elements are defined by shifted particle/hole positions:

\[
\begin{align*}
x_i^{[k]} &:= \begin{cases} 
p + \sigma_{k-1,\alpha}, & r \equiv (p, \alpha) \in I_{k-1}, \\
-p + \sigma_{k,\alpha}, & r \equiv (-p, \alpha) \in J_k, \end{cases} \\
y_j^{[l]} &:= \begin{cases} 
-q + \sigma_{j-1,\beta}, & j \equiv (-q, \beta) \in J_{k-1}, \\
q + \sigma_{j,\beta}, & j \equiv (q, \beta) \in I_k. \end{cases}
\end{align*}
\]

**Lemma 4.26.** If \( r^{[k]} = 1 \), then \( Z_{I_{k-1}J_k}^{I_{k-1}J_k} (\mathcal{T}^{[k]}) \) can be written as

\[
Z_{I_{k-1}J_k}^{I_{k-1}J_k} (\mathcal{T}^{[k]}) = \pm \prod_{(p,\alpha) \in I_{k-1}} (\psi_{p,\alpha}^{[k]}) \prod_{(-p,\alpha) \in J_{k-1}} (\bar{\psi}_{p,\alpha}^{[k]}) \prod_{(p,\alpha) \in J_k} (\varphi_{p,\alpha}^{[k]})^{-p,\alpha} \prod_{(p,\alpha) \in I_k} (\bar{\varphi}_{p,\alpha}^{[k]})^{-p,\alpha} \times \\
\prod_{i,j \in I_{k-1} \cup J_k; i < j} (x_i^{[k]} - x_j^{[k]}) \prod_{i,j \in J_{k-1} \cup J_k; i < j} (y_i^{[l]} - y_j^{[l]}) \\
\times \prod_{i \in I_{k-1} \cup J_k; j \in J_{k-1} \cup J_k} (x_i^{[k]} - y_j^{[l]}) \\
\times \prod_{i \in I_{k-1} \cup J_k; j \in J_{k-1} \cup J_k} (x_i^{[k]} - y_j^{[l]}). 
\]

(4.70)
Proof. The diagonal factors in (4.61) produce the first line of (4.70). It remains to compute the determinant
\[
\det \begin{pmatrix}
1 & (p,\alpha) \in I_{k-1} & (p,\alpha) \in I_{k-1} \\
\frac{1}{p + \sigma_{k-1,\alpha} + q - \sigma_{k,\beta}} & -q,\beta \in J_{k-1} & p + \sigma_{k-1,\alpha} - q - \sigma_{k,\beta} \\
\frac{1}{-p + \sigma_{k,\alpha} + q - \sigma_{k-1,\beta}} & -q,\beta \in J_{k-1} & -p + \sigma_{k,\alpha} - q - \sigma_{k-1,\beta}
\end{pmatrix}
\]
which already includes the sign \((-1)^{|I_k|}\) in (4.63a). The \(\pm\) sign in (4.70) depends on the ordering of rows and columns of the determinant (4.63a). This ambiguity does not play any role as the relevant sign appears twice in the full product (4.64).

On the other hand, the notation introduced above allows to rewrite (4.71) as a \((|I_{k-1}| + |I_k|) \times (|I_{k-1}| + |I_k|)\) Cauchy determinant
\[
\det \left( \frac{1}{x^i_k - y^j_k} \right)_{i \in |I_{k-1}| \sqcup J_k, j \in |J_{k-1}| \sqcup I_k},
\]
and the factorized expression (4.70) easily follows. □

We now restrict ourselves to the case \(N = 2\), where the condition \(v^{[1]} = \ldots = v^{[n-2]} = 1\) does not lead to restrictions on monodromy. Let us start by preparing a suitable notation.

- The color indices will take values in the set \(+, -\) and will be denoted by \(\epsilon, \epsilon'\).

- Recall that the spectrum of \(A^{[k]}\) coincides with that of \(\Theta_k\). According to Assumption 4.25, the diagonal matrix \(\Theta_k\) has a zero eigenvalue for \(k = 0, \ldots, n - 2\). Its second eigenvalue will be denoted by \(-2\theta_k\). Obviously, there is a relation
  \[2\theta_k a_k = v^{[k]} \cdot u^{[k]}, \quad k = 1, \ldots, n - 2,\]
  where \(v \cdot u = v_+ u_+ + v_- u_-\) is the standard bilinear form on \(\mathbb{C}^2\). The eigenvalues of the remaining local monodromy exponent \(\Theta_{n-1}\) may be parameterized as
  \[\theta_{n-1,\epsilon} = \sum_{k=0}^{n-2} \theta_k + \epsilon \theta_{n-1}, \quad \epsilon = \pm.\]

- Also, the spectra of \(A^{[k]}_0\) and \(A^{[k]}_\infty = -A^{[k]}_0 - A^{[k]}_1\) coincide with the spectra of \(\mathcal{S}_{k-1}\) and \(-\mathcal{S}_k\). Since furthermore \(\text{Tr} \mathcal{S}_k = \sum_{j=0}^k \text{Tr} \Theta_j\), we may write the eigenvalues of \(\mathcal{S}_k\) as
  \[\sigma_{k,\epsilon} = -\sum_{j=0}^k \theta_j + \epsilon \sigma_k, \quad \epsilon = \pm, \quad k = 0, \ldots, n - 2, \quad (4.72)\]
  where \(\sigma_0 \equiv \theta_0\) and \(\sigma_{n-2} \equiv -\theta_{n-1}\).
The non-resonancy of monodromy exponents and Assumption 4.3 imply that
\[2\theta_k \not\in \mathbb{Z} \setminus \{0\}, \quad k = 0, \ldots, n-1,\]
\[|\Re \sigma_k| \leq \frac{1}{2}, \quad \sigma_k \neq \pm \frac{1}{2}, \quad k = 1, \ldots, n-3.\]
To simplify the exposition, we add to this extra genericity conditions.

**Assumption 4.27.** For \(k = 1, \ldots, n-2\), we have
\[\sigma_{k-1} + \sigma_k \pm \theta_k \not\in \mathbb{Z}, \quad \sigma_{k-1} - \sigma_k \pm \theta_k \not\in \mathbb{Z}.
It is also assumed that \(\sigma_k \neq 0\) for \(k = 0, \ldots, n-2\).

Let us introduce the space\[\mathcal{M}_\Theta = \{[M_0, \ldots, M_{n-1}] \in (\text{GL}(N, \mathbb{C}))^n / \sim | M_0 \ldots M_{n-1} = 1, M_k \in [e^{2\pi i \theta_k}] \text{ for } k = 0, \ldots, n-1\}\]
of conjugacy classes of monodromy representations of the fundamental group with fixed local exponents. The parameters \(\sigma_1, \ldots, \sigma_{n-3}\) are associated to annuli \(A_1, \ldots, A_{n-3}\)
and provide \(n-3\) local coordinates on \(\mathcal{M}_\Theta\) (that is, exactly one half of \(\text{dim } \mathcal{M}_\Theta = 2n-6\)).
The remaining \(n-3\) coordinates will be defined below.

Our task is now to find the 3-point solution \(\Psi^{[k]}\) explicitly. The freedom in the choice of its normalization allows to pick any representative in the conjugacy class \([A_0^{[k]}, A_1^{[k]}]\) for the construction of the 3-point Fuchsian system (4.16). An important feature of the \(N = 2\) case is that this conjugacy class is completely fixed by local monodromy exponents \(\mathcal{G}_{k-1}, \Theta_k\) and \(-\mathcal{G}_k\). We can set in particular
\[A_0^{[k]} = \text{diag } \{\sigma_{k-1,+}, \sigma_{k-1,-}\}, \quad a_k A_1^{[k]} = -u^{[k]} \otimes v^{[k]},\]
with \(\sigma_{k-1,\pm}\) parameterized as in (4.72) and
\[u^{[k]} = \frac{(\sigma_{k-1} \pm \theta_k) - \sigma_k^2}{2\sigma_{k-1}} a_k, \quad v^{[k]} = \pm 1.\]

As in Subsection 4.2.5, one may first construct the solution \(\tilde{\Phi}^{[k]}\) of the rescaled system
\[\partial_z \tilde{\Phi}^{[k]} = \tilde{\Phi}^{[k]} \left( \frac{A_0^{[k]}}{z} + \frac{A_1^{[k]}}{z-1} \right),\]
having the same monodromy around 0, 1, \(\infty\) as the solution \(\Phi^{[k]}\) of the original system (4.16) has around 0, \(a_k\) and \(\infty\). To write it explicitly in terms of the Gauss hypergeometric function \(\Hypergeometric{2}{1}{a,b}{c}{z}\), we introduce a convenient notation,
\[
\begin{align*}
\chi \left[ \begin{array}{cc} \theta_2 & \theta_3 \\ \theta_1 & \end{array} ; z \right] & := \Hypergeometric{2}{1}{\theta_2 + \theta_3, \theta_1 + \theta_2 - \theta_3}{2\theta_1}{\theta_1 + \theta_2 + \theta_3}{z}, \\
\phi \left[ \begin{array}{cc} \theta_2 & \theta_3 \\ \theta_1 & \end{array} ; z \right] & := \frac{\theta_3^2 - (\theta_1 + \theta_2)^2}{2\theta_1 (1 + 2\theta_1)} \Hypergeometric{2}{1}{1 + \theta_1 + \theta_2 + \theta_3, 1 + \theta_1 + \theta_2 - \theta_3}{2 + 2\theta_1}{\theta_1 + \theta_2 + \theta_3}{z}.
\end{align*}
\]
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

The solution of (4.73) can then be written as
\[ \tilde{\Phi}^{[k]}(z) = S_{k-1}(-z)^{\tilde{\theta}_{k-1}} \tilde{\Psi}^{[k]}_{\text{in}}(z), \] (4.75)
where \( S_{k-1} \) is a constant connection matrix encoding the monodromy (cf (4.15)), and \( \tilde{\Psi}^{[k]}_{\text{in}} \) is given by
\[ \begin{pmatrix} \tilde{\Psi}^{[k]}_{\text{in}} \end{pmatrix} \pm (z) = \chi \begin{pmatrix} \theta_k \\ \sigma_{k-1} \sigma_k ; z \end{pmatrix}, \]
(4.76)

It follows that \( \Phi^{[k]}(z) = \tilde{\Phi}^{[k]} \left( \frac{z}{a_k} \right) \) and
\[ \Psi^{[k]}_{+}(z) = a_k^{-\tilde{\theta}_{k-1}} \tilde{\Psi}^{[k]}_{\text{in}} \left( \frac{z}{a_k} \right), \quad z \in C_{in}^{[k]}. \] (4.77a)

Let us also note that \( \det \tilde{\Phi}^{[k]}(z) = \text{const} \cdot (-z)^{\text{Tr} A^0} (1 - z)^{\text{Tr} A^k} \) implies that \( \det \tilde{\Psi}^{[k]}_{\text{in}}(z) = (1 - z)^{-2\theta_k} \), which in turn yields a simple representation for the inverse matrix
\[ \Psi^{[k]}_{+}(z)^{-1} = \left( 1 - \frac{z}{a_k} \right) 2^{\theta_k} \begin{pmatrix} \tilde{\Psi}^{[k]}_{\text{in}} - \frac{z}{a_k} & \tilde{\Psi}^{[k]}_{\text{in}} + \frac{z}{a_k} \\ \tilde{\Psi}^{[k]}_{\text{in}} - \frac{z}{a_k} & \tilde{\Psi}^{[k]}_{\text{in}} + \frac{z}{a_k} \end{pmatrix} a_k^{\tilde{\theta}_{k-1}}, \quad z \in C_{in}^{[k]}. \] (4.77b)

The equations (4.75)–(4.76) are adapted for the description of local behavior of \( \Psi^{[k]}(z) \) inside the disk around 0 bounded by the circle \( C_{in}^{[k]} \), cf left part of Fig. 4.9. To calculate \( \Psi^{[k]}_{+}(z) \) inside the disk around \( \infty \) bounded by \( C_{out}^{[k]} \), let us first rewrite (4.75) using the well-known \( _2F_1 \) transformation formulas. One can show that
\[ \tilde{\Phi}^{[k]}(z) = S_{k-1} C_{out}^{[k]} (-z)^{\tilde{\theta}_k} \tilde{\Psi}^{[k]}_{\text{out}}(z) G_{\infty}^{[k]}, \] (4.78)
where
\[ \begin{pmatrix} \tilde{\Psi}^{[k]}_{\text{out}} \end{pmatrix} \pm (z) = \chi \begin{pmatrix} \theta_k \\ \mp \sigma_{k-1} \sigma_k ; z^{-1} \end{pmatrix}, \]
(4.79)
and
\[ G_{\infty}^{[k]} = \frac{1}{2\sigma_k} \begin{pmatrix} -\theta_k + \sigma_{k-1} + \sigma_k & \theta_k + \sigma_{k-1} - \sigma_k \\ -\theta_k + \sigma_{k-1} - \sigma_k & \theta_k + \sigma_{k-1} + \sigma_k \end{pmatrix}, \] (4.80)

\[ C_{\infty}^{[k]} = \begin{pmatrix} \frac{\Gamma(2\sigma_{k-1})\Gamma(1 + 2\sigma_k)}{\Gamma(1 + \sigma_{k-1} + \sigma_k - \theta_k)\Gamma(\sigma_{k-1} + \sigma_k + \theta_k)} & \frac{\Gamma(2\sigma_{k-1})\Gamma(1 - 2\sigma_k)}{\Gamma(1 + \sigma_{k-1} - \theta_k)\Gamma(\sigma_{k-1} - \sigma_k + \theta_k)} \\ \frac{\Gamma(-2\sigma_{k-1})\Gamma(1 + 2\sigma_k)}{\Gamma(1 - \sigma_{k-1} + \sigma_k - \theta_k)\Gamma(\sigma_{k-1} + \sigma_k + \theta_k)} & \frac{\Gamma(-2\sigma_{k-1})\Gamma(1 - 2\sigma_k)}{\Gamma(1 - \sigma_{k-1} - \sigma_k + \theta_k)\Gamma(\sigma_{k-1} - \sigma_k - \theta_k)} \end{pmatrix}. \] (4.81)

As a consequence,
\[ \Psi^{[k]}_{+}(z) = D_{\infty}^{[k]} a_k^{-\tilde{\theta}_k} \tilde{\Psi}^{[k]}_{\text{out}} \left( \frac{z}{a_k} \right) G_{\infty}^{[k]}, \quad z \in C_{out}^{[k]}. \] (4.82a)
where $D^{[k]}_\infty = \text{diag} \left\{ d^{[k]}_{\infty,+}, d^{[k]}_{\infty,-} \right\}$ is a diagonal matrix expressed in terms of monodromy as

$$D^{[k]}_\infty = S^{-1}_k S_{k-1} C^{[k]}_\infty.$$  

Analogously to (4.77b), it may be shown that for $z \in C^{[k]}_{\text{out}}$

$$\hat{\Psi}^+_{[k]} (z) \left( z \right)^{-1} = \left( 1 - \frac{a_k}{z} \right) 2\theta_k \left( \Psi^{[k]}_{\text{out}} \right)^{-1} \left( \begin{array}{cc} \bar{z} & \bar{z} \\ a_k & a_k \end{array} \right) - \left( \begin{array}{cc} \hat{\Psi}^+_{[k]} \left( z \right) & \hat{\Psi}^+_{[k]} \left( z \right) \\ a_k & a_k \end{array} \right) \right) \left( \begin{array}{cc} z & z \\ a_k & a_k \end{array} \right) \right) \hat{a}^{[k]}_k D^{[k]}_\infty^{-1}. \quad (4.82b)$$

Lemma 4.28. For $N = 2$, the integral kernels (4.57) can be expressed as

$$a^{[k]}(z, z') = -a_k^{-\Theta_{k-1}} \left( 1 - \frac{a_k}{z} \right) 2\theta_k \left( \begin{array}{cc} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{-+}(z) \end{array} \right) \left( \begin{array}{cc} K_{--}(z') & K_{+-}(z') \\ K_{++}(z') & K_{++}(z') \end{array} \right) \right) - \left( \begin{array}{cc} z & z \\ a_k & a_k \end{array} \right) \right) D^{[k]}_\infty^{-1}, \quad (4.83a)$$

$$\hat{b}^{[k]}(z, z') = -a_k^{-\Theta_{k-1}} \left( 1 - \frac{a_k}{z} \right) 2\theta_k \left( \begin{array}{cc} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{-+}(z) \end{array} \right) G^{[k]}_{\infty} \left( \begin{array}{cc} K_{--}(z') & K_{+-}(z') \\ K_{++}(z') & K_{++}(z') \end{array} \right) \right) - \left( \begin{array}{cc} z & z \\ a_k & a_k \end{array} \right) \right) D^{[k]}_\infty^{-1}, \quad (4.83b)$$

$$\hat{c}^{[k]}(z, z') = D^{[k]}_\infty a_k^{-\Theta_{k-1}} \left( 1 - \frac{a_k}{z} \right) 2\theta_k \left( \begin{array}{cc} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{-+}(z) \end{array} \right) G^{[k]}_{\infty} \left( \begin{array}{cc} K_{--}(z') & K_{+-}(z') \\ K_{++}(z') & K_{++}(z') \end{array} \right) \right) - \left( \begin{array}{cc} z & z \\ a_k & a_k \end{array} \right) \right) D^{[k]}_\infty^{-1}, \quad (4.83c)$$

$$\hat{d}^{[k]}(z, z') = D^{[k]}_\infty a_k^{-\Theta_{k-1}} \left( 1 - \frac{a_k}{z} \right) 2\theta_k \left( \begin{array}{cc} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{-+}(z) \end{array} \right) \left( \begin{array}{cc} K_{--}(z') & K_{+-}(z') \\ K_{++}(z') & K_{++}(z') \end{array} \right) \right) - \left( \begin{array}{cc} z & z \\ a_k & a_k \end{array} \right) \right) D^{[k]}_\infty^{-1}, \quad (4.83d)$$

where we introduced a shorthand notation $K(z) = \hat{\Psi}^+_{[k]}(z/a_k)$, $\hat{K}(z) = \hat{\Psi}^+_{[k]}(z/a_k)$; the matrices $\hat{\Psi}^+_{\text{in,out}}(z)$ and $G^{[k]}_{\infty}$ are defined by (4.76), (4.78) and (4.80).

Proof. Straightforward substitution. \hfill \Box

Lemma 4.29. Under genericity assumptions on parameters formulated above, the
Fourier coefficients which appear in (4.60) are given by

\[
(\psi^{[k]}_{m,n;\bar{\psi}})_{\mu\nu} = \frac{\Gamma(\sigma \cdot p + 1 + 2\theta_{k-1} + \alpha)}{\Gamma(\sigma \cdot p + 1 + 2\theta_{k-1} + \alpha)} \frac{\Gamma(p + 1 + 2\theta_{k-1} + \alpha)}{\Gamma(p + 1 + 2\theta_{k-1} + \alpha)} \left(\begin{array}{c} \mu \\ v \end{array}\right) \frac{\Gamma(p + 1 + 2\theta_{k-1} + \alpha)}{\Gamma(p + 1 + 2\theta_{k-1} + \alpha)} \left(\begin{array}{c} \nu \\ v \end{array}\right)
\]

where \( c = \pm \) and \( (c)_l := \frac{\Gamma(c + l)}{\Gamma(c)} \) denotes the Pochhammer symbol.

**Proof.** From the first equation in (4.60a), the representation (4.77a) for \( \Psi^{[k]}_{\pm}(z) \) on \( C_{l,n} \), and hypergeometric contiguity relations such as

\[
\begin{aligned}
2F1\left[ a, b; c \right] &+ (z - 1) 2F1\left[ a + 1, b + 1; c + 1 \right] \Rightarrow \frac{(c - a)(c - b)}{c(c + 1)} 2F1\left[ a + 1, b + 1; c + 2 \right],
\end{aligned}
\]

it follows that

\[
\sum_{p \in \mathbb{Z}_+} (\psi^{[k]}_{m,n;\bar{\psi}})_{p} \cdot \frac{z^{p - \frac{1}{2}}}{\Gamma\left(\sigma \cdot p + \alpha \right)} = \frac{\Gamma\left(\sigma \cdot p + 1 + 2\theta_{k-1} + \alpha \right)}{\Gamma\left(\sigma \cdot p + 1 + 2\theta_{k-1} + \alpha \right)} \left(\begin{array}{c} \mu \\ v \end{array}\right) \frac{\Gamma\left(p + 1 + 2\theta_{k-1} + \alpha \right)}{\Gamma\left(p + 1 + 2\theta_{k-1} + \alpha \right)} \left(\begin{array}{c} \nu \\ v \end{array}\right).
\]

This in turn implies the equation (4.84a). The proof of three other identities is similar.

The Cauchy determinant in (4.70) remains invariant upon simultaneous translation of all \( x^{[k]}_{i} \) and \( y^{[k]}_{j} \) by the same amount. Let us use this to replace the notation (4.69) in the case \( N = 2 \) by

\[
x^{[k]}_{i} := \begin{cases} p + \epsilon \sigma_{k-1}, & i \equiv (p, \epsilon) \in I_{k-1}; \\ -p - \theta_{k} + \epsilon \sigma_{k}, & i \equiv (-p, \epsilon) \in J_{k}; \end{cases}
\]

\[
y^{[k]}_{j} := \begin{cases} -q + \epsilon \sigma_{k-1}, & j \equiv (-q, \epsilon) \in J_{k-1}; \\ q - \theta_{k} + \epsilon \sigma_{k}, & j \equiv (q, \epsilon) \in I_{k}. \end{cases}
\]

Define a notation for the charges

\[
m_{k} := |(\cdot, +) \in I_{k}| - |(\cdot, +) \in J_{k}| = |(\cdot, -) \in J_{k}| - |(\cdot, -) \in I_{k}|, \quad k = 1, \ldots, n-3,
\]

and combine them into a vector \( \mathbf{m} := (m_{1}, \ldots, m_{n-3}) \in \mathbb{Z}^{n-3} \). We will also write \( \sigma := (\sigma_{1}, \ldots, \sigma_{n-3}) \in \mathbb{C}^{n-3} \) and further define

\[
\eta := (\eta_{1}, \ldots, \eta_{n-3}), \quad \epsilon^{\eta \mathbf{m}} := \frac{d^{[k]}_{\eta \mathbf{m}}}{d^{[k]}_{\eta \mathbf{m}+1}}.
\]

The parameters \( \eta \) provide the remaining \( n - 3 \) local coordinates on the space \( \mathcal{M}_{\Theta} \) of monodromy data. The main result of this section may now be formulated as follows.
Theorem 4.30. The isomonodromic tau function of the Garnier system admits the following multivariate combinatorial expansion:

$$\tau_{\text{Garnier}}(a) = \text{const} \cdot \prod_{k=1}^{n-2} \prod_{1 \leq k < l \leq n-2} \left( 1 - \frac{a_k}{a_l} \right)^{-2g_k \theta_k} \times \sum_{m \in \mathbb{Z}^{n-3}} e^{im \eta} \sum_{\vec{Y}_1, \ldots, \vec{Y}_{n-3} \in \mathcal{Y}} \prod_{k=1}^{n-3} \left( \frac{a_k}{\theta_{k+1}} \right)^{(\sigma_k + m_k)^2 + |Y_k|} \prod_{k=1}^{n-2} Z_{\vec{Y}_{k-1}m_{k-1}}^{\vec{Y}_{k}m_{k}}(T^{[k]})$$

(4.87)

where $\vec{Y}_k$ stands for the pair of charged Young diagrams associated to $(I_k, J_k)$, the total number of boxes in $\vec{Y}_k$ is denoted by $|\vec{Y}_k|$, and

$$Z_{\vec{Y}_{k-1}m_{k-1}}^{\vec{Y}_{k}m_{k}}(T^{[k]}) = \prod_{(p, \epsilon) \in I_{k-1}} \prod_{\epsilon' = \pm} \frac{(\theta_k + c\sigma_{k-1} + \epsilon' \sigma_k)_{p + \frac{1}{2}}}{(p - \frac{1}{2})! (2c\sigma_{k-1})_{p + \frac{1}{2}}} \prod_{(p, \epsilon) \in J_{k-1}} \prod_{\epsilon' = \pm} \frac{(1 - \theta_k - c\sigma_{k-1} + \epsilon' \sigma_k)_{p - \frac{1}{2}}}{(p - \frac{1}{2})! (1 - 2c\sigma_{k-1})_{p - \frac{1}{2}}} \times \prod_{(p, \epsilon) \in I_{k}} \prod_{\epsilon' = \pm} \frac{(\theta_k + c\sigma_{k-1} - \epsilon' \sigma_k)_{p + \frac{1}{2}}}{(p - \frac{1}{2})! (2c\sigma_{k})_{p + \frac{1}{2}}} \prod_{(p, \epsilon) \in J_{k}} \prod_{\epsilon' = \pm} \frac{(1 - \theta_k + c\sigma_{k-1} + \epsilon' \sigma_k)_{p - \frac{1}{2}}}{(p - \frac{1}{2})! (1 + 2c\sigma_{k})_{p - \frac{1}{2}}} \times \prod_{j \in I_{k-1} \cup I_{k}} \prod_{i \in I_{k-1} \cup I_{k}} \frac{(x_i^k - x_j^k)}{(y_i^k - y_j^k)} \prod_{i, j \in I_{k-1} \cup I_{k}, i \neq j}.$$  

(4.88)

Proof. Consider the product in the first line of (4.70). The balance conditions $|I_k| = |J_k|$ imply that the factors such as $e^{\sum_{j=0}^{n-1} \theta_k}$ in (4.84) cancel out from $Z_{\vec{Y}_{k-1}m_{k-1}}^{\vec{Y}_{k}m_{k}}(T^{[k]})$. The factors of the form $\pm \epsilon$ also compensate each other in the product of elementary determinants in (4.65). The factors $d_{\infty, \epsilon}^{[k]}$, $\pm 1$ in (4.84c) and (4.84d) produce the exponential $e^{im \eta}$ in (4.87).

The total power in which the coordinate $a_k$ appears in (4.70) is equal to

$$2m_k \sigma_k - 2m_{k-1} \sigma_{k-1} - \sum_{(p, \epsilon) \in I_{k-1}} p - \sum_{(p, \epsilon) \in I_{k-1}} p + \sum_{(p, \epsilon) \in I_{k}} p + \sum_{(p, \epsilon) \in I_{k}} p =$$

$$= \left( 2m_k \sigma_k + m_k^2 + |\vec{Y}_k| \right) - \left( 2m_{k-1} \sigma_{k-1} + m_{k-1}^2 + |\vec{Y}_{k-1}| \right).$$

The last equality is demonstrated graphically in Fig. 4.13. The prefactor in the first line of (4.87) comes from two sources: i) the shifts of (initially traceless) Garnier monodromy exponents $\Theta_k$ by $-\theta_k \mathbb{1}$ making one of their eigenvalues equal to 0 and ii) the prefactor $\Upsilon(a)$ from Theorem 4.11. □

In the Appendix, we show that the formula (4.88) can be rewritten in terms of Nekrasov functions. In the Painlevé VI case $(n = 4)$, this transforms Theorem 4.30 into Theorem 4.2 of the Introduction.
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

Figure 4.13: A charged Maya diagram $m$ and the associated partition $Y(m)$ for positive and negative charges $Q(m)$. Given the positions $p(m) = (p_1, \ldots, p_r)$ and $q(m) = (-q_1, \ldots, -q_s)$ of particles and holes, the red and green areas represent the sums $\sum p_k$ and $\sum q_k$. We clearly have $\sum p_k + \sum q_k = \frac{Q(m)^2}{2} + |Y(m)|$ in both cases.

**Hypergeometric kernel**

Recall that the matrices $\Theta_0, \ldots, \Theta_{n-1}$ are by convention diagonal with eigenvalues distinct modulo non-zero integers. However, all of the results of Section 4.2 remain valid if the diagonal parts corresponding to the degenerate eigenvalues are replaced by appropriate Jordan blocks.

In this subsection we will consider in more detail a specific example of this type by revisiting the 4-point tau function. We will thus follow the notational conventions of Subsection 4.2.5. Fix $n = 4$, $N = 2$ and assume furthermore that the monodromy representation $\rho^{[L]} : \pi_1(\mathbb{P}^1 \setminus \{0, t, \infty\}) \to \text{GL}(2, \mathbb{C})$ associated to the internal trinion $\mathcal{T}^{[L]}$ is reducible, whereas its counterpart $\rho^{[R]} : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to \text{GL}(2, \mathbb{C})$ for the external trinion $\mathcal{T}^{[R]}$ remains generic. For instance, one may set

$$\Theta_0 = \mathcal{S} = \begin{pmatrix} 0 & 0 \\ 0 & -2\sigma \end{pmatrix}, \quad \Theta_t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so that the monodromy matrices $M_0$, $M_t$ can be assumed to have the lower triangular form

$$M_0 = \begin{pmatrix} 1 & 0 \\ -2\pi ike^{-2\pi i\sigma} & 0 \end{pmatrix}, \quad M_t = \begin{pmatrix} 1 & 0 \\ 2\pi ike^{2\pi i\sigma} & 1 \end{pmatrix}, \quad M_0 M_t = e^{2\pi i\mathcal{S}}. \quad (4.89)$$

The solution $\Phi^{[L]}(z)$ of the appropriate internal 3-point RHP may be constructed from the fundamental solution of a Fuchsian system

$$\partial_z \Phi^{[L]}(z) = \Phi^{[L]} \begin{pmatrix} 0 & 0 \\ \frac{\varrho t}{z(z-t)} & -2\sigma/z \end{pmatrix}, \quad (4.90)$$

with a suitably chosen value of the parameter $\varrho$. Taking into account the diagonal monodromy around $\infty$, such a solution $\Phi^{[L]}(z)$ on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ can be written as

$$\Phi^{[L]}(z) = \begin{pmatrix} 1 & 0 \\ \frac{\varrho t}{1 + 2\sigma} l_{2\sigma} \left( \frac{z}{t} \right) & (z)^{-2\sigma} \end{pmatrix} = \tilde{C}_0 \begin{pmatrix} \varrho (z)^{-2\sigma} & 0 \\ \frac{\varrho (z)^{-2\sigma}}{2\sigma} l_{-1-2\sigma} \left( \frac{z}{t} \right) & (z)^{-2\sigma} \end{pmatrix}, \quad (4.91)$$
where $l_a(z) := \binom{1 + a}{2 + a} Z_1 \left[ \frac{1}{1 + a}, \frac{1}{2 + a} ; z \right]$, and the modified connection matrix $\tilde{C}_0$ is lower-triangular:

$$\tilde{C}_0 = \begin{pmatrix}
1 & 0 \\
\pi \rho t^{-2\sigma} & 0 \\
\sin 2\pi \sigma & 1
\end{pmatrix}.$$

The monodromy matrix around 0 is clearly equal to $M_0 = \tilde{C}_0 e^{2\pi \Theta_0 \tilde{C}_0^{-1}}$. This allows to relate the monodromy parameter $\kappa$ to the coefficient $\rho$ of the Fuchsian system (4.90) as

$$\kappa = \rho t^{-2\sigma}.$$ (4.92)

The 3-point RHP solution $\Psi^{[L]}(z)$ inside the annulus $A$ is thus explicitly given by

$$\Psi^{[L]}(z) \big|_A = \begin{pmatrix} 1 & 0 \\ 0 & (-z)^{2\sigma} \end{pmatrix} \Phi^{[L]}(z) = \begin{pmatrix} 1 & 0 \\ -\rho t \frac{(2\sigma + 1)}{z} & 1 \end{pmatrix}.$$ (4.93)

This formula leads to substantial simplifications in the Fredholm determinant representation (4.48) of the tau function $\tau_{\text{JMU}}(t)$. It follows from the structure of (4.93) and (4.49b) that

$$d_{-+}(z, z') = \frac{\theta}{1 + 2\sigma} \frac{l_{2\sigma} \left( \frac{t}{z} \right) - \frac{\theta}{1 + 2\sigma} l_{2\sigma} \left( \frac{t}{z'} \right)}{z - z'}.$$ (4.94)

is the only non-zero element of the $2 \times 2$ matrix integral kernel $d(z, z')$ (note that the lower indices here are color and should not be confused with half-integer Fourier modes). This in turn implies that the only entry of $a(z, z')$ contributing to the determinant is

$$a_{-+}(z, z') = \frac{1}{\det \Psi^{[R]}(z')} \Psi^{[R]}_{-+}(z) \Psi^{[R]}_{++}(z') - \Psi^{[R]}_{++}(z) \Psi^{[R]}_{++}(z').$$ (4.95)

Therefore, (4.48) reduces to

$$\tau_{\text{JMU}}(t) = \det \left( 1 - a_{-+} d_{-+} \right).$$ (4.96)

The action of the operators $a_{-+}, d_{-+}$ involves integration along a circle $C \subset A$. The kernel $a_{-+}(z, z')$ extends to a function holomorphic in both arguments inside $C$. Therefore in the computation of contributions of different exterior powers to the determinant one may try to shrink all integration contours to the branch cut $B := [0, t] \subset \mathbb{R}$. The latter comes from two branch points 0, $t$ of $d_{-+}(z, z')$ defined by (4.94).

Lemma 4.31. Let $|\Re \sigma| < \frac{1}{2}$. For $m \in \mathbb{Z}_{\geq 0}$, denote $X_m = \text{Tr} (a_{-+} d_{-+})^m$. We have

$$X_m = \text{Tr} K_{\text{F}}^m,$$

where $K_{\text{F}}$ denotes an integral operator on $L^2(B)$ with the kernel

$$K_{\text{F}}(z, z') = -\kappa (z z')^{\sigma} a_{-+}(z, z').$$ (4.97)
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

**Proof.** Let us denote by $B_{up}$ and $B_{down}$ the upper and lower edge of the branch cut $B$. After shrinking of the integration contours in the multiple integral $I_k$ to $B$, the operators $a_{+-}, d_{-+}$ should be interpreted as acting on $W = L^2(B_{up}) \oplus L^2(B_{down})$ instead of $L^2(C)$. Here $L^2(B_{up,down})$ arise as appropriate completions of spaces of boundary values of functions holomorphic inside $D_C \backslash B$, where $D_C$ denotes the disk bounded by $C$. The space $W$ can be decomposed as $W = W_+ \oplus W_-$, where the elements of $W_+$ are continuous across the branch cut, whereas the elements of $W_-$ have opposite signs on its two sides:

$$W_\pm = \{ f \in W : f(z+i0) = \pm f(z-i0), z \in B \}.$$

We will denote by $pr_\pm$ the projections on $W_\pm$ along $W_\mp$.

Since $a_{+-}(z,z')$ is holomorphic in $z, z'$ inside $C$, it follows that $\text{Im} a_{+-} \subseteq W_+ \subseteq \ker a_{+-}$. Therefore $X_k$ remains unchanged if $a_{+-}$ is replaced by $pr_- \circ a_{+-} \circ pr_+$. This is in turn equivalent to replacing $d_{-+}$ by $pr_- \circ d_{-+} \circ pr_+$. Given $f = g \oplus g \in W_+$ with $g \in L^2(B)$, the action of $d_{-+}$ on $f$ is given by

$$(d_{-+}f)(z) = \frac{1}{2\pi i} \int_0^t [d_{-+}(z,z' - i0) - d_{-+}(z,z' + i0)] g(z') dz' = \frac{gt}{2\pi i (1 + 2\sigma)} \int_0^t \frac{l_{2\sigma}(\frac{t}{z' + i0}) - l_{2\sigma}(\frac{t}{z' - i0})}{z' (z' - z')} g(z') dz'.$$

An important consequence of the lower triangular monodromy is that the jump of $l_{2\sigma}(\frac{t}{z'})$ on $B$ yields an elementary function, cf (4.91):

$$l_{2\sigma}(\frac{t}{z' + i0}) - l_{2\sigma}(\frac{t}{z' - i0}) = -2\pi i (2\sigma + 1) \left( \frac{z'}{t} \right)^{2\sigma + 1}.$$

Substituting this jump back into the previous formula and using (4.92), one obtains

$$(d_{-+}f)(z) = \kappa \int_0^t \frac{z'^{2\sigma} g(z') dz'}{z' - z}, \quad z \in D_C \backslash B.$$

Next we have to compute the projection $pr_-$ of this expression onto $W_-$. Write $pr_- \circ d_{-+} f = h \oplus (-h)$, with $h \in L^2(B)$. Then

$$h(z) = \frac{1}{2} [(d_{-+} f)(z+i0) - (d_{-+} f)(z-i0)] = \pi i \kappa z^{2\sigma} g(z), \quad z \in B.$$

Finally, write $a_{+-} \circ pr_- \circ d_{-+} f$ as $\tilde{g} \oplus \tilde{g} \in W_+$. It follows from the previous expression for $h(z)$ that

$$\tilde{g}(z) = -\kappa \int_0^t a_{+-}(z,z') z'^{2\sigma} g(z') dz', \quad z \in B.$$

The minus sign in front of the integral is related to orientation of the contour $C$ in the definition of $a$. We have thereby computed the action of $a_{+-} \circ pr_- \circ d_{-+}$ on $W_+$. Raising this operator to an arbitrary power $k \in \mathbb{Z}_{\geq 0}$ and symmetrizing the factors $z'^{2\sigma}$ under the trace immediately yields the statement of the lemma. \qed
Theorem 4.32. Given complex parameters $\theta_1, \theta_\infty, \sigma$ satisfying previous genericity assumptions, let

$$\varphi (x) := x^\sigma (1 - x)^{\theta_1} \binom{\sigma + \theta_1 + \theta_\infty, \sigma + \theta_1 - \theta_\infty}{2 \sigma} F_1 ; x, \quad (4.98a)$$

$$\psi (x) := x^{1+\sigma} (1 - x)^{\theta_1} \binom{1 + \sigma + \theta_1 + \theta_\infty, 1 + \sigma + \theta_1 - \theta_\infty}{2 + 2 \sigma} F_1 ; x. \quad (4.98b)$$

Define the continuous $2F_1$ kernel by

$$\tilde{K}_F (x, y) := \frac{\psi (x) \varphi (y) - \varphi (x) \psi (y)}{x - y}, \quad (4.99)$$

and consider Fredholm determinant

$$D (t) := \det \left( 1 - \lambda \tilde{K}_F \right)_{(0, t)}, \quad \lambda \in \mathbb{C}. \quad (4.100)$$

Then $D (t)$ is a tau function of the Painlevé VI equation with parameters $\tilde{\theta} = (\theta_0 = \sigma, \theta_1 = 0, \theta_\infty)$. The conjugacy class of monodromy representation for the associated 4-point Fuchsian system is generated by the matrices $(4.89)$ and

$$M_1 = \frac{e^{-2\pi i \theta_1}}{i \sin 2 \pi \sigma} \binom{\cos 2 \pi \theta_\infty - e^{-2\pi i \sigma} \cos 2 \pi \theta_1}{\cos 2 \pi (\theta_1 + \sigma) - \cos 2 \pi \theta_1 \theta_\infty} \binom{s^{-1} e^{-2\pi i \sigma} [\cos 2 \pi \theta_\infty - \cos 2 \pi (\theta_1 - \sigma)]}{e^{-2\pi i \sigma} \cos 2 \pi \theta_1 - \cos 2 \pi \theta_\infty} \binom{1}{1}, \quad (4.101a)$$

$$M_\infty = \frac{e^{-2\pi i \theta_\infty}}{i \sin 2 \pi \sigma} \binom{\cos 2 \pi \theta_1 - e^{-2\pi i \sigma} \cos 2 \pi \theta_\infty}{s \cos 2 \pi \theta_\infty - \cos 2 \pi (\theta_1 + \sigma)} \binom{s^{-1} [\cos 2 \pi (\theta_1 - \sigma) - \cos 2 \pi \theta_\infty]}{e^{2\pi i \sigma} \cos 2 \pi \theta_\infty - \cos 2 \pi \theta_1} = M_1^{-1} e^{-2\pi i \theta \tilde{\sigma}}. \quad (4.101b)$$

where

$$\lambda = \kappa (\theta_1 + \sigma)^2 - \theta_\infty^2, \quad (4.102)$$

$$s = \frac{\Gamma (1 - 2 \sigma) \Gamma (\theta_1 + \sigma + \theta_\infty) \Gamma (\theta_1 + \sigma - \theta_\infty)}{\Gamma (1 + 2 \sigma) \Gamma (\theta_1 - \sigma + \theta_\infty) \Gamma (\theta_1 - \sigma - \theta_\infty)}. \quad (4.103)$$

Proof. To prove that $D (t)$ is a Painlevé VI tau function with $\lambda$ and $\kappa$ related by (4.102), it suffices to combine the determinant representation $(4.96)$ with Lemma 4.31, and substitute into the formula $(4.95)$ for $a_{++} (z, z')$ explicit hypergeometric expressions $(4.76)$.

The formula $(4.101b)$ follows from $M_\infty = C_\infty e^{2\pi i \theta \tilde{\sigma}} C_\infty^{-1}$, where $C_\infty$ is obtained from the connection matrix $(4.81)$ by replacements $(\theta_k, \sigma_{k-1}, \sigma_k) \rightarrow (\theta_1, \sigma, -\theta_\infty)$. The expression $(4.101a)$ for $M_1$ is then most easily deduced from the diagonal form of the product $M_1 M_\infty = e^{-2\pi i \theta \tilde{\sigma}}$. \hfill \Box

Remark 4.33. The $2F_1$ kernel is related to the so-called ZW-measures [BO05] arising in the representation theory of the infinite-dimensional unitary group $U (\infty)$. It produces various other classical integrable kernels (such as sine and Whittaker) as limiting cases. The first part of Theorem 4.32, namely the Painlevé VI equation for $D (t)$, was proved by Borodin and Deift in [BD]. Monodromy data for the associated Fuchsian system
have been identified in [Lis]. To facilitate the comparison, let us note that introducing instead of \( \lambda \) and \( \kappa \) a new parameter \( \bar{\sigma} \) defined by

\[
\lambda = \frac{\sin \pi (\bar{\sigma} - \theta_1) \sin \pi (\bar{\sigma} + \theta_1)}{\pi^2} \frac{\prod_{e, e'} (1 + \sigma + \epsilon \theta_1 + \epsilon' \theta_\infty)}{\Gamma (1 + 2 \sigma) \Gamma (2 + 2 \sigma)},
\]

we have in particular that \( \text{Tr} M_\infty M_0 = 2 e^{2 \pi i (\sigma + \theta_\infty)} \cos 2 \pi \bar{\sigma} \). The relation between parameters \( z, z', w, w' \) of [BD] and ours is

\[
(z, z', w, w')_{[BD]} = (\bar{\sigma} + \theta_1, \bar{\sigma} - \theta_1, \sigma - \bar{\sigma} + \theta_\infty, \sigma - \bar{\sigma} - \theta_\infty).
\]

**Appendix**

**Relation to Nekrasov functions**

Here we demonstrate that the formula (4.88) can be rewritten in terms of Nekrasov functions. This rewrite is conceptually important for identification of isomonodromic tau functions with dual partition functions of quiver gauge theories [NO]. It is also useful from a computational point of view: naively, the formula (4.88) may produce poles in the tau function expansion coefficients when \( \theta_k \pm \sigma_k \pm \sigma_{k-1} \in \mathbb{Z} \). Our calculation shows that these poles actually cancel.

The statement we are going to prove\(^3\) is the relation

\[
Z_{\bar{Y}'}^{Y',m'}(T) = (-1)^{\text{sgn}(\bar{Y}',m')} + \text{sgn}(\bar{Y},m) \tilde{Z}_{\bar{Y}',\bar{Q}'}^{Y',\bar{Q}'}(T), \tag{4.104}
\]

where

\[
\tilde{Z}_{\bar{Y}',\bar{Q}'}^{Y',\bar{Q}'}(T) = \frac{\prod_{\alpha<\beta}^N C(\sigma'_\alpha - \sigma_\beta|Q'_\alpha, Q_\beta)}{\prod_{\alpha<\beta}^N C(\sigma_\alpha' - \sigma'_\beta|Q'_\alpha, Q_\beta)} \times
\]

\[
\times \prod_{\alpha}^N |Z_{\text{bif}}(0|Y_\alpha, Y_\alpha)|^2 |Z_{\text{bif}}(0|Y'_\alpha, Y'_\alpha)|^2 \times
\]

\[
\times \prod_{\alpha<\beta}^N Z_{\text{bif}}(\sigma'_\alpha + Q'_\alpha - \sigma_\beta - Q_\beta|Y'_\alpha, Y'_\beta) \times
\]

\[
\times \prod_{\alpha<\beta}^N Z_{\text{bif}}(\sigma'_\alpha + Q'_\alpha - \sigma'_\beta - Q'_\beta|Y'_\alpha, Y'_\beta) Z_{\text{bif}}(\sigma_\alpha + Q_\alpha - \sigma_\beta - Q_\beta|Y_\alpha, Y_\beta).
\]

The notation used in these formulas means the following:

- \( \bar{Q} = (m, -m), \bar{Q}' = (m', -m') \), though the right side of (4.105) is defined even without this specialization.

- \( Y' \) and \( Y \) are identified, respectively, with \( Y_{k-1} \) and \( Y_k \) in (4.88). Similar conventions will be used for all other quantities. We denote, however, \( \sigma'_\pm = \pm \sigma_{k-1} \) and \( \sigma_\pm = -\theta_k \pm \sigma_k; T \) stands for \( T^{[k]} \).

\(^3\)In the present chapter we do it only for \( N = 2 \) but the generalization is relatively straightforward.
4.5. Relation to Nekrasov functions

- \( \text{lsgn} \left( Y, m \right) \in \mathbb{Z}/2\mathbb{Z} \) means the “logarithmic sign”,

\[
\text{lsgn} \left( Y, m \right) := \left| q_+ \right| \cdot \left| p_+ \right| + \sum_i \left( q_{+,i} + \frac{1}{2} \right) + \sum_i \left( p_{-,i} + \frac{1}{2} \right). \tag{4.106}
\]

Here, for example, \( \left| p_+ \right| \) denotes the number of coordinates \( p_{+,i} \) of particles in the Maya diagram corresponding to the charged partition \( (Y_+, m) \). The logarithmic signs cancel in the product \( \prod_{k=1}^{n-2} Z_{Y_k, m_k}^{Y_{k-1}, m_{k-1}} \) which appears in the representation \( (4.87) \) for the Garnier tau function.

- \( \delta \vec{\eta} \) and \( \delta \vec{\eta}' \) are some explicit functions which are computed below. They just shift Fourier transformation parameters and their relevant combinations are explicitly given by

\[
e^{i\delta \vec{\eta}_{+} - i\delta \vec{\eta}_{-}} = \frac{1}{2 \sigma_{k-1}} \left( \theta_k + \sigma_{k-1} \right)^2 - \sigma_k^2,
\]

\[
e^{i\delta \vec{\eta}_{+} - i\delta \vec{\eta}_{-}} = \frac{-1}{2 \sigma_k} \left( \theta_k + \sigma_k \right)^2 - \sigma_{k-1}^2. \tag{4.107}
\]

- \( Z_{\text{bif}} (\nu|Y', Y) \) is the Nekrasov bifundamental contribution

\[
Z_{\text{bif}} (\nu|Y', Y) := \prod_{\Box \in Y'} (\nu + 1 + a_{Y'} (\Box) + l_{Y'} (\Box)) \prod_{\Box \in Y} (\nu - 1 - a_Y (\Box) - l_Y (\Box)). \tag{4.108}
\]

In particular, we have \( |Z_{\text{bif}} (0|Y, Y)|^2 = \prod_{\Box \in Y} h_Y (\Box) \).

- The three-point function \( C (\nu|Q', Q) \) is defined by

\[
C (\nu|Q', Q) \equiv C (\nu|Q' - Q) = \frac{G \left( 1 + \nu + Q' - Q \right)}{G \left( 1 + \nu \right) \Gamma (1 + \nu) Q' - Q}, \tag{4.109}
\]

where \( G (x) \) is the Barnes \( G \)-function. The only property of this function essential for our purposes is the recurrence relation \( G (x + 1) = \Gamma (x) G (x) \).

- Using the formula \( (4.88) \), we assume a concrete ordering: \( p'_{+, p'_-, q_{+}, q_{-}} \), \( p_1 > p_2 > \ldots \), and in \( (4.105) \) we suppose that \( + < - \).

An important feature of the product \( (4.105) \) is that the combinatorial part in the 2nd line depends only on combinations such as \( \sigma_{1} + Q_a, \sigma_{1}' + Q'_a \). This is most crucial for the Fourier transform structure of the full answer for the tau function \( \tau_{\text{Garnier}} (a) \).

Let us now present the plan of the proof, which will be divided into several self-contained parts. Most computations will be done up to an overall sign, and sometimes we will omit to indicate this. In the end we will consider the limit \( \theta_k \to +\infty, \sigma_{k}, \sigma_{k-1} \ll \theta_k, \sigma_k, \sigma_{k-1} \to +\infty \) to recover the correct sign.

1. First we will rewrite the formula \( (4.88) \) as

\[
Z_{Y', m'} (\mathcal{T}) = \pm e^{i\delta \vec{\eta} \cdot \vec{Q} + i\delta \vec{\eta} \cdot \vec{Q}' \tilde{Z}_{Y', Q'} (\mathcal{T})},
\]

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where \( \tilde{Z}_{Y,Q}^{
u, Q'} (T) \) is expressed in terms of yet another function \( \tilde{Z}_{\text{bif}} (\nu \mid Q', Y' ; Q, Y) \),
\[
\tilde{Z}_{Y,Q}^{
u, Q'} (T) = \prod_{\alpha}^{N} | \tilde{Z}_{\text{bif}} (0 \mid Q_{\alpha}, Y_{\alpha}, Q_{\alpha}, Y_{\alpha}) |^{-\frac{1}{2}} | \tilde{Z}_{\text{bif}} (0 \mid Q'_{\alpha}, Y'_{\alpha}, Q'_{\alpha}, Y'_{\alpha}) |^{-\frac{1}{2}} \times \\
\times \prod_{\alpha<\beta}^{N} \tilde{Z}_{\text{bif}} (\sigma_{\alpha} - \sigma_{\beta} \mid Q'_{\alpha}, Y'_{\alpha}, Q'_{\beta}, Y'_{\beta}) \tilde{Z}_{\text{bif}} (\sigma_{\alpha} - \sigma_{\beta} \mid Q_{\alpha}, Y_{\alpha}, Q_{\beta}, Y_{\beta}) ,
\]
(4.110)

which is defined as
\[
\tilde{Z}_{\text{bif}} (\nu \mid Q', Y' ; Q, Y) = \prod_{i}^{\nu} (-\nu)_{q'_{i}+\frac{1}{2}} \prod_{i}^{\nu+1} \prod_{i}^{\nu+\frac{1}{2}} (-\nu)_{p_{i}+\frac{1}{2}} \prod_{i}^{\nu+1} (\nu+1)_{p_{i}-\frac{1}{2}} \times \\
\times \prod_{i,j} \prod_{i,j} (\nu - q'_{i} - p_{j}) \prod_{i,j} (\nu + p'_{i} + q_{j}) \prod_{i,j} (\nu + p_{i} - p_{j}) ;
\]
(4.111)

2. At the second step, we prove that \( \tilde{Z}_{\text{bif}} (\nu \mid 0, Y' ; 0, Y) \equiv \tilde{Z}_{\text{bif}} (\nu \mid Y', Y) = \pm Z_{\text{bif}} (\nu \mid Y', Y) \).

3. Next it will be shown that
\[
\tilde{Z}_{\text{bif}} (\nu \mid Q', Y' ; Q, Y) = C (\nu \mid Q', Q) Z_{\text{bif}} (\nu + Q' - Q \mid Y', Y) .
\]
(4.112)

4. Finally, we check the overall sign and compute extra contribution to \( \tilde{\eta} \) to absorb it.

A realization of this plan is presented below.

**Step 1**

It is useful to decompose the product (4.88) into two different parts: a “diagonal” one, containing the products of functions of one particle/hole coordinate, and a “non-diagonal” part containing the products of pairwise sums/differences. Careful comparison of the formulas (4.88) and (4.110) shows that their non-diagonal parts actually coincide. Further analysis of (4.110) shows that its diagonal part is given by
\[
\prod_{(p',e) \in \Gamma} \psi_{p',e} \prod_{(-q',e) \in \Phi} \tilde{\psi}_{q',e} \prod_{(q,e) \in \Omega} \varphi_{q,e} \prod_{(p,e) \in \Gamma} \tilde{\varphi}_{p,e},
\]
with
\[
\psi_{p',e} = (1 + \epsilon \sigma_{k-1} + \theta_{k} - \sigma_{k})_{p'-\frac{1}{2}} \epsilon = : (1 + 2\sigma_{k-1})_{p'-\frac{1}{2}} \left( \frac{p'}{2} \right) !,
\]
\[
\tilde{\psi}_{q',e} = \left[ \epsilon = + : (\epsilon \sigma_{k} - \epsilon \sigma_{k} + 1)_{q'+\frac{1}{2}} \left( \frac{q'}{2} \right) ! \right],
\]
\[
\varphi_{q,e} = (\sigma_{k-1} + \theta_{k} - \epsilon \sigma_{k} + 1)_{q'-\frac{1}{2}} \epsilon = - : (1 + 2\sigma_{k})_{q'-\frac{1}{2}} \left( \frac{q'}{2} \right) !,
\]
\[
\tilde{\varphi}_{p,e} = (\sigma_{k-1} - \theta_{k} + \epsilon \sigma_{k})_{p'-\frac{1}{2}} \epsilon = - : (1 + 2\sigma_{k})_{p'+\frac{1}{2}} \left( \frac{p'}{2} \right) !.
\]
(4.113)
The notation $[\epsilon = + : X; \epsilon = - : Y]$ means that we should substitute this construction by $X$ when $\epsilon = +$ and by $Y$ when $\epsilon = -$. Comparing these expressions with (4.88), we may compute the ratios of diagonal factors which appear in $Z_{Y,m}^{\tilde{Y}^{\mu}, m^\prime} / \hat{Z}_{Y, \hat{Q}}^{\tilde{Y}^{\mu}, \hat{Q}^\prime}$:

$$
\delta \psi_{\mu, \epsilon} = \frac{(e\sigma_{k-1} + \theta_k - \sigma_k)(e\sigma_{k-1} + \theta_k + \sigma_k)}{[\epsilon = + : 2\sigma_k; \epsilon = - : 1]},
$$

$$
\hat{\delta} \psi_{\mu, \epsilon} = \frac{(-e\sigma_{k-1} - \theta_k + \sigma_k)^{-1}(-e\sigma_{k-1} - \theta_k - \sigma_k)^{-1}}{[\epsilon = + : (2\sigma_k)^{-1}; \epsilon = - : 1]},
$$

$$
\delta \varphi_{\mu, \epsilon} = \frac{(\sigma_{k-1} + \theta_k - \sigma_k)(-\sigma_{k-1} + \theta_k - \sigma_k)}{[\epsilon = + : 1; \epsilon = - : 2\sigma_k]},
$$

$$
\hat{\delta} \varphi_{\mu, \epsilon} = \frac{(\sigma_{k-1} - \theta_k + \sigma_k)^{-1}(\sigma_{k-1} - \theta_k + \sigma_k)^{-1}}{[\epsilon = + : 1; \epsilon = - : (-2\sigma_k)^{-1}]}.
$$

Since $|p_\pm| - |q_\pm| = Q_\pm$, these formulas allow to determine the corrections $\delta_1 \eta_\pm$:

$$
e^{i\delta_1 \eta_+} = \frac{(\theta_k + \sigma_{k-1})^2 - \sigma_k^2}{2\sigma_k}, \quad e^{-i\delta_1 \eta_+} = \frac{(\theta_k - \sigma_k)^2 - \sigma_{k-1}^2}{2\sigma_k},
$$

$$
e^{i\delta_1 \eta_-} = \frac{(\theta_k - \sigma_{k-1})^2 - \sigma_k^2}{2\sigma_k}, \quad e^{-i\delta_1 \eta_-} = \frac{(\theta_k + \sigma_k)^2 - \sigma_{k-1}^2}{2\sigma_k}.
$$

One could notice that some minus signs should also be taken into account, so that

$$
Z_{Y,m}^{\tilde{Y}^{\mu}, m^\prime} (T) = (-1)^{[a_1^\mu]_+ + [p^\mu]_-} e^{i\delta_1 \eta} \hat{Q}^\epsilon + i\delta_1 \eta \hat{Q}^\epsilon Z_{Y, \hat{Q}}^{\tilde{Y}^{\mu}, \hat{Q}^\prime} (T).
$$

This is however not essential, as these signs will be recovered at the last step. A more important thing to note is that in the reference limit described by $\theta_k \to +\infty$, $\sigma_k, \sigma_{k-1} \ll \theta, \sigma_k, \sigma_{k-1} \to +\infty$ one has

$$
\text{sgn}(e^{i\delta_1 \eta}) = \text{sgn}(e^{i\delta_1 \eta'}) = 1.
$$

**Step 2**

Let us now formulate and prove combinatorial

**Theorem 4.34.** $\tilde{Z}_{\text{bif}} (\nu|0, Y'; 0, Y) \equiv \hat{Z}_{\text{bif}} (\nu|Y', Y) = \pm Z_{\text{bif}} (\nu|Y', Y)$.

This statement follows from the following two lemmas.

**Lemma 4.35.** Equality $Z_{\text{bif}} = \pm \tilde{Z}_{\text{bif}}$ holds for the diagrams $Y', Y \in \mathbb{Y}$ iff it holds for $Y', Y$ with added one column of admissible height $L$.

**Proof.** Let us denote the new value of $Z_{\text{bif}}$ by $Z_{\text{bif}}^*$, then

$$
Z_{\text{bif}}^* = \frac{(1 + \nu)_{L} \prod_i (L + p_i + \frac{1}{2} + \nu)}{\prod_i (L - q_i + \frac{1}{2} + \nu)} \frac{(1 - \nu)_{L} \prod_i (L + p_i + \frac{1}{2} - \nu)}{\prod_i (L - q_i + \frac{1}{2} - \nu)} Z_{\text{bif}}.
$$

The extra factor comes only from the product over $2L$ new boxes. To explain how its expression is obtained, we will use the conventions of Fig. 4.14.

---

4 Everywhere in this appendix $X^*$ denotes the value of a quantity $X$ after appropriate transformation.
4. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions

Figure 4.14: A Young diagram $Y^{\prime*}$ obtained from $Y^{\prime} = \{6, 4, 4, 2, 2\}$ by addition of a column of length $L = 7$.

To compute the contribution from the red boxes it is enough just to multiply the corresponding shifted hook lengths, which yields $\prod_i (L + p_{i}^{\prime} + \frac{1}{2} + \nu)$. To compute the contribution from the green boxes one has to first write down the product of numbers from $\nu + L$ to $\nu + 1$ (i.e. the Pochhammer symbol $(1 + \nu)^L$ in the numerator), keeping in mind that each step down by one box decreases the leg-length of the box by at least one. Then one has to take into account that some jumps in this sequence are greater than one: this happens exactly when we meet some rows of the transposed diagram. We mark with the green crosses the boxes whose contributions should be cancelled from the initial product: they produce the denominator.

Next let us check what happens with $\tilde{Z}_{\text{bif}}$. We have

$$\tilde{Z}_{\text{bif}} (\nu|Y^{\prime}, Y) = \prod_i (\nu-q_i^{*}+\frac{1}{2}) \prod_i (\nu-p_i^{*}+\frac{1}{2}) \prod_i (\nu-p_i^{\prime*}+\frac{1}{2}) \times$$

$$\times \frac{\prod_{i,j} (\nu-q_i^{*} - p_j^{*}) \prod_{i,j} (\nu+p_i^{*} + q_j^{*})}{\prod_{i,j} (q_i^{*} - q_j^{*} - \nu) \prod_{i,j} (p_i^{*} - p_j^{*} + \nu)},$$

where

$$\{q_i^{*}\} = \{(L-1/2), (q_1 - 1), \ldots, (q_{d-1} - 1), (q_d - 1)\},$$

$$\{p_i^{*}\} = \{(p_1 + 1), \ldots, (p_d + 1), \frac{1}{2}\},$$

$$\{q_i^{\prime*}\} = \{(L-1/2), (q_1^{\prime} - 1), \ldots, (q_{d'-1}^{\prime} - 1), (q_d^{\prime} - 1)\},$$

$$\{p_i^{\prime*}\} = \{(p_1^{\prime} + 1), \ldots, (p_{d'}^{\prime} + 1), \frac{1}{2}\},$$

and $d, d'$ denote the number of boxes on the main diagonals of $Y, Y^{\prime}$. The above notation means that one has either to simultaneously include or not to include the coordinates tilded in the same way. These numbers are included in the case when both of them are positive (it implies that $q_d \neq \frac{1}{2}$ or $q_{d'}^{\prime} \neq \frac{1}{2}$). Fig. 4.15 below illustrates the difference between these two cases.

We may now consider one by one four possible options, namely: i) $q_d \neq \frac{1}{2}, q_{d'}^{\prime} \neq \frac{1}{2}$; ii) $q_d = q_{d'}^{\prime} = \frac{1}{2}$; iii) $q_d \neq \frac{1}{2}, q_{d'}^{\prime} = \frac{1}{2}$; iv) $q_d = \frac{1}{2}, q_{d'}^{\prime} \neq \frac{1}{2}$. For instance, for $q_d \neq \frac{1}{2}$,
4.5. Relation to Nekrasov functions

Figure 4.15: Possible mutual configurations of main diagonals of $Y, Y^*$; $q_d = \frac{1}{2}$ (left) and $q_d \neq \frac{1}{2}$ (right).

$q_d' \neq \frac{1}{2}$ after massive cancellations one obtains

$$
\frac{Z_{\text{bif}}'}{Z_{\text{bif}}} = \prod_{i=1}^{d'} \frac{1}{-\nu + q_i' - \frac{1}{2}} \frac{1}{\nu + q_i - \frac{1}{2}} \nu^{-1} (\nu)_L \prod_{i=1}^{d} \left( -\nu + p_i + \frac{1}{2} \right) \nu^1 (\nu)_L \prod_{i=1}^{d} \left( \nu + p_i' + \frac{1}{2} \right) \times \\
\times \prod_i (\nu - L - \frac{1}{2} - p_i) \prod_i (\nu - q_i' + \frac{1}{2}) \prod_i (\nu - \frac{1}{2} + q_i) \prod_i (\nu + p_i' + L + \frac{1}{2}) \frac{(\nu - L)(\nu + L)}{\nu^2} = \\
\times = (1 - \nu)_L (1 + \nu)_L \prod_i (\nu - L - \frac{1}{2} - p_i) \prod_i (\nu + p_i' + L + \frac{1}{2}) = Z_{\text{bif}}'/Z_{\text{bif}},
$$

where the first line of the first equality corresponds to the ratio of diagonal parts and the second to non-diagonal ones. The proof in the other three cases is analogous. □

Corollary 4.36. $Z_{\text{bif}} = \tilde{Z}_{\text{bif}}$ for arbitrary $Y, Y' \in \mathbb{Y}$ iff $Z_{\text{bif}} = \pm Z_{\text{bif}}'$ for diagrams with $\{q_i\} = \{\frac{1}{2}, \ldots, L - \frac{1}{2}\}$ (that is, for the diagrams containing a large square on the left).

Lemma 4.37. The equality $Z_{\text{bif}} = \tilde{Z}_{\text{bif}}$ holds for given diagrams $Y, Y' \in \mathbb{Y}$ with a large square iff it holds for the diagrams with a large square and one deleted box.

Proof. Suppose that we have added one box to the $i$th row of $Y'$. The only boxes

Figure 4.16: A pair of Young diagrams (red and green) with a large square (black) and added box (blue square).

whose contribution to $Z_{\text{bif}}$ depends on the added box lie on its left in the diagram $Y'$ and above it in the diagram $Y$, see Fig. 4.16. The contribution from the boxes on the left (green circles) was initially given by

$$
Z_{\text{bif}}^{\text{left}} = \frac{(\nu)^p_i + L + \frac{1}{2}}{\prod_{j \geq i} (p_i' - p_j + \nu) \cdot (\nu)_{j-1+i}},
$$

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where \( \hat{j} = \min \{ \lfloor j/p_j + j \leq p'_j + i + 1 \rfloor \cup \{ L \} \} \) (notice that we can move \( \hat{j} \) in the range where \( p_j + j = p'_j + i + 1 \)). The contribution from the boxes on the top (red circles) was \( Z_{\text{top}}^{\text{bif}} = \prod_{j<j} (-\nu + p_j - p'_j - 1) \). After addition of one box (blue square) it transforms into \( Z_{\text{bif}}^{\text{step}} = \prod_{j<j} (-\nu + p_j - p'_j) \), whereas the previous part becomes

\[
Z_{\text{bif}}^{\text{left}} = \frac{(\nu)_{p'_j + L + \frac{1}{2}}}{\prod_{j \geq j} (p'_i - p_j + \nu) \cdot (\nu)_{j-i+1}}.
\]

The ratio of the transformed and initial functions is then given by

\[
\frac{Z_{\text{bif}}^*}{Z_{\text{bif}}} = \frac{(p'_j + L + \frac{1}{2} + \nu) \prod_{j<j} (p'_j - p_j + \nu) \prod_{j \geq j} (p'_i - p_j + \nu)}{(p'_i + \frac{1}{2} + L + \nu) \prod_{j} (p'_i - p_j + \nu)} = \frac{Z_{\text{bif}} - Z_{\text{bif}}}{Z_{\text{bif}}}.
\]

On the other hand, the ratio \( \frac{Z_{\text{bif}}^*}{Z_{\text{bif}}} \) is easier to compute since the addition of one box to the \( i \)th row of \( Y' \) simply shifts one coordinate, \( p'_i \mapsto p'_i + 1 \). From (4.111) and the large square condition \( \{ q_i \} = \{ \frac{1}{2}, \ldots, L - \frac{1}{2} \} \) it follows that

\[
\frac{Z_{\text{bif}}^*}{Z_{\text{bif}}} = (p'_i + \frac{1}{2} + \nu) \prod_{j} (p'_i - p_j + \nu) = \frac{Z_{\text{bif}}}{Z_{\text{bif}}},
\]

which finishes the proof.

Using two inductive procedures described above, any pair of diagrams \( Y, Y' \in \mathcal{Y} \) can be reduced to equal squares, in which case the statement of Theorem 4.34 can be checked directly.

**Step 3**

Let us move to the third part of our plan and prove

**Theorem 4.38.** \( Z_{\text{bif}} (\nu|Q', Y'; Q, Y) = C (\nu|Q' - Q) Z_{\text{bif}} (\nu + Q' - Q|Y', Y) \).

**Proof.** It is useful to start by computing \( Z_{\text{bif}} \) for the “vacuum state”

\[
p_\alpha = p_\alpha^Q := \{ \frac{1}{2}, \frac{3}{2}, \ldots, Q^{(\alpha)} - \frac{1}{2} \}, \quad q_\alpha = \emptyset \quad \text{for } Q^{(\alpha)} > 0,
\]

\[
p_\alpha = \emptyset, \quad q_\alpha = q_\alpha^Q := \{ \frac{1}{2}, \frac{3}{2}, \ldots, -Q^{(\alpha)} - \frac{1}{2} \} \quad \text{for } Q^{(\alpha)} < 0.
\]

One obtains

\[
\hat{Z}_{\text{bif}} (\nu|p^Q, 0; p^Q, 0) = (-1)^{Q(Q+1)/2} \prod_{i=1}^{Q'} \nu^{-1} (\nu) \prod_{i=1}^{Q} (-\nu) \prod_{i=1}^{Q'} \prod_{j=1}^{Q} (\nu + i - j)^{-1} =
\]

\[
= (-1)^{Q(Q+1)/2} \prod_{i=1}^{Q'} \frac{\Gamma (\nu + i)}{\Gamma (\nu + 1)} \prod_{i=1}^{Q} \frac{\Gamma (i - \nu)}{\Gamma (-\nu)} \prod_{j=1}^{Q} \frac{\Gamma (\nu - j + 1)}{\Gamma (\nu - j + Q') + 1} =
\]

\[
= \frac{G (1 + \nu + Q') G (1 - \nu + Q)}{G (1 - \nu)} \frac{(-1)^{Q(Q+1)/2}}{\Gamma (\nu + 1)} \frac{G (1 + 1 - Q) G (\nu + 1 - Q)}{G (\nu + Q' + 1)}
\]

\[
= (-1)^{Q(Q+1)/2} \frac{G (1 + \nu + Q) G (1 + \nu + Q' - Q)}{G (1 + \nu - Q) G (1 - \nu) \Gamma (1 + \nu) \Gamma (\nu + Q)^{-Q} \Gamma (-\nu)^Q}.
\]

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Using the recurrence relation
\[ \frac{G(1 - \nu + Q)}{G(1 + \nu - Q)} = (-1)^{Q(Q-1)/2} \frac{G(1 - \nu)}{G(1 + \nu)} \left( \frac{\pi}{\sin \pi \nu} \right)^Q, \]
and the reflection formula \( \Gamma(-\nu) \Gamma(1 + \nu) = -\frac{\pi}{\sin \pi \nu} \), the last expression can be rewritten as
\[ C(\nu|Q' - Q) := \tilde{Z}_{\text{bif}}\left( \nu | p'_Q, \emptyset; p'_Q, \emptyset \right) = \frac{G(1 + \nu + Q' - Q)}{G(1 + \nu) \Gamma(1 + \nu)^{Q'-Q}}. \]

Next let us rewrite the expression for \( \tilde{Z}_{\text{bif}}(\nu|Y', Q'; Y, Q) \) for charged Young diagrams in terms of uncharged ones. To do this, we will try to understand how this expression changes under the following transformation, shifting in particular all particle/hole coordinates associated to \( Y' \):
\[ p'_i \mapsto p'_i + 1, \quad q'_i \mapsto q'_i - 1, \quad \nu \mapsto \nu - 1. \]

It should also be specified that if we had \( q' = \frac{1}{2} \), then this value should be dropped from the new set of hole coordinates; if not, we should add a new particle at \( p' = \frac{1}{2} \).

Looking at Fig. 4.12, one may understand that this transformation is exactly the shift \( Q' \mapsto Q' + 1 \) preserving the form of the Young diagram.

Now compute what happens with \( \tilde{Z}_{\text{bif}}(\nu|Y', Q'; Y, Q) \). One should distinguish two cases:

1. If there is no hole at \( q' = \frac{1}{2} \) in \( (Y', Q') \), then it follows from (4.111) that
\[ \frac{\tilde{Z}_{\text{bif}}(\nu - 1|Q' + 1, Y'; Q, Y)}{\tilde{Z}_{\text{bif}}(\nu|Q', Y'; Q, Y)} = \prod_i \frac{(\nu - \frac{1}{2} + q_i)}{(\nu - \frac{1}{2} - p_i)} \prod_i \nu - p_i + \frac{1}{2} - q_i \nu - q_i \nu - |q'| = \nu^{Q'-Q}. \]

2. Similarly, if there is a hole at \( q' = \frac{1}{2} \) to be removed, then
\[ \frac{\tilde{Z}_{\text{bif}}(\nu - 1|Q' + 1, Y'; Q, Y)}{\tilde{Z}_{\text{bif}}(\nu|Q', Y'; Q, Y)} = \nu^{-1} \prod_i \frac{(\nu - \frac{1}{2} + q_i)}{(\nu - \frac{1}{2} - p_i)} \times \nu^{\nu - |q'| + 1} \prod_i \nu - p_i - \frac{1}{2} + q_i \nu - q_i \nu - |q'| = \nu^{Q'-Q}. \]

The computation of the shift of \( Q \) is absolutely analogous thanks to the symmetry properties of \( \tilde{Z}_{\text{bif}} \).

Introducing
\[ \tilde{Z}_{\text{bif}}^\nu(\nu|Q', Y'; Q, Y) = \frac{Z_{\text{bif}}(\nu|Q', Y'; Q, Y)}{C(\nu|Q' - Q)}, \]
it is now straightforward to check that
\[ \frac{\tilde{Z}_{\text{bif}}(\nu - 1|Q' + 1, Y'; Q, Y)}{\tilde{Z}_{\text{bif}}(\nu|Q', Y'; Q, Y)} = \frac{\tilde{Z}_{\text{bif}}(\nu + 1|Q', Y'; Q + 1, Y)}{\tilde{Z}_{\text{bif}}(\nu|Q', Y'; Q, Y)} = 1, \]

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and therefore \( \tilde{Z}_{\text{bif}}^\nu (\nu|Q', Y'; Q, Y) = \tilde{Z}_{\text{bif}}^\nu (\nu + Q' - Q|0, Y'; 0, Y) \). Finally, combining this recurrence relation with \( C(\nu|0) = 1 \), one obtains the identity

\[
\frac{\tilde{Z}_{\text{bif}}^\nu (\nu|Q', Y'; Q, Y)}{C(\nu|Q' - Q)} = \tilde{Z}_{\text{bif}}^\nu (\nu + Q' - Q|Y', Y'),
\]

which is equivalent to the statement of the theorem. \( \square \)

**Step 4**

At this point, we have already shown that

\[
Z_{\tilde{Y}'}^{\text{m}', \text{m}} (T) = \pm e^{i(\delta_{\eta'_+ - \delta_{\eta'_-}}\text{m}') + i(\delta_{\eta'_+ - \delta_{\eta'_-}}\text{m})} Z_{\tilde{Y}'}^{\text{m}', \hat{\text{m}}'} (T).
\]

It remains to check the signs in the reference limit described above. Note that \( \text{sgn} (\tilde{Z}) = 1 \), since \( \text{sgn} (C(\nu|Q', Q)) = 1 \) and \( \text{sgn} (Z_{\text{bif}} (\nu|Y', Y')) = 1 \) as \( \nu \to \infty \). Everywhere in this subsection the calculations are done modulo 2.

First let us compute the sign of the non-diagonal part of \( Z \). To do this, one has to fix the ordering as

\[
x_i: p'_+ + \sigma_{k-1}, p'_- - \sigma_{k-1}, -q'_+ - \theta_k + \sigma_k, -q'_- - \theta_k - \sigma_k,
\]

\[
y_i: -q'_+ + \sigma_{k-1}, -q'_- - \sigma_{k-1}, p'_+ - \theta_k + \sigma_k, p'_- - \theta_k - \sigma_k.
\]

The variables in each of these groups are ordered as \( p_1 > p_2 > \ldots \) where \( p_1 > p_2 > \ldots \). This gives

\[
\text{lsgn} \left( Z_{\text{non-diag}} \right) = |p'_+| \cdot |q'_+| + |q'_+| \cdot (|q'_+| + |p'_+|) + |q'_-| \cdot (|q'_-| + |p'_-|) + |p'_+| \cdot (|p'_+| + |p'_-|) + |q'_+| \cdot (|q'_+| - 1) + \frac{|q'_-|}{2} \cdot (|q'_-| - 1) + |p'_+| \cdot (|p'_+| - 1) + \frac{|p'_-|}{2} \cdot (|p'_-| - 1) + |q'_-| \cdot |p'_+| + |p'_-| + |q'_-| \cdot |p'_+| + |p'_-| + |q'_-|.\]

Using the charge balance conditions

\[
|p'_+| - |q'_+| = |q'_-| - |p'_-| = m,
\]

\[
|p'_+| - |q'_+| = |q'_-| - |p'_-| = m',
\]

the above expression can be simplified to

\[
\text{lsgn} \left( Z_{\text{non-diag}} \right) = m + m' + m|p'_+| + m'|p'_+| + |p'_+| + |p'_-|.
\]

Next compute the sign of the diagonal part,

\[
\text{lsgn} \left( Z_{\text{diag}} \right) = \sum (p'_- + q'_+ + q'_- + p_-) + \frac{|p'_-| - |q'_+| + |q'_-| - |p'_-|}{2}.
\]

Combining these two expressions, after some simplification we get

\[
\text{lsgn} (Z) = |p'_+| \cdot |q'_+| + |q'_+| \cdot |q'_-| + \sum \left( q'_+ + \frac{1}{2} \right) + \sum \left( q'_+ + \frac{1}{2} \right) + \sum \left( p'_- + \frac{1}{2} \right) + \sum \left( p'_- + \frac{1}{2} \right) + m.
\]

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This expression can be represented as

\[ \text{lsgn}(Z) = \text{lsgn}(p,q) + \text{lsgn}(p',q') + m. \]

To get the desired formula, one has to absorb \( m \) by adding extra shift \( e^{i2\eta_+} = -1 \). Combining this shift with the previous formulas (4.115), we deduce the full shift (4.107) of the Fourier transformation parameters. The final formula for the relative sign is

\[ \text{lsgn}\left(\frac{Z}{\hat{Z}}\right) = \text{lsgn}(p,q) + \text{lsgn}(p',q'), \]

which completes our calculation.
Exact conformal blocks for the W-algebras, twist fields and isomonodromic deformations

Abstract

We consider the conformal blocks in the theories with extended conformal W-symmetry for the integer Virasoro central charges. We show that these blocks for the generalized twist fields on sphere can be computed exactly in terms of the free field theory on the covering Riemann surface, even for a non-abelian monodromy group. The generalized twist fields are identified with particular primary fields of the W-algebra, and we propose a straightforward way to compute their W-charges. We demonstrate how these exact conformal blocks can be effectively computed using the technique arisen from the gauge theory/CFT correspondence. We discuss also their direct relation with the isomonodromic tau-function for the quasipermutation monodromy data, which can be an encouraging step on the way of definition of generic conformal blocks for W-algebra using the isomonodromy/CFT correspondence.

Introduction

An interest to conformal field theories (CFT) with extended nonlinear W-symmetry generated by the higher spin holomorphic currents has long history, starting from the original work [ZamW]. These theories resemble many features of ordinary CFT (with only Virasoro symmetry), like free field representation and degenerate fields [FZ, FL], but it already turns to be impossible to construct in generic situation their conformal blocks [BW] (or the blocks for the algebra of higher spin W-currents) which are the main ingredients in the bootstrap definition of the physical correlation functions.

This interest has been seriously supported in the context of rather nontrivial correspondence between two-dimensional CFT and four-dimensional supersymmetric gauge theory [LMN, NO, AGT], where the conformal blocks have to be compared with the Nekrasov instanton partition functions [Nek, NP] producing in the quasiclassical limit the Seiberg-Witten prepotentials [SW]. This correspondence meets serious difficulties beyond the level of the $SU(2)$ gauge quivers on gauge theory side, i.e. for the higher rank gauge groups, which should correspond to the not yet defined generic blocks of
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the W-conformal theories. It is already clear, however, that the technique developed in two-dimensional CFT can be applied to four-dimensional gauge theories, and vice versa. Following [KriW, Mtau, GMqui] we are going to demonstrate how it can save efforts for the computation of the exact conformal blocks for the twist fields in theories with W-symmetry.

Even in the Virasoro case generic conformal block is a very nontrivial special function [BPZ], but there exists two important particular cases where the answer is known almost in explicit form – the correlation functions containing degenerate fields (which are related to the integrals of hypergeometric type) and the exact Zamolodchikov blocks for a nontrivial (though \(c = 1\)) theory [ZamAT86, ZamAT87, ApiZam]. The first class can be generalized to the case of W-algebras, where similar hypergeometric formulas arise in the case of so-called completely degenerate fields [FLitv07]. The algebraic definition still exists when degeneracy is not complete, and in this case the most effective way of computation comes from use of the gauge theory Nekrasov functions.

Below we are going to study the W-analogs of the Zamolodchikov conformal blocks, which do not belong to the class of algebraic ones. They can be nevertheless computed exactly, partially using the methods of gauge theories and corresponding integrable systems. We are going to demonstrate also their direct relations with exactly known isomonodromic \(\tau\)-functions [SMJ], which confirms therefore their role as an important example of a generic W-block which can be possible defined (for integer central charges) in terms of corresponding isomonodromic problem [Gav].

The exact conformal blocks of the W-algebras are closely related to the correlation functions of the twist fields, studied long ago in the context of perturbative string theory (see e.g. [Knizhnik, BR, DFMS]). However, unlike [ZamAT87], the correlators of the twist fields in these papers were not really expressed through the conformal blocks, and therefore their relation to the W-algebras remained out of interest, so we are going to fill partially this gap.

The chapter is organized as follows. In sect. 6.4 we define the correlators of currents on sphere in presence of the twist fields, and show how they can be computed in terms of free conformal field theory on the cover. In sect. 5.3 we identify the twist fields with the primary fields of the W-algebra and propose a way to extract the values of their quantum numbers from the previously computed correlation functions of the currents. We also show there that these W-charges have obvious meaning in terms of the eigenvalues of the quasipermutation monodromy matrices. In sect. 5.4 we define the result for the exact conformal block in terms of integrable systems. In particular, we show that the main classical contribution to the result satisfies the well-known Seiberg-Witten (SW) period equation [SW, KriW], moreover, in this case they can be immediately solved, which gives the most effective way to express the answer through the period matrix and the prime form on the covering surface. Next, in sect. 5.5 we discuss the connection of the W-algebra conformal blocks with the \(\tau\)-function of the isomonodromic problem, and show that the W-blocks we have constructed correspond in this context to the \(\tau\)-function for the case of quasipermutation monodromy data. In sect. 5.6 we construct some explicit examples, and some extra technical information

\footnote{Strictly speaking the CFT-Painlevé correspondence [GIL12] gives rise to a collection of new exact conformal blocks, coming from the algebraic solutions of Painlevé VI.}

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(the recursion procedure we have used for construction of correlators of the higher W-currents, the discussion of their OPE with the stress-tensor, and the computation of the asymptotics of the period matrix on the cover and its relation with the structure constants in the expansion of the isomonodromic τ-function) is located in the Appendix.

Twist fields and branched covers

Definition

We start now with the construction of the conformal blocks of $W(sl_N) = W_N$ algebra at integer Virasoro central charges $c = N - 1$ following the lines of [ZamAT87, Knizhnik, BR]. It is well-known [FL] that $W_N$ algebra has free-field representation in terms of $N - 1$ bosonic fields with the currents $J^a(z) = i \partial \phi^a(z)$ satisfying operator product expansion (OPE)

$$J^a(z)J^b(z') = \frac{K^{ab}}{(z - z')^2} + \text{reg.} \quad (5.1)$$

where $K^{ab}$ is the scalar product in the Cartan subalgebra $h \subset g = sl_N$. For the current $J(z) = \sum_{a=1}^{N-1} h_a J^a(z) = i \partial \phi(z)$, where $h_a$ is the basis in $h$, it is useful to introduce explicit components

$$J_i(z) = (e_i, J(z)), \quad i = 1, \ldots, N \quad (5.2)$$

with $\{e_i\}$ being the weights of the first fundamental or vector representation, so that

$$J_i(z)J_j(z') = \frac{(e_i, e_j)}{(z - z')^2} + \text{reg.} = \frac{\delta_{ij} - \frac{1}{N}}{(z - z')^2} + \text{reg.} \quad (5.3)$$

All high-spin currents of the $W_N$-algebra at $c = N - 1$ are elementary symmetric polynomials of $J_i(z) \ (\sum_i J_i(z) = 0)$, e.g. the first three are

$$T(z) = -W_2(z) = \frac{1}{2} : (J(z), J(z)) : = \frac{1}{2} \sum_i : J_i(z)^2 :$$

$$W(z) = W_3(z) = \sum_{i<j<k} : J_i(z)J_j(z)J_k(z) : = \frac{1}{3} \sum_i : J_i(z)^3 :$$

$$W_4(z) = \sum_{i<j<k<l} : J_i(z)J_j(z)J_k(z)J_l(z) : = \frac{1}{8} : \left( \sum_i J_i^2(z) \right)^2 : - \frac{1}{4} \sum_i : J_i^4(z) :$$

(5.4)

and the primary fields for the current algebra are exponentials of $\phi(z) \in h$

$$V_{\theta}(z) = e^{i(\theta, \phi(z))} \quad (5.5)$$

with the corresponding eigenvalues $w_k(\theta)$ of the zero modes of the $W_k(z)$-generators given by symmetric functions of $(e_i, \theta)$. 

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Now we are going to introduce new fields $O_s(z)$, which are still primary for all high-spin currents $\{W_k(z)\}$, but not for the currents $J_s(z)$. They can be realized as monodromy fields

$$\gamma_q : J(z)O_s(q) \rightarrow s(J(z))O_s(q)$$

(5.6)

for some contours $\gamma_q$ encircling the point $q$ on the base curve, where $s \in W_{stw} = S_N$ is an element of the corresponding Weyl group. The particular cases of this construction were known for the Abelian monodromy group of the cover [BR, ZamAT'87], but even there in the cases with $N > 2$ they were not identified with $W_N$ primary fields.

Now we are going to construct the particular conformal block (on $\mathbb{P}^1$ with global coordinate $z$), where all monodromy fields can be grouped as $O_s(q_{2i+1})O_{s^{-1}}(q_{2i+2})$ at $q_{2i+1} \rightarrow q_{2i+2}$, so that one can take an OPE

$$O_s(z)O_{s^{-1}}(z') = \sum_\theta C_{s,\theta}(z-z')^{\Delta(\theta)-2\Delta(s)}(V_\theta(z') + \text{descendants})$$

(5.7)

and fix the quantum numbers in the intermediate channels, where there are only the fields with definite $h = u(1)^{N-1}$ charges $\frac{1}{2\pi} \oint zJ(z)V_\theta(z) = \theta \cdot \nu_\theta(z)$. In order to do this consider $G_0(q) = G_0(q_1, \ldots, q_{2L})$, together with 1-form $G_1(z|q)dz = G_1(z|q_1, \ldots, q_{2L})dz$ and bidifferential $G_2^ij(z, z'|q)dzdz' = G_2^ij(z, z'|q_1, \ldots, q_{2L})dzdz'$, where

$$G_0(q_1, \ldots, q_{2L}) = \langle O_{s_1}(q_1)O_{s_2^{-1}}(q_2)\ldots O_{s_L^{-1}}(q_{2L-1})O_{s_L^{-1}}(q_{2L}) \rangle$$

$$G_1(z|q_1, \ldots, q_{2L}) = \langle \langle J(z)O_{s_1}(q_1)O_{s_2^{-1}}(q_2)\ldots O_{s_L^{-1}}(q_{2L-1})O_{s_L^{-1}}(q_{2L}) \rangle \rangle$$

$$G_2^ij(z, z'|q_1, \ldots, q_{2L}) = \langle \langle J_i(z)J_j(z')O_{s_1}(q_1)O_{s_2^{-1}}(q_2)\ldots O_{s_L^{-1}}(q_{2L-1})O_{s_L^{-1}}(q_{2L}) \rangle \rangle$$

(5.8)

which become single-valued on the cover $\pi : \mathcal{C} \rightarrow \mathbb{P}^1$ with the branch points $q_a$ and corresponding monodromies $s_a$. The indices $i, j$ are just labels of the sheets of this cover, so the multi-valued differentials (5.8) on $\mathbb{P}^1$ are now expressed in terms of the single-valued $G_1(\xi;q_1, \ldots, q_{2L})d\xi$ and $G_2(\xi, \xi'|q_1, \ldots, q_{2L})d\xi d\xi'$ on the covering surface $\mathcal{C}$:

$$G_1^i(z|q_1, \ldots, q_{2L})dz = G_1(z^i|q_1, \ldots, q_{2L})dz^i$$

$$G_2^{ij}(z, z'|q_1, \ldots, q_{2L})dzdz' = G_2(z^i, z'^j|q_1, \ldots, q_{2L})dz^idz'^j$$

(5.9)

where $z^i = \pi^{-1}(z)^i$ is the coordinate at $i$'th preimage of the point $z$, not the power (note that number $i$ is not defined globally due to the presence of monodromies). We should also point out that only local deformations of the positions of the branch points $\{q_\alpha\}$ are allowed, since the global ones – due to nontrivial monodromies – can change the global structure of the cover $\pi : \mathcal{C} \rightarrow \mathbb{P}^1$. This leads in particular to the fact that in the case of non-abelian monodromy group the positions of the branch points $\{q_\alpha\}$ cannot play the role of the global coordinates on the corresponding Hurwitz space.\footnote{Although, sometimes the Hurwitz space of our interest occurs to be rational, and in this case one can choose some global coordinates – but not the positions of the branch points. An explicit example is considered below in sect.5.6.}

The picture of the 3-sheeted cover with the most simple branch cuts looks like at fig.5.1, where we have shown explicitly three (dependent) cycles in $H_1(\mathcal{C})$ corresponding to the cuts between the positions of the fields, labeled by mutually inverse permutations. To understand our notations better we present also at fig.5.2 the picture of the vicinity of the branch-point (of the 6-sheeted cover) of the cyclic type $s \sim [3, 1, 2]$ with several independent permutation cycles.
5.2. Twist fields and branched covers

Figure 5.1: Covering Riemann surface $\mathcal{C}$ with simplest cuts between the positions of colliding twist-fields. Sum of the shown cycles of A-type vanishes in $H_1(\mathcal{C})$.

Figure 5.2: Vicinity of a ramification point of a general type.
5. Exact conformal blocks for the W-algebras, twist fields and isomonodromic deformations

Correlators with the current

Consider a permutation of the cyclic type \( s = [l_1, \ldots, l_k] \), which corresponds to the ramification at \( z = q \) (for simplicity we put \( q = 0 \)) with \( k \) preimages \( q^i, \pi(q^i) = q \) with multiplicities \( l_i \). The coordinates in the vicinity of these points can be chosen as \( \xi_i = z^{1/l_i} \). One can write down a general expression for the expansion of current \( J(z) \) on the cover

\[
J(z) = \sum_{i=1}^{k} \sum_{v_i=1}^{l_i-1} \sum_{n \in \mathbb{Z}} a_{n-v_i/l_i}^{(i)} \cdot h_{i,v_i} + \sum_{j=1}^{k-1} \sum_{n \in \mathbb{Z}} b_{n}^{(j)} \cdot H_j
\]

where \( h_{i,v_i} \) and \( H_j \) form the orthogonal basis in \( \mathfrak{h} \) out of the eigenvectors of the permutation \( s \), and in coordinates (related to the weights \( \{e_i\} \)) they have the form

\[
\begin{align*}
    h_{1,v_1} &= (1, e^{2\pi i v_1/l_1}, \ldots, e^{2\pi i (l_1-1)v_1/l_1}, 0, \ldots, 0; 0, \ldots, 0, 0) \\
    h_{2,v_2} &= (0, \ldots, 0, 1, e^{2\pi i v_2/l_2}, \ldots, e^{2\pi i (l_2-1)v_2/l_2}, 0, \ldots, 0; 0, \ldots, 0, 0) \\
    H_j &= (y_j^{(1)}, \ldots, y_j^{(1)}; y_j^{(2)}, \ldots, y_j^{(2)}; \ldots; y_j^{(k)}, \ldots, y_j^{(k)}) \\
    &\quad \quad \sum_i l_i y_j^{(i)} = 0
\end{align*}
\]

with \( h_{i,v_i} \), corresponding to non-zero eigenvalues of the permutation cycles \( s_i \), while \( H_j \) to the trivial permutations.

The expansion modes satisfy usual Heisenberg commutation relations \([a_u^{(i)}, a_v^{(j)}] = u \delta_{u+v, \delta_{ij}}, [b_u^{(i)}, b_v^{(j)}] = u \delta_{u+v, \delta_{ij}}\), up to possible inessential numerical factors which can be extracted from the singularity of the OPE \( J(z)J(z') \). The condition that field \( \mathcal{O}_s(q) \) is primary for the W-currents means in terms of the corresponding state that

\[
a_{u_i}^{(i)}|s\rangle = b_{n}^{(j)}|s\rangle = 0, \quad u_i > 0, \ n > 0, \ \forall \ i, j
\]

and this state is also an eigenvector of the zero modes \( b_{0}^{(j)} \ \forall \ j \). The corresponding eigenvalues are extra quantum numbers – the charges, which have to be included into the definition of the state \( |s\rangle \rightarrow |s, r\rangle \) (and \( \mathcal{O}_s(q) \rightarrow \mathcal{O}_{s,r}(q) \)) and fixed by expansion of the \( \mathfrak{h} \)-valued 1-form \( dzJ(z)|s\rangle \) at \( z \rightarrow 0 \), i.e.

\[
\frac{dz}{z} J(z)|s, r\rangle = \frac{dz}{z} \sum_{i=1}^{N} r_i e_i |s, r\rangle + \text{reg.}
\]

where \( r^1 = \ldots = r^{l_1}, \ r^{l_1+1} = \ldots = r^{l_1+l_2}, \ldots \) the \( U(1) \) charges are obviously the same for each point of the cover, they also satisfy the \( \mathfrak{sl}_N \) condition

\[
\sum_{i=1}^{N} r^\alpha_i = 0, \ \forall \ \alpha
\]

for each branch point \( q \in \{q_a\} \). It means that \( G_1(z)dz \) on the cover \( \mathcal{C} \) has only poles with prescribed by \( (5.13) \) singularities, so one can write

\[
\frac{G_1(\xi|q)d\xi}{G_0(q)} = \sum_{\alpha=1}^{2L} d\Omega_{r_\alpha} + \sum_{l=1}^{q} a_l d\omega_l = dS
\]
5.2. Twist fields and branched covers

and we shall call this 1-form as the Seiberg-Witten (SW) differential, since its periods over the cycles in $H_1(C)$ play important role in what follows. Here $\{d\omega_I\}$, $I = 1, \ldots, g$ are the canonically normalized first kind Abelian holomorphic differentials

$$\frac{1}{2\pi i} \oint_{A_I} d\omega_J = \delta_{IJ}$$

(in slightly unconventional normalization of [Dub] as compare to [Fay, Mumford]), while

$$d\Omega_{\tau_i} = \sum_{i=1}^{N} r^i_{\alpha} d\Omega_{q^i_{\alpha},p_0}$$

is the third kind meromorphic Abelian differential with the simple poles at all preimages of $q_{\alpha}$ (with the expansion $d\Omega_{\tau_i} = \frac{l^i_{\alpha}}{p_{\alpha}} \frac{dz}{z^\alpha} + \text{reg.}$ in corresponding local coordinates) and vanishing A-periods. We denote by $q^i_{\alpha} = \pi^{-1}(q_{\alpha})$, $i = 1, \ldots, N$ the preimages on $C$ of the point $q_{\alpha}$, with such conventions the point of multiplicity $l^i_{\alpha}$ has to be counted $l^i_{\alpha}$ times ($\text{Res}_{p_{\alpha}} d\Omega_{\tau_i} = l^i_{\alpha} r^i_{\alpha}$).

The A-periods of the differential (5.15)

$$a^I = \frac{1}{2\pi i} \oint_{A_I} dS = \frac{1}{2\pi i} \oint_{A_I} \frac{d\xi G_1(\xi|q)}{G_0(q)}, \quad I = 1, \ldots, g$$

(5.16)

are determined by fixed charges in the intermediate channels due to (5.7). The number of these constraints is ensured by the Riemann-Hurwitz formula $\chi(C) = N \cdot \chi(\mathbb{P}^1) - \#BP$ for the cover $\pi : C \to \mathbb{P}^1$, or

$$g = \sum_{\alpha=1}^{L} \sum_{j=1}^{k_{\alpha}} (l^i_{\alpha} - 1) - N + 1 = \sum_{\alpha=1}^{L} (N - k_{\alpha}) - N + 1$$

(5.17)

where $k_{\alpha}$ stands for the number of cycles in the permutation $s_{\alpha}$. One can easily see this in the “weak-coupling” regime, when we can apply (5.7) in the limit $q_{2\alpha-1} \to q_{2\alpha}$, so that

$$G_0(q_1, \ldots, q_{2L})|_{\theta} = \langle O_{s_1}(q_1)O_{s_1^{-1}}(q_2) \cdots O_{s_L}(q_{2L-1})O_{s_L^{-1}}(q_{2L}) \rangle_{\theta_L} \sim$$

$$\sim_{q_{2\alpha-1} \to q_{2\alpha}} \langle \prod_{\alpha=1}^{L} V_{\theta_{\alpha}}(q_{2\alpha}) \rangle + \ldots$$

(5.18)

and the charge conservation law $\sum_{\alpha=1}^{L} \theta_{\alpha} = 0$ gives exactly $N - 1$ constraints to the parameters $\{\theta_{\alpha}\}$, whose total number is $\sum_{\alpha=1}^{L} (N - k_{\alpha})$, since for each pair of colliding ends of the cut (i.e. $\alpha = 1, \ldots, L$) there are $k_{\alpha}$ linear relations for the $N$ integrals over the contours, encircling two colliding ramification points, see fig.5.1 (this procedure also gives a way to choose convenient basis in $H_1(C)$ as shown on this picture). For the dual B-periods of (5.15) one gets

$$a^I_B = \oint_{B_I} dS = T_{IJ} a_J + U_I, \quad I = 1, \ldots, g$$

(5.19)
5. Exact conformal blocks for the W-algebras, twist fields and isomonodromic deformations

where the last term can be transformed using the Riemann bilinear relations (RBR) as

\[
U_J = \sum_{\alpha} \oint_{B_J} d\Omega_{\alpha} = \sum_{\alpha,m} v^m_\alpha A_J(q^m_\alpha), \quad J = 1, \ldots, g
\]  

(5.20)

where \( A_J(p) = \int_{p_0}^p d\omega_J \) is the Abel map of a point \( p \in \mathcal{C} \), and \( U_J \) do not depend on the reference point \( p_0 \in \mathcal{C} \) due to (5.14).

**Stress-tensor and projective connection**

Similarly the 2-differential from (5.8) is fixed by its analytic properties and one can write

\[
\frac{\mathcal{G}_2(p', p | q)}{\mathcal{G}_0(q)} d\xi' d\xi_p = dS(p') dS(p) + K(p', p) - \frac{1}{N} K_0(p', p)
\]

(5.21)

where

\[
K(p', p) = d\xi' d\xi_p \log E(p', p) = \frac{d\xi' d\xi_p}{(\xi' - \xi_p)^2} + \text{reg.}, \quad \oint_{A_J} K(p', p) = 0
\]

(5.22)

is the canonical meromorphic bidifferential on \( \mathcal{C} \) (the double logarithmic derivative of the prime form, see [Fay]), normalized on vanishing A-periods in each of two variables, while

\[
K_0 = \frac{d\pi(\xi) d\pi(\xi')}{(\pi(\xi) - \pi(\xi'))^2}
\]

(5.23)

is just the pull-back \( \pi^* \) of the bidifferential \( \frac{d\xi' d\xi}{(\xi' - \xi)^2} \) from \( \mathbb{P}^1 \). Formula (5.21) is fixed by the following properties: in each of two variables it has almost the same structure as \( \mathcal{G}_1(\xi) d\xi \), but with extra singularity on diagonal \( p' = p \), which comes from (5.3), it also satisfies an obvious condition \( \sum_i \mathcal{G}^i(z, z') = \sum_j \mathcal{G}^j(z, z') = 0 \)

Now one can define [Fay] the projective connection \( t_x(p) \) by subtracting the singular part of (5.22)

\[
t_x(p) dx^2 = \frac{1}{2} \left( K(p', p) - \frac{dx(p') dx(p)}{(x(p') - x(p))^2} \right) \bigg|_{p' = p}
\]

(5.24)

It depends on the choice of the local coordinate \( x(p) \), and it is easy to check that

\[
t_x(p) dx^2 - t_\xi(p) d\xi^2 = \frac{1}{12} \{\xi, x\} dx^2
\]

(5.25)

where \( \{\xi, x\} = (S\xi)(x) = \frac{\xi x + \xi}{\xi x} - \frac{3}{2} \left( \frac{\xi x}{\xi} \right)^2 \) is the Schwarzian derivative.

It is almost obvious that expression (5.24) is directly related with the average of the Sugawara stress-tensor \( T(z) \) (5.4) of conformal field theory (with extended W-symmetry), since normal ordering of free bosonic currents exactly results in subtraction of its singular part. One gets in this way from (5.21) that

\[
\langle \frac{1}{2} J_1(z) J_1(z) : \mathcal{O}_{s_1}(q_1) \mathcal{O}_{s_1^{-1}}(q_2) \ldots \mathcal{O}_{s_L}(q_{2L-1}) \mathcal{O}_{s_L^{-1}}(q_{2L}) \rangle

(\mathcal{O}_{s_1}(q_1) \mathcal{O}_{s_1^{-1}}(q_2) \ldots \mathcal{O}_{s_L}(q_{2L-1}) \mathcal{O}_{s_L^{-1}}(q_{2L}))

=t_x(z') + \frac{1}{2} \left( \frac{dS(z')}{dz} \right)^2
\]

(5.26)

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where \( z = z(p) \) is the global coordinate on \( \mathbb{P}^1 \), and we have used that after subtraction (5.24) one can substitute \( K \mapsto 2 t_z(p) dz^2 \) and \( K_0 \mapsto 0 \), leading to

\[
\langle T(z) \rangle_O = \frac{\langle T(z) \mathcal{O}_{s_1}(q_1) \mathcal{O}_{s_1'}(q_2) \ldots \mathcal{O}_{s_L}(q_{2L-1}) \mathcal{O}_{s_L'}(q_{2L}) \rangle}{\langle \mathcal{O}_{s_1}(q_1) \mathcal{O}_{s_1'}(q_2) \ldots \mathcal{O}_{s_L}(q_{2L-1}) \mathcal{O}_{s_L'}(q_{2L}) \rangle} = \\
= \sum_{\pi(p) = z} \left( t_z(p) + \frac{1}{2} \left( \frac{dS(p)}{dz} \right)^2 \right)
\]

(5.27)

where sum in the r.h.s. computes the pushforward \( \pi_* \), appeared here as a result of summation in (5.4).

**W-charges for the twist fields**

**Conformal dimensions for quasi-permutation operators**

Using the OPE with the stress-tensor \( T(z) \)

\[
T(z) \mathcal{O}_{s,r}(q) = \frac{\Delta(s,r) \mathcal{O}_{s,r}(q)}{(z-q)^2} + \frac{\partial_q \mathcal{O}_{s,r}(q)}{z-q} + \text{reg.}
\]

(5.28)

one can extract from the singularities of (5.27) the dimensions of the twist fields. Following [BR] we first notice from (5.24) that near the branch point (e.g. at \( q = 0 \)) the local coordinate is \( \xi_i = z^{1/l_i} \), so that

\[
t_z(p) = t_\xi(p) \left( \frac{d\xi_i}{dz} \right)^2 + \frac{1}{12} \{\xi, z\} = t_\xi(p) z^{2/l_i - 2} + \frac{l_i^2 - 1}{24l_i^2} z^2
\]

(5.29)

The first term in the r.h.s. cannot contain \( \frac{1}{z^2} \)-singularity, since \( t_\xi(p) \) is regular in local coordinate on the cover \( \mathcal{C} \). The second source of the second-order pole in (5.27) comes from the poles of the Seiberg-Witten differential (5.15), which look as

\[
dS \approx r_i l_i \frac{d\xi_i}{\xi_i} + \text{reg.} = r_i \frac{dz}{z} + \text{reg.}
\]

(5.30)

Taking them into account together with (5.29) one comes finally to the formula

\[
\Delta(s,r) = \sum_{i=1}^k \frac{l_i^2 - 1}{24l_i} + \sum_{i=1}^k \frac{1}{2} l_i r_i^2
\]

(5.31)

which gives the full conformal dimension for the twist fields with \( r \)-charges.

Since we are going to use this formula intensively below, let us illustrate first, how it works in the first two nontrivial cases:

- \( N = 2 \): there are only two possible cyclic types:
  - \( s \sim [1,1] \), then \( l_1 = l_2 = 1 \), \( r_1 = -r_2 = r \), so \( \Delta(s,r) = r^2 \) is only given by the \( r \)-charges;
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- $s \sim [2]$, then the only $l_1 = 2$, the single $r$-charge must vanish, so one just gets here the original Zamolodchikov’s twist field with $\Delta(s, r) = \frac{1}{16}$.

- $N = 3$: here one has three possible cyclic types:
  - $s \sim [1, 1, 1]$, then $l_1 = l_2 = l_3 = 1$, $r_1 + r_2 + r_3 = 0$, $\Delta(s, r) = \frac{1}{2} (r_1^2 + r_2^2 + r_3^2)$
  - $s \sim [2, 1]$, then $l_1 = 2, l_3 = 1$, $r_1 = r_2 = r$, $r_3 = -2r$, $\Delta(s, r) = \frac{1}{16} + 3r^2$
  - $s \sim [3]$, then $l_1 = 3$, the single $r$-charge again should vanish, so that the dimension is $\Delta(s, r) = \frac{1}{9}$.

Quasipermutation matrices

The hypothesis of the isomonodromy-CFT correspondence [Gav] relates the constructed above twist fields to the quasipermutation monodromies (we return to this issue in more details later). This correspondence relates the $W_N$ charges of the twist fields to the symmetric functions of eigenvalues of the logarithms of the quasipermutation monodromy matrices

$$M_\alpha \sim e^{2\pi i \theta_\alpha}, \quad \alpha = 1, \ldots, 2L,$$

being the elements of the semidirect product $S_N \ltimes (\mathbb{C}^\times)^N$ (here we consider only the matrices with $\det M_\alpha = 1$). An example of the quasipermutation matrix of cyclic type $s \sim [3, 2]$ is

$$M = \begin{pmatrix}
0 & a_1 e^{2\pi i r_1} & 0 & 0 & 0 \\
0 & 0 & a_2 e^{2\pi i r_1} & 0 & 0 \\
a_3 e^{2\pi i r_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_1 e^{2\pi i r_2} & 0 \\
0 & 0 & b_2 e^{2\pi i r_2} & 0 & 0
\end{pmatrix} \quad (5.33)$$

where $a_1 a_2 a_3 = 1$, $b_1 b_2 = -1$, $3r_1 + 2r_2 = 0$ to get $\det M = 1$. A generic quasipermutation is decomposed into several blocks of the sizes $\{l_i\}$, each of these blocks is given by

$$e^{2\pi i r_i} \times e^{\frac{2\pi i}{l_i} \epsilon(l_i)} s_{l_i}, \quad i = 1, \ldots, k$$

where $s_l$ is the cyclic permutation of length $l$, $\epsilon(l) = 0$ for $l$-odd and $\epsilon(l) = 1$ for $l$-even. It is easy to check that eigenvalues of such matrices are

$$\lambda_{i,v} = e^{2\pi i \theta_{i,v}} = e^{2\pi i \left( r_i + \frac{v_i}{l_i} \right)}, \quad i = 1, \ldots, k$$

$$v_i = \frac{1 - l_i}{2}, \frac{1 - l_i}{2} + 1, \ldots, \frac{l_i - 1}{2} - 1, \frac{l_i - 1}{2}$$

According to relation (5.32) the conformal dimension of the corresponding field is

$$\Delta(M) = \frac{1}{2} \sum_i \theta^2_{i,v} = \frac{1}{2} \sum_i \left( r_i + \frac{v_i}{l_i} \right)^2 = \sum_{i=1}^k \frac{l_i^2 - 1}{24 l_i} + \sum_{i=1}^k \frac{1}{2} l_i r_i^2$$

(5.35)
where we have used that $\sum v_i = 0$ for any fixed $i = 1, \ldots, k$, and

$$l(l^2 - 1) \frac{12}{-1} = \begin{cases} 
\sum_{l=-(l-1/2)}^{l=-(l-1/2)} v^2 & l = 2m + 1 \ (v \in \mathbb{Z}) \\
\sum_{l=-(l-1/2)}^{l=-(l-1/2)} v^2 & l = 2m \ (v \in \mathbb{Z} + \frac{1}{2})
\end{cases} \quad (5.36)$$

for both even or odd $l \in \{l_i\}$. The calculation (5.35) for the quasipermutation matrices reproduces exactly the CFT formula (5.31), confirming the correspondence.

**W_3 current**

One can also perform a similar relatively simple check for the first higher $W_3$-current. An obvious generalization of (5.35) gives

$$w_3(M) = \sum_{a< b< c} (r_a + \frac{v_a}{l_a})(r_b + \frac{v_b}{l_b})(r_c + \frac{v_c}{l_c}) = \frac{1}{3} \sum_a (r_a + \frac{v_a}{l_a})^3 =$$

$$= \frac{1}{3} \sum_a r_a^3 + \sum_a r_a v_a^2 l_a = \sum_{i=1}^k \frac{1}{3} l_i r_i^3 + \sum_{i=1}^k r_i l_i^2 - \frac{1}{12} l_i \quad (5.37)$$

To extract such formulas from conformal field theory one has to analyze the multicomponent correlation functions in presence of twist operators and action of the corresponding modes of the $W_k(z)$ currents. For $W = W_3(z)$, following (5.8) one can first define

$$G^{i j k}_{3}(z, z', z''|q) dz dz' dz'' = G^{i j k}_{3}(z, z', z''|q_1, ..., q_{2L}) dz dz' dz'' =$$

$$= \langle J_i(z) J_j(z') J_k(z'') \mathcal{O}_{s_1}(q_1) \mathcal{O}_{s_1^{-1}}(q_2) ... \mathcal{O}_{s_L}(q_{2L-1}) \mathcal{O}_{s_L^{-1}}(q_{2L}) \rangle dz dz' dz'' \quad (5.38)$$

and write, similarly to (5.21)

$$\frac{G_3(p'', p', p|q)}{G_0(q)} d\xi_{p''} d\xi_{p'} d\xi_p = dS(p''|dS(p')dS(p) +$$

$$+dS(p'') (K(p', p') - \frac{1}{N} K_0(p', p') + dS(p') (K(p'', p) - \frac{1}{N} K_0(p'', p) +$$

$$+dS(p) (K(p'', p') - \frac{1}{N} K_0(p'', p') \quad (5.39)$$

where the r.h.s. has appropriate singularities at all diagonals and correct $A$-periods in each of three variables. Extracting singularities and using (5.4), (5.24) one can write

$$\langle W(z) \rangle_\mathcal{O} = \sum_{\pi(p) = z} \left( \frac{1}{3} \left( \frac{dS(p)}{dz(p)} \right)^3 + 2 t_2 z(p) \frac{dS(p)}{dz(p)} \right) \quad (5.40)$$

It is easy to see that due to (5.29), (5.30) this formula gives the same result as (5.37).
5. Exact conformal blocks for the $W$-algebras, twist fields and isomonodromic deformations

Formula (5.34) also shows, how the charges of the twist fields can be seen within the context of $W$-algebras. It is important, for example, that for the complete cycle permutation one would get its $W_N$ charges $w_2(\theta), w_3(\theta), \ldots, w_N(\theta)$, where

$$\theta = \rho_N = \frac{1}{N} \left( \frac{N-1}{2}, \frac{N-1}{2}, \ldots, \frac{1-N}{2} + 1, \frac{1-N}{2} \right)$$

(5.41)

i.e. the vector of charges is proportional to the Weyl vector of $\mathfrak{g} = \mathfrak{sl}_N$. Such fields are non-degenerate from the point of view of the $W_N$ algebra, since for degenerate fields the charge vector always satisfy the condition $(\theta, \alpha) \in \mathbb{Z}$ for some root $\alpha$. It means that here we are beyond the algebraically defined conformal blocks, and further investigation of descendants $W_{-1} \mathcal{O}$ etc can shed light on the structure of generic conformal blocks for the $W$-algebras. We are going to return to this issue elsewhere.

**Higher $W$-currents**

For the higher $W$-currents ($W_k(z)$ with $k > 3$) the situation becomes far more complicated. We discuss here briefly only the case of $W_4(z)$, which already gives a hint on what happens in generic situation. An analog of (5.35), (5.37) gives for the quasipermutation matrices

$$w_4(M) = \sum_{a<b<c<d} (r_a + v_{la})(r_b + v_{lb})(r_c + v_{lc})(r_d + v_{ld}) = \frac{1}{2} \Delta(M)^2 - \frac{1}{4} A$$

(5.42)

with $\Delta(M)$ given by (5.35) and

$$A = \sum_{a=1}^N \left( r_a + \frac{v_a}{l_a} \right)^4 = \sum_{i=1}^k (r_i^4 + 6 \sum_{i=1}^k r_i^2 l_i - 1) \frac{l_i^2 - 1}{12 l_i} + \sum_{i=1}^k \frac{(l_i^2 - 1)(3 l_i^2 - 7)}{240 l_i^3}$$

(5.43)

To get this from CFT one needs just the most singular part of the correlation function

$$\langle W_4(z) \rangle_{\mathcal{O}}(dz)^4 = w_4 \left( \frac{dz}{z - q} \right)^4 + \ldots$$

(5.44)

which is a particular case of the current correlators

$$\mathcal{R}_{i_1 \ldots i_n}(z_1, \ldots z_n) = \langle : J_{i_1}(z_1), \ldots J_{i_n}(z_n) : \rangle_{\mathcal{O}} dz_1 \ldots dz_n$$

(5.45)

and the technique of calculation of such expressions is developed in Appendix 5.8.

From the definition of the $W_4(z)$ current (5.4) it is clear, that one should take only the most singular parts of the correlation functions of four currents

$$\mathcal{R}_{i_1 i_2 i_3 i_4}(z_1, z_2, z_3, z_4) = \frac{i}{i} \cdot \frac{i}{i} + 6 \cdot \frac{i}{i} \cdot \frac{i}{i} + 3 \cdot \frac{i}{i} \cdot \frac{i}{i} =$$

$$dS(z)^4 + 6dS(z)^2 \tilde{K}_{i_1 i_2}(z, z) + 3 \tilde{K}_{i_1 i_2}(z, z)^2$$

(5.46)

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\[ R_{iijj}(z, z, z, z) = \begin{align*}
&= i \cdot j + i \cdot j + i \cdot j + i \cdot j \\
&+ 4 \cdot i \cdot j + i \cdot j + 2 \cdot i \cdot j
\end{align*} \]

\[ = dS(z')^2 dS(z')^2 + \hat{K}_{ii}(z, z) dS(z')^2 + \hat{K}_{jj}(z, z) dS(z')^2 + 4 \hat{K}_{ij}(z, z) \hat{K}_{jj}(z, z) + 2 \hat{K}_{ij}(z, z)^2 \]  \hspace{1cm} (5.47)

taken at the coinciding values of all arguments. It means, that one has to substitute

\[ dS(z') = r_i \frac{dz}{z} + \ldots \]  \hspace{1cm} (5.48)

(we again put here \( q = 0 \) for simplicity) and do the same for the propagator \( \hat{K}_{ij}(z_1, z_2) = \hat{K}(z_1^i, z_2^j) - \delta_{ij} \Delta(z_1, z_2) \) (see Appendix 5.8 for details), i.e. to substitute into (5.46), (5.47)

\[ \hat{K}_{ii}(z, z) = \frac{dz^{1/l} \hat{d}z^{1/l}}{(z^{1/l} - \hat{z}^{1/l})^2} - \frac{dz \hat{d}z}{(z - \hat{z})^2} + \ldots = \frac{l^2 - 1 (dz)^2}{12l^2} \frac{z^2}{z^2} + \ldots \]  \hspace{1cm} (5.49)

\[ \hat{K}_{ij}(z, z) = \frac{\zeta^i dz^{1/l} \zeta^j \hat{d}z^{1/l}}{(\zeta^i z^{1/l} - \zeta^j \hat{z}^{1/l})^2} + \ldots = \frac{1}{l^2} \frac{\zeta^i - j (dz)^2}{(1 - \zeta^i-j)^2} z^2 + \ldots \]

where \( \zeta = \exp \left( \frac{2\pi}{l} \right) \). In order to compute \(-\frac{1}{4} \sum_i R_{iii}(z, z, z, z) + \frac{1}{8} \sum_{ij} R_{ijij}(z, z, z, z)\) it is useful to move the term \( K_{ij}(z, z)^2 \) from the second expression to the first one, which gives

\[ \sum_i dS(z')^4 + 6 \sum_i dS(z')^2 \hat{K}_{ii}(z, z) + 3 \sum_i \hat{K}_{ii}(z, z)^2 \]

\[ - \sum_{ij} \hat{K}_{ij}(z, z)^2 \xrightarrow{\text{5.48, 5.49}} A \left( \frac{dz}{z} \right)^4 \]  \hspace{1cm} (5.50)

while the rest from (5.47) gives rise to

\[ \left( \sum_i dS(z')^2 + \sum_i \hat{K}_{ii}(z, z) \right)^2 + 4 \sum_{ij} \hat{K}_{ij}(z, z) dS(z') dS(z') \xrightarrow{\text{5.48, 5.49}} 4 \Delta_{N}^2 A \left( \frac{dz}{z} \right)^4 \]  \hspace{1cm} (5.51)

after using (5.48), (5.49) and several nice formulas like

\[ \frac{1}{l} \sum_{j=1}^{l-1} \frac{\zeta^j}{(1 - \zeta^j)^2} = \frac{1}{l} \sum_{j=1}^{l-1} \frac{e^{2\pi ij/l}}{(1 - e^{2\pi ij/l})^2} = - \sum_{v=(1-l)/2}^{(l-1)/2} \frac{v^2}{l^2} \]

\[ \frac{1}{l^3} \sum_{j=1}^{l-1} \frac{\zeta^{2j}}{(1 - \zeta^j)^4} = \frac{1}{l^3} \sum_{j=1}^{l-1} \frac{e^{4\pi ij/l}}{(1 - e^{2\pi ij/l})^4} = \frac{2}{l} \left( \sum_{v=(1-l)/2}^{(l-1)/2} \frac{v^2}{l^2} \right)^2 - \sum_{v=(1-l)/2}^{(l-1)/2} \frac{v^4}{l^4} \]  \hspace{1cm} (5.52)
Here the sum over the roots of unity can be performed using the contour integral
\[
\sum_{j=1}^{l-1} \frac{\zeta_j^m}{(1 - \zeta_j)^{2m}} = \frac{1}{2\pi i} \oint_{\gamma \neq 1} \frac{dz}{z-1} \cdot \frac{z^m}{(1-z)^{2m}} = \text{Res}_{z=1} d\log z \cdot \frac{z^m}{(1-z)^2}
\]
and the result indeed allows to identify the coefficients at maximal singularities in (5.50), (5.51) with the expressions (5.43). It means that the conformal charge (5.44) of the twist field indeed coincide with the corresponding symmetric function (5.42) of the eigenvalues of the permutation matrix, but it comes here already from a nontrivial computation.

It is known from long ago that already a definition of the higher W-currents is a nontrivial issue (see e.g. [FZ, FL, Bil, MarMor, FLitv12]). Here it was important to consider the particular (normally ordered) symmetric function of the currents (5.4), since, for example, another natural choice \[ \sum_i J^4_i(z) : \] is even not contained in the algebra generated by \( T(z), W_3(z) \) and \( W_4(z) \). However, the so defined \( W_4(z) \)-current is not a primary field of conformal algebra, we discuss this issue in Appendix 5.9.

**Conformal blocks and \( \tau \)-functions**

Consider now the next singular term from the OPE (5.28), which immediately allows to extract from (5.27) the accessory parameters
\[
\frac{\partial}{\partial q_\alpha} \log G_0(q_1, \ldots, q_{2L}) = \sum_{\pi(q_\alpha) = q_\alpha} \text{Res}_{q_\alpha} t_z dz + \frac{1}{2} \sum_{\pi(q_\alpha) = q_\alpha} \text{Res}_{q_\alpha} \frac{(dS)^2}{dz}
\]
Computing residues in the r.h.s. one gets the set of differential equations (\( \alpha = 1, \ldots, 2L \)), which define the correlation function of the twist fields \( G_0(q_1, \ldots, q_{2L}) \) itself. A non-trivial statement [KriW, GMqui, KK04, KK06] is that these equations are compatible, moreover (5.54) defines actually two different functions \( \tau_{SW}(q) \) and \( \tau_B(q) \), where
\[
\frac{\partial}{\partial q_\alpha} \log \tau_{SW}(q_1, \ldots, q_{2L}) = \frac{1}{2} \sum_{\pi(q_\alpha) = q_\alpha} \text{Res}_{q_\alpha} \frac{(dS)^2}{dz}
\]
and
\[
\frac{\partial}{\partial q_\alpha} \log \tau_B(q_1, \ldots, q_{2L}) = \sum_{\pi(q_\alpha) = q_\alpha} \text{Res}_{q_\alpha} t_z dz
\]
so that \( G_0(q) = \tau_{SW}(q) \cdot \tau_B(q) \), and the claim of [Mtau, KK04, KK06] is that both them are well-defined separately.

**Seiberg-Witten integrable system**

Let us concentrate attention on \( \tau_{SW} = \tau_{SW}(a, q) \) or the Seiberg-Witten prepotential \( F = \log \tau_{SW} \), which is the main contribution to conformal block, and the only one,
which depends on the charges in the intermediate channel. According to [Mtau, GMqui] $\mathcal{F}(a,q)$, up to some possible only $a$-dependent term, satisfies also another set of equations

$$\frac{\partial}{\partial a_I} \log \tau_{SW} = a_I^D, \quad I = 1, \ldots, g$$

(5.57)

where the dual periods $a_I^D$ are defined in (5.19). The total system of equations (5.55), (5.57) is also integrable [KriW, Mtau, GMqui] due to the Riemann bilinear relations. Moreover, in our case this system of equations can be easily solved due to

**Theorem 5.1.** Function

$$\log \tau_{SW} = \frac{1}{2} \sum_{I,J} a_I T_{IJ} a_J + \sum_I a_I U_I + \frac{1}{2} Q(r)$$

(5.58)

solves the system (5.55), iff $Q(r)$ solves the system $\frac{\partial Q(r)}{\partial q_\alpha} = \sum_{\pi(q_\alpha')} \text{Res} q_\alpha' \frac{(\partial q)^2}{dx}$ for $\alpha = 1, \ldots, 2L$, $d\Omega = \sum_{\alpha} \Omega_{r_\alpha}$ and other ingredients in the r.h.s. are given by (5.16), (5.20) and the period matrix of $\mathcal{C}$.

One can check this statement explicitly, using the definitions (5.15) and (5.20)

$$\sum_{\pi(q_\alpha')} \text{Res} q_\alpha' \frac{d\omega_I dq_J}{dz} = - \sum_{\pi(q_\alpha')=q_\alpha} \text{Res} q_\alpha' \frac{d\omega_I}{dq_\alpha} d\omega_J = - \frac{\partial}{\partial q_\alpha} \oint_{B_I} d\omega_J = \frac{\partial}{\partial q_\alpha} T_{IJ},$$

(5.59)

where we have first applied the formula $\frac{d\omega_I}{dq_\alpha} = - \frac{d\omega_I}{dz} + \text{hol.}$ and then the RBR. Similarly, for the second term:

$$\sum_{\pi(q_\alpha')} \text{Res} q_\alpha' \frac{d\Omega_{r_\alpha}}{dz} = - \sum_{\pi(q_\alpha')=q_\alpha} \text{Res} q_\alpha' \frac{\partial \Omega_{r_\alpha}}{\partial q_\alpha} d\omega_I = - \oint_{\partial C} \frac{\partial \Omega_{r_\alpha}}{\partial q_\alpha} d\omega_I = \frac{\partial}{\partial q_\alpha} \oint_{B_I} d\Omega_{r_\alpha} = U_I,$$

(5.60)

while the last term $Q(r)$, vanishing after taking the $a$-derivatives, should be computed separately, and the proof will be completed in next section.

**Quadratic form of r-charges**

In the limit $a_I = 0$ equation (5.55) gives us the formula

$$\frac{\partial}{\partial q_\alpha} Q(r) = \sum_{q_\alpha' \in \pi^{-1}(q_\alpha)} \text{Res} q_\alpha' \frac{d\Omega^2}{dz}$$

(5.61)

where

$$d\Omega = \sum_{\alpha} d\Omega_{r_\alpha} = \sum_{\alpha,i} r^i_\alpha d\Omega_{q^{\alpha}_i,p_0}$$

(5.62)
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Figure 5.3: Integration path for $Q(r)_{\epsilon}$

**Theorem 5.2. Regularized expression for $Q(r)$**

$$Q(r)_{\epsilon} = \sum_{\alpha,i} r_i^\alpha \int_{\tilde{p}_0}^{(q^\alpha_i)_{\epsilon_\alpha}} d\Omega$$

(5.63)

satisfies (5.61) in the limit $\epsilon \to 0$

**Proof:** It is useful to introduce the differential with shifted poles

$$d\Omega_{\epsilon} = \sum_{\alpha,i} r_i^\alpha d\Omega((q^\alpha_i)_{\epsilon_\alpha})$$

(5.64)

Note that due to conditions (5.14) nothing depends on the reference points $p_0, \tilde{p}_0$. The regularized points $(q^\alpha_i)_{\epsilon_\alpha}$ are defined in such a way that

$$z((q^\alpha_i)_{\epsilon_\alpha}) = z(q^\alpha_i) - \epsilon_\alpha = q_\alpha - \epsilon_\alpha$$

(5.65)

and this is the only place where the coordinate $z$ on $\mathbb{P}^1$ enters the definition of $Q(r)$. All other parts of $\tau_{SW}$ do not depend explicitly on the choice of the coordinate $z$ because they are given by the periods of some meromorphic differentials on the covering curve. Expression (5.63) can now be rewritten equivalently

$$Q(r)_{\epsilon} = \frac{1}{2\pi i} \oint_C \Omega_{\epsilon} d\Omega$$

(5.66)

where contour $C$ (see fig.5.3) encircles the branch-cuts of $\Omega_{\epsilon}$, while the poles of $d\Omega$ are left outside. Taking the derivatives one gets

$$\frac{\partial}{\partial q_\alpha} Q(r)_{\epsilon} = \frac{1}{2\pi i} \oint_C \left[ \frac{\partial \Omega}{\partial q_\alpha} d\Omega_{\epsilon} - \frac{\partial \Omega_{\epsilon}}{\partial q_\alpha} d\Omega \right]$$

(5.67)

where each of the terms in r.h.s. contains only the poles at the points $q^\alpha_i$ and $(q^\alpha_i)_{\epsilon_\alpha}$ correspondingly. One can therefore shrink the contour of integration in the first term onto the points $q^\alpha_i$ (up to the integration over the boundary of cut Riemann surface, which vanishes due to the Riemann bilinear relations for the differentials with vanishing $A$-periods), and in the second – to the points $(q^\alpha_i)_{\epsilon_\alpha}$, hence

$$\frac{\partial}{\partial q_\alpha} Q(r)_{\epsilon} = -\sum_i \text{Res}_{q^\alpha_i} \frac{\partial \Omega}{\partial q_\alpha} d\Omega_{\epsilon} - \sum_i \text{Res}_{(q^\alpha_i)_{\epsilon_\alpha}} \frac{\partial \Omega_{\epsilon}}{\partial q_\alpha} d\Omega$$

(5.68)
Near the point $p_i^z$ one can choose the local coordinate $\xi$ such that $z = q_\alpha + \xi^1$, so that expansion of Abelian integrals can be written as
\[
\begin{align*}
\Omega &= r_i \log(z - q_\alpha) + c_0(q) + c_1(q)(z - q_\alpha)^{1/l} + c_2(q)(z - q_\alpha)^{2/l} + \ldots \\
\Omega_{\epsilon} &= \tilde{c}_0(q) + \tilde{c}_1(q)(z - q_\alpha)^{1/l} + \tilde{c}_2(q)(z - q_\alpha)^{2/l} + \ldots 
\end{align*}
\] (5.69)
giving rise to
\[
\begin{align*}
\frac{\partial \Omega}{\partial q_\alpha} &= -\frac{d\Omega}{dz} + \frac{\partial c_0(q)}{\partial q_\alpha} + O \left((z - q_\alpha)^{1/l}\right) \\
\frac{\partial \Omega_{\epsilon}}{\partial q_\alpha} &= -\frac{d\Omega_{\epsilon}}{dz} + \frac{\partial \tilde{c}_0(q)}{\partial q_\alpha} + O \left((z - q_\alpha - \epsilon_\alpha)^{1/l}\right)
\end{align*}
\] (5.70)
Since the differential $d\Omega_{\epsilon}$ is regular near $z = q_\alpha$, one can ignore the regular part when computing the residues:
\[
\begin{align*}
\frac{\partial}{\partial q_\alpha} Q(r)_{\epsilon} &= \sum_i \text{Res}_{q_i^\alpha} \frac{d\Omega}{dz} d\Omega_{\epsilon} + \sum_i \text{Res}_{(q_i^\alpha),\alpha} \frac{d\Omega_{\epsilon}}{dz} d\Omega = \\
&= \sum_i \frac{1}{2\pi i} \oint_{q_i^\alpha,(q_i^\alpha),\alpha} \frac{d\Omega d\Omega_{\epsilon}}{dz}
\end{align*}
\] (5.71)
The r.h.s. of this formula has a limit when $\epsilon_\alpha \to 0$, so extracting the singular part from $Q(r)_{\epsilon}$ (easily found from the explicit formula below)
\[
Q(r) = Q(r)_{\epsilon} - 2 \sum \Delta_\alpha \log \epsilon_\alpha
\] (5.72)
one gets from (5.71) exactly the formula (5.61). This also completes (together with (5.59), (5.60)) the proof of (5.58).

Using the integration formula for the third kind Abelian differentials [Fay]
\[
\int_a^b d\Omega_{c,d} = \log \frac{E(c,b)E(d,a)}{E(c,a)E(d,b)}
\]
one gets from (5.63) an explicit expression
\[
Q(r)_{\epsilon} = \sum_{\alpha,\beta,i,j} r^i_\alpha r^j_\beta \log \frac{E((q_i^\alpha,\epsilon_\alpha),q_j^\beta)}{E((q_i^\alpha,\epsilon_\alpha),p_0,\epsilon_\alpha)} = \sum_{\alpha,\beta,i,j} r^i_\alpha r^j_\beta \log E((q_i^\alpha,\epsilon_\alpha),q_j^\beta) = \\
= \sum_{q_i^\alpha \neq q_j^\beta} r^i_\alpha r^j_\beta \log E(q_i^\alpha, q_j^\beta) + \sum_{\alpha,i} (r^i_\alpha)^2 l^i_\alpha \log E((q_i^\alpha,\epsilon_\alpha),q_i^\alpha)
\] (5.73)
The first term in the r.h.s. is regular, while for the second one can use
\[
E((q_i^\alpha,\epsilon_\alpha),q_i^\alpha) = \left. \frac{\sqrt{d(z - q_\alpha + \epsilon_\alpha)^{1/l} - (z - q_\alpha)^{1/l}}}{d(z - q_\alpha + \epsilon_\alpha)^{1/l} d(z - q_\alpha)^{1/l}} \right|_{z \rightarrow q_\alpha}
\] (5.74)
Therefore
\[
Q(r) = \sum_{q_i^\alpha \neq q_j^\beta} r^i_\alpha r^j_\beta \log E(q_i^\alpha, q_j^\beta) - \sum_{\alpha,i} (r^i_\alpha)^2 l^i_\alpha \log d[(z - q_\alpha)^{1/l}]_{z \rightarrow q_\alpha}
\] (5.75)
Substituting expression of the prime form

$$E(p, p') = \frac{\Theta_*(A(p) - A(p'))}{h_*(p) h_*(p')} \quad (5.76)$$

in terms of some odd theta-function $\Theta_*$, the already defined above Abel map $A(p)$, and holomorphic differential

$$h_2^2(p) = \sum \frac{\partial \Theta_*(0)}{\partial A_I} d\omega_I(p) \quad (5.77)$$

one can write more explicitly

$$Q(r) = \sum_{q_i \neq q_j} r_i^i r_j^j \log \Theta_*(A(q_i^j) - A(q_j^i)) - \sum_{q_i} (r_i^i)^2 \ell_i^i \log \left. \frac{d(z(q) - q_i)^{1/\ell_i}}{h_2^2(q)} \right|_{q = q_i^i} \quad (5.78)$$

If cover $C$ has zero genus $g(C) = 0$ itself, the prime form is just

$$E(\xi, \xi') = \xi - \xi' \sqrt{d\xi / d\xi'}$$

in terms of the globally defined coordinate $\xi$, and formula (5.78) acquires the form

$$Q(r) = \sum_{q_i \neq q_j} r_i^i r_j^j \log(\xi_i^i - \xi_j^j) - \sum_{\xi_i} (r_i^i)^2 \ell_i^i \log \left. \frac{d(z(\xi) - q_i)^{1/\ell_i}}{d\xi} \right|_{\xi = \xi_i^i} \quad (5.79)$$

Below we are going to apply this formula to explicit calculation of a particular example for a genus zero cover, but with a non-abelian monodromy group. The result of the computation clearly shows that $\tau$-function (5.79) cannot be expressed already in such case as a function of positions of the ramification points $z = q_i$ on $\mathbb{P}^1$, which means that the corresponding formula for $Q(r)$ from [K04] can be applied only in the case of Abelian monodromy group.

### Bergman $\tau$-function

The Bergman $\tau$-function, was studied extensively for the different cases [Knizhnik, BR, ZamAT87] from early days of string theory, mostly using the technique of free conformal theory. Modern results and formalism for this object can be found in [KK04, KK06]. Already from its definition (5.56) $\tau_B$ can be identified with the variation w.r.t. the moduli of the complex structure of the one-loop effective action in the free field theory on the cover.

We are not going to present here an explicit formula for the general Bergman $\tau$-function, it can be found in [KK06, formula 1.7]. We would like only to point out, that for our purposes of studying the conformal blocks this is the less interesting part, since it does not depends on quantum numbers of the intermediate channels (it means in particular, that it can be computed just in free field theory). Below in sect. 5.6 we present the result of its direct computation in the simplest case with non-abelian monodromy group. The result shows that it arises just as some quasiclassical renormalization of the term (5.79) in the classical part.
However, as for the SW tau-function, the definition (5.56) is easily seen to be consistent. Taking one more derivative one gets from this formula

$$\frac{\partial^2 \log \tau_B(q)}{\partial q_\alpha \partial q_\beta} = \frac{\partial}{\partial q_\beta} \sum_{\pi(p) = q_\alpha} \frac{1}{d(p)} \lim_{p' \to p} \left( K(p', p) - \frac{dz(p)dz(p')}{(z(p') - z(p))^2} \right) =$$

$$= \sum_{\pi(p) = q_\alpha} \frac{1}{d(p)} \lim_{p' \to p} \frac{\partial K(p', p)}{\partial q_\beta} = \sum_{\pi(p) = q_\alpha} \frac{1}{d(p)} \times$$

$$\times \lim_{p' \to p} \sum_{\pi(p'') = q_\beta} \frac{K(p', p'')K(p, p'')}{d(z(p''))} = \sum_{\pi(p) = q_\alpha} \frac{K(p, p'')^2}{d(z(p))d(p'')},$$

where we have used the Rauch variational formula [Fay92, formula 3.21] for the canonical meromorphic bidifferential, computed in the points $p$ and $p'$ with fixed projections

$$\frac{\partial K(p', p)}{\partial q_\beta} = \sum_{\pi(p) = q_\beta} \frac{K(p', P)K(p, P)}{d(z(P))}$$

so that the expression in r.h.s. of (5.80) is symmetric w.r.t. $\alpha \leftrightarrow \beta$.

This is certainly a well-known fact, but we would like just to point out here, that the Rauch formula (5.81), which ensures integrability of (5.56) can be easily derived itself from the Wick theorem, using the technique, developed in sect. 6.4 and Appendix 5.8. Indeed,

$$\frac{\partial K(z^i, z^j)}{\partial q_\beta} = \frac{\partial}{\partial q_\beta} \frac{G_2^{ij}(z', z|q)}{G_0(q)}dz^i dz^j =$$

$$= \left( \frac{\partial}{\partial q_\beta} G_2^{ij}(z', z|q) \frac{G_0(q)}{G_0(q)} - \frac{G_2^{ij}(z', z|q)}{G_0(q)} \frac{\partial}{\partial q_\beta} G_0(q) \right) dz^i dz^j$$

as follows from (5.21) for the conformal block with two currents inserted $G_2^{ij}(z', z|q) = G_2^{ij}(z', z|q) = \langle J_i(z')J_j(z)\mathcal{O}(q) \rangle_0$ when projected to the vanishing $a$-periods (5.16) or the charges in the intermediate channels (note, that the Bergman tau-function does not depend on these charges). Proceeding with (5.82) and using $\frac{\partial}{\partial q_\beta} = L_{-1}^\beta$ one gets therefore

$$\frac{\partial K(z^i, z^j)}{\partial q_\beta} = \left( \langle J_i(z')J_j(z)L_{-1}^\beta\mathcal{O}(q) \rangle_0 - \langle J_i(z')J_j(z)\mathcal{O}(q) \rangle_0 \frac{L_{-1}^\beta\mathcal{O}(q)}{\mathcal{O}(q)} \right) dz^i dz^j$$

where we have used the obvious notations

$$\langle \mathcal{O}(q) \rangle_0 = \prod_{\alpha=1}^{2L} \mathcal{O}_\alpha(q_\alpha) = \langle \mathcal{O}_{s_1}(q_1)\mathcal{O}_{s_2}(q_2)\cdots\mathcal{O}_{s_{2L}}(q_{2L}) \rangle_0$$

$$\langle L_{-1}^\beta\mathcal{O}(q) \rangle_0 = \langle (L_{-1}\mathcal{O}(q_\beta)) \prod_{\alpha \neq \beta} \mathcal{O}_\alpha(q_\alpha) \rangle_0 = \frac{1}{2} \oint_{q_\beta} \sum_k d\zeta \langle J_k^2(\zeta) : \mathcal{O}(q) \rangle_0$$

$$\langle J_i(z')J_j(z)L_{-1}^\beta\mathcal{O}(q) \rangle_0 = \frac{1}{2} \oint_{q_\beta} \sum_k d\zeta \langle J_i(z')J_j(z) : J_k^2(\zeta) : \mathcal{O}(q) \rangle_0$$

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where the integration $\oint_{q_{\beta}} d\zeta$ is performed on the base $\mathbb{P}^1$. Applying now in the r.h.s. the Wick theorem (see Appendix 5.8 for details), one gets

$$\frac{1}{2} \langle J_i(z') J_j(z) : J_k^2(\zeta) : \mathcal{O}(q) \rangle_0 \mathcal{O}(q) \rangle_0 = \frac{1}{2} \langle J_i(z') J_j(z) \mathcal{O}(q) \rangle_0 \mathcal{O}(q) \rangle_0 + \langle J_i(z') J_k(\zeta) \mathcal{O}(q) \rangle_0 \langle J_j(z) J_k(\zeta) \mathcal{O}(q) \rangle_0$$

which means for (5.83), that

$$\frac{\partial K(z^n, z^j)}{\partial q_{\beta}} = \oint_{q_{\beta}} \sum_k d\zeta \langle J_i(z') J_k(\zeta) \mathcal{O}(q) \rangle_0 \langle J_i(z) J_k(\zeta) \mathcal{O}(q) \rangle_0 d\zeta^n d\zeta^j =$$

$$= \oint_{q_{\beta}} \sum_k K(z^n, \zeta^k) K(z^j, \zeta^k) \frac{d\zeta}{d\zeta} = \sum P \text{ Res}_P \frac{K(z^n, P) K(z^j, P)}{d\zeta(P)}$$

where we have used that $\oint_{q_{\beta}} \sum_k = \sum_{\pi(P)\equiv q_{\beta}} \text{ Res}_P$. Hence, the same methods, which give rise to explicit formula for the main part $\tau_{SW}(a, q)$ of the exact conformal block, ensure also the consistency of definition of the quasiclassical correction $\tau_B(q)$.

**Isomonodromic $\tau$-function**

The full exact conformal block equals therefore

$$\mathcal{G}_0(q|a) = \tau_B(q) \exp \left( \frac{1}{2} \sum_{IJ} a_I T_{IJ}(q) a_J + \sum_I a_I U_I(q, r) + \frac{1}{2} Q(r) \right)$$

(5.87)

According to [GIL12, Gav] the $\tau$-functions of the isomonodromy problem [SMJ] on sphere with four marked points 0, q, 1, $\infty$ can be decomposed into a linear combination of the corresponding conformal blocks. This expansion looks as

$$\tau_{IM}(q) = \sum_{w \in Q(sf_N)} e^{(b, w)} C^{(0 q)}_w(\theta_0, \theta_q, a, \mu_0, \nu_0) C^{(1 \infty)}_w(\theta_1, \theta_\infty, a, \mu_1, \nu_1) \times$$

$$\times q^{\frac{1}{2}(\sigma_0 + w, \sigma_0 + w) - \frac{1}{2}(\theta_0, \theta_0)} B_w(\{\theta_i\}, a, \mu_0, \nu_0, \mu_1, \nu_1; q)$$

(5.88)

and can be tested, both numerically and exactly for some degenerate values of the $W$-charges $\theta$ of the fields [Gav, GavIL]. In (5.88) the normalization of conformal block $B_w(\bullet; q)$ is chosen to be $B_w(\bullet; q) = 1 + O(q)$ and $C_w(\bullet) as usually denote the corresponding 3-point structure constants (all these quantities in the case of $W(sf_N) = W_N$ blocks with $N > 2$ depend on extra parameters $\{\mu, \nu\}$, being the coordinates on the moduli space of flat connections on 3-punctured sphere, and for their generic values the conformal blocks $B_w(\bullet; q)$ are not defined algebraically, see [Gav] for more details).

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3This relation has been predicted in [Knizhnik], see also [Nov] for a slightly different observation of the same kind.
We now conjecture that such decomposition exists also for conformal blocks considered above. Moreover, then a natural guess is, that the structure constants have such a form that

\[
C^{(0q)}_w(\{\theta_0, \theta_q, a, \mu_0q, \nu_0q\})C^{(1\infty)}_w(\{\theta_1, \theta_{\infty}, a, \mu_{1\infty}, \nu_{1\infty}\})q^{\frac{1}{2}(a+w,a+w)-\frac{1}{2}(\theta_0,\theta_0)-\frac{1}{2}(\theta_q,\theta_q)}.
\]

\[\mathcal{B}_w(\{\theta_1\}, a, \mu_0q, \nu_0q, \mu_{1\infty}, \nu_{1\infty}; q) = \mathcal{G}_0(\{\theta_1\}, a + w; q) \]

(5.89)

i.e. they are absorbed into our definition of the W-block of the twist fields, and this can be extended from four to arbitrary number of even 2L points on sphere. This conjecture can be easily checked in the \(N=2\) case, where the structure constants for the values, corresponding to the Picard solution [GIL12, ILTe], coincide exactly with given by degenerate period matrices in (5.87), when applied to the case of the Zamolodchikov conformal blocks [GMqui] (see sect. 5.6 and Appendix 5.10).

It means that in order to get isomonodromic \(\tau\)-functions from the exact conformal blocks (5.87) one has just to sum up the series (for the arbitrary number of points one has to replace the root lattice of \(Q(\mathfrak{sl}_N) = \mathbb{Z}^{N-1}\) by the lattice \(\mathbb{Z}^g\), where \(g = g(C)\) is the genus of the cover)

\[
\tau_{IM}(q|a, b) = \sum_{n \in \mathbb{Z}^g} \mathcal{G}_0(q|a + n)e^{(n,b)} = \tau_B(q) \exp \left(\frac{1}{2}Q(\mathbf{r})\right) \times
\]

\[\times \sum_{n \in \mathbb{Z}^g} \exp \left(\frac{1}{2} (a + n, \mathcal{T}(a + n)) + (U, a + n) + (b, n)\right) = \tau_B(q) \exp \left(\frac{1}{2}Q(\mathbf{r})\right) \Theta \left[\begin{array}{c} a \\ b \end{array}\right] (U) = \tau_{B}(q) \exp \left(\frac{1}{2}Q(\mathbf{r})\right) \right]
\]

(5.90)

which is easily expressed through the theta-function. One gets in this way exactly the Korotkin isomonodromic \(\tau\)-function, where the only difference of this expression with proposed in [K04, formula 6.10] is in the term \(Q(\mathbf{r})\), which is not expressed globally through the coordinates of the branch points in the case of non-abelian monodromy group. This fact supports both our conjectures: about the form of the structure constants, and about the general correspondence between the isomonodromic deformations and conformal field theory.

Formula (5.90) has also clear meaning in the context of gauge theory/topological string correspondence. It has been noticed yet in [NO], that the CFT free fermion representation exists only for the dual partition function, which is obtained from the gauge-theory matrix element (conformal block) by a Fourier transform \(^4\). We plan to return to this issue separately in the context of the free fermion representation for the exact W-conformal blocks.

## Examples

There are several well-known examples of the conformal blocks corresponding to Abelian monodromy groups. All of them basically come from the Zamolodchikov

\(^4\)The fact, that only the Fourier-Legendre transformed quantity can be identified with partition function in string theory has been established recently in quite general context from their transformation properties in [CWM].
5. Exact conformal blocks for the W-algebras, twist fields and isomonodromic deformations

exact conformal block [ZamAT87, formula 3.29] for the Ashkin-Teller model, defined on the families of hyperelliptic curves

$$y^2 = \prod_{\alpha=1}^{2L} (z - q_\alpha)$$  \hspace{1cm} (5.91)

with projection $\pi : (y, z) \mapsto z$. Parameters $r$ are absent here, so the result is just $G_0(q) = \tau_B(q) \exp \left( \frac{1}{2} \sum_{I>J} a_I T_{IJ}(q) a_J \right)$, where for the hyperelliptic period matrices one gets from (5.59) the well-known Rauch formulas (see e.g. [GMqui] and references therein).

When the hyperelliptic curve degenerates (see Appendix 5.10), this formula gives

$$G_0(q) \approx 4^{-\sum a^2 - (\sum a_I)^2} \prod_{l=1}^{g} (q_{2l} - q_{2l-1})^{\frac{a^2}{2} - \frac{1}{8}} \prod_{l>j}^{g} (q_{2l} - q_{2j})^{2a_la_j} R^{-\sum a_I^2} \approx$$

$$\approx 4^{-\sum a^2 - (\sum a_I)^2} \cdot \prod_{l=1}^{g} \epsilon_{l}^{-\frac{1}{8}} R^{-\sum a_I^2} \cdot \prod_{l>j}^{g} (q_{2l} - q_{2j})^{2a_la_j}$$  \hspace{1cm} (5.92)

Here in the r.h.s. the second factor comes from the OPE (5.7), i.e. $O(q_{2l} - \epsilon_l) O(q_{2l}) \sim \epsilon_{l}^{-\frac{1}{8}} V_{a_I}(q_{2l}) + \ldots$, while the third one is just the correlator $(\prod V_{a_I}(q_{2l}))$. Hence, the first most important factor corresponds to the non-trivial product of the structure constants in (5.89), which acquires here a very simple form. The main point of this observation is that normalization of (5.87) automatically contains not only $q^#$ factors, but also the structure constants, and we have already exploited such conjecture for general situation in sect. 5.5, since the argument with degenerate tau-function can be easily extended.

These observations have an obvious generalization for the $\mathbb{Z}_N$-curves

$$y^N = \prod_{\alpha=1}^{2L} (z - q_\alpha)^{k_\alpha}$$  \hspace{1cm} (5.93)

with the same projection $\pi : (y, z) \mapsto z$. The main contribution to the answer $\tau_{SW} = \exp \left( \frac{1}{2} \sum_{I>J} a_I T_{IJ} a_J \right)$ comes just from a general reasoning as in sect. 5.6 and to make it more explicit one can use the Rauch formulas for $\mathbb{Z}_N$-curves, which express everything in terms of the coordinates $\{q\}$ on the projection, since there is no summing over preimages in formulas like (5.59).

Let us now turn to an elementary new example with non-abelian monodromy group. Notice, first, that a simple genus $g(C) = 0$ curve

$$y^3 = x^2(1 - x)$$  \hspace{1cm} (5.94)

gives rise to the curve with non-abelian monodromy group if one takes a different (from $\mathbb{Z}_3$-option $\pi_x : (y, x) \mapsto x$) projection $\pi_y : (y, x) \mapsto y$. For the curve $C$, which is just a sphere or $\mathbb{P}^1$ itself, one gets here two essentially different (and unrelated!) setups, corresponding to differently chosen functions $x$ or $y$. 

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In the first case our construction leads, for example, to the formulas

$$\langle T(x) \rangle_0 = \frac{\langle T(x) \mathcal{O}_s(0) \mathcal{O}_{s-1}(1) \rangle}{\langle \mathcal{O}_s(0) \mathcal{O}_{s-1}(1) \rangle} = \frac{1}{4} \{ \xi; x \} = \frac{1}{9x^2(x-1)^2}$$  \hspace{1cm} (5.95)

where \( x = \frac{1}{1+\xi^2} \) in terms of the global coordinate \( \xi \) on \( \mathcal{C} \), and this formula fixes the insertions at \( x = 0, 1 \) to be the twist operators for \( s = (123) \), with \( \Delta(s) = \frac{p_i-1}{2d_i} = \frac{1}{5} \).

However, for a similar correlator on \( y \)-sphere

$$\langle T(y) \rangle_0 = \frac{\langle T(y) \prod_A \mathcal{O}(y_A) \rangle}{\langle \prod_A \mathcal{O}(y_A) \rangle} = \frac{1 + 54y^3}{(27y^3 - 4)^2y^2} = \sum_{A=0,1,2,3} \left( \frac{1}{16(y - y_A)^2} + \frac{u_A}{y - y_A} \right)$$  \hspace{1cm} (5.96)

\( y_0 = u_0 = 0, \quad 3y_k = -8u_k = 2^{2/3}e^{2\pi i(k-1)/3}, \quad k = 1, 2, 3 \)

one has to insert the twist operators for \( \tilde{s} = (12)(3) \) of dimension \( \Delta(\tilde{s}) = \frac{p_i-1}{2d_i} = \frac{1}{16} \).

The r.h.s. here follows from summation of

$$\frac{1}{12} \{ \xi; y \} = \frac{\xi(\xi^3 + 4)(1 + \xi^3)^4}{2(2\xi^3 - 1)^4} = \frac{\xi^5(3y + \xi)}{2y(2\xi - 3y)^4}$$  \hspace{1cm} (5.97)

where

$$y = \frac{\xi}{1 + \xi^2}, \quad \xi \in \mathcal{C} = \mathbb{P}^1$$  \hspace{1cm} (5.98)

To get (5.96) one has to sum (5.97) over \( \pi(\xi) = y \), or three solutions of the equation \( R(\xi) = \xi^3 - \frac{1}{y} \xi + 1 = 0 \), i.e.

$$\langle T(y) \rangle_c = \frac{1}{12} \sum_{\beta} \{ \xi^{(\beta)}; y \} = \sum_{\beta} \text{res}_{\xi=\xi^{(\beta)}} \left( \frac{\xi^5(3y + \xi)}{2y(2\xi - 3y)^4} \right) d \log R(\xi) = \frac{1 + 54y^3}{(27y^3 - 4)^2y^2}$$  \hspace{1cm} (5.99)

in contrast to the sum over three sheets of the cover \( \pi(\xi) = x \), which gives only a factor of \( 3 \cdot \{ \xi; x \} / 12 \).

To analyze the simplest nontrivial \( \tau \)-function for non-abelian monodromy group, let us consider the deformation of the formula from (5.98) for \( z = 1/y = \xi^2 + 1/\xi \), i.e. the cover \( \pi : \mathcal{C} = \mathbb{P}^1_{\xi} \to \mathbb{P}^1_z \) given by 1-parametric family

$$z = \frac{(2\xi - t^2 + 1)^2(\xi - 4)}{(t - 3)^2(t^2 - 2t - 3)}$$  \hspace{1cm} (5.100)

The parametrization is adjusted in the way that the branching points \( dz = 0 \) are at

$$\xi = \frac{1}{2}(t^2 - 1), \quad z = 0$$

$$\xi = 1 + t, \quad z = 1$$

$$\xi = 1 - t, \quad z = q(t) = \frac{(t + 3)(t - 1)}{(t - 3)(t + 1)}$$  \hspace{1cm} (5.101)

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Together with $\xi = \infty$, $z = \infty$.

One also has non-branching points above the branched ones $\xi = 4$, $z = 0$; $\xi = (t - 1)^2$, $z = 1$; $\xi = (t + 1)^2$, $z = q(t)$. Now we rewrite these points in our notation

$$
\begin{align*}
G_1^0 &= 4, & \xi_0^2 &= \frac{1}{2}(t^2 - 1), & \xi_0^3 &= \frac{1}{2}(t^2 - 1) \\
G_1^q &= (t + 1)^2, & \xi_0^2 &= 1 - t, & \xi_0^3 &= 1 - t \\
G_1^1 &= (t - 1)^2, & \xi_0^2 &= 1 + t, & \xi_0^3 &= 1 + t \\
G_\infty^1 &= 0, & \xi_\infty^2 &= \infty, & \xi_\infty^3 &= \infty
\end{align*}
$$

(5.102)

Using an explicit formula (5.79) and the definition (5.56) of $\tau_B$ one can write down the result for the $\tau$-function

$$
\tau(t) = \tau_B(t) \exp \left( \frac{1}{2} Q_r(t) \right) = (t - 3)^{\delta_3} \left( t - 1 \right)^{\delta_1} \left( t + 1 \right)^{\delta_0 + \frac{1}{2}} (t + 3)^{\delta_{-3} + \frac{1}{2}}
$$

(5.103)

where $\delta_i = \delta_i(r)$ are given by some particular quadratic forms

$$
\begin{align*}
\delta_3 &= 9r_0^2 - 9r_\infty^2 \\
\delta_1 &= r_0^2 - 4r_0r_1 + 4r_1^2 + 8r_0r_q - 4r_1r_q + r_q^2 - 4r_0r_\infty + 8r_1r_\infty - 4r_qr_\infty + 4r_\infty^2 \\
\delta_0 &= -9r_1^2 - 9r_q^2 \\
\delta_{-1} &= 4r_0^2 + 8r_0r_1 + 4r_1^2 - 4r_0r_q + 7r_q^2 - 4r_0r_\infty - 4r_1r_\infty + 8r_qr_\infty + r_\infty^2 \\
\delta_{-3} &= -9r_0^2 - 9r_q^2
\end{align*}
$$

(5.104)

while their “semiclassical” shifts come from the Bergman $\tau$-function. Notice that isomonodromic function (5.103) looks very similar to the tau-functions of algebraic solutions of the Painlevé VI equation [GIL12, examples 5-7], but depends on essentially more parameters.

An interesting, but yet unclear observation is that in this example $\tau_B(t)$ itself can be represented as

$$
\tau_B(t) = \exp \left( \frac{1}{2} Q(\bar{r}) \right)
$$

(5.105)

for several particular choices of parameters $\bar{r}$, e.g.

$$
\begin{align*}
(\bar{r}_0, \bar{r}_q, \bar{r}_1, \bar{r}_\infty) &= \left( \frac{\sqrt{7}}{12\sqrt{3}}, -\frac{\sqrt{7}}{12\sqrt{3}}, \frac{i}{4\sqrt{3}}, \frac{\sqrt{7}}{12\sqrt{3}} \right) \\
(\bar{r}_0, \bar{r}_q, \bar{r}_1, \bar{r}_\infty) &= \left( -\frac{i}{12\sqrt{3}}, \frac{i}{12\sqrt{3}}, \frac{i}{12\sqrt{3}}, \frac{\sqrt{5}}{12} \right)
\end{align*}
$$

(5.106)

whereas all other (altogether eight) solutions are obtained after the action of the Galois group generated by $\sqrt{3} \mapsto -\sqrt{3}$, $\sqrt{5} \mapsto -\sqrt{5}$ and $i \mapsto -i$. Notice that this statement is nevertheless nontrivial because we express five variables $\delta_i$ in terms of only four variables $\bar{r}_i$. 

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Conclusions

We have presented above an explicit construction of the conformal blocks of the twist fields in the conformal theory with integer central charges and extended W-symmetry. We have computed the W-charges of these twist fields and show that their Verma modules are non-degenerate from the point of view of W-algebra representation theory. The obtained exact formulas for the corresponding conformal blocks were derived intensively using the correspondence between two-dimensional conformal and four-dimensional supersymmetric gauge theory. We also checked that so constructed exact conformal blocks, when considered in the context of isomonodromy/CFT correspondence, give rise to the isomonodromic τ-functions of the quasipermutation type.

We believe that it is only the beginning of the story and, finally, would like to present a list (certainly not complete) of unresolved yet problems. For the conformal field theory side these obviously include:

- What is the algebraic structure of the W-algebra representations corresponding to the twist-field vertex operators, and in particular – what are the form-factors or matrix elements of these operators?

- Already for the twist fields representations the analysis of this chapter should be supplemented by study of the W-analogs of the higher-twist representations [ApiZam] and of the W-representations at “dual values” of the central charges (an example of such block for the Virasoro case can be found in [ZamAT86]).

- Finally, perhaps the most intriguing question is – what is the constructive generalization of these vertex operators to non-exactly-solvable case?

However, the main intriguing part still corresponds to the side of supersymmetric gauge theory, where the resolution of these problems can help to understand their properties in the “unavoidable” regime of strong coupling, where even the Lagrangian formulation is not known. We are going to return to these questions elsewhere.

Appendix

Diagram technique

In order to compute the correlators of the currents (5.45) the first useful observation is that one can embed $\mathfrak{sl}_N \subset \mathfrak{gl}_N$ and introduce an extra current $h(z)$, commuting with $J_i(z)$, such that

$$h(z)h(z') = \frac{1/N}{(z-z')^2} + \text{reg.}, \quad h(z)\mathcal{O}(q) = \text{reg.}$$

(5.107)

Introduce the $\mathfrak{gl}_N$ currents

$$\tilde{J}_i(z) = J_i(z) + h(z)$$

(5.108)

which satisfy the OPE

$$\tilde{J}_i(z)\tilde{J}_k(z') = \frac{\delta_{jk}}{(z-z')^2} + \text{reg.}$$

(5.109)
and their normally-ordered averages are the same as for \( J_i(z) \) since
\[
\langle : J_{i_1}(z_1) \ldots J_{i_m}(z_m) h(z_{m+1}) \ldots h(z_n) : \rangle_\mathcal{O} =
\langle : J_{i_1}(z_1) \ldots J_{i_m}(z_m) : \rangle_\mathcal{O} \cdot \langle : h(z_{m+1}) \ldots h(z_n) : \rangle = 0
\]  
(5.110)

Hence, one can simply to replace \( J_i(z) \rightarrow \tilde{J}_i(z) \) in (5.45), so below we just compute the averages for the \( \mathfrak{gl}_N \) currents.

The normal ordering for two currents at colliding points is given by
\[
: J_i(z) J_j(z') : dz_1dz_2 = J_i(z) J_j(z')dz_1dz_2' - \frac{\delta_{ij}dz_1dz_2'}{(z-z')^2} = J_i(z) J_j(z')dz_1dz_2' - \delta_{ij}K_0(z,z')
\]  
(5.111)
i.e. it is defined by subtracting the canonical meromorphic bidifferential on the \textit{base curve}, since it corresponds to the \textit{vacuum} expectation value of the Gaussian fields. Normal ordering for the correlators of many currents is defined, as usual, by the Wick theorem.

Similarly to (5.8) consider now
\[
\langle J_{i_1}(z_1) : J_{i_2}(z_2) \ldots J_{i_n}(z_n) : \rangle_\mathcal{O} \ dz_1 \ldots dz_n =
\langle dS(z_i^1) : J_{i_2}(z_2) \ldots J_{i_n}(z_n) : \rangle_\mathcal{O} \ dz_2 \ldots dz_n +
\sum_{j=2}^{n} \ K(z_{i_1}^i, z_{i_j}^j) \langle J_{i_2}(z_2) \ldots \hat{J}_{i_j}(z_j) \ldots J_{i_n}(z_n) : \rangle_\mathcal{O} \ dz_2 \ldots \hat{dz}_j \ldots dz_n
\]  
(5.112)
where by \( z_k^i = \pi^{-1}(z_k) \) we have denoted the preimages on the cover. This formula is again obtained just from the analytic structure of this expression as 1-form in the first variable. The next formula comes from the application of the Wick theorem and (5.111)
\[
\langle J_{i_1}(z_1) : J_{i_2}(z_2) \ldots J_{i_n}(z_n) : \rangle_\mathcal{O} dz_1 \ldots dz_n =
\langle : J_{i_1}(z_1) \ldots J_{i_n}(z_n) : \rangle_\mathcal{O} dz_1 \ldots dz_n +
\sum_{j=2}^{n} \delta_{ij}K_0(z_1,z_j) \langle J_{i_2}(z_2) \ldots \hat{J}_{i_j}(z_j) \ldots J_{i_n}(z_n) : \rangle_\mathcal{O} dz_2 \ldots \hat{dz}_j \ldots dz_n
\]  
(5.113)
Subtracting them, one gets the recurrence relation
\[
\langle : J_{i_1}(z_1) \ldots J_{i_n}(z_n) : \rangle_\mathcal{O} dz_1 \ldots dz_n =
\langle dS(z_i^1) : J_{i_2}(z_2) \ldots J_{i_n}(z_n) : \rangle_\mathcal{O} dz_2 \ldots dz_n +
\sum_{j=2}^{n} \hat{K}_{i_1i_j}(z_1,z_j) \langle J_{i_2}(z_2) \ldots \hat{J}_{i_j}(z_j) \ldots J_{i_n}(z_n) : \rangle_\mathcal{O} dz_2 \ldots \hat{dz}_j \ldots dz_n
\]  
(5.114)
where we have introduced the “propagator”
\[
\hat{K}_{ij}(z_1,z_2) = K(z_1^i, z_2^j) - \delta_{ij}K_0(z_1,z_2)
\]  
(5.115)
Graphically for the result this recurrence produces one can write
\[ R_i(z_1) = \bullet i = dS(z'_1) \]
\[ R_{ij}(z_1, z_2) = \bullet i \bullet j + i_j = dS(z'_1)dS(z'_2) + \tilde{K}_{ij}(z'_1, z'_2) \]
\[ R_{ijk}(z_1, z_2, z_3) = \begin{array}{c}
\bullet i \\
\bullet j \\
\bullet k \\
\end{array} + \begin{array}{c}
\bullet i \\
\bullet j \\
\bullet k \\
\end{array} + \begin{array}{c}
\bullet i \\
\bullet j \\
\bullet k \\
\end{array} = \\
= dS(z'_1)dS(z'_2)dS(z'_3) + \tilde{K}_{ij}(z_1, z_2)dS(z'_3) + \tilde{K}_{jk}(z_2, z_3)dS(z'_1) + \tilde{K}_{ik}(z_1, z_3)dS(z'_2) \]

These expressions have very simple meaning: the full correlation function is expressed through the only possible connected parts \( R^c \), which are \( R^c_i(z_1) = dS(z'_1) \), \( R^c_{ij}(z_1, z_2) = \tilde{K}_{ij}(z_1, z_2) \), while \( R^c_{ijk}(z_1, z_2, z_3) = 0 \) and all higher connected parts vanish. The so constructed four point functions \( R_{ijkl}(z_1, z_2, z_3, z_4) \) at coinciding arguments (and at least pairwise coinciding labels of the sheets of the cover) were used in sect. 5.3.4 for computation of the higher W-charges.

**W_4(z) and the primary field**

Here we study the OPE of \( W_4(z) \) with \( T(z) \) and show an explicit correction which makes this field primary.

\[ W_4(z) = \sum_{ijkl} C^{ijkl} : J_i(z) J_j(z) J_k(z) J_l(z) : \] (5.116)

where \( C^{ijkl} \) is completely symmetric tensor, \( C^{ijkl} = \frac{1}{24} \) when \( i \neq j \neq k \neq l \) and \( C^{ijkl} = 0 \) otherwise.

\[ T(z)W_4(z') = \frac{6}{(z - z')^4} \sum_{ijkl} \left( \delta_{ij} - \frac{1}{N} \right) C^{ijkl} : J_k(z) J_l(z) : + \]
\[ + 4W_4(z') \frac{(z - z')}{(z - z')^2} + \frac{\partial W_4(z')}{z - z'} + \text{reg.} \] (5.117)

The first sum equals

\[ 6 \sum_{ij} \left( \delta_{ij} - \frac{1}{N} \right) C_{ijkl} = -\frac{(N - 2)(N - 3)}{4N} (1 - \delta_{ij}) \] (5.118)

Using now the fact that \( \sum_i J_i(z) = 0 \) we get

\[ T(z)W_4(z') = \frac{(N - 2)(N - 3)}{8N} \frac{T(z')}{(z - z')^4} + \frac{4W_4(z')}{(z - z')^2} + \frac{\partial W_4(z')}{z - z'} + \text{reg.} \] (5.119)
5. Exact conformal blocks for the $W$-algebras, twist fields and isomonodromic deformations

Figure 5.4: Degenerate hyperelliptic curve with chosen basis in $H_1$.

There is also another well-known field $\Lambda(z) = (TT)(z) - \frac{3}{10} \partial^2 T(z)$, where $(TT)(z) = \oint \frac{dw}{w-z} T(w) T(z)$, with the OPE

$$T(z)\Lambda(z') = \left(c + \frac{22}{5}\right) \frac{T(z')}{(z-z')^4} + \frac{4\Lambda(z')}{(z-z')^2} + \frac{\partial \Lambda(z')}{z-z'} + \text{reg.} \quad (5.120)$$

One can therefore cancel an anomalous term in (5.119) just introducing

$$\tilde{W}_4(z) = W_4(z) - \frac{(N-2)(N-3)}{8(N+17)} \Lambda(z) \quad (5.121)$$

which is already a primary conformal field. Its charge therefore is given by the formula

$$\tilde{w}_4 = w_4 - \frac{(N-2)(N-3)}{8(5N+17)} \Delta(5\Delta + 1) \quad (5.122)$$

**Degenerate period matrix**

Here we compute the period matrix of the genus $g$ hyperelliptic curve (see fig. 5.4)

$$y^2 = (z - R) \prod_{l=1}^{g} (z - q_{2l})(z - q_{2l} + \epsilon_l) = (z - R) \prod_{l=1}^{g} (z - q_{2l})(z - q_{2l-1}) \quad (5.123)$$

in the degenerate limit $\epsilon_l \to 0$, $R \to \infty$ up to the terms of order $O(\epsilon_l)$ and $O\left(\frac{1}{R}\right)$ (this equivalence will be denoted by "≈"). The normalized first kind Abelian differentials with such accuracy are

$$d\omega_I = \sqrt{q_{2l} - R} \prod_{K \neq I} (z - q_{2K}) \frac{dz}{y}, \quad \frac{1}{2\pi i} \oint_{A_j} d\omega_I \approx \delta_{1,I} \quad (5.124)$$

since $\sqrt{z-q_I} \approx 1$ when $z$ goes far from $q_I$. First we compute the off-diagonal matrix element $T_{I,J}$ for $J > I$

$$T_{I,J} = \oint_{B_J} d\omega_I \approx -2 \int_{q_J}^{R} \sqrt{\frac{q_{2l} - R}{z - R}} \frac{dz}{z - q_{2l}} \approx -2 \log 4 + 2 \log \frac{q_{2J} - q_{2l}}{R} \quad (5.125)$$
and then a little bit more complicated diagonal element

\[
\mathcal{T}_{II} = \oint_{B_I} d\omega_I \approx -2 \int_{q_{2I}}^R \frac{q_{2I} - R}{z - R} \frac{dz}{\sqrt{(z - q_{2I})(z - q_{2I} + \epsilon_I)}} \approx 2 \int_{q_{2I}}^R \left(1 - \sqrt{\frac{q_{2I} - R}{z - R}}\right) \frac{dz}{z - q_{2I}} - 2 \int_{q_{2I}}^R \frac{dz}{\sqrt{(z - q_{2I})(z - q_{2I} + \epsilon_I)}} = \frac{2R}{\epsilon_I} = -2 \log 4 - 2 \log 4 + 2 \log \frac{\epsilon_I}{R}
\]

(5.126)

where we have used the fact, that for our purposes in the expressions

\[
\frac{f(z)}{\sqrt{(z - q_{2I})(z - q_{2I} + \epsilon_I)}}
\]

one can drop \(\epsilon_I\) if \(f(q_{2I}) = 0\).

Now using (5.87) we can compute in this limit

\[
\tau_{SW} = \exp \left(\frac{1}{2} \sum_{I < J} a_I T_{IJ} a_J\right) \approx \frac{1}{2} \sum_{I < J} a_I T_{IJ} a_J R^{-(\sum_{I J}^2)^2}
\]

(5.127)

The result for \(\tau_B(q)\) in this simple hyperelliptic example can be taken from [ZamAT87]

\[
\tau_B(q) = \prod_{i < j} (q_i - q_j)^{-\frac{1}{2}} \times \left[\det_{I J} \frac{1}{2\pi i} \oint_{A_I} z^{-1} dz \right]^{-\frac{1}{2}}
\]

(5.128)

where the determinant can be easily computed using (5.123)

\[
\det_{I J} \frac{1}{2\pi i} \oint_{A_I} z^{-1} dz \approx \det_{I J} \frac{q_{2I}^{-1} - q_{2J}^{-1}}{\sqrt{R} \prod_{j \neq I} (q_{2I} - q_{2J})} = R^{-\frac{1}{2}} \prod_{I < J} (q_I - q_J)^{-1}
\]

(5.129)

Altogether this gives the formula (5.92) for the degenerate form of the hyperelliptic Zamolodchikov exact conformal block.
Abstract

We study twist-field representations of the W-algebras and generalize the construction of the corresponding vertex operators to D- and B-series. We demonstrate how the computation of characters of such representations leads to the nontrivial identities involving lattice theta-functions. We propose a construction of their exact conformal blocks, which for D-series express them in terms of geometric data of corresponding Prym variety.

Introduction

Representation theory for the W-algebras [ZamW, FZ, FL] is still the subject with many open questions. These questions often arise in the context of a two-dimensional conformal theory (CFT) with extended symmetry, and due to a nontrivial recently found correspondence [LMN, NO, AGT, Nek15] may be important for multidimensional supersymmetric gauge theories.

The main object of this study is a conformal block, which for generic W-algebra beyond the Virasoro case is not fixed by its algebraic properties. Actually there are at least two different meanings of the term “conformal block” in the literature:

- Space of all functionals on the product of Verma modules at given points of the Riemann surface that solve Ward identities (see e.g. [FBZv] where the language of coinvariants was used). Such spaces form a bundle over moduli space of curves with fixed points. We prefer to call this a ”space of conformal blocks”, and reserve the word ”conformal block” for another meanings following [BPZ].

- Concrete section of this bundle. Usually the corresponding functionals are computed on the highest weights in each of the Verma modules. For the Virasoro algebra the conformal blocks are specified by definite intermediate dimensions, or, equivalently, by the asymptotic behavior of the conformal block on the boundary of the moduli space;
The first object (which we call the “space of conformal blocks”) is defined for any vertex algebra, but the problem is to specify a concrete section of this bundle, it is no longer enough to do this by fixing quantum numbers in the intermediate channels. Even for three points on sphere, the vector space of conformal blocks becomes infinite dimensional for \( W_N \) algebras with \( N > 2 \).

However, for certain particular cases this conformal block can be constructed explicitly applying some extra machinery. In what follows we first restrict ourselves to the case of integer and sometimes half-integer Virasoro central charges, when representations \( W \)-algebras are more directly related to the representations of the corresponding Kac-Moody (KM) algebras (of level \( k = 1 \), and the corresponding field theories can be directly described by free fields [FK].

Even in such situation the most general case cannot be formulated explicitly, one of the recent methods [GMfer] reduces the problem here to a Riemann-Hilbert problem, arising in the context of the isomonodromy/CFT correspondence [GIL12, ILTe, Gav]. Below, following [GMtw], we are going to restrict ourselves to the case of so-called twist fields [ZamAT87, ZamAT86, ApiZam], when the representations of the \( W \)-algebras become related to the twisted representations of the corresponding KM algebras \(^1\) [KacBook, BK].

The chapter is organized as follows. We start from the formulation of the representations of KM and \( W \)-algebras in terms of free bosons and fermions, remind first the \( GL(N) \) case and extend it to the \( D \)- and \( B \)-series, using real fermions. We define then the twist representations, and show that they are parameterized by the conjugacy classes in the correspondent Cartan’s normalizer \( N_G(h) \). We classify the conjugacy classes \( g \in N_G(h) \) for \( G = GL(N) \) and \( G = O(n) \) (for \( n = 2N \) and \( n = 2N + 1 \)) and define the twist fields \( O_g \) in terms of the boundary conditions in corresponding free theory.

Bosonization rules allow to compute easily the characters \( \chi_g(q) \) of the corresponding representations. For the twist fields of “\( GL(N) \) type” this goes back to the old results of Al. Zamolodchikov and V. Knizhnik, and we develop here similar technique in the case of real fermions and another class of twist fields, arising in \( D \)- and \( B \)-series. The character formulas include summations over the root lattices, reflecting the fact that we deal here with the class of lattice vertex algebras. Dependently of the conjugacy class \( g \in N_G(h) \) of a twist field the lattice can be reduced to its projection to the Weyl-invariant part, in this case the “smaller” lattice theta functions show up, or we find even a kind of “exchange” between those for \( D \)- and \( B \)-series.

If two different classes \( g_{1,2} \in N_G(h) \) are nevertheless conjugated \( g_1 \sim g_2 \) in \( G \) (but not in \( N_G(h) \)) this gives a nontrivial identity \( \chi_{g_1}(q) = \chi_{g_2}(q) \) between two characters, involving lattice theta-functions. Such character identities go back to 1970’s (see [Mac], [Kac78]) and even to Gauss, but our derivation gives probably the new ones, involving in particular the theta functions for \( D \)- and \( B \)-root lattices.

We propose construction of the exact conformal blocks of the twist fields for \( W \)-algebras of \( D \)-series, generalizing approach of [ZamAT87, ZamAT86, ApiZam, GMtw], and obtain an explicit formula, expressing multipoint blocks in terms of the algebro-

\(^1\) To prevent the reader’s confusion we should notice that “twist field representation” is different from “twisted representation”: the latter one implies that the algebra itself is changed (twisted), whereas the first one only reflects the way – how this representation was constructed.
geometric objects on the branched cover with extra involution.

## W-algebras and KM algebras at level one

### Boson-fermion construction for GL(N)

We start from standard complex fermions

\[
\psi^*_\alpha(z) = \sum_{p \in \frac{1}{2} + \mathbb{Z}} \frac{\psi^*_{\alpha,p}}{z^{p + \frac{1}{2}}}, \quad \psi_\alpha(z) = \sum_{p \in \frac{1}{2} + \mathbb{Z}} \frac{\psi_{\alpha,p}}{z^{p + \frac{1}{2}}}
\]

with the operator product expansions (OPE’s)

\[
\psi^*_\alpha(z)\psi_\beta(w) = -\psi_\beta(w)\psi^*_\alpha(z) = \frac{\delta_{\alpha\beta}}{z - w} + \text{reg.}
\]

\[
\psi_\alpha(z)\psi_\beta(w) = \psi^*_\alpha(z)\psi^*_\beta(w) = \text{reg.}
\]

(6.2)

equivalent to the following anticommutation relations

\[
\{\psi^*_\alpha, p; \psi_\beta, q\} = \delta_{\alpha\beta}\delta_{p+q,0}, \quad \{\psi_\alpha, p; \psi^*_\beta, q\} = \{\psi^*_\alpha, p; \psi^*_\beta, q\} = 0, \quad p, q \in \frac{1}{2} + \mathbb{Z}
\]

(6.3)

One can introduce the Kac-Moody \(\hat{\mathfrak{gl}}(N)_1\) algebra by the currents

\[
J_{\alpha\beta}(z) = :\psi^*_\alpha(z)\psi_\beta(z):
\]

(6.4)

where the free fermion normal ordering moves all \(\{\psi_r\}\) and \(\{\psi^*_r\}\) with \(r > 0\) to the right. These currents have standard OPE’s:

\[
J_{\alpha\beta}(z)J_{\gamma\delta}(w) = \frac{\delta_{\beta\gamma}\delta_{\alpha\delta}}{(z - w)^2} + \frac{\delta_{\beta\gamma}J_{\alpha\delta}(w) - \delta_{\alpha\delta}J_{\beta\gamma}(w)}{z - w} + \text{reg.}
\]

(6.5)

and when expanded into the (integer!) powers of \(z\)

\[
J_{\alpha\beta}(z) = \sum_{n \in \mathbb{Z}} \frac{J_{\alpha\beta,n}}{z^{n+1}}
\]

(6.6)

we get the standard Lie-algebra commutation relations

\[
[J_{\alpha\beta,n}, J_{\gamma\delta,m}] = n\delta_{n+m,0}\delta_{\beta\gamma}\delta_{\alpha\delta} + \delta_{\beta\gamma}J_{\alpha\delta,n+m} - \delta_{\alpha\delta}J_{\beta\gamma,m+n}, \quad n, m \in \mathbb{Z}
\]

(6.7)

This set contains zero modes \(J_{\alpha\beta,0}\), generating the subalgebra \(\mathfrak{gl}(N) \subset \hat{\mathfrak{gl}}(N)_1\). The \(W(\mathfrak{gl}(N)) = W_N \oplus H\) algebra can be defined in a standard way - as a commutant of \(\mathfrak{gl}(N)\) in the (completion of the) universal enveloping \(U(\hat{\mathfrak{gl}}(N)_1)\).

This basis of the generators of \(W(\mathfrak{gl}(N)) = W_N \oplus H\) algebra can be chosen in several different ways. In what follows the most convenient for our purposes is to use fermionic bilinears

\[
\sum_{\alpha=1}^{N} \psi^*_\alpha \left(z + \frac{1}{2}t\right) \psi_\alpha \left(z - \frac{1}{2}t\right) = \frac{N}{t} + \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} U_k(z)
\]

(6.8)
6. Twist-field representations of W-algebras, exact conformal blocks and character identities

or, using the Hirota derivative $D^n f(z) \cdot g(z) = (\partial_{z_1} - \partial_{z_2})^n f(z_1)g(z_2)|_{z_1=z_2=z}$,

$$U_k(z) = D_z^{k-1} \sum_{\alpha=1}^N :\psi_\alpha^*(z) \cdot \psi_\alpha(z):$$

(6.9)

while another useful basis is the bosonic representation

$$W_k(z) = \sum_{\alpha_1<\cdots<\alpha_k} :J_{\alpha_1\alpha_1}(z) \cdots J_{\alpha_k\alpha_k}(z): \equiv \sum_{\alpha_1<\cdots<\alpha_k} :J_{\alpha_1}(z) \cdots J_{\alpha_k}(z):$$

(6.10)

The formula (6.10) is equivalent to quantum Miura transform from [ZamW, FZ, FL]. To explain that the formula (6.9) is actually equivalent to (6.10) one can use description of $W_{gl_{N}}$ as centralizer of screening operators which coincide with $gl_{N}$ in this case. It is already proven, that (6.10) is centralizer of screening operators [FF], so it remains to show that (6.9) is centralizer as well, what can be done in several steps:

1. Consider all normally-ordered fermionic monomials: $\prod_i \partial^k_i \psi_{\alpha_i}(z) \prod_j \partial^l_j \psi^*_\alpha_j(z)$, which transform as tensors under the action of $GL(N)$. By First fundamental theorem of invariant theory [Weyl] the only invariants in such representation are given by all possible contractions, so they can be written as: $\prod_i (\sum_{\alpha} \partial^k_i \psi_{\alpha}(z) \partial^l_j \psi^*_\alpha(z))$.

2. Any such expression can be obtained by taking regular products of the “elementary elements” $\sum_{\alpha} \partial^k_i \psi_{\alpha}(z) \partial^l_j \psi^*_\alpha(z)$ since

$$\prod_i \left( \sum_{\alpha} \partial^k_i \psi_{\alpha}(z) \partial^l_j \psi^*_\alpha(z) \right) = \sum_{\alpha} \partial^k \psi_{\alpha}(z) \partial^l \psi^*_\alpha(z) :=$$

(6.11)

$$= \prod_i \left( \sum_{\alpha} \partial^k_i \psi_{\alpha}(z) \partial^l_j \psi^*_\alpha(z) \right) \sum_{\beta} \partial^k \psi_{\beta}(z) \partial^l \psi^*_\beta(z) := + \text{lower terms}$$

Therefore one can perform this procedure iteratively and express everything as regular products of bilinears.

3. Any element $\sum_{\alpha} \partial^k_i \psi_{\alpha}(z) \partial^l_j \psi^*_\alpha(z)$ can be expressed as a linear combination of $\partial^{l'+1}U_{k'}(z)$ for different $l'$ and $k'$ with $l'+k' = l+k$.

Hence, the generators $\{U_k(z)\}$ are expressible in terms of $\{W_k(z)\}$ (and vice versa) by some non-linear triangular transformations, but we do not need here these explicit formulas.

Formally there is an infinite number of generators in (6.8) and (6.9), since all of them are expressed in terms of $N$ generators (6.10) they are not independent: we have

$$U_{N+n}(z) = P_n(\{\partial^k U_{l\leq N}\})$$

(6.12)

2The fact, that nonlinear W-algebra generators can be expressed through just bilinear fermionic expressions is well-known, and was already exploited in [LMN, NO] (see also [GMfer] and references therein).
for some polynomials \( \{P_n\} \), and this is the origin of the non-linearity of the W-algebra [FKRW]. The relation between fermions and bosons are given by well-known [FK, KVdL] bosonization formulas

\[
\psi^*(\alpha)(z) = \exp \left( -\sum_{n<0} J_{\alpha,n} \frac{z^n}{n!} \right) \exp \left( -\sum_{n>0} J_{\alpha,n} \frac{z^{-n}}{n!} \right) e^{Q_{\alpha} z} J_{\alpha,0} e_\alpha(J_0) = e^{i\varphi_{\alpha,-}(z)} e^{i\varphi_{\alpha,+}(z)} e^{Q_{\alpha} z} J_{\alpha,0} e_\alpha(J_0) (6.13)
\]

\[
\psi_\alpha(z) = \exp \left( \sum_{n<0} J_{\alpha,n} \frac{z^n}{n!} \right) \exp \left( \sum_{n>0} J_{\alpha,n} \frac{z^{-n}}{n!} \right) e^{-Q_{\alpha} z} J_{\alpha,0} e_\alpha(J_0) = e^{-i\varphi_{\alpha,-}(z)} e^{-i\varphi_{\alpha,+}(z)} e^{-Q_{\alpha} z} J_{\alpha,0} e_\alpha(J_0)
\]

where \( e_\alpha(J_0) = \prod_{\beta=1}^{\alpha-1} (-1)^{J_{0,\beta}} \) and diagonal \( J_{\alpha,n} \equiv J_{\alpha,n} \) form the Heisenberg algebra

\[
[J_{\alpha,n}, J_{\beta,m}] = n \delta_{\alpha\beta} \delta_{m+n,0}, \quad [J_{0,\alpha}, Q_{\beta}] = \delta_{\alpha\beta} (6.14)
\]

One can also express all other generators in terms of (positive and negative parts of) the bosons

\[
i\varphi_{+,\alpha}(z) = -\sum_{n>0} J_{\alpha,n} \frac{z^n}{n!}, \quad i\varphi_{-,\alpha}(z) = -\sum_{n<0} J_{\alpha,n} \frac{z^{-n}}{n!} (6.15)
\]

namely

\[
J_{\alpha\beta}(z) = e^{i\varphi_{-,\alpha}-i\varphi_{-,\beta}} e^{i\varphi_{+,\alpha}-i\varphi_{+,\beta}} e^{Q_{\alpha}-Q_{\beta} z} J_{\alpha,0} J_{\beta,0} (-1)^{\gamma-1} \sum_{\gamma=1}^{\beta-1} J_{\gamma,0} + \theta(\beta-\alpha), \quad \alpha \neq \beta (6.16)
\]

**Real fermions for D- and B- series**

Now we can almost repeat the same construction for the orthogonal series, \( B_N \) and \( D_N \), which correspond to the W-algebras \( W(\mathfrak{so}(2N+1)) \) and \( W(\mathfrak{so}(2N)) \), respectively. The corresponding Kac-Moody algebras at level one can be realized in terms of the real fermions (see e.g. [AWM]) with the OPE’s

\[
\Psi_i(z)\Psi_j(w) = \frac{\delta_{ij}}{z-w} + \text{reg.}, \quad i, j = 1, \ldots, n (6.17)
\]

(here dependently on the case we put either \( n = 2N \) or \( n = 2N+1 \), which corresponds to anti-commutation relations

\[
\{\Psi_{i,p}, \Psi_{j,q}\} = \delta_{ij} \delta_{p+q,0}, \quad p, q \in \frac{1}{2} + \mathbb{Z} (6.18)
\]

One can say that these OPE’s and commutation relations are defined by the metrics on \( n \)-dimensional space given by \( \delta_{ij} \), or symbolically by \( ds^2 = \sum_{i=1}^{n} d\Psi_i^2 \). The Kac-Moody currents are again expressed by bilinear combinations

\[
J_{ik}^{(1)}(z) = :\Psi_i(z)\Psi_j(z): (6.19)
\]
6. Twist-field representations of W-algebras, exact conformal blocks and character identities

and satisfy usual commutation relations together with $J_{ij}^{(1)}(z) = -J_{ji}^{(1)}(z)$. It is also convenient to pass to the complexified fermions ($\alpha = 1, \ldots, N$)

$$
\psi^*_\alpha(z) = \frac{1}{\sqrt{2}} (\Psi_{2\alpha-1}(z) + i\Psi_{2\alpha}(z)), \quad \psi_\alpha(z) = \frac{1}{\sqrt{2}} (\Psi_{2\alpha-1}(z) - i\Psi_{2\alpha}(z))
$$

(6.20)

which due to (6.17) have the standard OPE’s given by (6.2). Let us point out that $B_N$-series ($i, j = 1, \ldots, 2N+1$) differs from $D_N$-series ($i, j = 1, \ldots, 2N$) by remaining single real fermion $\Psi_{2N+1}(z) = \Psi(z)$.

Using the complexified fermions the generators (6.19) can be re-written as

$$
J_{\alpha\beta} =: \psi^*_\alpha(z)\psi_\beta(z) := \frac{1}{2} (J_{2\alpha-1,2\beta-1}^{(1)} + J_{2\alpha,2\beta}^{(1)}) + \frac{i}{2} (J_{2\alpha,2\beta-1}^{(1)} + J_{2\beta,2\alpha-1}^{(1)})
$$

(6.21)

together with

$$
J_{\alpha\bar{\beta}} = \psi^*_\alpha(z)\psi^*_{\bar{\beta}}(z), \quad J_{\bar{\alpha}\beta} = \psi_\alpha(z)\psi_\beta(z)
$$

$$
J_{\alpha,\Psi} = \psi^*_\alpha(z)\Psi(z), \quad J_{\bar{\alpha},\Psi} = \psi_\alpha(z)\Psi(z)
$$

(6.22)

so that we see explicitly $\widehat{gl(N)}_1 \subset \widehat{so(n)}_1$. Note also, that elements $J_{\alpha\alpha}(z) = J_{\alpha}(z)$ again form the Heisenberg algebra, and its zero modes $J_{\alpha,0}$ correspond to the Cartan subalgebra of $so(n)$.

As before, we define the W-algebra $W(so(n))$ as commutant of $so(n) \subset \widehat{so(n)}_1$. In contrast to the simple-laced cases we find this commutant for B-series not in completion of the $U(so(2N+1))$, but in the entire fermionic algebra. An inclusion of algebras $gl(N) \subset so(2N)$, acting on the same space, leads to inverse inclusion

$$
W(so(2N)) \subset W(gl(N))
$$

(6.23)

Similarly to (6.9) one can present the generators of the $W(so(n))$-algebra explicitly, using the real fermions

$$
U_k(z) = \frac{1}{2} D_z^{k-1} \sum_{j=1}^n : \Psi_j(z) \cdot \Psi_j(z) : \quad V(z) = \prod_{j=1}^n \Psi_j(z)
$$

(6.24)

The last current is bosonic in $D_N$ case and fermionic for $B_N$. These expressions are obtained analogously to (6.9) with the help of invariant theory, the only important difference is that for $SO(n)$ case there is also completely antisymmetric invariant tensor. We can rewrite these expressions using complex fermions (for the $D_N$ case one should just put here $\Psi(z) = 0$ in the expressions for $U$-currents and $\Psi(z) = 1$ in the expressions for the $V$-current)

$$
U_k(z) = \frac{1}{2} D_z^{k-1} \sum_{\alpha=1}^N (\psi^*_\alpha(z) \cdot \psi_\alpha(z) + \psi_\alpha(z) \cdot \psi^*_\alpha(z)) + \frac{1}{2} D_z^{k-1} \Psi(z) \cdot \Psi(z)
$$

$$
V(z) = \prod_{\alpha=1}^N : \psi^*_\alpha(z)\psi_\alpha(z) : \Psi(z)
$$

(6.25)
6.4. Twist-field representations from twisted fermions

It is easy to see that odd generators vanish $U_{2k+1} = 0$, while even generators in $D_N$ case coincide with those in $W(\mathfrak{gl}(N))$ algebra

$$U_{2k}(z) = D_z^{2k-1} \sum_{\alpha=1}^N :\psi_\alpha^*(z) \cdot \psi_\alpha(z): + \frac{1}{2} D_z^{2k-1} \Psi(z) \cdot \Psi(z), \quad k = 1, 2, \ldots$$  (6.26)

So finally we have the following sets of independent generators:

$$U_1(z), U_2(z), \ldots, U_N(z) \quad \text{for} \quad W(\mathfrak{gl}(N))$$  
$$U_2(z), U_4(z), \ldots, U_{2N-2}(z), V(z) \quad \text{for} \quad W(\mathfrak{so}(2N))$$  
$$U_2(z), U_4(z), \ldots, U_{2N}(z), V(z) \quad \text{for} \quad W(\mathfrak{so}(2N+1))$$  (6.27)

**Twist-field representations from twisted fermions**

**Fermions and W-algebras**

For any current algebra, generated by currents $\{A_I(z)\}$, the commutation relations follow from their local OPE’s

$$A_I(z) A_J(w) \underset{z \to w}{=} \sum_K \frac{(A_I A_J)_K(w)}{(z-w)^K}$$  (6.28)

However, to define the commutation relations in addition to local OPE’s one should also know the boundary conditions for the currents: in radial quantization – the analytic behaviour of $A_I(z)$ around zero. Any vertex operator $V_g(0)$, e.g. sitting at the origin $^3$, can create nontrivial monodromy for our currents:

$$A_I(e^{2\pi i z}) V_g(0) = \sum_j g_{IJ} A_J(z) V_g(0)$$  (6.29)

for some linear automorphism of the current algebra.

Perhaps the simplest example of such nontrivial monodromy is the diagonal transformation of the fermionic fields

$$\psi_\alpha^*(e^{2\pi i z}) = e^{2\pi i \theta_\alpha} \psi_\alpha^*(z), \quad \psi_\alpha(e^{2\pi i z}) = e^{-2\pi i \theta_\alpha} \psi_\alpha(z), \quad \alpha = 1, \ldots, N$$  (6.30)

which just shifts the mode expansion index

$$\psi_\alpha^*(z) = \sum_{p \in \mathbb{Z}+\frac{1}{2}} \frac{\psi_\alpha^*(z)}{z^{p+\frac{1}{2}-\theta_\alpha}}, \quad \psi_\alpha(z) = \sum_{p \in \mathbb{Z}+\frac{1}{2}} \frac{\psi_\alpha(z)}{z^{p+\frac{1}{2}+\theta_\alpha}}$$  (6.31)

Instead of the OPE (6.2) one gets therefore

$$\psi_\alpha^*(z) \psi_\beta(w) \to z^{\theta_\alpha - \theta_\beta} \psi_\alpha^*(z) \psi_\beta(w) = \frac{z^{\theta_\alpha} w^{-\theta_\beta}}{z-w} + z^{\theta_\alpha} w^{-\theta_\beta} :\psi_\alpha^*(z) \psi_\beta(w): \quad =$$

$$= \frac{1}{z-w} + \frac{\theta_\alpha \delta_\alpha \beta}{w} + :\psi_\alpha^*(w) \psi_\beta(w): + \text{reg.}$$  (6.32)

$^3$For nontrivial boundary conditions we assume presence of such field by default, when obvious – not indicating it explicitly.
6. Twist-field representations of W-algebras, exact conformal blocks and character identities

which means that for the shifted fermions (6.31) one should use different normal ordering:

\[
(\psi^*_\alpha(z)\psi_\beta(z)) = \frac{\theta_\alpha\delta_{\alpha\beta}}{z} + :\psi^*_\alpha(z)\psi_\beta(z):
\]  

(6.33)

This implies that for the diagonal components \(\hat{\mathfrak{gl}}(N)_1\) algebra one has extra shift \(J_\alpha(z) \rightarrow J_\alpha(z) + \frac{\sigma_\alpha}{z}\), while for the non-diagonal currents we obtain

\[
J_{\alpha\beta}(z) = \sum_{n \in \mathbb{Z}} \frac{J_{\alpha\beta,n}}{z^{n+1} + g_{\alpha - \beta}}
\]  

(6.34)

so that the commutation relations for this “twisted” Kac-Moody algebra become

\[
[J_{\alpha\beta,n}, J_{\gamma\delta,m}] = (n - \theta_\alpha + \theta_\beta)\delta_{n+m,0}\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\beta\gamma}J_{\alpha\delta,m+n} - \delta_{\alpha\delta}J_{\beta\gamma,m+n}
\]  

(6.35)

We see that these commutation relations differ from the conventional ones (6.7) only by the extra shift which can be hidden into the Cartan generators \(J_{\alpha\alpha,0}\). However, in the twisted case \(\hat{\mathfrak{gl}}(N)_1\) does not contain zero modes, and we cannot think about the W-algebra as about commutant of some \(\mathfrak{gl}(N)\). But nevertheless we define the currents

\[
U_k(z) = D_z^{k-1} \sum_{\alpha=1}^N (\psi^*_\alpha(z) \cdot \psi_\alpha(z))
\]  

(6.36)

One can still use two basic facts:

- since \(U_k(e^{2\pi i}z) = U_k(z)\), they are expanded in integer powers of \(z\) as before;
- they satisfy the same algebraic relations for all values of monodromies \(\{\sigma_\alpha\}\), because the OPE’s of \(\psi_\alpha, \psi^*_\alpha\) (and so the OPE’s of \(U_k\)) do not depend on these monodromy parameters.

**Twist fields and Cartan’s normalizers**

Consider now more general situation, when

\[
\psi^*_\alpha(e^{2\pi i}z) = \sum_{\beta=1}^N g_{\alpha\beta}\psi^*_\beta(z), \quad \psi_\alpha(e^{2\pi i}z) = \sum_{\beta=1}^N g^{-1}_{\beta\alpha}\psi_\beta(z)
\]  

(6.37)

i.e. compare to (6.30) the monodromy is no longer diagonal \(^4\). It is clear that then the action on \(\hat{\mathfrak{gl}}(N)_1\) is

\[
J_{\alpha\beta}(z) \mapsto g_{\alpha\alpha'}g^{-1}_{\beta\beta'}J_{\alpha'\beta'}(z)
\]  

(6.38)

The most general transformation we consider in the \(O(n)\) case mixes \(\psi\) and \(\psi^*\):

\[
\psi_\alpha(e^{2\pi i}z) = \sum_{\beta=1}^N g_{\alpha\beta}\psi_\beta(z), \quad \alpha = -N, \ldots, N
\]  

(6.39)

\(^4\)This element should preserve the structure of the OPEs, so it should preserve symmetric form on fermions, and lies therefore in \(O(2N)\). Notice that it automatically implies that all even generators of the W-algebra \(U_{2k}(w)\) are also preserved. To preserve odd generators \(U_{2k+1}(z)\) one should have also \(g \in \text{Sp}(2N)\), but \(O(2N) \cap \text{Sp}(2N) = \text{GL}(N)\), so \(g \in \text{GL}(N)\).
where it is convenient to introduce conventions $\psi^*_\alpha = \psi_\alpha$, $\alpha > 0$, and $\psi_0$ can be absent. Matrix $g$ here should preserve the anticommutation relations.

**Definition 1.** We call the vertex operator $V_g = O_g$ a twist field when $g$ lies in the normalizer of Cartan $\mathfrak{h} \subset \mathfrak{g}$, i.e. $g \in N_G(\mathfrak{h})$ iff

$$g\mathfrak{h}g^{-1} = \mathfrak{h} \quad (6.40)$$

Such elements also preserve the Heisenberg subalgebra $\widehat{\mathfrak{h}} = \langle J_1(z), \ldots, J_{\text{rank } \mathfrak{g}}(z) \rangle \subset \widehat{\mathfrak{g}}$

$$g\widehat{\mathfrak{h}}g^{-1} = \widehat{\mathfrak{h}} \quad (6.41)$$

We are going now to discuss the structure of the Cartan normalizers $N_{GL(N)}(\mathfrak{h})$ and $N_{O(n)}(\mathfrak{h})$, which classify the twist fields for the $W_N = W(\mathfrak{gl}(N))$ and $W(\mathfrak{so}(n))$ (for even $n = 2N$ and odd $n = 2N + 1$) correspondingly.

**Structure of the Cartan normalizer for $\mathfrak{gl}(N)$.** Let us choose the Cartan subalgebra in a standard way $\mathfrak{h} \supset \text{diag}(x_1, \ldots, x_N)$, so conjugation (6.40) can only permute the eigenvalues. Therefore we conclude that

$$g = s \cdot (\lambda_1, \ldots, \lambda_N) \in S_N \ltimes (\mathbb{C}^\times)^N = N_{GL(N)}(\mathfrak{h}) \quad (6.42)$$

or just $g$ is a quasipermutation.

Let us now find the conjugacy classes in this group. Any element of $N_{GL(N)}(\mathfrak{h})$ has the form $g = (c_1 \ldots c_k, (\lambda_1, \ldots, \lambda_N))$, where $c_i$ are the cyclic permutations - their only parameters are lengths $l_j = l(c_i)$. It is enough to consider just a single cycle of the length $l = l(c)$

$$g = (c, (\lambda_1, \ldots, \lambda_l)) \quad (6.43)$$

since any $g$ can be decomposed into a product of such elements. Conjugation of this element by diagonal matrix is given by

$$(1, (\mu_1, \ldots, \mu_l)) \cdot (c, (\lambda_1, \ldots, \lambda_l)) \cdot (1, (\mu_1, \ldots, \mu_l))^{-1} = (c, (\lambda_1\mu_1, \lambda_2\mu_2, \ldots, \lambda_l\mu_l))$$

Therefore one can always adjust \{\mu_i\} to get rid of all \{\lambda_i\} except for one, e.g. to put $\lambda_i \mapsto \bar{\lambda} = \prod_{i=1}^l \lambda_i^{1/l} = e^{2\pi ir}$, these “averaged over a cycle” parameters have been called as $r$-charges in [GMtw]. Hence, all elements of $g \in N_{GL(N)}(\mathfrak{h})$ can be conjugated to the products over the cycles

$$[g] \sim \prod_{j=1}^K [l_j, \bar{\lambda}_j] = \prod_{j=1}^K [l_j, e^{2\pi ir}] \quad (6.45)$$

**Structure of $N_{O(n)}(\mathfrak{h})$.** Using complexification of fermions (6.20) we rewrite the quadratic form $ds^2 = \sum_{i=1}^n d\Psi_1^2$ as $ds^2 = \sum_{\alpha=1}^N d\psi_\alpha^* d\psi_\alpha + d\Psi^2 = \sum_{\alpha=1}^N d\psi_{-\alpha} d\psi_\alpha + d\Psi^2$ (the last term is present only for the $B_N$-series). In this basis the $\mathfrak{so}(n)$ algebra (the algebra, preserving this form) becomes just the algebra of matrices, which are antisymmetric.
under the reflection w.r.t. the anti-diagonal. In particular, the Cartan elements are given by
\[ h \ni \text{diag}(x_1, \ldots, x_N, 0, -x_N, \ldots, -x_1) \tag{6.46} \]
for \( B_N \)-series (and for the \( D_N \)-series 0 in the middle just should be removed). The action of an element from \( N_{O(n)}(h) \) should preserve the chosen quadratic form, and, when acting on the diagonal matrix (6.46), it can only permute some eigenvalues, also doing it simultaneously in the both blocks, or interchange \( x_\alpha \) with \(-x_\alpha\) (the same as to change the sign of \( x_\alpha \)). It is defined in this way up to a subgroup of diagonal matrices themselves. In other words
\[
N_{O(2N)}(h) = S_N \ltimes (\mathbb{Z}/2\mathbb{Z})^N \ltimes (\mathbb{C}^\times)^N
\]
\[
N_{O(2N+1)}(h) = N_{O(2N)}(h) \times \mathbb{Z}/2\mathbb{Z}
\tag{6.47}
\]
where the last factor \( \mathbb{Z}/2\mathbb{Z} \) comes from changing sign of the extra fermion \( \Psi \). This triple \((s, \vec{\sigma}, \vec{\lambda}) \in N_{O(n)}(h)\), with \( s \in S_N \), \( \sigma_\alpha \in \mathbb{Z}/2\mathbb{Z} \) and \( \lambda_\alpha \in \mathbb{C}^\times \), is embedded into \( O(n) \) as follows
\[
S_N : \{s(\alpha) \mapsto 1, 1\} = \{\psi_\alpha \mapsto \psi_{s(\alpha)} ; \psi_\alpha^* \mapsto \psi_{s(\alpha)}^*\}
\]
\[
(\mathbb{Z}/2\mathbb{Z})^N : (1, \vec{\sigma}, 1) = \{\psi_\alpha \mapsto \psi_{\sigma_\alpha}\}
\]
\[
(\mathbb{C}^\times)^N : (1, 1, \vec{\lambda}) = \{\psi_\alpha \mapsto \lambda_\alpha \psi_\alpha ; \psi_\alpha^* \mapsto \lambda_\alpha^{-1} \psi_\alpha^*\}
\tag{6.48}
\]
and in these formulas \( \psi_{-\alpha} = \psi_{\alpha}^* \) and \( \psi_{s(\alpha)}^* = \psi_\alpha \) is again implied. The structure of the actions in the semidirect product has the obvious from:
\[
\vec{\sigma} : \lambda_\alpha \mapsto \lambda_{\sigma_\alpha}^\alpha, \quad s : (\sigma_\alpha, \lambda_\alpha) \mapsto (\sigma_{s(\alpha)}, \lambda_{s(\alpha)})
\tag{6.49}
\]
Notice that normalizer of Cartan in \( SO(n) \)
\[
N_{SO(n)}(h) = SO(n) \cap N_{O(n)}(h)
\tag{6.50}
\]
is distinguished by condition that \( \prod_{\alpha=1}^N \sigma_\alpha = 1 \), and the Weyl group is given as the factor of this normalizer by the Cartan torus
\[
W(so(n)) = N_{SO(n)}(h)/H
\tag{6.51}
\]
Consider now the conjugacy classes in \( N_{O(n)}(h) \). First, conjugating an arbitrary element \((s, \vec{\sigma}, \vec{\lambda})\) by permutations, we again reduce the problem to the case when \( s = c \) is just a single cycle. Then one can further conjugate this element by \( (\mathbb{Z}/2\mathbb{Z})^N \):
\[
(1, \vec{\epsilon}, 1) \cdot (c, \vec{\sigma}, \bullet) \cdot (1, \vec{\epsilon}, 1)^{-1} \mapsto (c, (\sigma_1 \cdot \epsilon_1, \sigma_2 \epsilon_2 \epsilon_3, \ldots, \sigma_N \cdot \epsilon_1 \epsilon_N), \bullet)
\tag{6.52}
\]
and solving equations for \{\epsilon_\alpha\} remove all \( \sigma_\alpha = -1 \) except for, maybe, one. Hence:

- For \( \sigma = (1, \ldots, 1) \) the situation is the same as in \( gl(N) \) case: we can transform \( \vec{\lambda} \) to \((\lambda, \ldots, \lambda)\). These conjugacy classes are therefore (denoted by \([l, \lambda]_+\))
\[
(c, 1, \vec{\lambda}) \sim [l(c), \prod \lambda_i^{1/h(c)}]_+
\tag{6.53}
\]
6.4. Twist-field representations from twisted fermions

Let \( \sigma = (-1, 1, \ldots, 1) \) be the transposition that fixes \( 1 \) and interchanges the other elements. Let \( \tau = (1, \ldots, -1) \) be the transposition that interchanges the other elements and fixes \( 1 \). Then \( \tau \sigma = \sigma \tau \), and we can study the action of \( \tau \) on \( \sigma \) as well as on \( \psi \) and \( \psi^\ast \).

\[
(1, 1, \ldots, 1, -1) \mapsto \tau (1, 1, \ldots, 1, -1) = (1, -1, \ldots, 1, 1) \quad \text{and} \quad (1, 1, \ldots, 1, -1) \mapsto \tau (1, 1, \ldots, 1, -1) = (1, -1, \ldots, 1, 1) \end{equation}

(6.45)

In contrast to the previous case, here one can put all \( \lambda_i = 1 \), since one can put first \( \mu_1^2 = \prod_i \lambda_i^{-1} \), and then solve \( N - 1 \) equations \( \mu_{i+1} = \lambda_i \mu_i \) not being restricted by any boundary conditions. It means that

\[
(c, (-1, 1, \ldots, 1), \tilde{\lambda}) \sim [(c)]_+ \quad \text{(6.55)}
\]

Therefore we can formulate:

**Lemma 6.1.** One gets for the conjugacy classes

\[
N_{O(2N)}(\mathfrak{h}) : g \sim \prod_{j=1}^{K} [l_j, \lambda_j]_+ \cdot \prod_{j=1}^{K'} [l_j]_- \quad \text{(6.56)}
\]

\[
N_{O(2N+1)}(\mathfrak{h}) : g \sim [\epsilon] \cdot \prod_{j=1}^{K} [l_j, \lambda_j]_+ \cdot \prod_{j=1}^{K'} [l_j]_- \quad \text{(6.57)}
\]

and we are now ready to describe the twist fields in detail.

**Corollary:** Formulas (6.45) and (6.56) give also classification of the conjugacy classes in the Weyl groups \( \tilde{W}(\mathfrak{g}(N)) = S_N \) and \( \tilde{W}(\mathfrak{so}(n)) \). It is enough just to drop the \( \lambda \)-dependence and to consider only the even number of minus-cycles (the latter condition corresponds to the fact that (extension of) the Weyl group lies inside the connected component of identity in \( O(n) \), whereas another component corresponds to exterior automorphism of the Dynkin diagram). In \( [BK] \) the Weyl group has been extended to \( \tilde{W} = \tilde{W} \times (\mathbb{Z}/2\mathbb{Z})^{N-1} \subset \tilde{N}_G(\mathfrak{h}) \), which corresponds to breaking the Cartan torus \( H \subset G \) down to \( (\mathbb{Z}/2\mathbb{Z})^{N-1} \).

**Twist fields and bosonization for \( \mathfrak{g}(N) \)**

Take an element (6.45), whose action on fermions (in the fundamental and antifundamental representations), say for a single cycle, is

\[
g : (\psi^\ast_\alpha(z), \psi_\alpha(z)) \mapsto (e^{2\pi i r} \psi^\ast_{\alpha+1}(z), e^{-2\pi i r} \psi_{\alpha+1}(z)), \quad \text{mod } l \quad \text{(6.57)}
\]

while the corresponding (adjoint) action on the Cartan is just

\[
g_{\text{adj}} : J_\alpha(z) \mapsto J_{\alpha+1}(z), \quad \text{mod } l \quad \text{(6.58)}
\]

Such formulas have simple geometric interpretation [Knizhnik]: there is the branched cover in the vicinity of the point \( z = 0 \) given by \( \xi^l = z \), so that all these (fermionic and bosonic) fields are actually defined on different sheets \( \xi^{(\alpha)} = z^{1/l} e^{2\pi i \alpha/l} \) of the cover:

\[
\psi^\ast_\alpha(z) \sqrt{dz} = \tilde{\psi}^\ast(\xi^{(\alpha)}) \sqrt{d\xi^{(\alpha)}}, \quad \psi_\alpha(z) \sqrt{dz} = \tilde{\psi}(\xi^{(\alpha)}) \sqrt{d\xi^{(\alpha)}} \quad \text{(6.59)}
\]

\[
J_\alpha(z)dz = J(\xi^{(\alpha)})dz = \tilde{J}(\xi^{(\alpha)})d\xi^{(\alpha)}
\]

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Using these formulas one can write down expansions for the fields on the cover, whose OPE’s would be locally given by

\[
\hat{\psi}(z) \hat{\psi}(z') = \frac{1}{\xi - \xi'} + \text{reg.,} \quad \hat{J}(z) \hat{J}(z') = \frac{1}{(\xi - \xi')^2} + \text{reg.} \quad (6.60)
\]

Now one write for the mode expansion

\[
\psi(z) = \sqrt{\frac{dz}{d\xi}} \tilde{\psi}(\xi) = \frac{z^{\frac{1}{2} - \frac{r}{l}}}{\sqrt{l}} \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{\psi_p}{\xi^{p + \frac{1}{2} + \sigma}} = \frac{1}{\sqrt{l}} \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{\psi_p}{z^{\frac{1}{2} + \frac{1}{l}(p + \sigma)}}
\]

\[
\psi^*(z) = \frac{d\xi}{dz} \tilde{\psi}^*(\xi) = \frac{z^{\frac{1}{2} - \frac{r}{l}}}{\sqrt{l}} \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{\psi^*_p}{\xi^{p + \frac{1}{2} - \sigma}} = \frac{1}{\sqrt{l}} \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{\psi^*_p}{z^{\frac{1}{2} + \frac{1}{l}(p - \sigma)}} \quad (6.61)
\]

Due to (6.57) one should have \( \psi^*(e^{2\pi i l} z) = e^{2\pi il r} \psi^*(z) \) and \( \psi(e^{2\pi il} z) = e^{-2\pi il r} \psi(z) \), therefore one can take, for example, \( \sigma = lr + \frac{1-l}{2} \), so that:

\[
\psi(z) = \frac{1}{\sqrt{l}} \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{\psi_p}{z^{\frac{1}{2} + (p + \frac{1}{l}r)}} \quad \psi^*(z) = \frac{1}{\sqrt{l}} \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{\psi^*_p}{z^{\frac{1}{2} - (p - \frac{1}{l}r)}} \quad (6.62)
\]

or the mode expansion is shifted by the \( r \)-charges, corresponding to given cycles.

The same procedure gives for the twisted bosons

\[
J(z) = \frac{1}{l} z^{\frac{1}{l} - 1} \hat{J}(\xi) = \frac{1}{l} z^{\frac{1}{l} - 1} \sum_{n \in \mathbb{Z}} \frac{J_{n/l}}{z^{n/l + 1}} = \frac{1}{l} z^{\frac{1}{l} - 1} \sum_{n \in \mathbb{Z}} J_{n/l} z^{\frac{1}{l}(n+1)} = \frac{1}{l} \sum_{n \in \mathbb{Z}} J_{n/l} z_{n+1}^{\frac{1}{l}} \quad (6.63)
\]

with the commutation relations between the modes of these bosons being

\[
[J_{n/l}, J_{m/l}] = n \delta_{n+m,0} \quad n, m \in \mathbb{Z} \quad (6.64)
\]

These twisted bosons provide one of the convenient languages for the twist field representations. The other one is provided by bosonization of the constituent fermions with the fixed fractional parts of the power expansions in (6.62)

\[
\psi(z) = \frac{1}{\sqrt{l}} \sum_{a \in \mathbb{Z}/l\mathbb{Z}} \psi(a)(z), \quad \psi(a)(e^{2\pi i z}) = e^{-2\pi ir - 2\pi i a/l} \psi(a)(z)
\]

\[
\psi^*(z) = \frac{1}{\sqrt{l}} \sum_{a \in \mathbb{Z}/l\mathbb{Z}} \psi^*(a)(z), \quad \psi^*(a)(e^{2\pi i z}) = e^{2\pi ir + 2\pi i a/l} \psi^*(a)(z) \quad (6.65)
\]

The corresponding bosons (see (6.266) in Appendix)

\[
I_{(a)}(z) = (\psi^*_a(z) \psi_{(a)}(z)) = \sum_{n \in \mathbb{Z}} \frac{I_{n/l}^{(a)}}{z^{n+1}} + \frac{1}{z} \left( r + \frac{a}{l} \right) \quad (6.66)
\]

always have integer mode expansion.
Twist fields and bosonization for $\mathfrak{so}(n)$

Let us mention first, that there is a difference between the groups $N_{O(n)}(h)$ and $N_{SO(n)}(h)$, since the action of the first one can also map $V(z) \mapsto -V(z)$, so that one of the generators of the W-algebra $V(e^{2\pi i z}) = -V(z)$ becomes a Ramond field, and we allow this extra minus sign below \(^5\).

In addition to the conjugacy classes $[l, \lambda]_+$, similar to those of $\mathfrak{gl}(N)$, we now also have to study $[l]_-$'s. First one has to identify the action of $N_{O(n)}(h)$ on the fermions, where just by definition:

$$\sigma_\alpha = -1 : (1, \sigma, 1) : \psi_\alpha \mapsto \psi_\alpha^*$$  \quad (6.67)

This means that the element of our interest is the complete cycle

$$[l]_- : \psi_1 \mapsto \psi_2 \mapsto \ldots \mapsto \psi_l \mapsto \psi^*_1 \mapsto \ldots \mapsto \psi^*_l \mapsto \psi_1$$  \quad (6.68)

Therefore $2N$ complex fermions can be realized as a pushforward of a single real fermion $\eta(\xi)$, living on a $2l$-sheeted branched cover

$$\psi_\alpha(z) \sqrt{dz} = \eta(\xi^{(\alpha)}) \sqrt{d\xi}$$

$$\psi^*_\alpha(z) \sqrt{dz} = \eta(\xi^{(l+\alpha)}) \sqrt{d\xi}$$

(6.69)

Here the branched cover $z = \xi^{2l}$ can be realized as a sequence of two covers $\pi_2 : \xi \mapsto \zeta = \xi^2$ and $\pi_1 : \zeta \mapsto \zeta^l = z$, and it leads to more tricky global construction of the exact conformal blocks, see sect. 6.7 below.

An important fact is that there is an element $\sigma \in (N_{O(n)}(h)/H)$ in the center of this group

$$\sigma = (1, (-1, -1, \ldots, -1))$$  \quad (6.70)

which generates the global automorphism of the cover of order two, which is continued to the global automorphism of algebraic curve during the consideration of exact conformal blocks in sect. 6.7. It acts locally by $\xi \mapsto -\xi$. Using this element one can write the OPE of $\eta(\xi)$ in the form:

$$\eta(\xi)\eta(\sigma(\xi')) = \frac{1}{\xi - \xi'} + \text{reg.}$$

(6.71)

Now the analytic structure of this field can be obtained

$$\psi(z) = \sqrt{\frac{dz}{d\xi}} \eta(\xi) = \frac{z^{\frac{1}{2l} - \frac{1}{2}}}{\sqrt{2l}} \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{\eta_{p + \frac{1}{2}}}{z^{\frac{1}{2l}(p+\frac{1}{2}) + \frac{1}{2}}} = \frac{1}{\sqrt{2l}} \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{\eta_{p + \frac{1}{2}}}{z^{\frac{1}{2l}(p+\frac{1}{2}) + \frac{1}{2}}}$$

(6.72)

$$\psi^*(z) = \psi(e^{2\pi i l} z)$$

\(^5\)Note that in case of $n = 2N$ the action of $N_{SO(2N)}(h)$ on $h$ is given by Weyl group action, but additional element from $N_{O(2n)}(h)$ gives external (diagram) automorphism. Corresponding twisted representations could be viewed as a representation of twisted affine Lie algebra $D^{(2)}_N$. 
In order to ensure right monodromies (6.68) for $\psi, \psi^*$ one should get powers $\frac{1}{2l} \mathbb{Z}$ in the r.h.s., which means, that $\sigma \sim l - \frac{1}{2} \sim \frac{1}{2},$ and $\eta(\xi)$ turns to be a Ramond fermion with the extra ramification

$$\eta(\xi) = \sum_{n \in \mathbb{Z}} \frac{\eta_n}{\xi^{n+\frac{1}{2}}}$$

$$\psi(z) = \frac{1}{\sqrt{2l}} \sum_{n \in \mathbb{Z}} \frac{\eta_n}{z^{\frac{n}{2l} + \frac{1}{2}}}$$

$$\psi^*(z) = \frac{(-)^l}{\sqrt{2l}} \sum_{n \in \mathbb{Z}} \frac{(-)^n \eta_n}{z^{\frac{n}{2l} + \frac{1}{2}}}$$  \hspace{1cm} (6.73)

Let us now construct (a twisted!) boson from this fermion by

$$J(z) = (\psi^*(z)\psi(z)) = (\psi(e^{2\pi i l}z)\psi(z))$$  \hspace{1cm} (6.74)

This boson behaves like follows under the action of twist field:

$$J_1 \mapsto J_2 \mapsto \ldots \mapsto J_l \mapsto -J_1 \mapsto \ldots \mapsto -J_l$$  \hspace{1cm} (6.75)

To realize this situation we may take the Ramond boson on the cover in variable $\zeta$:

$$J(z) = \frac{d\zeta}{dz} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{J_{r/l}}{\zeta^{r+1}} = \frac{z^{\frac{1}{l}} - 1}{l} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{J_{r/l}}{z^{\frac{r+1}{l}}} = \frac{1}{l} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{J_{r/l}}{z^{\frac{r+1}{l}}}$$  \hspace{1cm} (6.76)

where commutation relations of modes are given by

$$[J_{r/l}, J_{r'/l}] = r\delta_{r+r',0} \quad r, r' \in \mathbb{Z} + \frac{1}{2}$$  \hspace{1cm} (6.77)

Inverse bosonization formula for this real fermion looks like

$$\sqrt{z}\psi(z) = \sum_{n \in \mathbb{Z}} \frac{\eta_n}{z^{\frac{n}{2}}} = \frac{\sigma_1}{\sqrt{2}} e^{i\phi_-} e^{i\phi_+}$$  \hspace{1cm} (6.78)

with the Pauli matrix $\sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$, and it is discussed in detail in Appendix (6.10.1).

**Characters for the twisted modules**

Now we turn directly to the computation of characters, using bosonization rules. In order to do this one has to apply the following heuristic “master formula” for the trace

$$\chi_g(q) = \operatorname{tr}_{\mathcal{H}_g} q^{L_0} = \frac{\chi_{ZM}(q)}{\prod_{k=1}^{\infty} \prod_{n=1}^{\infty} (1 - q^{\theta_{A_k,0}(g)+n})}$$  \hspace{1cm} (6.79)

over the space $\mathcal{H}_g$ which is the minimal space closed under the action of both W-algebra and twisted Kac-Moody algebra. For simply-laced cases, $\mathfrak{gl}(N)$ and $\mathfrak{so}(2N)$, $\mathcal{H}_g$ is the module of corresponding Kac-Moody algebra, whereas in the $\mathfrak{so}(2N+1)$ case it should be entire fermionic Fock module due to presence of the fermionic W-current. Explicit descriptions of $\mathcal{H}_g$ are the following: for $\mathfrak{gl}(N)$ it is the subspace with fixed total fermionic charge, for $\mathfrak{so}(2N)$ it is the subspace with fixed parity of total fermionic charge, and for $\mathfrak{so}(2N+1)$ it is entire space.
Denominator of (6.79) collects the contributions from the Fock descendants of twisted bosons (parameters $\theta_{\text{Adj},k}(g)$ are the eigenvalues of adjoint action of $g$ on the Cartan subalgebra), and the numerator – contribution of the zero modes. This formula is heuristic, moreover in some important cases we also get contribution from extra fermion, sometimes it is more informative to consider super-characters etc. Below we prove the following

**Theorem 6.2.** The characters of twisted representations are given by the formulas (6.85), (6.88), (6.95), (6.97).

**$\mathfrak{gl}(N)$ twist fields**

To be definite, let us fix an element $g = \prod_{j=1}^{K} [l_j, e^{2\pi i r_j}]$ from (6.45) which, according to (6.57) performs the permutation of fermions with simultaneous multiplication by $e^{\pm 2\pi i r_j}$. In this setting $N$ fermions can be bosonized in terms of $K$ twisted bosons (see detail in Appendix 6.10.3), and here we just present the final formulas

$$
\psi^*(\alpha)(z) = \frac{2^{1/4}}{\sqrt{l}} e^{i\phi_j(\alpha)} e^{(2\pi i a z)} e^{Q_j(\alpha)} e^{-Q_j(\alpha)} e^{J_0(j)} (-1)^{\sum \tilde{j}^k_{\leq j}}
$$

$$
\psi_\alpha(z) = \frac{2^{1/4}}{\sqrt{l}} e^{-i\phi_j(\alpha)} e^{(2\pi i a z)} e^{-Q_j(\alpha)} e^{Q_j(\alpha)} e^{-J_0(j)} (-1)^{\sum \tilde{j}^k_{\leq j}}
$$

(6.80)

for $\alpha \in \mathbb{Z}/l_j\mathbb{Z}$, labeling the fields within $[l_j]$-cycle. For the conformal dimension one gets therefore (see (6.262), and computation by alternative methods in (6.139), (6.191))

$$
L_0 = \sum_{j=1}^{K} \frac{l_j^2 - 1}{24l_j} + \sum_{j=1}^{K} \frac{1}{l_i} (J_0^{(j)})^2 + \ldots
$$

(6.81)

and since we are computing character of the space, obtained by the action of $\hat{\mathfrak{gl}}(N)_1$, we have to take into account all vacua arising after the action of the shift operators $e^{Q_j^{(n)} - Q_j^{(0)}}$, i.e. labeled by $A_{K-1}$ root lattice. Hence, the character (6.79) for this case is given by

$$
\chi_g(q) = q^{\sum_{j=1}^{K} \frac{r_j^2 - 1}{24l_j} \sum_{n_1 + \ldots + n_K = 0}^{\sum_{j=1}^{K} J_0^{(j)}} \frac{1}{l_i} (r_i n_i)^2} \prod_{j=1}^{K} \prod_{k=1}^{\infty} (1 - q^{k/l_j})
$$

(6.82)

In this formula the numerator collects contribution from the highest vectors $\chi_{ZM}$ (they differ by the value of zero modes $J_0^{(i)}$ of the Heisenberg algebras with generators $J_n^{(i)}$), whereas the denominator contains the contributions from the descendants.
so\((2N)\) twist fields, \(K' = 0\)

Consider now the twist fields (6.56) for \(g \in N_{O(2N)}(\mathfrak{h})\), and take first \(K' = 0\), so our twist has no minus-cycles

\[
g = \prod_{j=1}^{K} [l_j, e^{2\pi i r_j}]
\]

(6.83)

The only difference from the previous situation with the \(\mathfrak{gl}(N)\) case is that now one also has extra currents \(J_{\alpha\bar{\beta}} = \psi^\alpha(z)\psi^{\bar{\beta}}(z)\) and \(J_{\bar{\alpha}\beta} = \psi^\beta(z)\psi^{\bar{\alpha}}(z)\). It means that due to bosonization (6.266), (6.80) possible charge’s shifts now include \(e^{\pm (Q(i) + Q(j))}\), so the full lattice of the zero-mode charges (one zero mode for each cycle \([l_i, e^{2\pi i r_i}]\)) contains all points with

\[
\sum_{i=1}^{K} n_i \in 2\mathbb{Z}, \quad \{n_i\} \in \mathbb{Z}^K
\]

(6.84)

or is just the root lattice \(Q_{DK}\). After corresponding modification of numerator and the same contribution of the twisted Heisenberg algebra to denominator, the formula for the character in this case acquires the form

\[
\chi_g(q) = q^{\Delta_0 g} \prod_{j=1}^{K} \prod_{n=1}^{\infty} \left(1 - q^{n/l_j}\right)
\]

(6.85)

so\((2N)\) twist fields, \(K' > 0\)

Take

\[
g = \prod_{j=1}^{K} [l_j, e^{2\pi i r_j}]_+ \prod_{j=1}^{K'} [l'_j]_-
\]

(6.86)

Now we have extra cycles of type \([l'_j]_-\), so we have extra \(\eta\)-fermions that have to be bosonized in a different way (6.241):

\[
\eta_i(z) = z^{-\frac{1}{2}} \frac{e^{i\phi_i(z)}}{2\sqrt{l_i}} e^{i\phi_i'(z)} e^{\frac{1}{2}}(\sum_k J^{(k)}_i) \gamma_i
\]

(6.87)

where \(\{\gamma_i, \gamma_j\} = 2\delta_{ij}\) are gamma-matrices (or generators of the Clifford algebra \(Cl_{K'}(\mathbb{C})\)) in the smallest possible representation, which make different fermions anti-commuting. Due to presence of \(K'\) cycles of type \([l'_j]_-\), the zero-mode \(\chi_{ZM}(q)\) generating operators include now \(\gamma_j e^{Q(i)}\), which perform integer shifts of \(i\)-th bosonic zero mode together with inessential action on fermionic vacua – now we do not have to imply that the number of shifts by \(e^{Q(i)}\) should be even. Hence, instead of \(D_{K'}\)-lattice from (6.85) the numerator includes now summation over the root lattice \(Q_{BK}\), i.e.

\[
\chi_g(q) = q^{\Delta_0 g} \prod_{i=1}^{K} \prod_{k=1}^{\infty} (1 - q^{k/l_i}) \prod_{i=1}^{K'} \prod_{k=0}^{\infty} (1 - q^{(k+\frac{1}{2})/l'_i})
\]

(6.88)
where factor $2^{(K'+1)-1}$ corresponds to the dimension of the smallest representation of $\mathfrak{so}(K')$, generated by $\gamma_i \gamma_j$. Another simple factor $q^{\Delta_0}$ contains the minimal conformal dimension (without contribution of the “$r$-charges”)

$$\Delta_0 = \sum_{i=1}^{K} \frac{l_i^2 - 1}{24l_i} + \sum_{i=1}^{K'} \frac{2l_i^2 + 1}{48l_i}$$

(6.89)

which will be computed below in (6.142), (6.191). Numerator of (6.88) contains $K$ contributions from twisted bosons corresponding to plus-cycles, and $K'$ contributions from twisted Ramond bosons corresponding to minus-cycles.

### $\mathfrak{so}(2N+1)$ twist fields

The W-algebra $W(\mathfrak{so}(2N+1))$ contains fermionic operator $V(z) = \Psi_1(z) \ldots \Psi_{2N+1}(z)$, which cannot be expressed in terms of generators of $\mathfrak{so}(2N+1)$, since they are all even in fermions. It means that to construct a module of the W-algebra one should use entire fermionic algebra. Taking into account the fermionic nature of this W-algebra one can consider $\mathbb{Z}/2\mathbb{Z}$ graded modules and define two different characters

$$\chi^+(q) = \text{tr} q^{L_0}, \quad \chi^-(q) = \text{tr} (-1)^F q^{L_0}$$

(6.90)

where $F$ is the fermionic number:

$$(-1)^F U_k(z) = U_k(z)(-1)^F, \quad (-1)^F V(z) = -V(z)(-1)^F$$

(6.91)

One of the characters vanishes $\chi^-(q) = 0$ if at least one fermionic zero mode exists, since each state gets partner with opposite fermionic parity. Such fermionic zero modes are always present for the Ramond fermions and $\eta$-fermions, so the only case with non-trivial $\chi^-(q)$ corresponds to:

$$g = [1] \prod_{i=1}^{K} [l_i, e^{2\pi i r_i}]_+$$

(6.92)

In this case our computation works as follows: take bosonization for the $[l]_+$-cycles in terms of $K$ twisted bosons (6.266), (6.80), then the fermionic operators produce the zero-mode shifts $e^{\pm Q(i)}$ with the fermionic number $F = F^b + F^f = F^b = 1$, and the Heisenberg generators $J_n^{(i)}$, with the fermionic number $F = F^b = 0$. Moreover, we also have an extra “true” fermion $\Psi(z)$ with $F = F^f = 1$. Therefore the total trace can be computed, separating bosons and fermions, as

$$\chi^-(q) = \text{tr} q^{L_0} (-1)^F = \text{tr} q^{L_0} (-1)^{F^b} \cdot \text{tr} q^{L_0} (-1)^{F^f}$$

(6.93)

where the traces over bosonic and fermionic spaces are given by

$$\text{tr} q^{L_0} (-1)^{F^b} = \sum_{n_1, \ldots, n_K \in \mathbb{Z}} \frac{\sum_{i=1}^{K} \frac{1}{q^n} (n_i + l_i r_i)^2}{\prod_{i=1}^{K} \prod_{n=1}^{\infty} (1 - q^{n/l_i})} \quad (6.94)$$

$$\text{tr} q^{L_0} (-1)^{F^f} = \prod_{n=0}^{\infty} (1 - q^{n + \frac{1}{2}})$$

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Hence, the final answer for this character is given by

\[
\chi_g^{-}(q) = q^{\Delta_0^g} \left( \prod_{\vec{n} \in Q_{BK}} \frac{\sum_{i=1}^{K} \frac{1}{2} (n_i + l_i r_i)^2 - \sum_{\vec{n} \in Q'_{BK}} \sum_{i=1}^{K} \frac{1}{2} (n_i + l_i r_i)^2}{\prod_{k=0}^{K} (1 - q^{k + \frac{1}{2}})} \right) \prod_{i=1}^{K} \prod_{k=1}^{\infty} (1 - q^{k/l_i})
\]

(6.95)

where \(D\) and \(D'\)-lattices are defined in (6.205).

Let us now turn to the computation of \(\chi^+(q)\). Choose an element from \(N_{O(2N+1)}(\mathfrak{h})\)

\[
g = [(-1)^{a+1}] \prod_{i=1}^{K} [l_i, e^{2\pi i r_i}] + \prod_{i=1}^{K'} [l'_i]
\]

(6.96)

where \(a = 0, 1\). The bosonized fermions \(e^{i\varphi(z)}\) contain elements of \(e^{Q^{(i)}}\), generating the \(B_K\) root lattice, which together with contribution of the fermionic and Heisenberg modes finally give

\[
\chi_g^+(q) = q^{\Delta_0^g} \frac{2^{K'} |K'| \prod_{\vec{n} \in Q_{BK}} \sum_{i=1}^{K} \frac{1}{2} (n_i + l_i r_i)^2 \prod_{k=0}^{\infty} (1 + q^{k + \frac{1}{2}})}{\prod_{i=1}^{K} \prod_{k=1}^{\infty} (1 - q^{k/l_i}) \prod_{i=1}^{K'} \prod_{k=0}^{\infty} (1 - q^{(k + \frac{1}{2})/l'_i})}
\]

(6.97)

where

\[
\Delta_0^g = \frac{\delta_a, 0}{16} + \sum_{i=1}^{K} \frac{l_i^2 - 1}{24l_i} + \sum_{i=1}^{K'} \frac{2l'_i^2 + 1}{48l'_i}
\]

(6.98)

Here the only new part, compare to the \(D_N\)-case, is extra factor corresponding to \((R\) or \(NS))

\[
\chi_f(q) = q^{\frac{\delta_a, 0}{16}} \prod_{k=0}^{\infty} (1 + q^{2 + k})
\]

(6.99)

fermionic contribution.

**Character identities**

In sect. 6.4 we have classified the twist fields by conjugacy classes in \(N_G(\mathfrak{h})\). However it is possible, that two different elements \(g_1, g_2 \in N_G(\mathfrak{h})\) in the normalizer of Cartan are nevertheless conjugated in the group \(G\). Such twisted representations are isomorphic, and it gives an obvious

**Theorem 6.3.** If \(g_1 \sim g_2\) in \(G\) for different \(g_1, g_2 \in N_G(\mathfrak{h})\), then \(\chi_{g_1}(q) = \chi_{g_2}(q)\).

This leads sometimes to non-trivial identities and product formulas for the lattice theta-functions, and below we examine such examples.
6.5. Characters for the twisted modules

**$\mathfrak{gl}(N)$ case.** Here any element is conjugated to a product of cycles of length one:

$$[l, e^{2\pi ir}] \sim \prod_{j=0}^{l-1} [1, e^{2\pi iv_j}],$$  \hspace{1cm} (6.100)

where $v_j = r + \frac{1-l+2j}{2l}$. One gets therefore an identity

$$\sum_{k_1+\ldots+k_N=0} q^{\frac{1}{2}\sum_{i=1}^{N}(v_i+k_i)^2} \eta(q)^N \prod_{i=1}^{K} \eta(q^{1/l_i}) = \sum_{n_1+\ldots+n_K=0} \prod_{i=1}^{K} \eta(q^{1/l_i})$$  \hspace{1cm} (6.101)

where all conformal dimensions for vanishing $r$-charges are conveniently absorbed by the Dedekind eta-functions $\eta(q) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$.

This equality of characters can be checked by direct computation, see (6.221) in Appendix 6.9 for $S = \{0\}$. For a single cycle $K = 1$ this gives a product formula for the lattice $A_{N-1}$-theta function (6.220), which for $N = 2$

$$\frac{q^{1/4}}{\prod_{k\geq0}(1-q^{k+1/2})} = \frac{\sum_{n\in\mathbb{Z}} q^{(n+1/4)^2}}{\prod_{n>0}(1-q^n)}$$  \hspace{1cm} (6.102)

was known yet to Gauss and has been originally used by Al. Zamolodchikov in the context of twist-field representations of the Virasoro algebra.

**$\mathfrak{so}(2N)$ case.** For the conjugacy classes of the first type we have again (6.100), or

$$[l, e^{2\pi iv}]_+ \sim \prod_{j=0}^{l-1} [1, e^{2\pi iv_j}]_+$$  \hspace{1cm} (6.103)

which leads to very similar identities to the $\mathfrak{gl}(N)$-case. For example, one can easily rederive the product formula [Mac] for the $D$-lattice theta function

$$\sum_{\vec{n} \in Q_{D_N}} q^{\frac{1}{2}(\vec{n}+\vec{\rho})^2} = \Theta_{D_N}(\vec{v}|q) = \frac{\eta(q)^{N+1} \eta(q^{1/(N-1)})}{\eta(q^{1/2}) \eta(q^{1/(2(N-1))})}$$  \hspace{1cm} (6.104)

for $\vec{v} = \vec{\rho}$, where the structure of product in the r.h.s. again comes from the characteristic polynomial of the Coxeter element of the Weyl group $W(D_N)$. Here $h = 2(N-1)$ is the Coxeter number, and $\vec{\rho} = (N-1, N-2, \ldots, 1, 0)$ is the Weyl vector, corresponding to the twist field with dimension $\Delta = \Delta^0 = \frac{N(2N-1)}{48(N-1)}$, and the easiest way to derive (6.104) is to use (6.223) from Appendix 6.9.

For another type of the conjugacy classes $[l]_-$, the situation is more tricky. The corresponding $\eta$-fermion

$$\eta(z) = z^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} \frac{\eta_k}{z^{k}}$$  \hspace{1cm} (6.105)
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can be separated into the parts with fixed monodromies around zero:

\[ \eta_{(a)} = z^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} \frac{\eta_{a+2l}k}{z^{\frac{a^2}{2}+k}}, \quad (6.106) \]

so that the only non-trivial OPE is between \( \eta_{(a)} \) and \( \eta_{(2l-a)} \). In particular, \( \eta_{(0)} \) and \( \eta_{(l)} \) are self-conjugated Ramond (R) and Neveu-Schwarz (NS) fermions, which can be combined into new \( \bar{\eta} \) fermion, whereas all other components can be considered as charged twisted fermions \( \bar{\psi}, \bar{\psi}^\ast \):

\[
\bar{\psi}_{(a)}(z) = \eta_{(a)}(z), \quad \bar{\psi}^\ast(z) = \eta_{(2l-a)}(z), \quad a = 1, \ldots, l-1
\]

\[ \bar{\eta}(z) = \eta_{(0)}(z) + \eta_{(l)}(z) \quad (6.107) \]

Therefore one gets equivalence

\[ [l] \sim [1] \cdot \prod_{j=1}^{l-1} [1, e^{2\pi iv_j}], \quad (6.108) \]

where \( v_j = \frac{j}{2} \).

Moreover, if we take the product of two cycles \([1] \), then we can combine a pair of R-fermions and a pair of NS-fermions into two complex fermions with charges 0 and \( \frac{1}{2} \), therefore

\[ [1] \sim [1, 1]_+ [1, -1]_+ \quad (6.109) \]

This means literally that pair of \( \eta \)-fermions is equivalent to two charged bosons: one with charge \( v = 0 \) and another one with charge \( v = \frac{1}{2} \). Equivalence between these two representations leads to the simple identity (6.247), (6.248):

\[
\frac{2q^{\frac{1}{2}}}{\prod_{n=1}^{\infty} (1 - q^{n+\frac{1}{2}})^2} = \sum_{k,n \in \mathbb{Z}} q^{\frac{1}{2}n^2 + \frac{1}{2}(k+\frac{1}{2})^2} \prod_{n=1}^{\infty} (1 - q^n)^2 \quad (6.110)
\]

Using this identity we can remove a pair of \([1] \) cycles from (6.88) shifting \( K' \rightarrow K' - 2 \), and add two more directions to the lattice of charges \( B_K \rightarrow B_{K+2} \) with corresponding \( r \)-charges 0 and \( \frac{1}{2} \).

**so(2N) case, \( K' = 0 \).** We have the consequence of identity (6.221) for the case \( S = 2\mathbb{Z} \):

\[
\sum_{k \in Q_{DN}} q^{\frac{1}{2} \sum_{i=1}^{K} (n_i + k_i)^2} = \prod_{i=1}^{K} \frac{\eta(q)}{\eta(q^{r_i})} \cdot \sum_{\bar{n} \in Q_{DK}} \frac{K}{\bar{n}} q^{\frac{1}{2} \sum_{i=1}^{K} (n_i + l_i + \bar{n}_i)^2} \quad (6.111)
\]

**so(2N) case, \( K' > 0 \); so(2N + 1), \( K' > 0 \).** In these cases everything can be expressed in factorized form using (6.223) and checked explicitly, so these cases are not very interesting.
6.5. Characters for the twisted modules

\textbf{\textit{so}(2N + 1) case, NS fermion.} Here in addition to all identities that we had in the \textit{so}(2N) case, we have two more identities that appear because of the fact that we can combine \textit{NS} (or \textit{R}) fermion with a pair of \textit{NS}, \textit{R} fermions to get one complex fermion with twist 0 (or twist $\frac{1}{2}$) and one \textit{R}-fermion (or \textit{NS}-fermion). Thus

\begin{align*}
[1] \cdot [1]_+ & \sim [-1] \cdot [1, 1]_+ \\
[-1] \cdot [1]_- & \sim [1] \cdot [1, -1]_+
\end{align*}

(6.112)

Thanks to these identities in the cases $K' \neq 0$ we can transform any character with \textit{NS} fermion to a character with \textit{R} fermion, and vice versa.

\textbf{Twist representations and modules of W-algebras}

By definition, all our twisted representations are representations of the \textit{W}-algebra. As was explained in previous section it is sufficient to consider the case $g \in H$ (other elements of $N_G(h)$ are conjugated to $H$). In this case subspaces of $\mathcal{H}_g$ with all fixed fermion charges become representations of \textit{W}-algebra\footnote{This is a common well-known procedure, see e.g. [MMMO] and references therein.}. The $r$-charges of the corresponding representations are given by shifts of the vector $\vec{r} = \frac{\log g}{2\pi}$ by root lattice of $g$.

The explicit formulas are given below, but we want first to comment the irreducibility of representations. The Verma modules of \textit{W}-algebras are irreducible if

\[(\alpha, r) \notin \mathbb{Z},\]

(6.113)

see [FKW], [Arakawa] (in particular Theorem 6.6.1) or [FL] (eq (4.4)). For generic $r$ this condition is satisfied and all modules arising in the decomposition (subspaces of $\mathcal{H}_g$ with all fixed fermion charges) are Verma modules due to coincidence of the characters.

If $g$ comes from the element of $N_G(h)$ with nontrivial cyclic structure then $r$ is not necessarily generic. For \textit{gl}(N) case, as follows (6.100), the $r$-charges corresponding to a single cycle do satisfy (6.113), and for different cycles this condition also holds provided $r$ are generic (no relations between $r$ from different cycles). The same argument works for \textit{so}(N) with “plus-cycles”, but if we have at least two “minus-cycles” the corresponding $r$-charges can violate condition (6.113), and not only Verma modules arise in the decomposition over irreducible representations.

In any case we have an identity of characters

\[\chi_g(q) = \chi_0(q)\hat{\chi}_g(q)\]

(6.114)

where $\chi_0(q)$ is the character of Verma module, and $\hat{\chi}_g(q)$ is the character of the space of highest vectors. Hence, there is a non-trivial statement, that all coefficients of the power expansion of the ratios $\chi_g(q)/\chi_0(q)$ are positive integers, which can be proven using identities, derived in previous section.

The list of characters of the Verma modules, appeared above is:
6. Twist-field representations of $W$-algebras, exact conformal blocks and character identities

- **$\mathfrak{gl}(N)$, $\mathfrak{so}(2N)$ (NS sector).** Algebra is generated by $N$ bosonic currents, each of them producing \( \frac{1}{\prod_{n>0} (1-q^n)} \), so the character is

\[
\chi_0(q) = \frac{1}{\prod_{n=1}^{\infty} (1-q^n)^N} \tag{6.115}
\]

- **$\mathfrak{so}(2N)$ (R sector).** One of these currents, $V(z)$, becomes Ramond, with half-integer modes:

\[
\chi_0(q) = \frac{1}{\prod_{n=1}^{\infty} (1-q^n)^{N-1} \prod_{n=0}^{\infty} (1-q^{\frac{1}{2}+n})} \tag{6.116}
\]

- **$\mathfrak{so}(2N+1)$ (NS sector).** One current, $V(z)$ becomes Neveu-Schwarz fermion, so taking into account its parity we get

\[
\chi_0^\pm(q) = \prod_{n=0}^{\infty} (1 \pm q^{\frac{1}{2}+n}) \prod_{n=1}^{\infty} (1-q^n)^N \tag{6.117}
\]

- **$\mathfrak{so}(2N+1)$ (R sector).** In the case of Ramond fermion $V(z)$ character $\chi_0^-(q)$ vanishes because fermionic zero mode produces equal numbers of states with opposite fermionic parities:

\[
\chi_0^+(q) = 2 \prod_{n=1}^{\infty} (1+q^n) \prod_{n=1}^{\infty} (1-q^n)^N
\]

\[
\chi_0^-(q) = 0 \tag{6.118}
\]

**$\mathfrak{gl}(N)$ case.** Any element is conjugated to a product of cycles of length 1, so

\[
\hat{\chi}_g(q) = q^{\Delta_g^0} \sum_{\vec{\bar{n}} \in A_{N-1}} q^{\frac{1}{2}(v+\vec{\bar{n}})^2} \tag{6.119}
\]

**$\mathfrak{so}(2N)$ case, $K' = 0$.** Any element is conjugated to $\prod_{j=1}^{N} [1, e^{2\pi v_j}]_+$, so

\[
\hat{\chi}_g(q) = q^{\Delta_g^0} \sum_{\vec{n} \in D_N} q^{\frac{1}{2}(v+\vec{n})^2} \tag{6.120}
\]

**$\mathfrak{so}(2N)$ case, $K' > 0$, NS-sector.** Again, any element is conjugated to $\prod_{j=1}^{N} [1, e^{2\pi v_j}]_+$, so

\[
\hat{\chi}_g(q) = 2^{K'-1} q^{\frac{N}{2}} \sum_{\vec{n} \in B_N} q^{\frac{1}{2}(v+\vec{n})^2} \tag{6.121}
\]

**$\mathfrak{so}(2N)$ case, R-sector.** Here any element is conjugated to $[1]_- \prod_{j=1}^{N-1} [1, e^{2\pi v_j}]_+$, so

\[
\hat{\chi}_g(q) = 2^{K'} q^{\frac{N}{2}} \sum_{\vec{n} \in B_{N-1}} q^{\frac{1}{2}(v+\vec{n})^2} \tag{6.122}
\]

because contribution from the cycle $[1]_-$ to the denominator cancels contribution from the Ramond boson $V(z)$. 

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6.6. Characters from lattice algebras constructions

\textbf{so(2N + 1) case, }K’ = 0, \textbf{ NS fermion.} Here one has two non-trivial characters

\[
\hat{\chi}_g^+(q) = q^{\Delta_g^0} \sum_{\vec{n} \in B_N} q^{\frac{1}{2}(v + \vec{n})^2}
\]
\[
\hat{\chi}_g^-(q) = q^{\Delta_g^0} \left( \sum_{\vec{n} \in D_N} q^{\frac{1}{2}(v + \vec{n})^2} - \sum_{\vec{n} \in D_N'} q^{\frac{1}{2}(v + \vec{n})^2} \right)
\]  

\textbf{so(2N + 1) case, }K’ > 0 \textbf{ This case gives nothing interesting as compared to }D_N \textbf{ situation.}

\[
\hat{\chi}_g^+(q) = 2^{(K' - 1)/2} q^{\Delta_g^0} \sum_{\vec{n} \in B_N} q^{\frac{1}{2}(v + \vec{n})^2}
\]
\[
\hat{\chi}_g^-(q) = 0
\]  

\textbf{Characters from lattice algebras constructions}

\textbf{Twisted representation of }\hat{\mathfrak{g}}_1

Now we reformulate the results of previous sections using the notion of \textit{twisted representations of vertex algebras}. Recall the corresponding setting (following, for example, [BK]). Let \( V \) be a vertex algebra (equivalently vacuum representation of the vertex algebra), and \( \sigma \) be an automorphism of \( V \) of finite order \( l \). Then \( V = \oplus V_k \), where \( V_k = \{ v \in V | \sigma v = \exp(2\pi ik/l)v \} \). The \( \sigma \)-twisted module is a vector-space \( M \) endowed with a linear map from \( V \) to the space of currents

\[ v \mapsto A_v(z) = \sum_{m \in \frac{1}{k} \mathbb{Z}} a_m(v) z^{-m-1}, \quad v \in V, \quad a_m(v) \in \text{End}(M). \]  

(6.125)

Such correspondence should be \( \sigma \)-equivariant, namely

\[ A_{\sigma v}(z) = A_v(e^{2\pi i} z) \]  

(6.126)

giving the boundary conditions for the currents, and agree with the vacuum vector and relations in \( V \). In particular, it follows from the \( \sigma \)-equivariancy (6.126), that if \( v \in V_k \) then \( A_v(z) \in z^{-k/l} \mathbb{C}[z, z^{-1}] \).

Consider now a Lie group \( G \) (either \( GL(N) \) or \( SO(2N), N \geq 2 \)), with \( \mathfrak{g} = \text{Lie}(G) \) being the corresponding Lie algebra. Denote by \( V(\mathfrak{g}) \) the irreducible vacuum representation of \( \hat{\mathfrak{g}} \) of the level one. This space has a structure of the vertex algebra i.e. for any \( v \in V(\mathfrak{g}) \) one can assign the current \( A_v(z) \), this space of currents is generated by the currents \( J_{\alpha \beta}(z) \) from sect. 6.3.

The vertex algebra \( V(\mathfrak{g}) \) is a lattice vertex algebra. Let \( Q_\mathfrak{g} \) denote the root lattice of \( \mathfrak{g} \), and introduce rank of \( \mathfrak{g} \) bosonic fields with the OPE \( \varphi_i(z) \varphi_j(w) = -\delta_{ij} \log(z - w) + \text{reg} \), and the stress-energy tensor \( T(z) = -\frac{1}{2} \sum_j : \partial \varphi_j(z)^2 : \); then the currents of \( V(\mathfrak{g}) \) can be presented in the bosonized form

\[ : \prod_{i,m} \partial^{\alpha_i, m} \varphi_i \exp(i \sum \alpha_i \varphi_i(z)) : \]  

(6.127)
where $\alpha = (\alpha_1, \ldots, \alpha_n) \in Q_\theta$ and $a_{i,m}$ are any positive integers, while the stress-energy tensor corresponding to standard conformal vector $\frac{1}{2} \sum J_{j,1}^2 |0\rangle = \tau \in V(\mathfrak{g})$ (here $J_{j,n}$ are modes of the field $i\partial \varphi_j(z)$). The group $G$ acts on $V(\mathfrak{g})$, and in order to use lattice algebra description we consider only the subgroup $N_G(\mathfrak{h}) \subset G$ which preserves the Cartan subalgebra.

In [BK] the representations of the lattice vertex algebra, twisted by automorphisms, arise from isometries of the lattice $Q_\theta$. Here we restrict ourself to the isometries provided by action of the Weyl group $W$ (this case was actually considered in [KP] without language of twisted representations). Let $s \in W$ be an element of the Weyl group, by $g$ we denote its lifting to $G$, in other words $g \in N_G(\mathfrak{h})$ such that adjoint action $g$ on $\mathfrak{h}$ coincides with $s$. We consider representation twisted by such $g$. Setting of [BK] and [KP] works for special $g$, for example such $g$ should have finite order, but we will expand this to the generic $g \in N_G(\mathfrak{h})$. Clearly, the conformal vector $\tau$ is invariant under the adjoint action of $N_G(\mathfrak{h})$.

The $g$-twisted representations of $V(\mathfrak{g})$ in [BK] are defined as a direct sum of twisted representations of $\hat{\mathfrak{h}}$. By $\{e^{2\pi i \theta_{\text{adj},k}}\}$ we denote eigenvalues of $s$, or of the adjoint action $g_{\text{adj}}$ on $\mathfrak{h}$, we set $-1 < \theta_{\text{adj},k} \leq 0$, by $\{J_k \in \mathfrak{h}\}$ - the corresponding eigenvectors, and define the currents

$$J_k(z) = \sum_{n \in \mathbb{Z}} J_{k,\theta_{\text{adj},k} + n} z^{-\theta_{\text{adj},k} - n - 1} \quad (6.128)$$

A $g$-twisted representations of the Heisenberg algebra $\hat{\mathfrak{h}}$ is a Fock module $F_\mu$ with the highest weight vector $v_\mu$

$$J_{k,\theta_{\text{adj},k} + n} v_\mu = 0, \quad n > 0, \quad J_{k,0} v_\mu = \mu(J_k) v_\mu, \quad \text{if} \quad \theta_{\text{adj},k} = 0. \quad (6.129)$$

generated by creation operators $J_{k,\theta_{\text{adj},k} + n}$, $n \leq 0$. Here $\mu \in \mathfrak{h}_0^*$, where $\mathfrak{h}_0$ is $g_{\text{adj}}$-invariant subspace of $\mathfrak{h}$.

It has been proven in [BK] that twisted representations of $V(\mathfrak{g})$ have the structure

$$M(s, \mu_0) = \bigoplus_{\mu \in \mathfrak{h}_0 + \pi_s Q_\theta} F_\mu \otimes \mathbb{C}^{d(s)} \quad (6.130)$$

for certain finite set of $\mu_0 \in \mathfrak{h}_0^*$. Here $\pi_s$ denotes projection from $\mathfrak{h}^*$ to $\mathfrak{h}_0^*$, corresponding to the element $s \in W$ for the chosen adjoint action $g_{\text{adj}}$. For any root $\alpha$ the corresponding current $J_\alpha(z)$ acts from $F_\mu$ to $F_{\mu + \pi_s \alpha}$ and equals to the linear combination of vertex operators. Number $d(s)$ denotes the defect of the element $s \in W$, its square is defined by

$$d(s)^2 = |(Q_\theta \cap \mathfrak{h}_0^*)/(1 - s) P_\theta|. \quad (6.131)$$

Here $P_\theta$ denotes weight lattice of $\mathfrak{g}$, $\mathfrak{h}_0^*$ denotes the space of linear functions vanishing on $\mathfrak{h}_0$, $| \cdot |$ stands for number of elements in the group. It can be proven that for any $s$ the numbers $d(s)$ is integer. In our case ($GL(N)$ and $SO(n)$ groups) this number always equals to some power of 2.

Formula (6.130) allows to calculate the character of module $M$ i.e. the trace of $q^{L_0}$. First, notice that the character of the Fock module $F_\mu$ equals

$$\chi_\mu(q) = \frac{q^{\Delta_\mu}}{\prod_{n=1}^\infty (1 - q^{\theta_{\text{adj},k(n)}})} \quad (6.132)$$
where $\Delta_\mu$ is an eigenvalue of $L_0$ on the vector $v_\mu$. The value of $\Delta_\mu$ consists of two contributions. The first comes from the terms with $\theta_{\text{Adj}} = 0$, and, as follows from (6.129), is equal to $\frac{1}{2}(\mu, \mu)$. The second contribution comes from the normal ordering. The vectors $J_k \in \mathfrak{h}$, corresponding to $\theta_{\text{Adj},k} \neq 0$ can be always arranged into orthogonal pairs $(J_1, J_1')$, $(J_2, J_2')$, ... with complementary eigenvalues $\theta_{\text{Adj},k} + \theta_{\text{Adj},k'} = -1$. After normal ordering of the corresponding currents one gets

$$J_k(z)J_{k'}(w) = \sum_{n,m \in \mathbb{Z}} \frac{J_k,z^{n+\theta}}{w^{m-\theta+1}} + \sum_{n,m \in \mathbb{Z}, m < 0} \frac{J_{k'}(z)}{w^{m-\theta+1} z^{n+\theta+1}} \qquad (6.133)$$

$$\sum_{n>0} (n + \theta) \frac{w^{n+\theta-1}}{z^{n+\theta+1}} = \frac{(1 + \theta) w^\theta z^{-\theta} + (-\theta) w^{1+\theta} z^{-1-\theta}}{(z-w)^2} = \frac{1}{(z-w)^2} - \frac{\theta(1+\theta)}{2w^2} + \text{reg} \quad \qquad (6.134)$$

Altogether one gets

$$\Delta_\mu = \frac{1}{2}(\mu, \mu) - \sum_k \frac{\theta_{\text{Adj},k}(1 + \theta_{\text{Adj},k})}{4} \quad (6.135)$$

and therefore, finally for the character of (6.130)

$$\text{Tr} q^{L_0} \bigg|_{M(s,\mu_0)} = q^{-\frac{1}{4} \sum_k \theta_{\text{Adj},k}(1 + \theta_{\text{Adj},k})} \frac{d(s) \sum_{\mu_0 + \mu} \pi_w Q^{\frac{1}{2}(\mu, \mu)}}{\prod_{i=1}^N \prod_{n=1}^\infty (1 - q^{\theta_{\text{Adj},i} + n})} \quad (6.136)$$

Recall that in the the initial weight $\mu_0$ in the setting of [BK] should belong to the finite set in $\mathfrak{h}_0^*$ (or $\mathfrak{h}_0^*/\mathfrak{h}Q$). But we will generalize such representations and take any $\mu_0 \in \mathfrak{h}_0^*$. This can be viewed as a twisting by more general elements $g \in N_G(\mathfrak{h})$, which can have infinite order. Actually the corresponding elements are representatives of the conjugacy classes of $N_G(\mathfrak{h})$ used in sect. 6.4.

### Calculation of characters

#### $GL(N)$ case

The root lattice $Q_{g\mathfrak{h}(N)} = Q_{A_{N-1}}$ is generated by vectors $\{e_i - e_j\}$, where $\{e_1, \ldots, e_N\}$ denote the vectors of orthonormal basis in $\mathbb{R}^N$. Assume that $s \in \mathfrak{W}$ is product of disjoint cycles of lengths $l_1, \ldots, l_K$, then without loss of generality the action of such elements can be defined as $(e_1 \mapsto e_2 \mapsto \ldots \mapsto e_{l_1} \mapsto e_1), (e_{l_1+1} \mapsto e_{l_1+2} \mapsto \ldots \mapsto e_{l_1+l_2} \mapsto e_{l_1+l_2+1}), \ldots$

In this case $\mathfrak{h}_0^*$ (the $s$-invariant part of $\mathfrak{h}^*$) is generated by the vectors

$$f_1 = e_1 + \ldots + e_{l_1}, \quad f_2 = e_{l_1+1} + \ldots + e_{l_1+l_2}, \quad \ldots \quad (6.137)$$

7There is also “degenerate” case $J_k = J_k'$ for $\theta_{\text{Adj},k} = \theta_{\text{Adj},k'} = -1/2$. 

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while \( \pi_s Q_{\mathfrak{gl}(N)} \) is generated by vectors \( \frac{1}{l_i} f_i - \frac{1}{l_j} f_j \), so one can present any element of \( \pi_s Q_{\mathfrak{gl}(N)} \) as \( \sum \frac{1}{l_i} n_j f_j \) with \( \sum n_j = 0 \) and identify with that from \( Q_{\mathfrak{gl}(K)} \). Let \( \mu_0 = \sum_j r_j f_j \). Then the formula (6.136) takes here the form

\[
\text{Tr}(q^{L_0})\big|_{M(s,\mu_0)} = q^{\Delta^0} \sum_{j=1}^K q^{\frac{1}{l_j} (n_j + l_j r_j)^2} \prod_{i=1}^\infty (1 - q^{n_i/l_i}),
\]

(6.138)

where, since for any length \( l \) cycle \( \theta_{l,j,k} = -k/l \),

\[
\Delta^0 = \sum_{j=1}^K \sum_{i=1}^{l_j} \frac{i(l_j - i)}{4l_j^2} = \sum_{j=1}^K \frac{l_j^2 - 1}{24l_j}
\]

(6.139)

This formula coincides with (6.82), and the reason is that the corresponding element from \( N_{GL(N)}(\mathfrak{h}) \) is exactly (6.45), \( g = \prod_{j=1}^K [l_j, e^{2\pi i r_j}] \). Indeed, let \( \alpha = e_a - e_b \), where \( a \) belongs to the cycle \( j \) and \( b \) belongs to the cycle \( j' \) then the current \( J_\alpha(z) \) shifts \( L_0 \) grading by \( r_j - r_{j'} + \lfloor \text{rational number with denominator } l_j, l_{j'} \rfloor \).

\( SO(2N) \) case The root lattice \( Q_{so(2N)} = Q_{D_N} \) is generated by the vectors \( \{e_i - e_j, e_i + e_j\} \), where again \( e_1, \ldots, e_N \) denote the basis in \( \mathbb{R}^N \). As we already discussed in sect. 6.4, there are two types of the Weyl group elements, the first type just permutes \( e_i \), while the second type permutes \( e_i \) together with the sign changes.

The first case almost repeats the previous paragraph, without loss of generality we assume that the Weyl group element acts as \( (e_1 \mapsto e_2 \mapsto \ldots \mapsto e_1 \mapsto e_1), (e_{i+1} \mapsto e_{i+2} \mapsto \ldots \mapsto e_{i+2} \mapsto e_{i+1}), \ldots \), where \( l_1, \ldots, l_K \) are the lengths of the cycles. The \( s \)-invariant part of \( h_0^s \) is generated by the same “averaged” vectors (6.137), while \( \pi_s Q_{D_N} \) is generated by the vectors \( \frac{1}{l_i} f_i - \frac{1}{l_j} f_j, \frac{1}{l_i} f_i + \frac{1}{l_j} f_j \). In other words \( \pi_s Q_{D_N} \) consist of vectors \( \sum n_j f_j \), where \( (n_1, \ldots, n_k) \in Q_{so(2K)} \). Let \( \mu_0 = \sum j r_j f_j \), then the character formula (6.136) for this case acquires the form

\[
\text{Tr}(q^{L_0})\big|_{M(s,\mu_0)} = q^{\Delta^0} \sum_{j=1}^K q^{\frac{1}{l_j} (n_j + l_j r_j)^2} \prod_{i=1}^\infty (1 - q^{n_i/l_i}),
\]

(6.140)

and coincides with (6.85). Here \( \Delta^0 \) is defined in (6.139). The corresponding element from \( N_{SO(2N)}(\mathfrak{h}) \) has the form \( \prod_{j=1}^K [l_j, e^{2\pi i r_j}]_+ \) in the notations of sect. 6.4 (see (6.56)).

For the second type (the corresponding element from \( N_{SO(2N)}(\mathfrak{h}) \) has the form \( \prod_{j=1}^K [l_j, e^{2\pi i r_j}]_+ \cdot \prod_{j'=1}^{K'} [l_{j'}, e^{-2\pi i r_{j'}}]_- \) one can present the Weyl group element as product of \( K \) disjoint cycles of lengths \( l_1, \ldots, l_K \) which just permutes \( e_i \), and \( K' \) cycles of lengths \( l_1', \ldots, l_{K'} \) which permutes \( e_i \) with signs, see (6.56). Now, without loss of generality, we assume that \( s \) acts as \( (e_1 \mapsto e_2 \mapsto \ldots \mapsto e_1 \mapsto e_1), (e_{i+1} \mapsto e_{i+2} \mapsto \ldots \mapsto e_{i+2} \mapsto e_{i+1}), \ldots \mapsto e_{L+i}, \ldots \mapsto e_{L+i'}, \ldots \mapsto e_{L+i'), \ldots \mapsto e_{L+i'+l'} \mapsto e_{L+i'+l'}, \ldots \), where \( L = l_1 + \ldots + l_K \). The \( s \)-invariant part of \( h_0^s \) is generated by the same vectors (6.137), while \( \pi_s Q_{D_N} \) is generated by the vectors \( \frac{1}{l_i} f_i \). One can say that \( \pi_s Q_{D_N} \) consists of the vectors \( \sum n_j f_j \), where \( (n_1, \ldots, n_k) \in \ldots \).
6.6. Characters from lattice algebras constructions

\[ Q_{so(2K+1)} = Q_{B_N}, \]  
so that for the character formula one gets

\[ \text{Tr}(q^{L_0})|_{M(s,\mu_0)} = q^{\Delta_0} \frac{2^{K'/2 - 1}}{\prod_{j=1}^{K} \prod_{n=1}^{\infty} (1 - q^{n/l_j})} \prod_{j=1}^{K'} \prod_{n=1}^{\infty} (1 - q^{2n-1}/2l_j'), \tag{6.141} \]

where, since in addition to \([l]_+\)-cycles with \(\theta_{\text{Adj},k} = -k/l\) one now has \([l']_+\)-cycles with \(\theta'_{\text{Adj},k} = -(k - \frac{1}{2})/l'\),

\[ \Delta_0^s = \sum_{j=1}^{K} \sum_{i=1}^{l_j} \frac{i(l_j - i)}{4l_j^2} + \sum_{j=1}^{K'} \sum_{i=1}^{l_j'} \frac{(2i - 1)(2l_j' - 2i + 1)}{16l_j'^2} = \sum_{j=1}^{K} \frac{l_j^2 - 1}{24l_j} + \sum_{j=1}^{K'} \frac{2l_j'^2 + 1}{48l_j'} \tag{6.142} \]

This formula coincides with (6.88). The number \(2^{K'/2 - 1}\) equals to \(d(\sigma)\), this is the first case where this number is nontrivial. Note, that we consider here only internal automorphisms, i.e. \(K'\) is even.

Recall also (see sect. 6.5.5) that if \(g, g' \in N_G(\mathfrak{h})\) are conjugate in \(G\) then corresponding characters \(\text{Tr}(q^{L_0})|_{M(s,\mu_0)}\) and \(\text{Tr}(q^{L_0})|_{M(s',\mu_0')}\) are equal.

Characters from principal specialization of the Weyl-Kac formula

Fix element \(g \in G\) of finite order \(l\). The \(g\)-twisted representations of \(V(\mathfrak{g})\) are representations if the affine Lie algebra twisted by \(g\). Recall that these twisted affine Lie algebras \(\widehat{\mathcal{L}}(\mathfrak{g}, g)\) are defined in [KacBook, Sec 8] as \(g\) invariant part of \(\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}k\) where \(g\) acts as

\[ g(t^l \otimes J) = e^{-i}t^l \otimes (gJg^{-1}), \quad \text{where } \epsilon = \exp(2\pi i/l), \quad g(k) = k. \tag{6.143} \]

By definition \(g\) is an internal automorphism, therefore the algebra \(\widehat{\mathcal{L}}(\mathfrak{g}, g)\) is isomorphic to \(\widehat{\mathfrak{g}}\) (see Theorem [KacBook, 8.5]), though natural homogeneous grading on \(\widehat{\mathcal{L}}(\mathfrak{g}, g)\) differs from the homogeneous grading on \(\widehat{\mathfrak{g}}\).

Therefore the \(g\)-twisted representations of \(V(\mathfrak{g})\) as a vector spaces are integrable representations of \(\widehat{\mathfrak{g}}\). Their characters can be computed using the Weyl-Kac character formula. This formula has simplest form in the principal specialization, i.e. computed on the element \(q^{\rho'} \in \widehat{G}\). Here \(\rho' \in \mathfrak{h} \oplus \mathbb{C}k\) such that \(\alpha_i(\rho') = 1\), for all affine simple roots \(\alpha_i\) (including \(\alpha_0\)). Then the character of integrable highest weight module with the highest weight \(\Lambda\) equals (see [KacBook, eq. (10.9.4)])

\[ \text{Tr}(q^{\rho'/h})|_{L_\Lambda} = q^{\Lambda(\rho')/h} \prod_{\alpha^\vee \in \Delta_+^\vee} \left( \frac{1 - q^{(\Lambda + \rho, \alpha^\vee)/h}}{1 - q^{(\rho, \alpha^\vee)/h}} \right)^{\text{mult}(\alpha^\vee)}, \tag{6.144} \]

\(^8\)Note that we get only level 1 integrable representation of \(\widehat{\mathfrak{g}}\) since \(V(\mathfrak{g})\) was defined above as a lattice vertex algebra i.e. vacuum representation of the level \(k = 1\)
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where $\Delta^\vee$ is the set of all positive (affine) coroots. Here $h$ is the Coxeter number, it will be convenient to use $q^{\rho/\hbar}$ instead of $q^{\rho/\hbar}$. The weight $\rho$ is defined by $(\rho, \alpha^\vee) = 1$ for all simple coroots $\alpha^\vee$ (including affine root $\alpha_0$).

The grading above in this section was the $L_0$ grading and it was obtained using the twist by the element $g \in N_q \mathfrak{h}$. Now we take certain $g$ such that $g$-twisted $L_0$ grading coincides with principal grading in (6.144). We take $g$ in Cartan subgroup $H$ and as was explained above choice $g$ corresponds to the choice of $\mu_0$ in (6.136).

In the principal grading used in (6.144) $\deg E_{\alpha_i} = 1/\hbar$ for all simple roots $E_{\alpha_i}$ (including affine root $\alpha_0$). Therefore $\mu_0 \in P_\mathfrak{h} + 1/\hbar \mathfrak{p}$, where $P_\mathfrak{h}$ is the weight lattice for $\mathfrak{g}$, $\mathfrak{p}$ is defined by the formula $(\mathfrak{p}, \alpha_i) = 1$ for all simple roots.$^9$

Below we write explicit formulas for characters of twisted representation corresponding to such $g$ (and such $\mu$). In the simply laced case, computing the characters using two formulas (6.136) and (6.144) one gets an identity, which is actually the Macdonald identity [Mac].

In notation for root system we follow [Mac] and [KacBook]. Below we consider roots as vectors in the linear space, generated by $e_1, \ldots, e_n, \delta, \Lambda_0$, and coroots – in the space generated by $e^\vee_1, \ldots, e^\vee_n, K, d$. The pairing between these dual spaces given by $(e_i, e^\vee_j) = \delta_{ij}$, $(\Lambda_0, K) = (\delta, d) = 1$ while all other vanish.

**GL(N) case.** Root system is $A_{N-1}^{(1)}$ (affine $A_{N-1}$) dual root system is also $A_{N-1}^{(1)}$. 

Simple roots: $\alpha_0 = \delta - e_1 + e_N$, $\alpha_i = e_i - e_{i+1}$, $1 \leq i \leq N - 1$

Simple coroots: $\alpha_0^\vee = K + e^\vee_N - e^\vee_1$, $\alpha_i^\vee = e^\vee_i - e^\vee_{i+1}$, $1 \leq i \leq N - 1$

Real coroots: $mK + e^\vee_i - e^\vee_j$, $m \in \mathbb{Z}, i \neq j$

Imaginary coroots: $mK$ of multiplicity $N, m \in \mathbb{Z}$.

Level $k = 1$ weights: $\Lambda_0, \Lambda_j = \Lambda_0 + \sum_{i=1}^j e_i, 1 \leq j \leq N - 1$

$h = N$, $\rho = 1/2 \sum_{i=1}^N (N - 2i + 1)e_i + NA_0$, $\mathfrak{p} = 1/2 \sum_{i=1}^N (N - 2i + 1)e_i$.

Note the multiplicity of imaginary roots in $N$ instead on $N - 1$ since we consider $G = GL(N)$ instead of $SL(N)$, and the corresponding affine algebra differs by one additional Heisenberg algebra.

The computation of the denominator in (6.144), using (6.145) gives

$$
\prod_{\alpha^\vee \in \Delta^\vee} (1 - q^{(\rho, \alpha^\vee)/\hbar})^{\text{mult}(\alpha^\vee)} = \prod_{k=1}^\infty (1 - q^{k/N})^N
$$

while for the numerator (the same for all level $k = 1$ weights) one gets

$$
\prod_{\alpha^\vee \in \Delta^\vee} (1 - q^{(\Lambda_0 + \rho, \alpha^\vee)/\hbar})^{\text{mult}(\alpha^\vee)} = \prod_{k=1}^\infty (1 - q^{k/N})^{N-1}
$$

$^9$Note the difference between $\rho$ and $\mathfrak{p}/\hbar$: first was defined by pairing with simple coroots (including affine one) and the second is defined by scalar products with (non affine) roots. In the simply laced case conditions in terms of roots and coroots are equivalent and we have $\rho = \mathfrak{p} + h\Lambda_0$
so that the character (6.144) in principal specialization
\[ q^{-\Lambda(p \gamma)/h} \text{Tr}(q^{p \gamma'/h})_{1_A} = \frac{1}{\prod_{k=1}^{\infty} (1 - q^{k/N})} \quad (6.148) \]

One can compare the last expression with the formula (6.136) using the choice of \( \mu_0 \), as explained above. We get an identity
\[ \sum_{\alpha \in Q_{sl(N)}} q^{\alpha} = \frac{1}{\prod_{k=1}^{\infty} (1 - q^{k/N})} \quad (6.149) \]

which is a particular case of formula (6.101) from sect. 6.5.5, and again reproduces the product formula for the lattice \( A_{N-1} \)-theta function (6.220).

Recall that the r.h.s. of (6.149) also has an interpretation of a character of the twisted Heisenberg algebra. This twist of the Heisenberg algebra emerges in the representation twisted by \( g \) with \( g_{\text{Adj}} \) acting as the Coxeter element of the Weyl group, hence r.h.s. of (6.149) equals to the r.h.s. of (6.138) for a single cycle \( K = 1, l = N \). This \( g \) is conjugate to used above in computing of l.h.s., therefore the characters of the twisted modules should be the same. The construction of level one representations in terms of principal Heisenberg subalgebra is well-known, see [LW, KKLW]. Another interpretation of the l.h.s in (6.149) is the sum of characters of the \( W \)-algebra namely \( W \) algebra of \( gl(N) \), (see sect. 6.5.6).

**SO(2N) case.** Root system \( D_N^{(1)} \) (affine \( D_N \)), the dual root system is also \( D_N^{(1)} \).

Simple roots: \( \alpha_0 = \delta-e_1-e_2, \quad \alpha_i = e_i-e_{i+1}, 1 \leq i < N, \alpha_N = e_{N-1}+e_N \)

Simple coroots: \( \alpha_0^\vee = K-e_1^\vee-e_2^\vee, \quad \alpha_i^\vee = e_i^\vee-e_{i+1}^\vee, 1 \leq i < N, \quad \alpha_N^\vee = e_{N-1}^\vee+e_N^\vee \)

Real coroots: \( mK+e_i^\vee-e_i^\vee, \quad mK+e_i^\vee+e_j^\vee, \quad mK-e_i^\vee-e_j^\vee, m \in \mathbb{Z}, i \neq j \)

Imaginary coroots: \( mK \) of multiplicity \( N, m \in \mathbb{Z} \)

\( k = 1 \) weights: \( \Lambda_0, \quad \Lambda_i = e_i+\Lambda_0, \quad \Lambda_{N-1} = \frac{1}{2} \sum_{i=1}^{N} e_i+\Lambda_0, \quad \Lambda_N = \frac{1}{2} \sum_{i=1}^{N} e_i-e_N+\Lambda_0 \)

\( h = 2N-2, \quad \rho = \sum_{i=1}^{N} (N-i)e_i + (2N-2)\Lambda_0, \quad \rho = \sum_{i=1}^{N} (N-i)e_i. \)

Now we again just compute the denominator
\[ \prod_{\alpha^\vee \in \Delta^\vee} (1 - q^{(\Lambda+\rho,\alpha^\vee)/h})^{\text{mult}(\alpha^\vee)} = \prod_{k=1}^{\infty} (1 - q^{k/(2N-2)})^{N} \quad (6.151) \]

and the numerator (the same for all \( k = 1 \) weights)
\[ \prod_{\alpha^\vee \in \Delta^\vee} (1 - q^{(\rho,\alpha^\vee)/h})^{\text{mult}(\alpha^\vee)} = \prod_{j=1}^{N-1} \prod_{k=1}^{\infty} (1 - q^{k-\frac{2j+1}{2N-2}})^{N+1} (1 - q^{k-\frac{2j}{2N-2}})^{N} \prod_{k=1}^{\infty} (1 - q^{k-\frac{1}{2}}) \quad (6.152) \]

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in (6.144), giving for the character

\[ q^{-\Lambda_{\rho^V}/h} \text{Tr}(q^{\rho^V/h}) \mid_{\Lambda} = \frac{1}{\prod_{j=1}^{N-1} \prod_{k=1}^{\infty} (1 - q^{-k - \frac{1}{2N}}) \cdot \prod_{k=1}^{\infty} (1 - q^{k - \frac{1}{2}})} . \tag{6.153} \]

As in previous case, comparing this with the formula (6.130), one gets an identity

\[ \sum_{\alpha \in Q_D^N} q^{\frac{1}{2} (\langle \alpha^V + \frac{1}{2} \rho, \alpha + \frac{1}{2} \rho \rangle)} \prod_{k=1}^{\infty} (1 - q^k)^N = \frac{q^{\frac{1}{2N} \langle \rho, \rho \rangle}}{\prod_{j=1}^{N-1} \prod_{k=1}^{\infty} (1 - q^{-k - \frac{1}{2N}}) \cdot \prod_{k=1}^{\infty} (1 - q^{k - \frac{1}{2}})} . \tag{6.154} \]

where the r.h.s. side can be interpreted as a character of the representation Heisenberg algebra twisted by \( g \) such that \( g_{\text{Ad}_0} \) is Coxeter element. Again, this is the same as construction of level \( k = 1 \) representation in terms of principal Heisenberg subalgebra from [LW, KKLW]. The l.h.s formula (6.154) can be also interpreted as the sum of characters of the \( W(\mathfrak{so}(2N)) \)-algebra, (see sect. 6.5.6).

By now in this section we have considered only the simply laced case – the only one, when the algebra \( V(\mathfrak{g}) \) is lattice algebra or, in other words, when the level \( k = 1 \) representations can be constructed as a sum of representations of the Heisenberg algebra. However, the formula (6.144) is valid for any affine Kac-Moody algebra. Below we consider the case \( G = SO(2N + 1) \), where the level \( k = 1 \) representations can be constructed using free fermions.

\( SO(2N + 1), N > 2 \) case. Root system is \( B_N^{(1)} \) (affine \( B_N \)), the dual root system is \( B_N^{(1,\vee)} = A_{2N-1}^{(2)} \) (affine twisted \( A_{2N-1} \))

Simple roots: \( \alpha_0 = -e_1 - e_2, \alpha_i = e_i - e_{i+1}, 1 \leq i \leq N - 1, \alpha_N = e_N. \)

Simple coroots: \( \alpha_0^V = K - e_1^V - e_2^V, \alpha_i^V = e_i^V - e_{i+1}^V, 1 \leq i \leq N - 1, \alpha_N = 2e_N^V. \)

Real coroots: \( 2mK \pm 2e_i, mK \pm e_i \mp e_j, mK \pm e_i \pm e_j, 1 \leq i < j \leq N, m \in \mathbb{Z}. \)

Imaginary coroots: \( (2m - 1)K \) of multiplicity \( N - 1, m \in \mathbb{Z} \)

\[ 2mK \) of multiplicity \( N, m \in \mathbb{Z} \backslash \{0\}. \]

\( k = 1 \) weights: \( \Lambda_0, \Lambda_1 - \Lambda_0 = e_1, \Lambda_N = \Lambda_0 + \frac{1}{2} \sum_{i=1}^{N} e_i \)

\( h = 2N, \rho = \sum_{j=1}^{N} (N - j + \frac{1}{2})e_j + (2N - 1)\Lambda_0, \quad \bar{\rho} = \sum_{j=1}^{N} (N - j + 1)e_j. \tag{6.155} \)

Compute again the denominator

\[ \prod_{\alpha^V \in \Delta_+} \left( 1 - q^{-\frac{\langle \rho, \alpha^V \rangle}{2N}} \right)^{\text{mult}(\alpha^V)} = \prod_{k=1}^{\infty} (1 - q^{2N})^N \cdot \prod_{k=1}^{\infty} (1 - q^{2k-\frac{1}{2N}}) . \tag{6.156} \]

and the numerator in the formula (6.144). Now the numerator for \( \Lambda = \Lambda_0 \) and \( \Lambda = \Lambda_1 \) is the same

\[ \prod_{\alpha^V \in \Delta_+} \left( 1 - q^{-\frac{\langle \rho + \Lambda_0, \alpha^V \rangle}{2N}} \right) = \prod_{\alpha^V \in \Delta_+} \left( 1 - q^{-\frac{\langle \rho + \Lambda_1, \alpha^V \rangle}{2N}} \right) = \prod_{k=1}^{\infty} (1 - q^{2N})^N \cdot \prod_{k=1}^{\infty} (1 + q^k) . \tag{6.157} \]
but for \( \Lambda = \Lambda_N \) it is different

\[
\prod_{\alpha^x \in \Delta_k^N} \left( 1 - q^{(\rho + \Lambda_N - \alpha^y)/2N} \right) = \prod_{k=1}^{\infty} (1 - q^{2k/N}) \prod_{k=1}^{\infty} (1 + q^{k-\frac{1}{2}}), \tag{6.158}
\]

Here we used the identities (6.235) and \( \prod_{k=1}^{\infty} (1 - q^{2k/2}) = \prod_{k=1}^{\infty} (1 + q^{k-1/2}) \). It is convenient to consider the direct sums of two representations \( L_{\Lambda_0} \oplus L_{\Lambda_1} \) and \( L_{\Lambda_N} \oplus L_{\Lambda_N} \) since these sums have construction in terms of fermions. Using (6.144) one gets

\[
q^{-\Lambda_0(\rho^\vee)/h} \text{Tr}(q^{\rho^\vee}/h) \big|_{L_{\Lambda_0}} + q^{-\Lambda_1(\rho^\vee)/h} \text{Tr}(q^{\rho^\vee}/h) \big|_{L_{\Lambda_1}} = 2 \prod_{k=1}^{\infty} \frac{(1 + q^k)}{(1 - q^{2k-1})},
\]

\[
q^{-\Lambda_N(\rho^\vee)/h} \text{Tr}(q^{\rho^\vee}/h) \big|_{L_{\Lambda_N}} = 2 \prod_{k=1}^{\infty} \frac{(1 + q^{k-\frac{1}{2}})}{(1 - q^{2k-1})}. \tag{6.159}
\]

The r.h.s. of these equations suggest the existence of the construction of these representation in terms of \( N \)-component twisted (principal) Heisenberg algebra and additional fermion (in NS and R sectors correspondingly), exactly this construction has been considered in sect. 6.4.4.

On the other hand, these characters can be rewritten in terms of the simplest \( B \)-lattice theta-functions just using the Jacobi triple product identity

\[
2 \prod_{k=1}^{\infty} \frac{(1 + q^k)}{(1 - q^{2k+1})} = \prod_{k=1}^{\infty} \prod_{i=0}^{2N} (1 + q^{2k-1}) = \sum_{n_1, \ldots, n_N \in \mathbb{Z}} q^{\frac{1}{2} \sum_{j=1}^{N} (n_j^2 + \frac{1}{2} n_j)} \prod_{k=1}^{\infty} \frac{(1 + q^{k-\frac{1}{2}})}{(1 - q^k)^N} = \tag{6.160}
\]

\[
qu^{-\frac{(N+1)(2N+1)}{48N}} \sum_{\alpha \in Q_{BN}} q^{\frac{1}{2} (\alpha + \frac{1}{2N} \sigma, \alpha + \frac{1}{2N} \sigma)} \prod_{k=1}^{\infty} \frac{(1 + q^{k-\frac{1}{2}})}{(1 - q^k)^N},
\]

and

\[
2 \prod_{k=1}^{\infty} \frac{(1 + q^{k-\frac{1}{2}})}{(1 - q^{2k+1})} = 2 \prod_{k=1}^{\infty} \prod_{i=0}^{2N-1} (1 + q^{k-\frac{1}{2}}) \prod_{k=1}^{\infty} (1 + q^{k-\frac{1}{2}}) = \sum_{n_1, \ldots, n_N \in \mathbb{Z}} q^{\frac{1}{2} \sum_{j=1}^{N} (n_j^2 + \frac{1}{2} n_j) - \frac{1}{4N} n_j} \prod_{k=1}^{\infty} \frac{(1 + q^{k-1})}{(1 - q^k)^N} = \tag{6.161}
\]

\[
qu^{-\frac{(N-1)(2N+1)}{48N-6}} \sum_{\alpha \in Q_{BN} + \Lambda_N - \Lambda_0} q^{\frac{1}{2} (\alpha + \frac{1}{2N} \sigma, \alpha + \frac{1}{2N} \sigma)} \prod_{k=1}^{\infty} \frac{(1 + q^{k-1})}{(1 - q^k)^N},
\]

where \( \Lambda_N - \Lambda_0 \) is the highest weight of the spinor representation of \( SO(2N + 1) \). The r.h.s. of these formulas are the characters of sums of nontwisted representations of \( N \)-component Heisenberg algebra with additional infinite-dimensional Clifford algebra (or real fermion). Another point of view that the r.h.s. are the characters of sums of representations of \( W(B_N) \)-algebra [Luk].
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Finally, let us point out, that for the root system $B_2^{(1)} = C_2^{(1)}$ (affine $B_2$), the dual roots system is $C_2^{(1),\vee} = D_3^{(2)}$ (affine twisted $D_3$).

Simple roots: $\alpha_0 = \delta - 2e_1, \quad \alpha_1 = e_1 - e_2, \quad \alpha_2 = 2e_2$.
Simple coroots: $\alpha_0^\vee = K - e_1^\vee, \quad \alpha_1^\vee = e_1^\vee - e_2^\vee, \quad \alpha_2^\vee = e_2^\vee$.
Real coroots: $mK \pm e_1^\vee, \quad mK \pm e_2^\vee, \quad 2mK \pm e_1^\vee \mp e_2^\vee, \quad 2mK \pm e_1^\vee \mp e_2^\vee, \quad m \in \mathbb{Z}$.
Imaginary coroots: $(2m-1)K$ of multiplicity 1, $m \in \mathbb{Z}$
$2mK$ of multiplicity 2, $m \in \mathbb{Z} \setminus \{0\}$.

$k = 1$ weights: $\Lambda_0, \quad \Lambda_1 = \epsilon_1 + \Lambda_0, \quad \Lambda_2 = \Lambda_0 + \epsilon_1 + \epsilon_2$
$h = 4, \quad \rho = 2e_1 + e_2 + 3\Lambda_0, \quad \bar{\rho} = \frac{3}{2}e_1 + \frac{1}{2}e_2$.

the computation leads to result, coinciding with formulas (6.156), (6.157), (6.158) for $N = 2$. Though the root system here has a bit different combinatorial structure, the fermionic construction is the same, using 5 real fermions.

**Exact conformal blocks of $W(so(2N))$ twist fields**

**Global construction**

It has been shown in [GMtw], that conformal block of the generic $W(gl(N))$ twist fields is given by explicit formula, analogous to the famous Zamolodchikov’s conformal blocks of the Virasoro twist fields with dimensions $\Delta = \frac{1}{16}$ [ZamAT87, ZamAT86, ApiZam]. To generalize the construction of [GMtw] to all twist fields $\{\mathcal{O}_g | g \in N_G(h)\}$ considered in this chapter, one needs to glue local data in the vicinity of all twist field to some global structure. We consider below such construction for $G = O(2N)$, since it can be entirely performed in terms of twisted bosons.

First, let us remind the local data in the vicinity of $\mathcal{O}_g(0)$ already discussed in sect. 6.4:

- $2l$-fold cover $z = \xi^{2l}$ with holomorphic involution $\sigma : \xi \mapsto -\xi$ without stable points except for the twist field position.
- Fermionic field $\eta(\xi)$ with exotic OPE $\eta(\xi)\eta(\sigma(\xi')) \sim \frac{1}{\xi - \xi'}$. On the sheets, connected to each other by $[l, e^{2\pi i r}]_+$, one can identify $\eta(\xi)$ with ordinary complex fermion $\psi(\xi) = \eta(\xi)$, $\eta(\sigma(\xi)) = \psi^*(\xi)$, in this case $\sigma$ permutes $\psi \leftrightarrow \psi^*$.
- Bosonic field $J(z) = (\eta(\sigma(z))\eta(z))$, which is antisymmetric $J(\sigma(z)) = -J(z)$ under the action of involution $\sigma$, and has first-order poles coming from zero-mode charges in the branch-points corresponding to cycles of type $[l, e^{2\pi i r}]_+$.

To compute spherical $2M$-point conformal block $\textsuperscript{10}$

$$\mathcal{G}_0(q_1, \ldots, q_{2M}) = \langle \mathcal{O}_{h_1g_1}(q_1)\mathcal{O}^{-1}_{g_1}(q_2) \ldots \mathcal{O}_{h_Mg_M}(q_{2M-1})\mathcal{O}^{-1}_{g_M}(q_{2M}) \rangle \quad (6.163)$$

\textsuperscript{10}In principle, we may choose any monodromies, though in this way we will get complicated twisted representations in the intermediate channels, but as in [GMtw] we restrict ourselves to simpler, but still quite general case of pairwise inverse (up to diagonal factors $h_i$) monodromies.
we forget about fermion and consider only the twisted boson with current \( J(z) \). Now let us list the field-theoretic properties which fix this conformal block uniquely.

Considering the action of 1-form \( J(z)dz \) onto the highest weight vector \( |0\rangle_g \) of the module of twist-field \( O_g \) of order \( l \), due to \( J_{k/l} |0\rangle_g = 0 \) one gets, that the most singular term
\[
J(z)dz \sim z \rightarrow 0 \frac{dz}{z} + \ldots
\]
(6.164)
in the vicinity of the twist field can be simple pole – in presence of \( r \)-charge or a zero mode.

Notice, that for two fields with opposite (up to diagonal factor \( h = \text{diag} (e^{2\pi i a_1}, \ldots, e^{2\pi i a_n}) \)) monodromies
\[
O_{h \cdot g}(z)O_{g^{-1}}(z') \sim (z - z')^{\Delta_h - 2\Delta_g} V_h(z') + \text{descendants}
\]
(6.165)
where \( V_h(z') \) is a field with fixed charges \( \vec{a} \in \mathfrak{h} \). Hence
\[
\frac{1}{2\pi i} \oint_{\mathcal{C}_{z,z'}} J(\xi) d\xi O_{h \cdot g}(z)O_{g^{-1}}(z') = a_j O_{g}(z)O_{g^{-1}}(z')
\]
(6.166)
where contour \( \mathcal{C}_{z,z'} \) is the \( j \)-th preimage of the contour encircling two points \( z, z' \) on the base. We identify such contours with the \( A \)-cycles on the cover, and corresponding \( a \)'s with \( A \)-periods of 1-form \( J(z)dz \).

The standard OPE of two currents
\[
J(z)J(z')dzdz' = \frac{dzdz'}{z - z'} + 4\bar{T}(z') + \ldots
\]
(6.167)
gives the stress-energy tensor
\[
T(z) = \sum_{\pi_{2N}(\xi) = z} \bar{T}(\xi)
\]
(6.168)
and non-standard coefficient (4 instead of 2) arises due to involution \( \sigma \). Summarizing these facts we get:

- 2\( N \)-sheet branched cover \( \pi_{2N} : \Sigma \rightarrow \mathbb{P}^1 \) with the branch points \( \{q_1, \ldots, q_{2M}\} \) and ramification structure defined by the elements \( \{g_1, g_1^{-1}, \ldots, g_M, g_M^{-1}\} \). In particular, \( \Sigma \) is a disjoint union of two curves when all \( \{g_i\} \) do not contain \( [l] \)-cycles.

- Involuntion of this cover \( \sigma : \Sigma \rightarrow \Sigma \) with the stable points coinciding with \( [l] \)-cycles
\[
\sigma \subset \Sigma \xrightarrow{\pi_{2N}} \tilde{\Sigma} \xrightarrow{\pi_N} \mathbb{C} \mathbb{P}^1
\]
(6.169)
Projections and involution are shown on the diagram: \( \pi_{2N} = \pi_N \circ \pi_2, \pi_2 \circ \sigma = \pi_2 \).
6. Twist-field representations of $W$-algebras, exact conformal blocks and character identities

- Odd meromorphic differential $dS(\sigma(\xi)) = -dS(\xi)$ with the poles in preimages of $q_i$ and residues given by corresponding $r$-charges.
- Symmetric bidifferential $d\Omega(\xi, \xi')$, satisfying $d\Omega(\sigma(\xi), \xi') = -d\Omega(\xi, \xi')$, with two poles:

$$d\Omega_2(\xi, \xi') \sim \frac{d\xi d\xi'}{\xi - \xi'} \frac{d\xi d\xi'}{(\xi - \sigma(\xi'))^2}$$

and vanishing $A$-periods.

Using this data one can write for two auxiliary correlators

$$G_1(\xi|q_1, \ldots, q_{2M}) = d\xi J(\xi)\mathcal{O}_{h_1-g_1}(q_1)\mathcal{O}_{g_1^{-1}}(q_2)\ldots \mathcal{O}_{h_M-g_M}(q_{2M-1})\mathcal{O}_{g_M^{-1}}(q_{2M})$$

and

$$G_2(\xi, \xi') = d\xi d\xi' J(\xi)J(\xi')\mathcal{O}_{h_1-g_1}(q_1)\mathcal{O}_{g_1^{-1}}(q_2)\ldots \mathcal{O}_{h_M-g_M}(q_{2M-1})\mathcal{O}_{g_M^{-1}}(q_{2M})$$

their explicit expressions

$$G_1(\xi)G_0^{-1} = dS(\xi), \quad G_2(\xi, \xi')G_0^{-1} = dS(\xi)dS(\xi') + d\Omega_2(\xi, \xi')$$

fixed uniquely by their analytic behaviour. Now let us study in detail the structure of the curve $\Sigma$ in order to construct all these objects.

**Curve with holomorphic involution**

Involution $\sigma$ defines the two-fold cover $\pi_2 : \Sigma \to \tilde{\Sigma}$ with the total number of branch points being $2K' = 2\sum_{i=1}^M K'_i$, or exactly the total number of $\lfloor l \rfloor$ cycles in all elements $\{g_i, g_i^{-1}\}$. The Riemann-Hurwitz formula $\chi(\Sigma) = 2 \cdot \chi(\tilde{\Sigma}) - \#BP$ then gives for the genus

$$g(\Sigma) = 2g(\tilde{\Sigma}) + K' - 1 \quad (6.173)$$

Then a natural way to specify the $A$-cycles on $\Sigma$ is the following [Fay]: first to take $A_1^{(1)}, \ldots, A_{\tilde{g}}^{(1)}, A_1^{(2)}, \ldots, A_{\tilde{g}}^{(2)}$ on each copy of $\tilde{\Sigma}$, where $\tilde{g} = g(\tilde{\Sigma})$; and second, all other $A$-cycles that correspond to the branch cuts of the cover, connecting the branch points of $\pi_2$: $A_1^{(0)}, \ldots, A_{K' - 1}^{(0)}$. The action of involution on these cycles is obviously given by

$$\sigma(A_i^{(1)}) = A_i^{(2)}, \quad \sigma(A_i^{(2)}) = A_i^{(1)}, \quad i = 1, \ldots, \tilde{g}$$

$$\sigma(A_j^{(0)}) = -A_j^{(0)}, \quad j = 1, \ldots, K' - 1 \quad (6.174)$$

thus we have the decomposition of the real-valued first homology group into the even and odd parts

$$H_1(\Sigma, \mathbb{R}) = H_1(\Sigma, \mathbb{R})^+ \oplus H_1(\Sigma, \mathbb{R})^-$$

$$\dim H_1(\Sigma, \mathbb{R})^+ = g(\tilde{\Sigma}) = \tilde{g}$$

$$\dim H_1(\Sigma, \mathbb{R})^- = \tilde{g} + K' - 1 = g_- \quad (6.175)$$

Compute now $\tilde{g} = g(\tilde{\Sigma})$, using the Riemann-Hurwitz formula for the cover of $\mathbb{P}^1$. Let $K = \sum_{i=1}^M K_i$ be the total number of $\lfloor l, e^{2\pi ir} \rfloor$-type cycles in all elements $\{g_i\}$, as well
as $K'$ serves for the type $[l']_-$. Then $\chi(\tilde{\Sigma}) = N \cdot \chi(\mathbb{P}^1) - \#BP$ gives (cf. with the formula (2.17) of [GMtw])

$$\tilde{g} = 1 - N + \sum_{i=1}^{K} (l_i - 1) + \sum_{i=1}^{K'} (l_i' - 1)$$

(6.176)

so that

$$g_- = \tilde{g} + K' - 1 = \sum_{i=1}^{K} (l_i - 1) + \sum_{i=1}^{K'} l_i' - N$$

(6.177)

and

$$g = 1 - 2N + 2 \sum_{i=1}^{K} (l_i - 1) + 2 \sum_{i=1}^{K'} (l_i' - \frac{1}{2})$$

(6.178)

For our purposes the most essential is the odd part $H_1(\Sigma, \mathbb{R})^-$ of the homology. One can see these $g_-$ $A$-cycles explicitly as follows: two mutually inverse permutations of type $[l]_+$ produce $l$ pairs of $A$-cycles $A_i^{1(2)}$ with constraints $\sum_i A_i^{(1,2)} = 0$. These cycles are permuted by $\sigma$ (6.174), so they actually form $l - 1$ independent odd combinations, giving contribution to the r.h.s. of (6.177). For two mutually inverse elements of the type $[l']_-$ one gets instead $2l'$ $A$-cycles with constraint $\sum_i A_i = 0$, and with the action of involution $\sigma : A_i \mapsto A_{i+l'}$, giving $l'$ independent odd combinations $\{A_i - A_{i+l'}\}$, arising in the r.h.s. of (6.177), while the extra term $-N$ corresponds to charge conservation in the infinity.

Hence, we got $g_-$ odd $A$-cycles, whose projections to $\mathbb{P}^1$ encircle pairs of the colliding twist fields $O_{h,g}(q_{2i-1})O_{g-1}(q_{2i})$ for $i = 1, \ldots, M$, so that the integrals of

$$\frac{1}{2\pi i} \oint_{A_I} dS = a_I, \quad I = 1, \ldots, g_-$$

(6.179)

give the W-charges in the intermediate channels of conformal block (6.163). Therefore $dS$ can be expanded

$$dS = \sum_{I=1}^{g_-} a_I d\omega_I + \sum_{i=1}^{2M} dS_{r_i}$$

(6.180)

ever the odd holomorphic differentials, and meromorphic differentials of the 3-rd kind corresponding to the nonvanishing $r$-charges.

Now, for the bidifferential $d\Omega_2(\xi, \xi')$ one can write

$$d\Omega_2(\xi, \xi') = K(\xi, \xi') - K(\sigma(\xi), \xi') = 2K(\xi, \xi') - \tilde{K}(\xi, \xi')$$

(6.181)

where $K(\xi, \xi')$ is the canonical meromorphic bidifferential on $\Sigma$ (the double logarithmic derivative of the prime form, see [Fay]), normalized on vanishing $A$-periods in each of two variables, and

$$\tilde{K}(\xi, \xi') = K(\xi, \xi') + K(\sigma(\xi), \xi')$$

(6.182)

is actually a pullback of the canonical meromorphic bidifferential on $\tilde{\Sigma}$. Indeed, consider

$$\delta K(\xi, \xi') = K(\xi, \xi') - K(\sigma(\xi), \sigma(\xi'))$$

(6.183)
6. Twist-field representations of W-algebras, exact conformal blocks and character identities

which is already holomorphic at $\xi = \xi'$, and $\int_{A_i} \delta K(\xi, \xi') = 0$, since due to (6.174) normalization conditions do not change under involution. Thus $\delta K(\xi, \xi') = 0$ and the canonical bidifferential is $\sigma$-invariant

$$K(\sigma(\xi), \sigma(\xi')) = K(\xi, \xi')$$ (6.184)

Moreover, since

$$\tilde{K}(\xi, \xi') = \tilde{K}(\sigma(\xi), \xi') = \tilde{K}(\xi, \sigma(\xi'))$$ (6.185)

expression (6.182) actually defines the canonical bidifferential on $\tilde{\Sigma}$.

**Computation of conformal block**

Now we use the technique from [ZamAT87, ZamAT86, ApiZam, GMtw] to compute the conformal block (6.163). For the vacuum expectation value of the stress-energy tensor (6.168) one gets from (6.172), (6.181)

$$\langle T(z) O_{h_1,g_1}(q_1) O_{g_1^{-1}}(q_2) \cdots O_{h_M, g_M}(q_{2M-1}) O_{g_M^{-1}}(q_{2M}) \rangle_{G_{0}} =$$

$$= \sum_{\pi_2 N(\xi) = z} t_z(\xi) - \sum_{\pi_N(\zeta) = z} \tilde{t}_z(\zeta) + \frac{1}{4} \left( \frac{dS}{dz} \right)^2$$ (6.186)

where $t_z$ and $\tilde{t}_z$ are the regularized parts of the bidifferentials $K$ and $\tilde{K}$ on diagonal in coordinate $z$:

$$t_z(\xi) d\xi^2 = \frac{1}{2} \left( \lim_{\xi \to \xi'} K(\xi', \xi) - \frac{d\pi_{2N}(\xi)d\pi_{2N}(\xi')}{(\pi_{2N}(\xi') - \pi_{2N}(\xi))^2} \right)$$

$$\tilde{t}_z(\zeta) d\zeta^2 = \frac{1}{2} \left( \lim_{\zeta \to \zeta'} \tilde{K}(\zeta', \zeta) - \frac{d\pi_{N}(\zeta)d\pi_{N}(\zeta')}{(\pi_{N}(\zeta') - \pi_{N}(\zeta))^2} \right)$$ (6.187)

Expanding (6.186) at $z \to q_i$ one gets

$$\tilde{t}_z(\zeta) \bigg|_{z \to q_i} = \frac{1}{12} \{\zeta; z\} + \text{reg.} = \frac{1}{(z-q_i)^2} \frac{l^2 - 1}{24l^2} + \text{reg.}$$

$$t_z(\xi) \bigg|_{z \to q_i} = \frac{1}{12} \{\xi; z\} + \text{reg.} = \frac{1}{(z-q_i)^2} \frac{4l^2 - 1}{96l^2} + \text{reg.}$$ (6.188)
Exact conformal blocks of $W(\mathfrak{so}(2N))$ twist fields

in local co-ordinates $\xi^{2l'} = \zeta^l = z - q_i$, which gives for the conformal dimensions $^{11}$ of the fields $O_g$ (with generic $\mathfrak{so}(2N)$ twist field of the type (6.86))

$$
\Delta_g = \sum_{j=1}^{K} \frac{t_j^2 - 1}{24t_j} + \sum_{j=1}^{K'} \frac{2t_j^2 + 1}{48t_j} + \sum_{i=1}^{K} \frac{1}{2} r_i^2 = \Delta_0 + \sum_{i=1}^{K} \frac{1}{2} r_i^2 ,
$$

(6.191)

where the last term in the r.h.s. comes from the expansion $dS \approx r_i \frac{dz}{z - q_i} + \ldots$. Without contributions of $r$-charges this formula is equivalent to (6.142), (6.191).

From the first order poles we obtain

$$
\partial_q \log G_0(q_1, \ldots, q_{2M}) = \sum_{\pi_{2N}(\xi) = q_i} \text{Res} t_z(\xi) d\xi - \sum_{\pi_{N}(\zeta) = q_i} \text{Res} \tilde{t}_z(\zeta) d\zeta + \frac{1}{4} \sum_{\pi_{2N}(\xi) = q_i} \text{Res} \frac{(dS)^2}{d\zeta}, \quad i = 1, \ldots, 2M
$$

(6.192)

This system of equations for conformal block is obviously solved, so that we can formulate:

**Theorem 6.4.** Conformal blocks (6.163) for generic $W(\mathfrak{so}(2N))$ twist fields are given by

$$
G_0(a, r, q) = \tau_B(\Sigma|q) \tau_B^{-1}(\tilde{\Sigma}|q) \tau_{SW}(a, r, q)
$$

(6.193)

where

$$
\partial_q \log \tau_B(\Sigma|q) = \sum_{\pi_{2N}(\xi) = q_i} \text{Res} t_z(\xi) d\xi
$$

$$
\partial_q \log \tau_B(\tilde{\Sigma}|q) = \sum_{\pi_{N}(\zeta) = q_i} \text{Res} \tilde{t}_z(\zeta) d\zeta
$$

(6.194)

and

$$
\partial_q \log \tau_{SW}(a, r, q) = \frac{1}{4} \sum_{\pi_{2N}(\xi) = q_i} \text{Res} \frac{(dS)^2}{dz}, \quad i = 1, \ldots, 2M
$$

(6.195)

$$
\frac{\partial}{\partial a_I} \log \tau_{SW} = \oint_{B_I^*} dS, \quad A_I \circ B_J = \delta_{IJ}, \quad I, J = 1, \ldots, g_-
$$

$^{11}$The counting here works as

$$
t_z - \tilde{t}_z \to 2 \sum_{j=1}^{K} \frac{t_j^2 - 1}{24t_j^2} - \sum_{j=1}^{K} \frac{t_j^2 - 1}{24t_j} = \sum_{j=1}^{K} \frac{t_j^2 - 1}{24t_j} \quad (6.189)
$$

for the $[l]_+$-cycles, and

$$
t_z - \tilde{t}_z \to \sum_{j=1}^{K'} \frac{2t'_j + 1}{96t'_j^2} - \sum_{j=1}^{K'} \frac{t'_j + 1}{48t'_j} = \sum_{j=1}^{K'} \frac{t'_j + 1}{48t'_j} \quad (6.190)
$$

for the $[l']_-$-cycles.
Equations (6.194) define so-called Bergmann tau-functions [KK04] for the curves \( \Sigma \) and \( \tilde{\Sigma} \) respectively, while the so-called Seiberg-Witten tau-function (6.195) can be read literally from [GMtw]

\[
\log \tau_{\text{SW}}(a, r, q) = \frac{1}{4} \sum_{I, J=1}^{g-} a_I T_{IJ} a_J + \frac{1}{2} \sum_{I=1}^{g-} a_I U_I(r) + \frac{1}{4} Q(r)
\]

(6.196)

where \( T_{IJ} \) is the \( g- \times g- \) “odd block” of the period matrix of \( \Sigma \), or the period matrix of corresponding Prym variety [Fay], the “odd” vector

\[
U_J(r) = \alpha B_J d\Omega = \sum_{i, \alpha} r_\alpha^i A_J(q_\alpha^i), \quad J = 1, \ldots, g-
\]

(6.197)

where \( q_\alpha^i \) are preimages of \( q_i \) and \( r_\alpha^i \) – corresponding \( r \)-charges, and

\[
Q(r) = \sum_{q_\alpha^i \neq q_\beta^j} r_\alpha^i r_\beta^j \log \theta_s(A(q_\alpha^i)) - A(q_\beta^j)) - \sum_{q_\alpha^i} l_\alpha^i (r_\alpha^i)^2 \log \left( \frac{d(z(q) - q_i)^{1/l_\alpha^i}}{h_s^2(q)} \right)_{q=q_\alpha^i}
\]

(6.198)

where \( \theta_s \) is some odd Riemann theta-function for the curve \( \Sigma \), \( A \) is the Abel map, and

\[
h_s^2(z) = \sum_{I=1}^{g} \frac{\partial \theta_s(0)}{\partial Z_I} d\omega_I(z)
\]

(6.199)

**Relation between \( W(\mathfrak{so}(2N)) \) and \( W(\mathfrak{gl}(N)) \) blocks**

It is interesting to compare the formulas from previous section with the formulas from [GMtw] for the exact \( W(\mathfrak{gl}(N)) \) conformal blocks. Since, as we already discussed \( W(\mathfrak{so}(2N)) \subset W(\mathfrak{gl}(N)) \), any vertex operator of the \( W(\mathfrak{gl}(N)) \) algebra is a vertex operator of its subalgebra \( W(\mathfrak{so}(2N)) \), and it is clear from our construction, that twist fields \( O_g \) for the elements \( g \sim \prod[l, e^{2\pi ir}]_+ \), are also the twist fields for \( W(\mathfrak{gl}(N)) \). Moreover, the corresponding Verma modules, generated by \( W(\mathfrak{so}(2N)) \) and by \( W(\mathfrak{gl}(N)) \), actually coincide \(^{12}\), and it means that corresponding conformal blocks of such fields in these two theories should coincide as well.

Indeed, in such a case \( \Sigma = \tilde{\Sigma} \sqcup \tilde{\Sigma} \), and therefore \( K(\xi, \xi') = 0 \) if \( \xi', \xi \) are on different components, and \( K(\xi, \xi') = \tilde{K}(\xi, \xi') \) if they are on the same component, hence

\[
t_z(z) = 2\tilde{t}_z(z)
\]

(6.200)

For holomorphic and meromorphic differentials, one has in this case in natural basis

\[
a_I = \oint_{A_I^{(1)}} dS = \oint_{A_I^{(2)}} dS, \quad I = 1, \ldots, \tilde{g}
\]

(6.201)

\[
r_\alpha^i = \text{Res}_{q_\alpha^i} dS = \text{Res}_{\sigma(q_\alpha^i)} dS
\]

\(^{12}\)These two modules coincide due to dimensional argument: they are both irreducible and have the same characters. Irreducibility follows from the fact that null-vector condition can be written as \((\alpha, \frac{\log 2}{2\pi r}) \in \mathbb{Z}\) for a simple root \( \alpha \) and generic \( r \)-s, see also comments in sect. 6.5.6.
for the preimages \(\{q_i^{\alpha}\}\) on \(\tilde{\Sigma}\), and the period matrix of \(\Sigma\) consists of two nonzero \(\tilde{g} \times \tilde{g}\) blocks:

\[
\mathcal{T}^{(11)} = \mathcal{T}^{(22)} = \tilde{\mathcal{T}}
\]

(6.202)

Under such conditions formula (6.4) turns into

\[
\mathcal{G}_0(a, r, q) = \tau_B(\tilde{\Sigma}|q)\tilde{\tau}_{SW}(a, r, q)
\]

(6.203)

where

\[
\log \tilde{\tau}_{SW}(a, r, q) = \frac{1}{2} \sum_{I,J=1}^{\tilde{g}} a_I T_{IJ} a_J + \sum_{I=1}^{\tilde{g}} a_I \tilde{U}_I(r) + \frac{1}{2} \tilde{Q}(r)
\]

(6.204)

with corresponding obvious modifications of formulas (6.197) and (6.198), which gives exactly the \(W(\mathfrak{gl}(N))\) conformal block in terms of the data on smaller curve \(\tilde{\Sigma}\).

**Conclusion**

We have considered in this chapter the twist fields for the \(W\)-algebras with integer Virasoro central charges, which are labeled by conjugacy classes in the Cartan normalizers \(N_G(h)\) of corresponding Lie groups. In addition to the most common \(W_N\)-algebras, corresponding to \(A\)-series (or \(W(\mathfrak{gl}(N)) = W_N \oplus H\), coming from \(G = GL(N)\)), we have extended this construction for the \(G = O(n)\) case, which includes in addition to \(D\)-series the non simply-laced \(B\)-case with the half-integer Virasoro central charge.

In terms of two-dimensional conformal field theory our construction is based on the free-field representation, where generalization to the \(D\)-series and \(B\)-series exploits the theory of real fermions, which in the odd \(B\)-case cannot be fully bosonized, so that in addition to modules of the twisted Heisenberg algebra one has to take into account those of infinite-dimensional Clifford algebra. This construction produces representations of the \(W\)-algebras (that are at the same time twisted representations of corresponding Kac-Moody algebras), which can be decomposed further into Verma modules. To find this decomposition we have computed the characters of twisted representations, using two alternative methods.

The first one comes from bosonization of the \(W\)-algebra or corresponding Kac-Moody algebra at level one, dependently on particular element from \(N_G(h)\) it identifies the representation space with a collection of the Fock modules for untwisted or twisted bosons. The essential new phenomenon, which appears in the case of orthogonal groups is presence of different \(l\) cycles in \(g \in N_G(h)\) and necessity to use in such cases “exotic” bosonization for the Ramond-type fermions with non-local OPE on the cover.

Alternative method for computation of the characters uses pure algebraic construction of the twisted Kac-Moody algebras and the Weyl-Kac formula in principal gradation.

There are examples of elements \(g_1, g_2\) that are not conjugated in \(N_G(h)\), but conjugated in \(G\). Since two different constructions with elements \(g_1\) and \(g_2\) give different formulations of the same representation, computation of corresponding characters \(\chi_{g_1}(q)\) and \(\chi_{g_2}(q)\) leads to some simple but nontrivial identities for the corresponding lattice theta-functions, \(\chi_{g_1}(q) = \chi_{g_2}(q)\), which have been also proven by direct methods.
We have also derived an exact formula for the general conformal block of the twist fields in \( D \)-case, which directly generalizes corresponding construction for common \( W_N \)-algebra. The result, as is usual for Zamolodchikov’s exact conformal block, is expressed in terms of geometry of covering curve (here with extra involution), and can be factorized into the classical “Seiberg-Witten” part, totally determined by the period matrix of the corresponding Prym variety, and the quasiclassical correction, expressed now in terms of two different canonical bi-differentials. In order to expand this method for the \( B \)-case one has to learn more about the theory of “exotic fermions” on Riemann surfaces, probably along the lines of [FSZ, DVV], and we postpone this for a separate publication.

Another set of open problems is obviously related with generalization to other series and twisted fields related with external automorphisms. Here only the \( E \)-cases seem to be straightforward, since standard bosonization can be immediately applied in the simply-laced case, and there should be not many problems with the fermion construction. However, it is not easy to predict what happens in the situation when Kac-Moody algebras at level \( k = 1 \) have fractional central charges, and the direct application of the methods developed in this chapter is probably impossible. It is still not very clear, what is the role of these exact conformal blocks in the context of multi-dimensional supersymmetric gauge theories, since generally there is no Nekrasov combinatorial representation in most of the cases. We hope to return to these issues in the future.

Finally, there is an interesting question of possible generalization of our approach to the twisted representations with \( k \neq 1 \), which has been already considered in [FSS]. Some overlap with our formulas with sect. 8 of this chapter suggests that such generalization could exist. We hope to return to this problem elsewhere.

### Appendix

#### Identities for lattice \( \Theta \)-functions

Here we present few rigorously proved identities, used to verify representation-theoretic considerations at the level of computations of characters.

**First identity for \( A_{N-1} \) and \( D_N \) \( \Theta \)-functions**

One can describe the lattices \( A_{N-1} \), \( D_N \) and \( D'_N \) in a similar way:

\[
A_{N-1} = \{k_1, \ldots, k_N \mid \sum_{i=1}^N k_i = 0\}
\]

\[
D_N = \{k_1, \ldots, k_N \mid \sum_{i=1}^N k_i \in 2\mathbb{Z}\} \quad (6.205)
\]

\[
D'_N = \{k_1, \ldots, k_N \mid \sum_{i=1}^N k_i \in 2\mathbb{Z} + 1\}
\]
The last lattice is actually just $D_N$ lattice, but shifted by vector $(1, 0, \ldots, 0)$. So all these definitions can be rewritten as

$$L_S = \{k_1, \ldots, k_N \mid \sum_{i=1}^N k_i \in S\} \quad (6.206)$$

where $S \subseteq \mathbb{Z}$: in our cases it should be chosen to be $\{0\}$, $2\mathbb{Z}$, and $2\mathbb{Z} + 1$, respectively.

By definition

$$\Theta_{L_S}(\vec{v}; q) = \sum_{k_1 + \ldots + k_N \in S} q^{\frac{1}{2}(\vec{v} + \vec{k})^2} \quad (6.207)$$

For our purposes we need this function computed for the vector

$$\vec{v} = (r_1 + \frac{l_1 - 1}{2l_1}, r_1 + \frac{l_1 - 3}{2l_1}, \ldots, r_1 + \frac{1 - l_1}{2l_1}) \oplus (r_2 + \frac{l_2 - 1}{2l_2}, r_2 + \frac{l_2 - 3}{2l_2}, \ldots, r_2 + \frac{1 - l_2}{2l_2}) \oplus \ldots \oplus (r_K + \frac{l_K - 1}{2l_K}, r_K + \frac{l_K - 3}{2l_K}, \ldots, r_K + \frac{1 - l_K}{2l_K})$$

where $l_1 + \ldots + l_K = N$. Let us parameterize vector $\vec{k}$ as follows:

$$\vec{k} = (n_1, \ldots, n_1) \oplus \ldots \oplus (n_K, \ldots, n_K) + \omega^{(l_1)}_{a_1} \oplus \ldots \oplus \omega^{(l_K)}_{a_K} + \left(\frac{a_1}{l_1}, \ldots, \frac{a_1}{l_1}\right) \oplus \ldots \oplus \left(\frac{a_K}{l_K}, \ldots, \frac{a_K}{l_K}\right) + \vec{m}_1 \oplus \ldots \oplus \vec{m}_K \quad (6.209)$$

where $\vec{m}_i \in A_{l_i - 1}$, and

$$\omega^{(l)}_a = \left(\frac{l - a}{l}, \ldots, \frac{l - a}{l}, -\frac{a}{l}, \ldots, -\frac{a}{l}\right) \quad (6.210)$$

so that the first number is repeated $a$ times, whereas the second one $l - a$ times. Hence, vectors $\vec{k} \in L_S$ are parameterized by vectors $\{\vec{m}_i \in A_{l_i - 1}\}$ and integer numbers $\{n_i \in \mathbb{Z}; a_i \in \mathbb{Z}/l_i\mathbb{Z}\}$, restricted by

$$\sum_{i=1}^K (n_i l_i + a_i) \in S \quad (6.211)$$

The algorithm of decomposition (6.209) works as follows: first we sum up all components of $\vec{k}$ inside each cycle – each number divided by $l_i$ gives $n_i$, whereas remainder gives $a_i$. Subtracting $(n_i, \ldots, n_i) + \omega^{(l_i)}_{a_i}$, we are left with the vectors $\{\vec{m}_i\}$ with vanishing sums of components.

Now it is easy to see that

$$\Theta(\vec{v} + \omega^{(l_1)}_{a_1} \oplus \omega^{(l_2)}_{a_2} \oplus \ldots \oplus \omega^{(l_K)}_{a_K}; q) = \Theta(\vec{v}; q) \quad (6.212)$$

**Notation** $\vec{v} \oplus \vec{u}$ means $(v_1, \ldots, v_k) \oplus (u_1, \ldots, u_m) = (v_1, \ldots, v_k, u_1, \ldots, u_m)$. 

---

13Notation $\vec{v} \oplus \vec{u}$ means $(v_1, \ldots, v_k) \oplus (u_1, \ldots, u_m) = (v_1, \ldots, v_k, u_1, \ldots, u_m)$. 

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which follows from the fact that \( \Theta(\vec{v}; q) = \Theta(\sigma(\vec{v}); q) \), where \( \sigma \) is a permutation. For example, take \( \sigma_a \) to be \( a \)-th power of the cyclic permutation, then:

\[
\sigma_a \left( \frac{1-l}{2l}, \ldots, \frac{l-1}{2l} \right) = \left( \frac{l+1-2a}{2l}, \frac{l+3-2a}{2l}, \ldots, \frac{l-1}{2l}, \frac{l-1-2a}{2l} \right) = \\
= \left( \frac{1-l}{2l}, \ldots, \frac{l-1}{2l} \right) + \omega_a^{(l)}
\]

(6.213)

and therefore any vector \( \vec{v} + \omega_a^{(l_1)} \oplus \omega_a^{(l_2)} \oplus \ldots \oplus \omega_a^{(l_K)} \) can be obtained by several permutation of components of \( \vec{v} \), so the corresponding \( \Theta \)-functions are equal. Thus

\[
\Theta_{L_S}(\vec{v}; q) = \sum_{\sum_{l_i} = \rho} q^\frac{1}{2} \prod_{i=1}^K (\vec{v}_i + \vec{m}_K + (n_1 + \frac{n_{i-1}}{2l} + \ldots + n_{i+1} + \frac{n_i}{2l}) + \ldots + n_{K+1} + \frac{n_K}{2l})^2
\]

(6.214)

turns into the sum over several orthogonal sublattices

\[
\Theta_{L_S}(\vec{v}; q) = \sum_{\vec{m}_i \in \Lambda_{i-1}} q^\frac{1}{2} \prod_{i=1}^K (\vec{v}_i + \vec{m}_1 + \ldots + \vec{m}_K)^2 \cdot \\
\sum_{\sum_{l_i} = \rho} q^\frac{1}{2} \prod_{i=1}^K (\vec{v}_i + \vec{m}_K + (n_1 + \frac{n_{i-1}}{2l} + \ldots + n_{i+1} + \frac{n_i}{2l}) + \ldots + n_{K+1} + \frac{n_K}{2l})^2
\]

(6.215)

where

\[
\hat{\rho}^{(l)} = \left( \frac{l-1}{2l}, \frac{l-3}{2l}, \ldots, \frac{1-l}{2l} \right)
\]

(6.216)

One can identify the last factor in the r.h.s. with the contribution of zero modes, related to the \( r \)-charges [GMtw].

**Product formula for \( A_{N-1} \) \( \Theta \)-functions**

Apply (6.215) to the simplest case of \( \Theta_{B_N}(\hat{\rho}^{(N)}; q) \) with \( S = \mathbb{Z} \)

\[
\Theta_{B_N}(\hat{\rho}^{(N)}; q) = \Theta_{A_{N-1}}(\hat{\rho}^{(N)}; q) \cdot \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2N}}
\]

(6.217)

Using definition (6.216) and Jacobi triple product formula we get

\[
\Theta_{B_N}(\hat{\rho}^{(N)}; q) = q^{\frac{N^2-1}{2N}} \prod_{a=0}^{N-1} q^{\frac{k^2}{2N}} = \\
= q^{\frac{N^2-1}{2N}} \prod_{k=1}^{\infty} (1 + \frac{q^k}{2N})^2 \prod_{n=1}^{\infty} (1 - q^n)^N
\]

(6.218)

as well as

\[
\sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2N}} = \prod_{k=1}^{\infty} (1 + \frac{q^k}{2N})^2 \prod_{n=1}^{\infty} (1 - q^n)
\]

(6.219)
Substituting into (6.217) one obtains

$$\Theta_{A_{N-1}}(\hat{\rho}^{(N)}; q) = q^{\frac{N^2-1}{24}} \frac{\prod_{k=1}^{\infty} (1 - q^k)^{N}}{\prod_{k=1}^{\infty} (1 - q^{m_k})} = \frac{\eta(q)^N}{\eta(q^{1/2^n})}$$ (6.220)

or the product formula [Mac] for $$\Theta_{A_{N-1}}(\hat{\rho}^{(N)}; q)$$, where the r.h.s. is expressed in terms of the Dedekind functions. Substituting this into (6.215) we get it in its final form

$$\Theta_{L_S}(\vec{v}; q) = \sum_{k_1 + \ldots + k_N \in S} q^{\frac{1}{2} \sum_{i=1}^{N} (v_i + k_i)^2} = \prod_{i=1}^{K} \eta(q^{l_i}) \sum_{n_1 + \ldots + n_K \in S} q^{\frac{1}{2} \sum_{i=1}^{K} (n_i + l_i r_i)^2}$$ (6.221)

**An identity for $$D_N$$ and $$B_N$$ $$\Theta$$-functions**

Here we show how $$\Theta_{D_N}(\vec{v}_s; q)$$ can be simplified if $$\vec{v}_s$$ contains at least one component $$\frac{1}{2}$$. One has then

$$\Theta_{D_N}(\vec{v}_s; q) = \Theta_{D_N}(\left(\frac{1}{2}, v_2, \ldots, v_n\right); q) = \sum_{k_1 + \ldots + k_N \in 2\mathbb{Z}} q^{\frac{1}{2}(\vec{v}_s + \vec{k})^2} = \Theta_{D_N}(\left(-\frac{1}{2}, v_2, \ldots, v_n\right); q) = \Theta_{D_n}(\vec{v}_s - (1, 0, \ldots, 0); q)$$ (6.222)

Since for the lattices $$D_N \sqcup \{D_N - (1, 0, \ldots, 0)\} = B_N$$, it follows from (6.222) that

$$\Theta_{D_N}(\vec{v}_s; q) = \frac{1}{2} \Theta_{B_N}(\vec{v}_s; q)$$ (6.223)

**Exotic bosonizations**

Here we present some details of the bosonization procedures, used in the main text.

**NS × R**

Consider, first, construction [Ber, BBT] relating pair (of NS and R!) fermions to a twisted boson

$$\tilde{\phi}(t) = i \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{J_r}{r \theta} \equiv i \sqrt{2} \sum_{n \in \mathbb{Z}} \frac{a_{2n+1}}{2n+1} \xi^{2n+1} = \phi(\xi)$$ (6.224)

with differently normalized oscillator modes $$[a_M, a_N] = M \delta_{M,N,0}$$ $$(M, N \in 2\mathbb{Z} + 1)$$. Compute the correlator

$$-\langle \phi(\xi) \phi(\zeta) \rangle = 2 \sum (a_{2n+1} a_{2m+1}) \xi^{-2n-1} \zeta^{-2m-1} = -2 \sum_{n=0}^{\infty} (\frac{\zeta}{\xi})^{2n+1} \frac{1}{2n+1} = 2 \log \left(1 - \frac{\zeta}{\xi}\right) - \log \left(1 - \frac{\zeta^2}{\xi^2}\right) = \log \frac{\xi + \zeta}{\xi - \zeta} = -[\phi_+ (\xi), \phi_- (\zeta)]$$ (6.225)

14 It is more convenient to use in this section coordinate $$\xi = \sqrt{t}$$, so analytic continuation in $$t$$ around 0 maps $$\xi$$ to $$-\xi$$. 

15
assuming $|\xi| > |\zeta|$. Now introduce
\[ \hat{\eta}(\xi) = \frac{1}{\sqrt{2}} : e^{i\phi(\xi)} := \frac{1}{\sqrt{2}} e^{i\phi_{-}(\xi)} e^{i\phi_{+}(\xi)} \]  
(6.226)
so that for $|\xi| > |\zeta|$\[
\hat{\eta}(\xi) \hat{\eta}(\zeta) = \frac{1}{2} e^{-(\phi_{-}(\xi)+\phi_{-}(\zeta))} e^{i(\phi_{+}(\xi)+\phi_{+}(\zeta))} e^{-[\phi_{+}(\xi),\phi_{-}(\zeta)]} = \frac{1}{2} : e^{i(\phi_{+}(\xi)+\phi_{+}(\zeta))} : \frac{\xi - \zeta}{\xi + \zeta} \]  
(6.227)
while for $|\xi| < |\zeta|$\[
\hat{\eta}(\zeta) \hat{\eta}(\xi) = \frac{1}{2} : e^{i(\phi_{+}(\xi)+\phi_{+}(\zeta))} : \frac{\zeta - \xi}{\zeta + \xi} \]  
(6.228)
It means that OPE of the $\hat{\eta}$-fields has fermionic nature:
\[ \hat{\eta}(\xi) \hat{\eta}(-\zeta) = \frac{1}{2} \xi + \zeta : e^{i(\phi(\xi)-\phi(\zeta))} : \sim \frac{1}{2} \xi + \zeta : e^{i(\phi_{+}(\xi)+\phi_{+}(\zeta))} : + \text{reg.} \sim \frac{\zeta}{\xi + \zeta} + \text{reg.} \]
(6.229)
and in the anticommutator of components $\hat{\eta}(\xi) = \sum_{k \in \mathbb{Z}} \eta_{k}\hat{\eta}^{k}$ \[ \{\eta_{k}, (-1)^{l}\eta_{l}\} = \oint \xi^{l-1} d\xi \int_{\xi - \zeta}^{\xi} \zeta^{k-1} d\xi = \delta_{k+l,0} \]  
(6.230)
one gets unusual sign factor.
It is interesting to point our that the Ramond zero mode $\eta_{0}^{2} = \frac{1}{2}$ has bosonic representation\[
\sqrt{2}\eta_{0} = \oint \frac{d\xi}{\xi} e^{i\phi_{-}(\xi)} e^{i\phi_{+}(\xi)} = \sqrt{2}\eta_{0} \cdot |0\rangle = |0\rangle, \quad \sqrt{2}\eta_{0} \cdot a_{-1}|0\rangle = -a_{-1}|0\rangle, \quad \sqrt{2}\eta_{0} \cdot a_{2}^{2}a_{1}^{3} - \frac{2}{9} (a_{-3} + a_{-1}^{3})(a_{3} + a_{1}^{3}) + \ldots \]
(6.231)
For example, the action of this operator on low-level vectors gives\[
\sqrt{2}\eta_{0} \cdot a_{-3}|0\rangle = \frac{1}{3} a_{-3}|0\rangle - \frac{2}{3} a_{-1}^{3}|0\rangle, \quad \sqrt{2}\eta_{0} \cdot a_{-1}^{3}|0\rangle = -\frac{4}{3} a_{-3}|0\rangle - \frac{1}{3} a_{-1}^{3}|0\rangle \]  
(6.232)
Here in the second line one gets the matrix $\frac{1}{3} \begin{pmatrix} 1 & -2 \\ -4 & -1 \end{pmatrix}$ with the eigenvalues $\pm 1$.
We also have
\[ \eta_{0}\eta_{k} = -\eta_{k}\eta_{0}, \quad k \neq 0 \]  
(6.233)
so one can identify $\sqrt{2}\eta_{0} = (-1)^{F-\eta_{0}}$, where $F$ is fermionic parity. Generally, algebra, generated by $\{\eta_{k}\}$, has two representations with the vacua $|0\rangle_{\pm}$, such that $\eta_{0}|0\rangle_{\pm} = \pm |0\rangle_{\pm}$. One can also take direct sum of such representations: bosonization formula in this representation looks as\[
\hat{\eta}(\xi) = \frac{\sigma_{1}}{\sqrt{2}} e^{i\phi_{-}(\xi)} e^{i\phi_{+}(\xi)} \]
(6.234)
6.10. Exotic bosonizations

Existence of this bosonization at the level of characters gives us obvious identity

$$\prod_{k=0}^{\infty} \frac{1}{1 - q^{2k+1}} = \prod_{k=1}^{\infty} (1 + q^k)$$  \hspace{1cm} (6.235)

Notice that above consideration actually concerns $R$ and $NS$ fermions because one can construct two combinations

$$\frac{1}{\sqrt{2}} (\hat{\eta}(z) - \hat{\eta}(-z)) = \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{\eta_{2p}}{z^p} = i \hat{\psi}_{NS}(z)$$  \hspace{1cm} (6.236)

$$\frac{1}{\sqrt{2}} (\hat{\eta}(z) + \hat{\eta}(-z)) = \sum_{n \in \mathbb{Z}} \frac{\eta_{2n}}{z^n} = \hat{\psi}_R(z)$$

then

$$J(z) = \frac{1}{z} \left( \hat{\psi}^*(\sqrt{z}) \hat{\psi}(\sqrt{z}) \right) = i \hat{\psi}_{NS}(z) \hat{\psi}_R(z) = \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{J_p}{z^{p+1}}$$  \hspace{1cm} (6.237)

$$J_p = i \sum_{n+q=p} \eta_{2q} \eta_{2n}$$

here $t^{-\frac{1}{2}} \hat{\psi}_{NS}(\sqrt{t})$ and $t^{-\frac{1}{2}} \hat{\psi}_R(\sqrt{t})$ are usual Ramond and Neveu-Schwarz fermions.

Here we consider fermion corresponding to the branch point of type $[\ell]_-$. This means that we should have

$$\eta(z) \eta(\sigma(w)) \sim \frac{1}{z - w},$$  \hspace{1cm} (6.238)

and such monodromy that $\eta(e^{4\pi i l}z) = \pm \eta(z)$. Let us use the construction form (6.10.1)

$$\eta(z) = \frac{z^{-\frac{1}{4}}}{\sqrt{2l}} \hat{\eta}(z^{\frac{1}{2l}})$$  \hspace{1cm} (6.239)

Therefore

$$\eta(z) \eta(\sigma(w)) \sim \frac{z^{-\frac{1}{4}}w^{-\frac{1}{4}}}{2l} \frac{w^{\frac{1}{4l}}}{z^{\frac{1}{4l}} - w^{\frac{1}{4l}}} \sim \frac{1}{z - w}$$  \hspace{1cm} (6.240)

So final construction states that one should have

$$\eta(z) = \sigma_1 \frac{z^{-\frac{1}{4}}}{2\sqrt{l}} e^{i\phi_- (z^{\frac{1}{2l}})} e^{i\phi_+ (z^{\frac{1}{4l}})}$$  \hspace{1cm} (6.241)

$R \times R$

Let us take two Ramond fermions $\psi^{(1)}, \psi^{(2)}$ and introduce

$$\psi(z) = \frac{1}{\sqrt{2}} (\psi^{(1)}(z) + i \psi^{(2)}(z)) = \sum_{n \in \mathbb{Z}} \frac{\psi_n}{z^{n+\frac{1}{4}}}$$  \hspace{1cm} (6.242)

$$\psi^*(z) = \frac{1}{\sqrt{2}} (\psi^{(1)}(z) - i \psi^{(2)}(z)) = \sum_{n \in \mathbb{Z}} \frac{\psi^*_n}{z^{n+\frac{1}{4}}}$$
Since there are two zero modes $\psi_0^*$ and $\psi_0$, one expects to have four vacua $|0\rangle$, $\psi_0|0\rangle$, $\psi_0^*|0\rangle$, $\psi_0\psi_0^*|0\rangle$.

We can mimic expansion (6.242) using fractional powers

$$
\psi(z) = \sum_{p \in \mathbb{Z}+\frac{1}{2}} \psi_{NS,p} z^{p+\frac{1}{2}+\sigma}, \quad \psi^*(z) = \sum_{p \in \mathbb{Z}+\frac{1}{2}} \psi_{NS,p}^* z^{p+\frac{1}{2}-\sigma},
$$

with $\sigma = \frac{1}{2}$, i.e. $\psi_n = \psi_{NS,n+\frac{1}{2}}$ and $\psi_n^* = \psi_{NS,n+\frac{1}{2}}^*$. It means that after standard bosonization

$$
\psi(z) = e^{-i\phi(z)} e^{-Q z - J_0}, \quad \psi^*(z) = e^{i\phi(z)} e^{Q z - J_0}
$$

one gets $\psi_0^*|0\rangle = 0$, and only one half of the vacuum states survive. To identify this representation with something well-known, consider the eigenvectors $\sqrt{2}\psi_0(1)|0\rangle_{\pm} = \pm|0\rangle_{\pm}$ of $\sqrt{2}\psi_0(1) = \psi_0 + \psi_0^*$:

$$
|0\rangle_{+} = \frac{1}{\sqrt{2}} (|0\rangle + \psi_0|0\rangle), \quad |0\rangle_{-} = \frac{i}{\sqrt{2}} (|0\rangle - \psi_0|0\rangle)
$$

Acting by $\sqrt{2}\psi_0(2) = i(\psi_0^* - \psi_0)$ one gets

$$
\sqrt{2}\psi_0(2)|0\rangle_{+} = |0\rangle_{-}, \quad \sqrt{2}\psi_0(2)|0\rangle_{-} = |0\rangle_{+}
$$

The character of such module is given by

$$
2 \prod_{k=1}^{\infty} (1 + q^k)^2 = q^{-\frac{1}{8}} \sum_{n \in \mathbb{Z}} q^{\frac{1}{8}(n+\frac{1}{2})^2} \prod_{k=1}^{\infty} (1 - q^k)^{-\frac{1}{8}}
$$

where in the l.h.s. we have two Ramond fermions with two vacuum states, whereas the r.h.s. corresponds to sum over bosonic modules with half-integer vacuum $J_0$ charges. This formula is a simple consequence of the Jacobi triple product identity. Analogously we have similar formula for the bosonization of $NS \times NS$ fermions

$$
\prod_{k=0}^{\infty} (1 + q^{\frac{1}{2} + k})^2 = \sum_{n \in \mathbb{Z}} q^{\frac{1}{4} n^2} \prod_{k=1}^{\infty} (1 - q^k)
$$

It is the consequence of Jacobi triple product identity as well.

**l twisted charged fermions**

For the twisted boson

$$
i\phi(z) = -\sum_{n \neq 0} \frac{J_{n/l}}{n z_{n/l}} + \frac{1}{l J_0} \log z + Q
$$

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With

\[ [J_{n/l}, J_{m/l}] = n\delta_{n+m,0} \quad [J_0, Q] = 1 \]  \hspace{1cm} (6.250)

one has for \(|z| > |w|\)

\[ \left[ \phi_+(z) - \frac{i}{l} J_0 \log z, \phi_-(w) - iQ \right] = \sum_{n>0} \frac{z^{-n/l} w^{n/l}}{n} - \frac{1}{l} \log z = -\log (z^{1/l} - w^{1/l}) \]  \hspace{1cm} (6.251)

where

\[ i\phi_+(z) = -\sum_{n>0} \frac{J_{n/l}}{nz^{n/l}} \quad i\phi_-(z) = -\sum_{n<0} \frac{J_{n/l}}{nz^{n/l}} \]  \hspace{1cm} (6.252)

Define two operators

\[ \hat{\psi}^* (z) = z^{\frac{1}{2l}} e^{i\phi(z)} := z^{\frac{1}{2l}} e^{i\phi_-(z)} e^{iQ(z) J_0/l} \]

\[ \hat{\psi}(z) = z^{\frac{1}{2l}} e^{-i\phi(z)} := z^{\frac{1}{2l}} e^{-i\phi_-(z)} e^{-iQ(z) J_0/l} \]

with the OPE

\[ \hat{\psi}^* (z) \hat{\psi} (w) = \frac{(zw)^{\frac{1}{2l}}}{z^{1/l} - w^{1/l}} : e^{i\phi(z) - i\phi(w)} := \]

\[ = \frac{(zw)^{\frac{1}{2l}}}{z^{1/l} - w^{1/l}} e^{i\phi_+(z) - i\phi_+(w)} e^{i\phi_-(z) - i\phi_-(w)} \left( \frac{z}{w} \right)^{J_0/l} \]  \hspace{1cm} (6.254)

Then for the modes of their expansion

\[ \hat{\psi}^* (z) = \sum_{k \in \frac{1}{2} + \mathbb{Z}} \frac{\psi^*_{k/l}}{z^{k/l}} \quad \hat{\psi} (z) = \sum_{k \in \frac{1}{2} + \mathbb{Z}} \frac{\psi_{k/l}}{z^{k/l}} \]  \hspace{1cm} (6.255)

one gets canonical anticommutation relations

\[ \{ \psi^*_a, \psi_b \} = \delta_{a+b,0} \]  \hspace{1cm} (6.256)

Now one can express the \(l\)-component fermions in terms of a single twisted boson

\[ \psi^*_\alpha (z) = \frac{1}{\sqrt{l}} z^{-\frac{1}{2}} \hat{\psi}^* (e^{2\pi i\alpha} z), \quad \psi_\alpha (z) = \frac{1}{\sqrt{l}} z^{-\frac{1}{2}} \hat{\psi} (e^{2\pi i\alpha} z), \quad \alpha \in \mathbb{Z}/l\mathbb{Z} \]  \hspace{1cm} (6.257)

and it follows from (6.254), that their OPE is indeed

\[ \psi^*_\alpha (z) \psi_\beta (w) = \frac{\delta_{\alpha\beta}}{z-w} + \text{reg.} \]  \hspace{1cm} (6.258)

The stress-energy tensor and \(U(1)\) current can be extracted from the expansion:

\[ \sum_{\alpha \in \mathbb{Z}/l\mathbb{Z}} \psi^*_\alpha (z + t/2) \psi_\alpha (z - t/2) = \frac{1}{l} + J(z) + tT(z) + \ldots \]  \hspace{1cm} (6.259)
Using (6.253), (6.254) and (6.257) one gets for the l.h.s.
\[
\sum_{\alpha \in \mathbb{Z}/l\mathbb{Z}} \frac{1}{l} \left( z + \frac{l}{2} \right)^{1/2} ( z - \frac{l}{2} )^{1/2} : e^{i\phi(e^{2\pi i\alpha}(z+t/2))} e^{i\phi(e^{2\pi i\alpha}(z-t/2))} : = \sum_{\alpha \in \mathbb{Z}/l\mathbb{Z}} \left( \frac{1}{l} + \frac{l^2 - 1}{24l^2z^2} \right) e^{i\partial \phi(e^{2\pi i\alpha}z)} + O(t^2) = l \left( \frac{1}{l} + \frac{l^2 - 1}{24l^2z^2} \right) \sum_{\alpha \in \mathbb{Z}/l\mathbb{Z}} : \partial \phi(e^{2\pi i\alpha}z) : + O(t^2)
\]

One finds from here
\[
J(z) = \sum_{\alpha \in \mathbb{Z}/l\mathbb{Z}} i\partial \phi(e^{2\pi i\alpha}z) = \sum_{k \in \mathbb{Z}} \frac{J_k}{z^{k+1}}
\]
\[
T(z) = \frac{l^2 - 1}{24l^2z^2} + \frac{1}{l} \sum_{k+n \in \mathbb{Z}} J_n J_k : \frac{1}{z^{n+k+2}}
\]

which already have expansions over integer powers of \( z \). Therefore
\[
L_0 = \frac{l^2 - 1}{24l} + \frac{1}{2l} J_0^2 + \frac{1}{l} \sum_{n > 0} J_{-n} J_n
\]
and the character of this module is given by
\[
\text{tr} \ q^{L_0 + r J_0} = q^{\frac{r^2}{24}} \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{24} + rn} \prod_{n=1}^{\infty} (1 - q^n)
\]

**l charged fermions – standard bosonization**

From the modes (6.255) of the operators \( \hat{\psi}(z) \), \( \hat{\psi}^*(z) \) we can construct another \( l \) fermions
\[
\psi_{(a)}(z) = \frac{1}{\sqrt{l}} \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{\psi_{a+p}}{z^{a+p+1}}, \quad \psi_{(a)}^*(z) = \frac{1}{\sqrt{l}} \sum_{p \in \mathbb{Z} + \frac{1}{2}} \frac{\psi_{-a+p}}{z^{-a+p+1}}
\]
where
\[
a \in \left\{ -\frac{l-1}{2l}, -\frac{l-3}{2l}, \ldots, -\frac{1}{2l} \right\}
\]

These fermions can be bosonized in terms of \( l \) “normal”, untwisted, bosons
\[
\psi_{(a)}^* \langle z \rangle = e^{i\varphi_{(a),\alpha}(z)} e^{i\varphi_{(a),\beta}(z)} e^{Q_{(a)} z - J_{(a),\beta} (z)} \sum_{b < a} J_{(b),\alpha} \langle 0 | a | 0 \rangle
\]
\[
\psi_{(a)} \langle z \rangle = e^{-i\varphi_{(a),\alpha}(z)} e^{-i\varphi_{(a),\beta}(z)} e^{-Q_{(a)} z - J_{(a),\beta} (z)} \sum_{b < a} J_{(b),\alpha} \langle 0 | a | 0 \rangle
\]

where
\[
J_{(a),\alpha} \langle 0 | a | 0 \rangle = a | 0 \rangle
\]
Computation of character in this case gives us

\[
\text{tr} q^{L_0 + r} \sum_{\mathcal{J}_{(a),0}} = \sum_{n_0, \ldots, n_l-1} (\frac{1}{2} - l + 2\sum_{k=0}^{l-1} n_k) + r \sum_{k=0}^{l-1} n_k
\]

(6.268)

One can easily see that equality between (6.263) and (6.268) follows from particular case of (6.223).

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Summary

In this thesis I consider the correspondence between two important objects in mathematical physics: isomonodromic deformations of linear differential equations and two-dimensional conformal field theory with extra nonlinear higher spin symmetry, the W-symmetry. This correspondence relates the isomonodromic tau-functions with correlation functions of the primary fields.

The isomonodromic side of this correspondence deals with systems of linear differential equations in complex plane, their monodromy, and the non-linear Schlesinger equations, describing monodromy-preserving deformations of these linear systems. Solutions of such equations are highly transcendental functions: the simplest representatives from this family are the Painlevé transcendents – solutions of some non-linear non-autonomous second order differential equations.

The field-theoretic side of the correspondence contains the computation of correlation functions and characters in two-dimensional holomorphic field theories with extended conformal symmetry, the W-symmetry. The mathematical structures, considered in the thesis, contain the vertex operator algebras (Kac-Moody, Virasoro, Zamolodchikov’s W-algebras and Clifford algebras), their characters and conformal blocks. The construction of the exact conformal blocks involves computations in the algebraic geometry of complex curves – two-dimensional real surfaces with a complex structure that are constructed as a ramified covers of the complex sphere.

Among the field-theoretic results I present explicit constructions of the vertex operators (primary fields) for the W-algebras, for particular cases their correlators are found in a closed form. Such exactly-solvable cases have been connected with the known tau-functions for isomonodromic problems. One part of results can be considered as a continuation of the work of Sato, Miwa and Jimbo, or as a generalization of the work of Gamayun, Iorgov, Lisovyy and Teschner to higher rank. Another part can be considered as a generalization of the work of Knizhnik and Zamolodchikov to higher rank.

On the isomonodromic side the explicit Fredholm determinant representation for generic isomonodromic tau-function is obtained, which in already known cases reproduces the field-theoretic expression. In some particular cases this result answers the question inspired by the existence of a large collection of Fredholm determinant solutions of Painlevé equations: “Is there a Fredholm determinant representation for general solutions of the Painlevé VI equation?”. Such a representation is given with the use of matrix integral kernels of size two with hypergeometric functions inside. The rigorous proof of this result about general Fredholm determinant does not use any field-theoretic technique and has been done using ideas in a spirit of Palmer’s work. Study of the minor expansion of the Fredholm determinant expression gives as a byproduct, for the case of integer central charges, an independent proof of the AGT correspondence, the correspondence between two-dimensional conformal field theories and four-dimensional supersymmetric field theories.
Samenvatting

In dit proefschrift bestudeer ik een verband tussen twee belangrijke objecten uit de mathematische fysica: aan de ene kant zijn dat deformaties van lineaire differentiaalvergelijkingen die de monodromie van het lineaire stelsel behouden, aan de andere kant vinden we de twee-dimensionale conforme veldentheorie die een extra niet-lineaire spin symmetrie bezitten, de zogenaamde W-symmetrie. Het gevonden verband koppelt de tau-functies behorende bij de isomonodromie met de correlatie functies van de primaire velden.

Aan de isomonodromie kant van de correspondentie heeft men te maken met lineaire differentiaalvergelijkingen in het complexe vlak, hun monodromie, d.w.z. de verandering van oplossingen van het lineaire stelsel bij analytische voortzetting langs gesloten krommen, en de niet-lineaire Schlesinger vergelijkingen, die de monodromie-behoudende deformaties beschrijven. De oplossingen van deze laatste vergelijkingen zijn i.h.a. transcendent funkties. De eenvoudigste voorbeelden zijn de zgn. Painlevé transcendenten, oplossingen van bepaalde niet-lineaire, niet-autonome tweede orde vergelijkingen die gevonden zijn door Painlevé.

Aan de veldentheorie kant van de correspondentie is het berekenen van correlatie functies en karakters in twee-dimensionale holomorfe veldentheorieën met W-symmetrie een centraal thema. Onder de wiskundige structuren die een rol spelen in dit proefschrift, vind je diverse vertex operator algebra’s (Kac-Moody, Virasoro, Zamolodchikov’s W-algebra’s en Clifford algebra’s), hun karakters en de conforme blokken. De constructie van de exacte conforme blokken vereist algebraisch meetkundige berekeningen voor complexe krommen. Denk hierbij aan twee-dimensionale reële oppervlakken die vertakte overdekkingen zijn van de complexe bol.


Voor het onderzoek naar isomonodromie levert dit proefschrift voor generieke, aan isomonodromie gekoppelde tau-functies een expliciete voorstelling als een Fredholm determinant. In reeds bekende gevallen reproduceert de gevonden uitdrukking, degene die reeds gevonden is in de veldentheorie. In speciale gevallen beantwoordt dit resultaat een vraag, geïnspireerd door het feit dat een groot aantal oplossingen van Painlevé vergelijkingen uit te drukken zijn als een Fredholm determinant: “Bestaat er voor een algemene oplossing van de Painlevé VI vergelijking een uitdrukking als een Fredholm determinant?”

Merk op dat het bewijs van de Fredholm determinant uitdrukking geen technieken uit de veldentheorie vereist, maar geïnspireerd is door werk van Palmer. De gevonden uitdrukking leidt voor geheeltallige centrale ladingen nog tot een interessant bijkomend resultaat: een analyse van de minor ontwikkeling van de Fredholm determinant uitdrukking levert een onafhankelijk bewijs van de “AGT correspondence”, een verband tussen twee-dimensionale conforme veldentheorieën en vier-dimensionale supersymmetrische veldentheorieën.